MPRI 2–2 Final exam, 5/3/2020

Authorized documents: all documents, no electronic devices. You may answer the questions in French or English.

NB:

- A few questions are more difficult, they are highlighted by a "*". Of course additional points will be associated with these questions.
- Questions are written in such a way that you can easily skip them if you wish. However for solving a question you may need results stated in earlier questions.

1) Let M be the following term of PCF:

$$M = \mathsf{fix}(\lambda x^{\iota}\, \underline{\mathsf{succ}}(x))$$

1.1) Provide a typing derivation showing that $\vdash M : \iota$.

1.2) Prove that $[M] = \emptyset$ (in the relational model).

1.3) Give a typing derivation and compute the relational semantics of the term

$$\lambda f^{\iota \to \iota}\, (f)\, M\,.$$

[*Hint:* You can use the "intersection type system" presented during the lectures for computing the semantics, Section 7.2.4 in the Lecture Notes.]

2) We record that a $t \in \mathbf{Rel}(E, F)$ is an isomorphism in $\mathbf{Rel}$ (that is there is $t' \in \mathbf{Rel}(F, E)$ such that $t'\, t = \mathsf{Id}$ and $t\, t' = \mathsf{Id}$) if and only if t is a bijection (identified with its graph, that is, there is a bijection $f : E \to F$ such that $t = \{(a, f(a)) \mid a \in E\}$).

Given a set E, we define $!^{\mathsf{T}}E$ as the least set such that

- $0 \in\, !^{\mathsf{T}}E$
- if $a \in E$ then $(1, a) \in\, !^{\mathsf{T}}E$
- and if $\sigma, \tau \in\, !^{\mathsf{T}}E$ then $(2, (\sigma, \tau)) \in\, !^{\mathsf{T}}E$.

To increase readability, we use the following notations: $\langle\rangle = 0$, $\langle a \rangle = (1, a)$ (for $a \in E$) and $\langle \sigma, \tau \rangle = (2, (\sigma, \tau))$. An element of $!^{\mathsf{T}}E$ can be seen as a binary tree with two kind of leaves: "empty leaves" $\langle\rangle$ and "singleton leaves" $\langle a \rangle$ labeled by an element a of E. The main tool of reasoning with such trees is of course induction on their size or structure.

The goal of this exercise is to show that $!^{\mathsf{T}}$ is "almost" an exponential on $\mathbf{Rel}$.

2.1) Given $t \in \mathbf{Rel}(E, F)$, we define $!^{\mathsf{T}}t$ as the least subset of $!^{\mathsf{T}}E \multimap\, !^{\mathsf{T}}F$ such that

- $(\langle\rangle, \langle\rangle) \in\, !^{\mathsf{T}}t$
- $(a, b) \in t \Rightarrow (\langle a \rangle, \langle b \rangle) \in\, !^{\mathsf{T}}t$
- $(\sigma_1, \tau_1), (\sigma_2, \tau_2) \in\, !^{\mathsf{T}}t \Rightarrow (\langle \sigma_1, \sigma_2 \rangle, \langle \tau_1, \tau_2 \rangle) \in\, !^{\mathsf{T}}t$.

Prove that $!^{\mathsf{T}}$ is a functor. [*Hint:* Let $s \in \mathbf{Rel}(E, F)$ and $t \in \mathbf{Rel}(F, G)$. By induction on $\sigma \in\, !^{\mathsf{T}}E$ prove that for any $\varphi \in\, !^{\mathsf{T}}G$ one has $(\sigma, \varphi) \in (!^{\mathsf{T}}t)\,(!^{\mathsf{T}}s) \Leftrightarrow (\sigma, \varphi) \in\, !^{\mathsf{T}}(t\, s)$. Of course one can also use an induction on φ.]

2.2) We define $\mathsf{der}^{\mathsf{T}}_E \in \mathbf{Rel}(!^{\mathsf{T}}E, E)$ by $\mathsf{der}^{\mathsf{T}}_E = \{(\langle a \rangle, a) \mid a \in E\}$. Prove that it is a natural transformation $!^{\mathsf{T}} \Rightarrow \mathsf{Id}$. [*Hint:* For this, consider $t \in \mathbf{Rel}(E, F)$ and $(\sigma, b) \in\, !^{\mathsf{T}}E \times F$. By induction on σ, prove that $(\sigma, b) \in \mathsf{der}^{\mathsf{T}}_F\,(!^{\mathsf{T}}t) \Leftrightarrow (\sigma, b) \in t\, \mathsf{der}^{\mathsf{T}}_E$. You will see in particular that when σ is not of shape $\langle a \rangle$ for some $a \in E$, the two sides of this equivalence are false and hence the equivalence holds trivially.]

