Finiteness spaces

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Abstract

We investigate a new denotational model of linear logic based on the purely relational model. In this semantics, webs are equipped with a notion of "finitary" subsets satisfying a closure condition and proofs are interpreted as finitary sets. In spite of a formal similarity, this model is quite different from the usual models of linear logic (coherence semantics, hypercoherence semantics, the various existing game semantics...). In particular, the standard fix-point operators used for defining the *general* recursive functions *are not* finitary, although the primitive recursion operators are. This model can be considered as a discrete version of the Köthe space semantics introduced in a previous paper: we show how, given a field, each finiteness space gives rise to a vector space endowed with a *linear topology*, a notion introduced by Lefschetz in 1942, and we study the corresponding model where morphisms are linear continuous maps (a version of Girard's quantitative semantics with coefficients in the field). We obtain in that way a new model of the recently introduced *differential lambda-calculus*.

Notations. If S is a set, we denote by $\mathcal{M}(S) = \mathbf{N}^S$ the set of all multi-sets over S. If $\mu \in \mathcal{M}(S)$, $|\mu|$ denotes the support of μ which is the set of all $a \in S$ such that $\mu(a) \neq 0$. A multi-set is finite if it has a finite support. If $a_1,\dots, a_n$ are elements of some given set S, we denote by $[a_1, \dots, a_n]$ the corresponding multi-set over S. The usual operations on natural numbers are extended to multi-sets pointwise.

If $(S_i)_{i \in I}$ are sets, we denote by π_i the i-th projection $\pi_i : \prod_{j \in I} S_j \to S_i$.

Introduction

In the purely relational model of linear logic, which is certainly the simplest denotational model of linear logic, formulae are interpreted as sets and proofs as relations between these sets. Additive connectives are interpreted as disjoint unions, multiplicative connectives as cartesian products and exponentials as the operation which maps a set S to the set of all finite multi-sets with domain included in S (the *finite multi-powerset* of S)[1]. In the category of finite-dimensional vector spaces over a given field, direct product, tensor product and linear function space give rise to similar operations on *bases*, e.g. one obtains a basis of the tensor product of two vector spaces E and F by taking the cartesian product of a basis of E and a basis of F. Remember also that, given a basis of a vector space, there is a canonical way of defining a basis of the same cardinality for the dual of the space, and this is compatible with the fact that in the purely relational model, the linear negation of a set S is S itself.

[1] Taking here finite *sets* instead of multi-sets, one does not obtain naturally a model: the problem is that the standard definition of dereliction does not give rise to a natural transformation.

Having these observations in mind, it becomes quite natural to think of the sets interpreting formulae in the purely relational model as bases of some vector spaces. However, due to the exponentials, the dimension of these spaces cannot be restricted to be finite (the finite multi-powerset of a non-empty set is always infinite). Endowing these sets with an additional simple structure, one can fortunately preserve the interpretation of the sets interpreting formulae as "bases"[2]: we present several aspects of this idea.

Let R be a field (or a unitary ring), given once and for all. We want to interpret formulae as R-vector spaces (or R-modules), and these spaces should admit as "bases" the sets interpreting the corresponding formulae in the purely relational setting. So if A is a formula and $|A|$ is its interpretation in this relational model, the space A^* interpreting A should be a subspace of $R^{|A|}$. The space $(A^\perp)^*$ should be similarly a subspace of $R^{|A|}$, isomorphic to the dual of the space A^*. Therefore, given $x \in A^*$ and $x' \in (A^\perp)^*$, we should be able to define a scalar $\langle x, x' \rangle \in R$, the application of the linear form x' to the vector x. If we keep in mind the fact that $|A|$ should be a kind of basis of A^* and should also represent the dual of this basis in $(A^\perp)^*$, it appears that the formula giving $\langle x, x' \rangle$ should be

$$\langle x, x' \rangle = \sum_{a \in |A|} x_a x'_a \,.$$

However, the set $|A|$ is infinite in general and so we must manage to keep this sum finite[3].

