MPRI 2–2 Models of programming languages: domains, categories, games

Exercises

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The signs (*) and (**) try to indicate more difficult and interesting questions. These are of course completely subjective indications!

1. Let $E$ be a coherence space. We use $\text{Cl}(E)$ for the set of all cliques of $E$. We say that $X \subseteq \text{Cl}(E)$ is summable if $\forall x, y \in X \land x \neq y \Rightarrow x \cap y = \emptyset$ and $\bigcup X \in \text{Cl}(E)$. Let $E, F$ be coherence spaces and $f : \text{Cl}(E) \to \text{Cl}(F)$ be a function. Prove that $f$ is linear iff for all summable $X \subseteq \text{Cl}(E)$, one has that $f(X) = \{ f(x) \mid x \in X \}$ is summable and $\bigcup f(X) = f(\bigcup X)$.

2. This exercise develops a somehow degenerate model of Linear Logic which does not satisfy *-autonomy but satisfies all the other requirements. A pointed set is a structure $X = (X, 0_X)$ where $X$ is a set and $0_X \in X$. Given pointed sets $X, X_1, X_2$ and $Y$,

- a morphism of pointed sets from $X$ to $Y$ is a function $f : X \to Y$ such that $f(0_X) = 0_Y$,
- and a bimorphism of pointed sets from $X_1, X_2$ to $Y$ is a function $f : X_1 \times X_2 \to Y$ such that $f(0_{X_1}, x_2) = f(x_1, 0_{X_2}) = 0_Y$ for each $x_1 \in X_1$ and $x_2 \in X_2$.

(a) Prove that pointed sets together with morphisms of pointed sets form a category $\text{Set}_0$. What are the isos in that category?

One sets $1 = \{(0, \ast)\}$ where $\ast$ and $0$ are are distinct chosen elements (for instance $0$ is the integer $0$ and $\ast$ is the integer $1$). Given pointed sets $X_1$ and $X_2$ one defines $X_1 \otimes X_2$ as follows:

$$X_1 \otimes X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 = 0_{X_1} \Leftrightarrow x_2 = 0_{X_2}\} \quad \text{and} \quad 0_{X_1 \otimes X_2} = (0_{X_1}, 0_{X_2}).$$

Given $x_i \in X_i$ for $i = 1, 2$, one defines

$$x_1 \otimes x_2 = \begin{cases} (0_{X_1}, 0_{X_2}) & \text{if } x_1 = 0_{X_1} \text{ or } x_2 = 0_{X_2} \\ (x_1, x_2) & \text{otherwise.} \end{cases}$$

(b) Prove that the function $(x_1, x_2) \mapsto x_1 \otimes x_2$ is a bimorphism from $X_1, X_2$ to $X_1 \otimes X_2$ which is surjective as a function $X_1 \times X_2 \to X_1 \otimes X_2$ and that for any bimorphism $f$ from $X_1, X_2$ to $Y$ there is exactly one morphism $f \in \text{Set}_0(X_1 \otimes X_2, Y)$ such that $f(x_1, x_2) = f(x_1 \otimes x_2)$ for all $x_1 \in X_1$ and $x_2 \in X_2$.

(c) Given $f_i \in \text{Set}_0(X_i, Y_i)$ for $i = 1, 2$, deduce from the above that there is exactly one morphism $f_1 \otimes f_2 \in \text{Set}_0(X_1 \otimes X_2, Y_1 \otimes Y_2)$ such that

$$\forall x_1 \in X_1 \forall x_2 \in X_2 \quad (f_1 \otimes f_2)(x_1 \otimes x_2) = f_1(x_1) \otimes f_2(x_2).$$

(d) Using again the universal property of Question (b) prove that the operation on morphisms defined in Question (c) is a functor.
(e) Exhibit isomorphisms \( \lambda_X \in \textbf{Set}_0(1 \otimes X, X) \) and \( \alpha_{X_1, X_2, X_3} \in \textbf{Set}_0((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3)) \).

So \( \textbf{Set}_0 \) is an SMC (there is a symmetry iso \( \gamma_{X_1, X_2} \in \textbf{Set}_0(X_1 \otimes X_2, X_2 \otimes X_1) \) such that \( \gamma_{X_1, X_2}(x_1 \otimes x_2) = x_2 \otimes x_1 \) which is quite easy to define, and the Mac Lane coherence diagrams commute).

(f) One defines \( X \rightarrow Y \) by \( X \rightarrow Y = \textbf{Set}_0(X, Y) \) and for \( 0 \rightarrow X \) we take the function such that \( 0_{X \rightarrow Y}(x) = 0_Y \) for all \( x \in X \). Let \( e : X \rightarrow Y \times X \rightarrow Y \) be defined by \( e(f, x) = f(x) \). Prove that \( e \) is a bimorphism and that the SMC \( \textbf{Set}_0 \) is closed.

(g) Prove that there is no object \( \bot \) of \( \textbf{Set}_0 \) which turns this symmetric monoidal closed category into a \( * \)-autonomous category.

(h) Given a family \( (X_i)_{i \in I} \) of objects of \( \textbf{Set}_0 \) we define an object \( X \) as follows: \( X = \prod_{i \in I} X_i \) and \( 0_X = (0_X)_{i \in I} \in \prod_{i \in I} X_i \) so that the the projections \( \pi_i : X \rightarrow X_i \) are obviously morphisms of \( \textbf{Set}_0 \).

Prove that \( X \), together with these projections, is the cartesian product of the family \( (X_i)_{i \in I} \) that we denote as \( \prod_{i \in I} X_i \).

Notice that the terminal object (which is the product of an empty family of objects) is \( \top = (\{0\}, 0_\cdot) \).

Contrarily to \( \textbf{Rel} \), the category \( \textbf{Set}_0 \) has all (projective) limits. It seems rather difficult to build \( * \)-autonomous categories which are at the same time complete. A noticeable exception is the category of complete lattices (next exercise).