2.3) We define a function $\mathsf{flat} : !^{\mathsf{T}}!^{\mathsf{T}}E \to !^{\mathsf{T}}E$ by induction on trees as follows:

- $\mathsf{flat}(\langle\rangle) = \langle\rangle$,
- $\mathsf{flat}(\langle\sigma\rangle) = \sigma$ and
- $\mathsf{flat}(\langle\Sigma_1, \Sigma_2\rangle) = \langle\mathsf{flat}(\Sigma_1), \mathsf{flat}(\Sigma_2)\rangle$.

We define $\mathsf{digg}^{\mathsf{T}}_E \in \mathbf{Rel}(!^{\mathsf{T}}E, !^{\mathsf{T}}!^{\mathsf{T}}E)$ by $\mathsf{digg}^{\mathsf{T}}_E = \{(\mathsf{flat}(\Sigma), \Sigma) \mid \Sigma \in !^{\mathsf{T}}!^{\mathsf{T}}E\}$. Prove that $\mathsf{digg}^{\mathsf{T}}$ is a natural transformation $!^{\mathsf{T}} \Rightarrow (!^{\mathsf{T}} \circ !^{\mathsf{T}})$. [*Hint:* Let $t \in \mathbf{Rel}(E, F)$ and $(\sigma, b) \in !^{\mathsf{T}}E \times F$. By induction on $\Theta \in !^{\mathsf{T}}!^{\mathsf{T}}F$, prove that for all $\sigma \in !^{\mathsf{T}}E$, one has $(\sigma, \Theta) \in \mathsf{digg}^{\mathsf{T}}_F\,(!^{\mathsf{T}}t) \Leftrightarrow (\sigma, \Theta) \in (!^{\mathsf{T}}!^{\mathsf{T}}t)\,\mathsf{digg}^{\mathsf{T}}_E$.]

2.4) Prove that $(!^{\mathsf{T}}, \mathsf{der}^{\mathsf{T}}, \mathsf{digg}^{\mathsf{T}})$ is a comonad.

2.5) We record that, given sets $(E_i)_{i\in I}$, their cartesian product $E = \&_{i\in I} E_i$ in $\mathbf{Rel}$ is defined by $E = \cup_{i\in I}(\{i\} \times E_i)$. For each $i \in I$ we define a function $\mathsf{p}^{\mathsf{T}}_i : !^{\mathsf{T}}E \to !^{\mathsf{T}}E_i$ by induction as follows:

- $\mathsf{p}^{\mathsf{T}}_i(\langle\rangle) = \langle\rangle$,
- $\mathsf{p}^{\mathsf{T}}_i(\langle(i, a)\rangle) = \langle a\rangle$,
- $\mathsf{p}^{\mathsf{T}}_i(\langle(j, a)\rangle) = \langle\rangle$ if $j \neq i$,
- and $\mathsf{p}^{\mathsf{T}}_i(\langle\sigma_1, \sigma_2\rangle) = \langle\mathsf{p}^{\mathsf{T}}_i(\sigma_1), \mathsf{p}^{\mathsf{T}}_i(\sigma_2)\rangle$.

Provide a counter-example showing that it is not true that the function $\mathsf{p}^{\mathsf{T}}_i$ coincides (as a graph) with $!^{\mathsf{T}}\mathsf{pr}_i \in \mathbf{Rel}(!^{\mathsf{T}}E, !^{\mathsf{T}}E_i)$, where $\mathsf{pr}_i \in \mathbf{Rel}(E, E_i)$ is the i-th projection of the cartesian product, that is $\mathsf{pr}_i = \{((i, a), a) \mid a \in E\}$.