Given an object \( X \) of \( \textbf{Set}_0 \), we define \( !X \) by \( !X = \{0, 0_\cdot\} \cup \{1\} \times X \) where \( 0_\cdot \) is a chosen element (for instance, a given integer) and \( 0_X = (0, 0_\cdot) \). Notice that \( (1, 0_X) \in !X \) but \( 0_X \neq (1, 0_X) \). So \( !X \) is just \( X \) to which we have added a new 0-element.

Given \( f \in \textbf{Set}_0(X, Y) \), we define \( !f \in \textbf{Set}_0(!X, !Y) \) by \( !f(0_X) = 0_Y \) and \( !f(1, x) = (1, f(x)) \). This obviously defines a functor \( \textbf{Set}_0 \rightarrow \textbf{Set}_0 \).

(i) We define \( \text{der}_X : \text{Set}_0(!X, X) \) by \( \text{der}_X(0_X) = 0_X \) and \( \text{der}_X(1, x) = x \). Prove that this is a natural transformation.

(j) We define \( \text{dig}_X \in \text{Set}_0(!X, !X) \) by \( \text{dig}_X(0, 0_\cdot) = (0, 0_\cdot) \), that is \( \text{dig}_X(0, 0_\cdot) = 0_{0, X}, \) and \( \text{dig}_X(1, x) = (1, (1, x)) \) which is easily seen to be a natural transformation. Prove that equipped with the natural transformations \( \text{der} \) and \( \text{dig} \) the functor \( !\_ \) is a comonad.

(k) Given two objects \( X \) and \( Y \) of \( \textbf{Set}_0 \), exhibit an isomorphism between \( !\,(X \& Y) \) and \( !X \otimes !Y \).

(l) Prove that the Kleisli category of “!” can be identified with the category whose objects are pointed sets and morphisms are arbitrary functions (not necessarily preserving the 0 element).

3. In this exercise we study a model of linear logic which is based on complete sup-semilattices and linear maps. A complete sup-semilattice (most often we will simply say “sup semilattice”) is a partially ordered set \( S \) (the order relation will always be denoted as \( \leq \) or \( \leq_S \) if required) such that any subset \( A \) of \( S \) has a least upper bound \( \bigvee A \in S \) (also called “lub”, “sup” or “supremum”). Remember that this means

\[
\forall x \in A \ x \leq \bigvee A
\]
\[
\forall x \in S \ (\forall y \in A \ y \leq x) \Rightarrow \bigvee A \leq x.
\]

In particular we have two elements \( 0 = \bigvee \emptyset \) which is the least element of \( S \) and \( 1 = \bigvee S \) which is the greatest element of \( S \). In particular, a sup-semilattice is never empty.

A subset \( A \) of \( S \) is called down-closed if for all \( x \in A \) and all \( y \in S \), if \( y \leq x \) then \( y \in A \). Given \( x \in S \) we set \( \downarrow x = \{ y \in S : y \leq x \} \).

A linear morphism of sup-semilattices from \( S \) to \( T \) is a function \( f : S \rightarrow T \) such that for all \( A \subseteq S \) \( f(\bigvee A) = \bigvee f(A) \) where we define as usual \( f(A) = \{ f(x) : x \in A \} \). Notice that this implies that \( f \) is monotone: given \( x \leq y \) in \( S \) we have \( f(y) = f(\bigvee \{ x, y \}) = f(x) \vee f(y) \), that is \( f(x) \leq f(y) \).

Let \( \textbf{Slat} \) be the category whose objects are the sup-semilattices and morphisms are the linear maps of sup-semilattices. We set \( \downarrow = \{ 0 < 1 \} \) for the object of \( \textbf{Slat} \) which has exactly two elements.
It is important to remember that any inf-semilattice, partially ordered set $S$ where each $A \subseteq S$ has an greatest lower bound (also called “glb”, “inf” of “infimum”) $\bigwedge A$, is also a sup-semilattice: $\bigvee A = \bigcup \{ x \in S \mid \forall y \in A \, y \leq x \}$.

It is easy to check that $\text{Slat}$ is cartesian. The product of a family $(S_j)_{j \in J}$ of objects of $\text{Slat}$ is the usual cartesian product $\prod_{j \in J} S_j$ equipped with the product order and projection defined in the usual way. We also use $S = \bigwedge_{j \in J} S_j$ for this product and $\pi_j \in \text{Slat}(S, S_j)$ for the projections. The terminal object is $\top = \{ \emptyset \}$.

(a) Show that the isomorphisms of $\text{Slat}$ are the linear morphisms which are bijections.

(b) Given a set $X$ we denote as $\mathcal{P}(X)$ its powerset (that is, the set of all of its subsets) ordered under inclusion, so that $\mathcal{P}(X)$ is a sup-semilattice for $\bigvee A = \bigcup A$ for any $A \subseteq \mathcal{P}(X)$. Given $t \in \text{Rel}(X, Y)$ we define $\tilde{t} : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\tilde{t}(x) = t \cdot x = \{ b \in Y \mid \exists a \in x (a, b) \in t \}$. Prove that $\tilde{t} \in \text{Slat}(\mathcal{P}(X), \mathcal{P}(Y))$ and that, for any $f \in \text{Slat}(\mathcal{P}(X), \mathcal{P}(Y))$ there is exactly one $t = \text{tr} f \in \text{Rel}(X,Y)$ such that $f = \tilde{t}$. In other words, the functor $L : \text{Rel} \to \text{Slat}$ which maps $X$ to $\mathcal{P}(X)$ and $t \to \tilde{t}$ is full and faithful. This is the categorical way of saying that $\text{Rel}$ is a “subcategory” of $\text{Slat}$.

(c) Prove that the category $\text{Slat}$ has all equalizers, in other words: given objects $S$ and $T$ of $\text{Slat}$ and $f, g \in \text{Slat}(S,T)$ there is an object $E$ of $\text{Slat}$ and a morphism $e \in \text{Slat}(E,S)$ such that $f e = g e$ and, for any object $V$ of $\text{Slat}$ and any morphism $h \in \text{Slat}(V,S)$ such that $f h = g h$, there is exactly one morphism $h_0 \in \text{Slat}(V,E)$ such that $h = e h_0$.

The Cantor space is the set $\{0,1\}^\omega$ of all infinites sequences $\alpha$ of 0 and 1 equipped with the following topology (which is the product topology of the discrete space $\{0,1\}$): a subset $U$ of $\{0,1\}^\omega$ is open if for any $\alpha \in U$ there is a finite prefix $w$ of $\alpha$ such that, for any $\beta \in \{0,1\}^\omega$, if $w$ is a prefix of $\beta$ then $\beta \in U$. In other words, a subset $F$ of $\{0,1\}^\omega$ is closed if it has the following property: if $\alpha \in \{0,1\}^\omega$ is such that, for any finite prefix $w$ of $\alpha$ there exists $\beta \in F$ such that $w$ is a prefix of $\beta$, then $\alpha \in F$. As in any topological spaces, if $F$ is a set of closed subsets then $\bigcap F$ is closed (you are advised to check this directly using the characterization above of closed subsets).

So the set of closed subsets of $\{0,1\}^\omega$ is an inf-semilattice and hence also a sup-semilattice: the sup of a set of closed sets is the closure of its union (= the intersection of all closed sets which contain this union).

(d) $^{(**)}$ Let $W = \{0,1\}^\omega$ be the set of all finite sequences of 0 and 1. If $w = \langle a_1, \ldots, a_n \rangle \in W$ is such a sequence and $a \in \{0,1\}$ let $w a = \langle a_1, \ldots, a_n, a \rangle$. Let $\theta = \{ \langle w, a \rangle \mid w \in W \text{ and } a \in \{0,1\} \} \in \text{Rel}(W,W)$. Let $(C, c)$ be the equalizer of $\text{Id}, \tilde{\theta} \in \text{Slat}(\mathcal{P}(W), \mathcal{P}(X))$ (so that $C$ is a sup-semilattice and $c \in \text{Slat}(C, \mathcal{P}(W))$). Exhibit an order isomorphism between $C$ and the set of all closed subsets of the Cantor.

Given a lattice $S$, we say that $x \in S$ is prime if

$$\forall A \subseteq S \quad x \leq \bigvee A \Rightarrow \exists y \in A \, x \leq y.$$  

(e) $^\star$ Prove that, for a set $X$, the prime elements of $\mathcal{P}(X) \in \text{Slat}$ are exactly the singletons.  

Prove that $C$, in sharp contrast with the previous case, has no prime elements. 

[Hint: prove first that if $F \in C$ is prime, it must be a singleton $\{\alpha\}$ and then prove that no such singleton is prime. For this notice that, for a collection $\mathcal{F}$ of closed subsets of $\{0,1\}^\omega$, the closed set $\bigvee \mathcal{F}$ is the closure of $\bigcup \mathcal{F}$ (the intersection of all closed sets which contain $\bigcup \mathcal{F}$). So consider a set $\mathcal{F}$ of shape $\mathcal{F} = \{ \langle \alpha(n) \rangle \mid n \in \mathbb{N} \}$ where $\alpha(n) \to_{n \to \infty} \alpha$ and $\forall n \in \mathbb{N} \, \alpha(n) \neq \alpha$.]

This example is a concrete illustration of the fact that the category $\text{Rel}$ is not complete, indeed it has no equalizer for the two maps $\theta, \text{Id} \in \text{Rel}(W,W)$ because the equalizer of $\tilde{\theta}$ and $\text{Id}$ in $\text{Slat}$ is not an object of $\text{Rel}$ since it is not a prime-algebraic sup-semilattice.

(f) Prove that the set of linear morphisms $S \to T$, equipped with the pointwise order (that is $f \leq g$ if $\forall x \in S \, f(x) \leq g(x)$), is a sup-semilattice. We denote it as $S \to_0 T$. 

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4. This problem is the sequel of the previous one. We deal now with a class of non-(multi)linear functions. Given two objects \( S, T \) of \( \mathbf{Slat} \) we define \( S \to_x T \) as the set of all Scott continuous functions \( S \to T \) that is, of all monotone functions \( f : S \to T \) such that, for any directed \( D \subseteq S \) one has \( f(\bigvee D) = \bigvee \{ f(x) \mid x \in D \} \). We equip this set with the following order relation: \( f \leq g \) is \( \forall x \in S \ f(x) \leq g(x) \). We recall that \( D \subseteq S \) is directed if \( D \) is non-empty and \( \forall x, y \in D \ \exists z \in D \ x \leq z \ and \ y \leq z \), equivalently: any finite subset of \( D \) has an upper bound in \( D \).

(g) Given \( x \in S \) define a function \( x^* : S \to \bot \) by

\[
x^*(y) = \begin{cases} 
1 & \text{if } y \not\leq x \\
0 & \text{if } y \leq x
\end{cases}
\]

Prove that \( x^* \in S \to \bot \).

(h) Given a sup-semilattice \( S \), we use \( S^{op} \) for the same set \( S \) equipped with the reverse order: \( x \leq y \) if \( y \leq x \). Prove that the map \( x \mapsto x^* \) is an order isomorphism from the poset \( S^{op} \) to \( S \to \bot \). Warning: one must prove that it is monotone in both directions because a monotone bijection is not necessarily an order isomorphism! Call \( k : (S \to \bot) \to S^{op} \) the inverse isomorphism.