2.6) Then we define $\mathsf{m}^{\mathsf{T}}_{E_1,E_2} \in \mathbf{Rel}(!^{\mathsf{T}}E_1 \otimes !^{\mathsf{T}}E_2, !^{\mathsf{T}}(E_1 \,\&\, E_2))$ as

$$\mathsf{m}^{\mathsf{T}}_{E_1,E_2} = \{((\mathsf{p}^{\mathsf{T}}_1(\theta), \mathsf{p}^{\mathsf{T}}_2(\theta)), \theta) \mid \theta \in !^{\mathsf{T}}(E_1 \,\&\, E_2)\}\,.$$

We admit that this morphism is natural in E_1 and E_2. Provide a counter-example showing that $\mathsf{m}^{\mathsf{T}}_{E_1,E_2}$ is not an isomorphism in general.

2.7) Prove that the following diagram commutes (lax monoidality).

$$\begin{array}{ccc}
(!^{\mathsf{T}}E_1 \otimes !^{\mathsf{T}}E_2) \otimes !^{\mathsf{T}}E_3 & \xrightarrow{\quad\alpha\quad} & !^{\mathsf{T}}E_1 \otimes (!^{\mathsf{T}}E_2 \otimes !^{\mathsf{T}}E_3) \\
\downarrow{\scriptstyle \mathsf{m}^{\mathsf{T}}_{E_1,E_2} \otimes \mathsf{Id}} & & \downarrow{\scriptstyle \mathsf{Id} \otimes \mathsf{m}^{\mathsf{T}}_{E_2,E_3}} \\
!^{\mathsf{T}}(E_1 \,\&\, E_2) \otimes !^{\mathsf{T}}E_3 & & !^{\mathsf{T}}E_1 \otimes !^{\mathsf{T}}(E_2 \,\&\, E_3) \\
\downarrow{\scriptstyle \mathsf{m}^{\mathsf{T}}_{E_1\&E_2,E_3}} & & \downarrow{\scriptstyle \mathsf{m}^{\mathsf{T}}_{E_1,E_2\&E_3}} \\
!^{\mathsf{T}}((E_1 \,\&\, E_2) \,\&\, E_3) & \xrightarrow{!^{\mathsf{T}}(\langle \mathsf{pr}_1\,\mathsf{pr}_1, \langle \mathsf{pr}_2\,\mathsf{pr}_1, \mathsf{pr}_2\rangle\rangle)} & !^{\mathsf{T}}(E_1 \,\&\, (E_2 \,\&\, E_3))
\end{array}$$

[*Hint:* Define two functions $f, g : !^{\mathsf{T}}(E_1 \,\&\, (E_2 \,\&\, E_3)) \to (!^{\mathsf{T}}E_1 \otimes !^{\mathsf{T}}E_2) \otimes !^{\mathsf{T}}E_3$ allowing to describe simply the two morphisms that have to be proven equal. Prove that these two functions are equal.]

2.8) We define a function $\mathsf{ms} : !^{\mathsf{T}}E \to !E$ (where $!E$ is the set $\mathcal{M}_{\mathrm{fin}}(E)$ of finite multisets of elements of E, the exponential on $\mathbf{Rel}$ presented during the lectures) as follows:

- $\mathsf{ms}(\langle\rangle) = []$,
- $\mathsf{ms}(\langle a\rangle) = [a]$ and
- $\mathsf{ms}(\langle\sigma, \tau\rangle) = \mathsf{ms}(\sigma) + \mathsf{ms}(\tau)$.

We define $\mathsf{ms}_E = \{(\sigma, \mathsf{ms}(\sigma)) \mid \sigma \in !^{\mathsf{T}}E\}$. Prove that this is a natural transformation $!^{\mathsf{T}} \Rightarrow !$.

2.9) Prove that the following diagrams are commutative

$$
\begin{array}{ccc}
!^{\top}!^{\top}E & \xrightarrow{\;!^{\top}\mathsf{ms}_E\;} & !^{\top}!E \\
{\scriptstyle \mathsf{ms}_{!^{\top}E}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{ms}_{!E}} \\
!!^{\top}E & \xrightarrow{\;!\mathsf{ms}_E\;} & !!E
\end{array}
\qquad\qquad
\begin{array}{ccc}
!^{\top}E & \xrightarrow{\;\mathsf{digg}^{\top}E\;} & !^{\top}!^{\top}E \\
{\scriptstyle \mathsf{ms}_E}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
!E & \xrightarrow{\;\mathsf{dig}_E\;} & !!E
\end{array}
$$

where m is the morphism $!^{\top}!^{\top}E \to !!E$ defined in two different ways by the left hand diagram.