(i) (*) Given \( f \in (S \to T) \) define \( f^* : (T \to \bot) \to (S \to \bot) \) by \( f^*(y) = y \cdot f \). Prove that \( f^* \in \mathbf{Slat}(T \to \bot, S \to \bot) \). Let \( f^\perp \in \mathbf{Slat}(T^{op}, S^{op}) \) be the associated morphism (through the iso \( k \) defined above, that is \( f^\perp(y) = k(f^*(y^*)) \)). Prove that

\[
\forall x \in S \forall y \in T \ f(x) \leq y \iff x \leq f^\perp(y).
\]

One says that \( f \) and \( f^\perp \) define a Galois connection between \( S \) and \( T \). Last prove that \( f^\perp \perp = f \).

(j) Given sup-semilattices \( S \) and \( T \) we define \( S \otimes T \) as the set of all \( I \subseteq S \times T \) such that

- \( I \) is down-closed
- and, for all \( A \subseteq S \) and \( B \subseteq T \), if \( A \) and \( B \) satisfy \( A \times B \subseteq I \) then \( (\vee A, \vee B) \in I \).

Prove that \( (S \otimes T, \subseteq) \) is an inf-semilattice (that is, is closed under arbitrary intersections). As a consequence, it is also a sup-semilattice: if \( I \subseteq S \otimes T \) then \( \bigvee I = \bigcap \{ U \in S \otimes T \mid \bigcup I \subseteq U \} \). But notice that in this sup-semilattice, the sups are not defined as unions in general.

(k) Prove that the least element of \( S \otimes T \) is \( 0_{S \otimes T} = \{(0, \{0\}) \} \). \([Hint: \text{Remember that } f(0, \emptyset) = 0 \text{ and that } f(0 \times B) = \emptyset \text{ for any } B.\])

(l) We say that a map \( f : S \times T \to U \) (where \( S, T, U \) are sup-semilattices) is bilinear if for all \( A \subseteq S \) and \( B \subseteq T \) we have \( f(\vee A \times B) = f(\vee A) \cdot f(\vee B) \). Prove that this condition is equivalent to the following:

- for all \( x \in S \) and \( y \in T \), one has \( f(x, \bigvee B) = \bigvee_{y \in B} f(x, y) \)
- and for all \( y \in T \) and \( A \subseteq S \), one has \( f(\bigvee A, y) = \bigvee_{x \in A} f(x, y) \)

that is, \( f \) is separately linear in both variables.

(m) (*) Given \( x \in S \) and \( y \in T \) let \( x \otimes y = 1 \cdot I \) if \( (x, y) \in S \times T \subseteq S \times T \). Prove that \( x \otimes y \in S \otimes T \) and that the function \( \tau : (x, y) \mapsto x \otimes y \) is a bilinear map \( S \times T \to S \otimes T \).

(n) Let \( (S, T) \xrightarrow{\tau} U \) be the set of all bilinear maps \( S \times T \to U \) ordered pointwise (that is \( f \leq g \) if \( \forall (x, y) \in S \times T \ f(x, y) \leq g(x, y) \)). Prove that \( (S, T) \xrightarrow{\tau} U \simeq (S \xrightarrow{\tau} (T \xrightarrow{\tau} U)) \). Deduce from this fact that \( (S, T) \xrightarrow{\tau} U \) is a sup-semilattice.

(o) Given \( I \in X \otimes Y \) let \( I^t : S \times T \to \bot \) be given by

\[
I^t(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in I \\
1 & \text{otherwise.}
\end{cases}
\]

Prove that \( I^t \) is bilinear. Conversely given \( f \in (S, T) \xrightarrow{\tau} \bot \) prove that \( \ker f = \{(x, y) \in S \times T \mid f(x, y) = 0\} \) belongs to \( S \otimes T \). Prove that these operations define an order isomorphism between \( S \otimes T \) and \((S, T) \xrightarrow{\tau} \bot)^{op} \).

4. This problem is the sequel of the previous one. We deal now with a class of non-(multi)linear functions. Given two objects \( S, T \) of \( \mathbf{Slat} \) we define \( S \to_x T \) as the set of all Scott continuous functions \( S \to T \), that is, of all monotone functions \( f : S \to T \) such that, for any directed \( D \subseteq S \) one has \( f(\bigvee D) = \bigvee \{ f(x) \mid x \in D \} \). We equip this set with the following order relation: \( f \leq g \) if \( \forall x \in S \ f(x) \leq g(x) \). We recall that \( D \subseteq S \) is directed if \( D \) is non-empty and \( \forall x, y \in D \ \exists z \in D \ x \leq z \ and \ y \leq z \), equivalently: any finite subset of \( D \) has an upper bound in \( D \).
(a) Given semi-lattices $S, T, U$ and $f : S \& T \rightarrow U$, prove that $f$ is Scott continuous iff it is separately Scott-continuous, that is: for all $x \in S$ the function $y \mapsto f(x,y)$ is Scott-continuous $T \rightarrow U$ and for any $y \in T$ the function $x \mapsto f(x,y)$ is Scott-continuous $S \rightarrow U$.

(b) Prove that sup-semilattice and Scott-continuous functions form a category, that we will denote as $\text{SlatC}$. Prove that this category has all products (defined as in $\text{Slat}$).

(c) Prove that $S \Rightarrow S$ is a sup-semilattice.

(d) Prove that the function $\text{Ev} : (S \Rightarrow S) \& S \rightarrow T$ which maps $(f,x)$ to $f(x)$ is Scott continuous.

(e) Prove that $\text{SlatC}$ is cartesian closed, with $(S \Rightarrow S, \text{Ev})$ as object of morphisms from $S$ to $T$.

(f) Let $S$ be an object of $\text{Slat}$. We define $\downarrow S$ as the set of all $I \subseteq \mathcal{P}(S)$ (the powerset of $S$) which are down-closed and such that, for any directed subset $D$ of $S$, if $D \subseteq I$ then $\bigvee D \in I$.

Prove that, equipped with the $\subseteq$ partial order relation, $\downarrow S$ is an inf-semilattice where infima are intersections. Therefore it is also a sup-semilattice (but suprema are not unions in general).

What is the least element of $\downarrow S$ (give a proof of your answer)?

(g) Prove that if $I \subseteq \downarrow S$ is directed then $\bigvee I = \bigcup I$. And prove that if $I \in \downarrow S$ then $\bigvee \{ \downarrow x \mid x \in I \} = \bigcup \{ \downarrow x \mid x \in I \} = I$. As a consequence show that if $\varphi \in \text{Slat}(\downarrow S, T)$ then $\forall I \in \downarrow S \ varphi(I) = \bigvee \{ \varphi(I) \mid I \in I \}$. Show that if $\varphi, \psi \in \text{Slat}(\downarrow S, T)$ satisfy $\forall x \in \varphi \varphi(x) = \psi(x)$ then $\varphi = \psi$.

(h) Let $\varphi \in \text{Slat}(S, T)$. If $I \in \downarrow S$ set $\downarrow \varphi(I) = \bigvee \{ \varphi(x) \mid x \in I \} = \bigcap \{ J \in \downarrow T \mid \varphi(I) \subseteq J \}$.

Prove that $\downarrow \varphi \in \text{Slat}(\downarrow S, \downarrow T)$ and that $\downarrow$ is a functor $\text{Slat} \rightarrow \text{Slat}$. Notice that $\downarrow \varphi$ is fully characterized by $\forall x \in S \ \downarrow \varphi(x) = \varphi(x)$.

(i) Let $f \in \text{SlatC}(S, T)$. For $y \in T$ let

$$\overline{f}(y) = \{ x \in S \mid f(x) \leq y \}.$$

Prove that $\overline{f} \in \text{Slat}(T^\text{op}, (\downarrow S)^\text{op})$, that is, prove first that $\forall y \in T \ \overline{f}(y) \in \downarrow S$ and then that, for any $B \subseteq T$, one has $\overline{f}(\bigwedge B) = \bigcap_{y \in B} \overline{f}(y)$.

(j) Let $\text{lin}(f) = \overline{f}$ so that

$$\forall I \in \downarrow S \forall y \in T \ \text{lin}(f)(I) \leq y \Rightarrow I \subseteq \overline{f}(y).$$

Prove that $\text{lin}(f)(I) = \bigwedge \{ y \in T \mid f(I) \subseteq \downarrow y \}$ where as usual $f(I) = \{ f(x) \mid x \in I \}$ [Hint: See Question (i) of Problem 3.]

(k) Let $\text{cnt} : S \rightarrow \mathcal{P}(S)$ be defined by $\text{cnt}(x) = \downarrow x$. Prove that $\text{cnt} \in \text{SlatC}(S, \downarrow S)$ and that $\text{cnt}$ is never linear [Hint: Consider $\text{cnt}(0)$]. Prove that $\text{lin}(\text{cnt}) = \text{id}_{\downarrow S}$.

(l) Prove that the function

$$\text{Slat}(\downarrow S, T) \rightarrow \text{SlatC}(S, T)$$

$$\varphi \mapsto \varphi \circ \text{cnt}$$

is the inverse of $\text{lin}$.

(m) We define $\text{der}_{S} = \text{lin}(\text{id}_{S}) \in \text{Slat}(\downarrow S, S)$ where $\text{id}_{S} \in \text{SlatC}(S, S)$. Prove that $\text{der}_{S}(I) = \bigvee I$.

(n) Let $f : S \rightarrow \downarrow S$ be the function given by $f(x) = \downarrow \downarrow x$. Prove that $f$ is Scott continuous. Let $\text{dig}_{S} = \text{lin}(f) \in \text{Slat}(\downarrow S, \downarrow \downarrow S)$.

(o) Prove that $\text{der}$ and $\text{dig}$ are natural transformations and that $\downarrow$ equipped with these two natural transformations, is a comonad.

(p) For any objects $S, T, U$ prove that there is a bijection $\text{Slat}(\downarrow S \& T, U) \rightarrow \text{Slat}(\downarrow S \otimes \downarrow T, U)$.

From this observation, we could deduce with a bit more work that there are also Scott isomorphisms, taking $U = \downarrow S \& T$ and $U = \downarrow S \otimes \downarrow T$.

5. The goal of this exercise is to study the properties of the objects of the Eilenberg Moore category $\text{Rel}^\dagger$ of $\text{Rel}$, the relational model of LL.

Let $P$ be an object of $\text{Rel}^\dagger$ (the category of coalgebras of $\downarrow$). Remember that $P = (P, h_P)$ where $P$ is an object of $\text{Rel}$ (a set) and $h_P \in \text{Rel}(P, \downarrow P)$ satisfies the following commutations:
(a) Check that these commutations mean:
- for all $a, a' \in P$, one has $(a, [a']) \in h_P$ iff $a = a'$
- and for all $a \in P$ and $m_1, \ldots, m_k \in P$, one has $(a, m_1 + \cdots + m_k) \in h_P$ iff there are $a_1, \ldots, a_k \in P$ such that $(a, [a_1, \ldots, a_k]) \in h_P$ and $(a_i, m_i) \in h_P$ for $i = 1, \ldots, k$.
Intuitively, $(a, [a_1, \ldots, a_k])$ means that $a$ can be decomposed into “$a_1 + \cdots + a_k$” where the “$+$” is the decomposition operation associated with $P$.

(b) Prove that if $P$ is an object of $\mathbf{Rel}^f$ such that $P \neq \emptyset$ then there is at least one element $e$ of $P$ such that $(e, []) \in h_P$. Explain why such an $e$ could be called a “coneful element of $P$”.