3) Remember that a coherence space E is a pair $(|E|, \mathrel{\overset{\frown}{\smile}}_E)$ where $|E|$ is a set (the web) and $\mathrel{\overset{\frown}{\smile}}_E$ is a binary symmetric and reflexive relation on $|E|$ (coherence relation), and that the cliques of E form a domain that we will denote as $\mathsf{Cl}(E)$. Remember that $\frown_E$ is the strict coherence relation: $a \frown_E b$ if $a \neq b$ and $a \frown_E b$.

Remember also that, given coherence spaces E and F one defines a coherence space $E \multimap F$ whose cliques are the linear morphisms from E to F ($|E \multimap F| = |E| \times |F|$, $(a_1, a_2) \mathrel{\overset{\frown}{\smile}}_{E \multimap F} (b_1, b_2)$ if $a_1 \mathrel{\overset{\frown}{\smile}}_E a_2 \Rightarrow b_1 \mathrel{\overset{\frown}{\smile}}_F b_2$ and $a_1 \frown_E a_2 \Rightarrow b_1 \frown_F b_2$). We use **Coh** for the category of coherence spaces and linear maps, composition being defined as relational composition and identities being the diagonal relations. We also write $t : E \multimap F$ when $t \in \mathsf{Cl}(E \multimap F) = \mathbf{Coh}(E, F)$.

Given coherence spaces E and F, we say that a function $f : |E| \to |F|$ is an *embedding* if

- f is injective
- and $\forall a, a' \in |E|\ a \mathrel{\overset{\frown}{\smile}}_E a' \Leftrightarrow f(a) \mathrel{\overset{\frown}{\smile}}_F f(a')$. [Warning: this has to be an equivalence, not a simple implication!]

We use $\mathbf{Coh}^{\mathsf{e}}$ for the category of coherence spaces and embeddings. We write $f : E \lhd F$ when $f \in \mathbf{Coh}^{\mathsf{e}}(E, F)$.

Let $S = (E_n, f_n)_{n \in \mathbb{N}}$ be a family where the E_n are coherence spaces and $f_n : E_n \lhd E_{n+1}$. Such a family will be called an *embedding system*. If $n, p \in \mathbb{N}$ with $n \leq p$, we set $f_{n,p} = f_{p-1} \circ \cdots \circ f_n : E_n \lhd E_p$. In particular $f_{n,n} = \mathsf{Id}$.

Let $A = \cup_{n \in \mathbb{N}}(\{n\} \times |E_n|)$. We say that an element (n, a) of A is *root* if $n = 0$, or if $n > 0$ and there is no $a' \in |E_{n-1}|$ such that $f_{n-1}(a') = a$, or equivalently $a \in |E_n| \setminus f_{n-1}(|E_{n-1}|)$. Let A_0 be the set of all root elements of A.

3.1) Prove that for any $(n, a) \in A$ there is exactly one $(p, b) \in A_0$ such that $p \leq n$ and $f_{p,n}(b) = a$. We set $\mathsf{root}(n, a) = (p, b)$. [*Hint:* By induction on n, prove that the property holds for all $a \in |E_n|$.]

We define a "limit" coherence space $E = \mathsf{Lim}\, S$ by taking $|E| = A_0$ and coherence specified as follows. Let $(n, a), (p, b) \in |E| = A_0$. We say that $(n, a) \mathrel{\overset{\frown}{\smile}}_E (p, b)$ if

- $n = p$ and $a \mathrel{\overset{\frown}{\smile}}_{E_n} b$
- or $n < p$ and $f_{n,p}(a) \mathrel{\overset{\frown}{\smile}}_{E_p} b$ (notice that, in that case, necessarily $f_{n,p}(a) \neq b$ because b is root)
- or $n > p$ and $a \mathrel{\overset{\frown}{\smile}}_{E_n} f_{p,n}(b)$ (similar remark).

3.2) For each $n \in \mathbb{N}$, prove that the function $g_n : |E_n| \to |E|$ defined by $g_n(a) = \mathsf{root}(n, a)$ is an injection.

3.3) Prove that $g_n : E_n \lhd E$.