(c) If $P$ and $Q$ are objects of $\mathbf{Rel}^f$, remember that an $f \in \mathbf{Rel}^f(P, Q)$ (morphism of coalgebras) is an $f \in \mathbf{Rel}(P, Q)$ such that the following diagram commutes

\[
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\
\downarrow h_P & & \downarrow h_Q \\
P & \xrightarrow{!} & Q
\end{array}
\]

Check that this commutation means that for all $a \in P$ and $b_1, \ldots, b_k \in Q$, the two following properties are equivalent
- there is $b \in Q$ such that $(a, b) \in f$ and $(b, [b_1, \ldots, b_k]) \in h_Q$
- there are $a_1, \ldots, a_k \in P$ such that $(a, [a_1, \ldots, a_k]) \in h_P$ and $(a_i, b_i) \in f$ for $i = 1, \ldots, k$.

(d) Remember that 1 (the set $\{\ast\}$) can be equipped with a structure of coalgebra (still denoted 1) with $h_1 = \{(\ast, k[\ast]) | k \in \mathbb{N}\}$. Prove that the elements of $\mathbf{Rel}^f(1, P)$ can be identified with the subsets $x$ of $P$ such that: for all $a_1, \ldots, a_k \in P$, one has $a_1, \ldots, a_k \in x$ if there exists $a \in x$ such that $(a, [a_1, \ldots, a_k]) \in h_P$. We call values of $P$ these subsets of $P$ and denote as $\text{val}(P)$ the set of these values.

Prove that an element of $\text{val}(P)$ is never empty and that $\text{val}(P)$, equipped with inclusion, is a complete partially ordered set (cpo), that is: the union of a set of values which is directed (with respect to $\subseteq$) is still a value.

(e) Remember that if $E$ is an object of $\mathbf{Rel}$ then $(I_E, \text{dig}_E)$ is an object of $\mathbf{Rel}^f$ (the free coalgebra generated by $E$, that we can identify with an object of the Kleisli category $\mathbf{Rel}_h$). Prove that, as a partial ordered set, $\text{val}(I_E, \text{dig}_E)$ is isomorphic to $\mathcal{P}(E)$.

(f) Is it always true that if $x_1, x_2 \in \text{val}(P)$ then $x_1 \cup x_2 \in \text{val}(P)$?

(g) We have seen (without proof) that $\mathbf{Rel}^f$ is cartesian. Remember that the product of $P_1$ and $P_2$ is $P_1 \otimes P_2$, the coalgebra defined by $P_1 \otimes P_2 = P_1 \otimes P_2$ and $h_{P_1 \otimes P_2}$ is the following composition of morphisms in $\mathbf{Rel}$:

\[
P_1 \otimes P_2 \xrightarrow{h_{P_1} \otimes h_{P_2}} \text{!}P_1 \otimes \text{!}P_2 \xrightarrow{\mu_{P_1 \otimes P_2}} \text{!(}P_1 \otimes P_2\text{)}
\]

where $\mu_{E_1, E_2} \in \text{Rel}(\text{!}E_1 \otimes \text{!}E_2, \text{!(}E_1 \otimes E_2\text{)})$ is the lax monoidality natural transformation of $\text{!}$, recall that in $\mathbf{Rel}$ we have

\[
\mu_{E_1, E_2} = \{(([(a_1, \ldots, a_k) \cdot [b_1, \ldots, b_k]), [(a_1, b_1), \ldots, (a_k, b_k)]) | k \in \mathbb{N} \text{ and } (a_1, b_1), \ldots, (a_k, b_k) \in E_1 \times E_2 \}
\]

Concretely, we have simply that $((a_1, a_2), [(a_1^2, a_2^2), \ldots, (a_1^2, a_2^2)]) \in h_{P_1 \otimes P_2}$ iff $((a_i, a_{i}^2, \ldots, a_{i}^2)) \in h_{P_i}$ for $i = 1, 2$.

Prove that $P_1 \otimes P_2$, equipped with suitable projections, is the cartesian product of $P_1$ and $P_2$ in $\mathbf{Rel}^f$. Prove also that 1 is the terminal object of $\mathbf{Rel}^f$. Warning: $\mathcal{L}^f$ is always cartesian when $\mathcal{L}$ is a model of LL; I’m not asking for a general proof, just for a verification that this is true in $\mathbf{Rel}^f$.\]
From Pierre Boudes who discovered this condition and the nice properties of these objects.

We shall show that the class of NUCS’s can be turned into a categorical model of \( \text{Rel} \) in such a way that all the operations on objects coincide with the corresponding operations on objects in \( \text{Rel} \). For instance we shall define \( |X| \) to \( \text{Fin}(|X|) \). Moreover, all the “structure morphisms” of this model will be defined exactly as in \( \text{Rel} \). For instance, the digging morphism from \( ![E|X] \) to \( ![E|X] \) will simply be \( \text{dig}_{[X]} \). Important: such definitions are impossible with ordinary coherence spaces. When defining \( |E| \) in ordinary coherence spaces one needs to restrict to the finite multisets (or finite sets) of elements of \( |E| \) which are cliques of \( E \). It is exactly for that reason that, in NUCS’s, the relation \( \equiv \) is not required to coincide with equality. Nevertheless, the weaker Boudes’ condition will be preserved by all of our constructions.

(a) Check that a NUCS can be specified by \(|X|\) together with any of the following seven pairs of relations.