We consider now three examples of embedding systems. Let the sequence $(E_n)_{n \in \mathbb{N}}$ of coherence spaces be defined as follows: $E_0 = \top$ (the coherence space such that $|\top| = \emptyset$) and $E_{n+1} = (1 \mathbin{\&} (1 \oplus E_n))$. In other words (up to an isomorphism) $|E_n| = \{1, \ldots, n\} \times \{1, -1\}$ and $(i, \varepsilon) \mathrel{\overset{\frown}{\smile}}_{E_n} (i', \varepsilon')$ if

- $i < i'$ and $\varepsilon = 1$
- or $i' < i$ and $\varepsilon' = 1$
- or $i = i'$.

Hence (with the notations above), $A = \{(n, i, \varepsilon) \mid n, i \in \mathbb{N},\ 1 \leq i \leq n \text{ and } \varepsilon \in \{1, -1\}\}$.

3.4) Let $n, p \in \mathbb{N}$ with $n \leq p$ and let $\varphi : \{1, \dots, n\} \to \{1, \dots, p\}$ be an injection. Let $f : |E_n| \to |E_p|$ be defined by $f(i, \varepsilon) = (\varphi(i), \varepsilon)$. Prove that $f : E_n \lhd E_p$ if and only if φ is monotone (that is $i \leq j \Rightarrow \varphi(i) \leq \varphi(j)$).

3.5) We define $S = (E_n, f_n)_{n \in \mathbb{N}}$ where $f_n(i, \varepsilon) = (i, \varepsilon)$ for all $(i, \varepsilon) \in |E_n|$. Prove that each f_n is an embedding and that an element (n, i, ε) is root if and only if $i = n$. [*Hint:* By induction on $n \in \mathbb{N}$ prove that for all $i \in \{1, \dots, n\}$, (n, i, ε) is root if and only if $i = n$.].

3.6) For $(n, i, \varepsilon) \in A$ (so that $1 \leq i \leq n$) prove that $\mathsf{root}(n, i, \varepsilon) = (i, i, \varepsilon)$.

3.7) Let $E = \mathsf{Lim}\, S$, we can identify $|E|$ with $\mathbb{N} \times \{1, -1\}$. With this identification, prove that $(i, \varepsilon) \coh_E (i', \varepsilon')$ if

- $i < i'$ and $\varepsilon = 1$
- or $i' < i$ and $\varepsilon' = 1$
- or $i = i'$.

3.8) We define another embedding system $T = (E_n, g_n)_{n \in \mathbb{N}}$ where $g_n(i, \varepsilon) = (i + 1, \varepsilon)$ for all $(i, \varepsilon) \in |E_n|$. Prove that each g_n is an embedding and that an element (n, i, ε) is root if and only if $i = 1$.

3.9) Prove that for any $(n, i, \varepsilon) \in A$, one has $\mathsf{root}(n, i, \varepsilon) = (n - i + 1, 1, \varepsilon)$.

3.10) Let $F = \mathsf{Lim}\, T$. We identify $|F|$ with $\mathbb{N} \times \{1, -1\}$ by mapping a root element $(n, 1, \varepsilon) \in A$ to $(n, \varepsilon) \in \mathbb{N} \times \{1, -1\}$. With this identification, prove that $(n, \varepsilon) \coh_F (n', \varepsilon')$ if and only if

- $n = n'$ or
- $n > n'$ and $\varepsilon = 1$ or
- $n' > n$ and $\varepsilon' = 1$.

*3.11)** Prove that $\mathsf{Lim}\, S$ and $\mathsf{Lim}\, T$ are not isomorphic (an isomorphism from a coherence space from E to F is the same thing as an embedding $E \lhd F$ which, as a function, is a bijection).

*3.12)** As in questions (3.5)–(3.7) and (3.8)–(3.10), work out the following example: let $H_n = E_{2^n}$ and $h_n : H_n \lhd H_{n+1}$ be defined by $h_n(i, \varepsilon) = (2i, \varepsilon)$. Let $U = (H_n, h_n)_{n \in \mathbb{N}}$. Prove that $G = \mathsf{Lim}\, U$ can be described as follows: $|G| = \{r \in \mathbb{D} \mid r > 0\} \times \{1, -1\}$ where $\mathbb{D}$ is the set of rational numbers which can be written $\frac{k}{2^n}$ (dyadic numbers) and $(r, \varepsilon) \coh_G (r', \varepsilon')$ if

- $r = r'$
- or $r < r'$ and $\varepsilon = 1$
- or $r' < r$ and $\varepsilon' = 1$.