- Two symmetric relations \( \bowtie_X \) and \( \bowtie_X \) on \(|X|\) such that \( \bowtie_X \subseteq \bowtie_X \). Then setting \( \bowtie_X = (|X| \times |X|) / \bowtie_X \), the relation \( \bowtie_X \) is the one canonically associated with the NUCS \((|X|, \bowtie_X, \bowtie_X)\).
- Two symmetric relations \( \bowtie_X \) and \( \bowtie_X \) on \(|X|\) such that \( \bowtie_X \subseteq \bowtie_X \). How should we define \( \bowtie_X \) in that case?
- Two symmetric relations \( \bowtie_X \) and \( \bowtie_X \) on \(|X|\) such that \( \bowtie_X \subseteq \bowtie_X \). How should we define \( \bowtie_X \) and \( \bowtie_X \) in that case?
- Two symmetric relations \( \bowtie_X \) and \( \bowtie_X \) on \(|X|\) such that \( \bowtie_X \subseteq \bowtie_X \). How should we define \( \bowtie_X \) and \( \bowtie_X \) in that case?

6. The goal of this exercise is to illustrate the fact that \( \text{Rel} \), the relational model of LL, can be equipped with additional structures of various kinds without modifying the interpretation of proofs and programs. As an example we shall study the notion of non-uniform coherence space (NUCS). A NUCS is a triple \( (|X|, \bowtie_X, \bowtie_X) \) where

- \(|X|\) is a set (the web of \( X \))
- and \( \bowtie_X \) and \( \bowtie_X \) are two symmetric relations on \(|X|\) such that \( \bowtie_X \cap \bowtie_X = \emptyset \). In other words, for any \( a, a' \in |X| \), one never has \( a \bowtie_X a' \) and \( a \bowtie_X a' \).

So we can consider an ordinary coherence space (in the sense of the first part of this series of lectures) as a NUCS \( X \) which satisfies moreover:

\[ \forall a, a' \in |X| \quad (a \bowtie_X a' \text{ or } a \bowtie_X a') \iff a = a'. \]

It is then possible to introduce three other natural symmetric relations on the elements of \(|X|\):

- \( a \equiv_X a' \) if it is not true that \( a \bowtie_X a' \) or \( a \bowtie_X a' \).
- \( a \bowtie_X a' \) if \( a \bowtie_X a' \) or \( a \equiv_X a' \).
- \( a \bowtie_X a' \) if \( a \bowtie_X a' \) or \( a \equiv_X a' \).

A clique of a NUCS \( X \) is a subset \( x \) of \(|X|\) such that \( \forall a, a' \in |X| \ a \bowtie_X a' \), we use \( \text{Cl}(X) \) for the set of cliques of \( X \).

We say that a NUCS \( X \) satisfies the Boudes’ Condition\(^1\) (or simply that \( X \) is Boudes) if

\[ \forall a, a' \in |X| \ a \equiv_X a' \Rightarrow a = a'. \]

We shall show that the class of NUCS’s can be turned into a categorical model of LL in such a way that all the operations on objects coincide with the corresponding operations on objects in \( \text{Rel} \). For instance we shall define \(|X|\) in such a way that \( |X| = |X| = \mathcal{M}_{\text{fin}}(|X|) \). Moreover, all the “structure morphisms” of this model will be defined exactly as in \( \text{Rel} \). For instance, the digging morphism from \( ![E|X] \) to \( ![E|X] \) will simply be \( \text{dig}_{[X]} \). Important: such definitions are impossible with ordinary coherence spaces. When defining \(|E|\) in ordinary coherence spaces one needs to restrict to the finite multisets (or finite sets) of elements of \(|E|\) which are cliques of \( E \). It is exactly for that reason that, in NUCS’s, the relation \( \equiv_X \) is not required to coincide with equality. Nevertheless, the weaker Boudes’ condition will be preserved by all of our constructions.

(h) Check directly that the partially ordered sets \( \text{val}(P_1 \oplus P_2) \) and \( \text{val}(P_1) \times \text{val}(P_2) \) are isomorphic.

(i) Remember also that we have defined \( P_1 \oplus P_2 = (P_1 \oplus P_2, h_{P_1 \oplus P_2}) \) where \( h_{P_1 \oplus P_2} \) is the unique element of \( \text{Rel}(P_1 \oplus P_2, (P_1 \oplus P_2)) \) such that, for \( i = 1, 2 \), the morphism \( h_{P_1 \oplus P_2} \pi_i \) coincides with the following composition of morphisms in \( \text{Rel} \):

\[ P_1 \xrightarrow{h_{P_1 \oplus P_2}} P_2 \xrightarrow{\pi_i} (P_1 \oplus P_2) \]

Describe \( h_{P_1 \oplus P_2} \) as simply as possible and prove that, equipped with suitable injections, \( P_1 \oplus P_2 \) is the coproduct of \( P_1 \) and \( P_2 \) in \( \text{Rel} \).

\(^1\)From Pierre Boudes who discovered this condition and the nice properties of these objects.
• Two symmetric relations $\sim_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \cap \sim_X = \emptyset$. How should we define $\sim_X$ in that case?
• Two symmetric relations $\sim_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \cap \sim_X = \emptyset$. How should we define $\sim_X$ in that case?
• Two symmetric relation $\subset_X$ and $\succeq_X$ such that $\subset_X \cup \succeq_X = |X| \times |X|$. How should we define $\sim_X$ and $\sim_X$ in that case?

(b) Given NUCS’s $X$ and $Y$, we define a NUCS $X \rightarrow Y$ by $|X \rightarrow Y| = |X| \times |Y|$ and

- $(a, b) \equiv_X Y (a', b')$ if $a \equiv_X a'$ and $b \equiv_Y b'$
- $(a, b) \subset_X Y (a', b')$ if $a \subset_X a'$ or $b \subset_Y b'$.

Check that we have defined in that way a NUCS. Prove that $\text{Id}_{|X|} = \{(a, a) | a \in |X|\} \in \text{Cl}(X \rightarrow Y)$. Prove that if $X$ and $Y$ are Boudes then $X \rightarrow Y$ is Boudes.