*3.13)** Prove that $\mathsf{Lim}\, U$ is neither isomorphic to $\mathsf{Lim}\, S$ nor to $\mathsf{Lim}\, T$.

4)

4.1) Given probabilistic coherence spaces (PCS for short) X, Y and Z and $t \in \mathbb{R}_{\geq 0}^{|(X \otimes !Y) \multimap Z|}$, prove that $t \in \mathbf{Pcoh}(X \otimes !Y, Z)$ if and only if

$$\forall u \in \mathsf{P}(X)\, \forall v \in \mathsf{P}(Y) \quad t \cdot (u \otimes v^{(!)}) \in \mathsf{P}(Z)\,.$$

We use e_i for the element of $\mathbb{R}_{\geq 0}^I$ such that $(\mathsf{e}_i)_j = \delta_{i,j}$ ($= 1$ if $i = j$ and 0 otherwise).

Given an at most countable set I, we use $\overline{I}$ for the probabilistic coherence space such that $|\overline{I}| = I$ and $\mathsf{P}(\overline{I}) = \{u \in \mathbb{R}_{\geq 0}^I \mid \sum_{i \in I} u_i \leq 1\}$. Notice that (up to a trivial isomorphism) $\overline{I} = \bigoplus_{i \in I} 1$. Let $B = \{0, 1\}$ and let $W = B^*$ (the set of finite sequences of elements of B).

4.2) For each $w \in W$ we define a function $f_w : \mathsf{P}(\overline{B}) \to \mathbb{R}_{\geq 0}^B$ by induction on w (using $\langle\rangle$ for the empty word and aw for prefixing $w \in W$ with $a \in B$): for all $u \in \mathsf{P}(\overline{B})$,

- $f_{\langle\rangle}(u) = \mathsf{e}_0$
- $f_{0w}(u) = u_0 f_w(x) + u_1 \mathsf{e}_1$
- $f_{1w}(u) = u_0 \mathsf{e}_0 + u_1 f_w(x)$

By induction on w, prove that there is a family $(t_w)_{w \in W}$ of elements of $\mathsf{P}(!\overline{B} \multimap \overline{B})$ such that

$$\forall u \in \mathsf{P}(\overline{B}) \quad f_w(u) = t_w \cdot u^{(!)} \,.$$

[*Hint:* Prove first that if $s \in \mathsf{P}(!\overline{B} \multimap \overline{B})$ and $i \in B$ then $s^{(i)} \in \mathbb{R}_{\geq 0}^{|!\overline{B} \multimap \overline{B}|}$ defined by $s^{(i)}_{m,b} = s_{m+[i],b}$ satisfies $s^{(i)} \in \mathsf{P}(!\overline{B} \multimap \overline{B})$ and $\forall u \in \mathsf{P}(B)\ s^{(i)} \cdot u^{(!)} = u_i(s \cdot u^{(!)})$.]

4.3) Prove that there is a morphism $t \in \mathbf{Pcoh}(\overline{W} \otimes !\overline{B}, \overline{B})$ such that $\forall w \in W\, \forall u \in \mathsf{P}(B) \quad t \cdot (\mathsf{e}_w \otimes u^{(!)}) = f_w(u)$.

4.4) Given a word $w \in W$ let $\mathsf{len}(w)$ be its length and $\mathsf{nb}(w)$ the number w represents in binary notation (if w is the word $a_{n-1} \cdots a_0$ then $\mathsf{len}(w) = n$ and $\mathsf{nb}(w) = \sum_{i=0}^{n-1} a_i 2^i$ so that $0 \leq \mathsf{nb}(w) \leq 2^n - 1$). Prove that

$$\forall w \in W \quad f_w(\frac{1}{2}\mathsf{e}_0 + \frac{1}{2}\mathsf{e}_1) = \frac{\mathsf{nb}(w)+1}{2^{\mathsf{len}(w)}}\mathsf{e}_0 + \left(1 - \frac{\mathsf{nb}(w)+1}{2^{\mathsf{len}(w)}}\right)\mathsf{e}_1$$

4.5) Explain the usefulness of the functions f_w in a programming language where the only available random number generator produces $\underline{0}$ with probability $\frac{1}{2}$ and $\underline{1}$ with probability $\frac{1}{2}$, for instance a version of our pPCF where $\mathsf{rand}(r)$ is available only for $r = \frac{1}{2}$.