(c) Prove that, if $s \in \text{Cl}(X \rightarrow Y)$ and $t \in \text{Cl}(Y \rightarrow Z)$ then $ts \in \text{Cl}(X \rightarrow Z)$. So we define a category $\text{Nucs}$ by taking the NUCS’s as object and by setting $\text{Nucs}(X, Y) = \text{Cl}(X \rightarrow Y)$.

(d) We define $X^\perp$ by $|X^\perp| = |X|$, $\sim_{X^\perp} = \sim_X$ and $\equiv_{X^\perp} = \equiv_X$. Then we set $X \otimes Y = (X \rightarrow Y^\perp)^\perp$.

Describe as simply as possible the NUCS structure of $X \otimes Y$. We set $1 = (\{1\}, ) \emptyset, 0)$ (in other words $* \equiv_1 *$). Prove that if $X$ and $Y$ are Boudes then $X^\perp$ and $X \otimes Y$ is Boudes.

(e) Given $s_i \in \text{Nucs}(X_i, Y_i)$ for $i = 1, 2$, prove that $s_1 \otimes s_2 \in \text{Rel}(|X_1| \otimes |X_2|, |Y_1| \otimes |Y_2|)$ (defined as in Rel) does actually belong to $\text{Nucs}(X_1 \otimes X_2, Y_1 \otimes Y_2)$.

(f) Check quickly that $\text{Nucs}$ (equipped with the $\otimes$ defined above and 1 as tensor unit, and $\bot = 1$ as dualizing object) is a $*$-autonomous category.

(g) Prove that the category $\text{Nucs}$ is cartesian and cocartesian, with $X = \biguplus_{i \in I} X_i$, given by $|X| = \bigcup_{i \in I} \{i\} \times |X_i|$, and

- $(i, a) \equiv_X (i', a')$ if $i = i'$ and $a \equiv_X a'$
- $(i, a) \sim_X (i', a')$ if $i = i'$ and $a \sim_X a'$.

and the associated operations (projections, tupling of morphisms) defined as in Rel.

Prove that if all $X_i$‘s are Boudes then $\biguplus_{i \in I} X_i$ is Boudes.

(h) We define $!X$ as follows. We take $|!X| = |X| \mathcal{M}_{\text{fin}}(|X|)$ and, given $m, m' \in |!X|

- we have $m \equiv_X m'$ if for all $a \in |X|$ and $a' \in |X|$ one has $a \equiv_X a'$
- and $m \equiv_X m'$ if $m \equiv_X m'$ and $m = [a_1, \ldots, a_k]$, $m' = [a'_1, \ldots, a'_k]$ with $a_i \equiv_X a'_i$ for each $i \in \{1, \ldots, k\}$.

Notice that $m \equiv_X m'$ iff there is $a \in |X|$ and $a' \in |X|$ such that $a \equiv_X a'$. Remember that $|X| = \mathcal{M}_{\text{fin}}(|X|) | m(a) \neq 0|$.

Let $s \in \text{Nucs}(X, Y)$. Prove that $Ds \in \text{Rel}(|!X|, |!Y|)$ actually belongs to $\text{Nucs}(|!X|, |!Y|)$.

(i) Prove that $\text{der}_{|X|} = \{(a, a) | a \in |X|\}$ belongs to $\text{Nucs}(|!X|, X)$.

(j) Prove that $\text{dig}_X = \{(m_1 + \cdots + m_k, [m_1, \ldots, m_k]) | m_1, \ldots, m_k \in \mathcal{M}_{\text{fin}}(|X|)\}$ is an element of $\text{Nucs}(|!X|, !X)$.

(k) Prove that if $X$ is Boudes then $!X$ is Boudes.

(l) Let $X = 1 + 1$, and let $t, f$ be the two elements of $|X|$ ($X$ is the “type of booleans”). Let $s \in \text{Rel}(|X| \otimes |X|, |X|)$ by $s = \{(t, f), t, ((t, f), t)\}$. Prove that $s \in \text{Nucs}(X \otimes X, X)$. Let then $t \in \text{Nucs}(|!X|, X)$ be defined by the following composition of morphisms in $\text{Nucs}$:

$$
!X \xrightarrow{c_X} !X \otimes !X \xrightarrow{\text{der}_{!X} \otimes \text{der}_{!X}} X \otimes X \xrightarrow{s} X
$$

We recall that contraction $c_X \in \text{Nucs}(|!X|, !X \otimes !X)$ is given by $c_X = \{(m_1 + m_2, [m_1, m_2]) | m_1, m_2 \in |!X|\}$ and dereliction $\text{der}_{!X} \in \text{Nucs}(|!X|, X)$ is given by $\text{der}_{!X} = \{(a, a) | a \in |X|\}$.

Prove that $(t, f), ((t, f), t) \in t$. So any notion of coherence on $|!X|$ must satisfy $t \sim_X t, f \sim_X t, f$ since we have $t \sim_X f$ by the definition of the NUCS 1 $\oplus$ 1 since we must have $(t, f), t \sim_X (t, f)$.
([t,f],f) because t is a clique. In particular it is impossible to endow ![|X|] with a notion of Girard’s coherence space since in such a coherence space we would have [t,f] ⊓ ![|X|] [t,f] and hence ([t,f],t) ≿ ![|X|→X] ([t,f],f).

As an illustration of the usefulness of this semantics, consider the language PCF studied during the lectures. Let M be a closed term such that ⊢ M : τ. By the Church Rosser Theorem for PCF (of which we have outlined the proof) we know that if M β* n and M β* p then n = p. This proof is completely syntactic and not very modular (if we modify the syntax, a lot of work has to be redone).

The PCF type τ is interpreted in any model of LL as N = 1 ⊕ 1 ⊕ ···. In Nucs, the only cliques of N are ∅ and the singletons. The semantics of any term is identical in Rel and in Nucs. Since the semantics of M in Nucs is a clique of N, this proves that if M β* n and M β* p then n = p.