A simple way to fulfill this requirement is to interpret formulae as sets equipped with a notion of *finitary* subsets. Such a pair $(I, \mathcal{F})$ (where $\mathcal{F} \subseteq \mathcal{P}(I)$) will then have $(I, \mathcal{F}^\perp = \{u' \subseteq I \mid \forall u \in \mathcal{F}\ u \cap u' \text{ is finite}\})$ as orthogonal (linear negation). Since linear negation should be an involutive operation, we must require $\mathcal{F} = \mathcal{F}^{\perp\perp}$ and this will be our sole constraint on $\mathcal{F}$. So we define a *finiteness space* as a pair $X = (|X|, \mathsf{F}(X))$ where $|X|$ is a set, the *web* of X, and $\mathsf{F}(X)$ is a collection of subsets of $|X|$ satisfying $\mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$. These subsets of $|X|$ will be called the *finitary sets* of X. The vector space associated with such a finiteness space X will be the collection $R\langle X \rangle$ of all vectors $x \in R^{|X|}$ whose support (the set of elements a of $|X|$ such that $x_a \neq 0$) is finitary in X. Therefore, by definition, for $x \in R\langle X \rangle$ and $x' \in R\langle X^\perp \rangle$ (where $X^\perp$ is obviously defined by $|X^\perp| = |X|$ and $\mathsf{F}(X^\perp) = \mathsf{F}(X)^\perp$), the sum $\sum_{a \in |X|} x_a x'_a$ will always have only finitely many non-zero terms, although both x and x' will in general have infinite supports.

Using finiteness spaces, we build first in Section 1 a new relational model of first order propositional linear logic, a *linear category* (see [Bie95]), which is at first sight similar to the model of coherence spaces ([Gir87]) for instance: the notion of clique is replaced here by the notion of finitary set. The analogy however is quite superficial since in a coherence space, the whole structure is known when all finite cliques (and even, all two elements cliques) are known. Here on the contrary, finite sets are always finitary, so that a finiteness space is described by the collection of all its *infinite* finitary sets and the notion of finiteness space is highly non. . . finitary, in sharp contrast with the usual notions of denotational semantics[4]. In particular, finitary sets are not closed under directed unions (unless $\mathsf{F}(X)$ contains all subsets of $|X|$) so that the situation is in some sense opposite to the usual domain-theoretic one. A finiteness space is not order-theoretically complete, but is complete in another topological sense. For this reason, interpreting recursion in finiteness spaces becomes a delicate issue; we shall show in Section 2 that tail-recursive iteration and thus, a tail-recursive version of primitive recursion can be interpreted in finiteness spaces, so that finiteness spaces provide a model for (a version of) Gödel's system $\mathcal{T}$, but *a priori* not a model of PCF as the standard interpretation of the fix-point operator Y is not a finitary set.

This observation that the fix-point is not finitary suggests some links between finiteness semantics and normalization properties of logical systems, and opens seemingly a new line of research in that direction. With this respect, the status of the empty set in finiteness spaces is interesting: usually (in coherence spaces

[2]In infinite dimension, the notion of basis becomes more complicated, and the standard algebraic notion (Hamel basis) is usually not suitable; one rather considers as bases linearly independent collections whose linear span is *dense* for some topology on the considered vector space.

[3]Or absolutely converging if R is the field of real or complex numbers, and this leads to the Köthe space approach. In the present setting, we make no topological assumption about R and so we shall require this sum to have only finitely many non-zero terms.

[4]With the concept of *totality* as a noticeable exception.

for instance), the empty set is thought of as representing the ever-looping program. But here, since the typical ever-looping program which is $(Y)\lambda x\, x$ cannot be interpreted, one should maybe rather consider the empty set as a kind of "daemon", in the sense of Girard's ludics [Gir01], that is, a pure termination, without resulting information.

One peculiarity of this model is that the finiteness space associated with a formula A of linear logic has as web the set interpreting A in the purely relational model of linear logic, and the interpretation of proofs in the two models coincide. The finiteness space structure is just a structure added to the purely relational model and this structure is respected by the interpretation of proofs, which gives rise only to finitary sets. Due to the exponentials, the same cannot be said of the usual coherence semantics. The situation is identical in the models introduced in [BE01].

In Section 3, we develop the algebraic theory of finiteness spaces, considering the R-vector spaces associated with them as explained above. Given a finiteness space X, we endow $R\langle X\rangle$ with a linear topology[5] by giving a basis of neighborhoods of 0: a subset U of $R\langle X\rangle$ belongs to this basis if for some $u' \in \mathsf{F}(X^\perp)$, U is the set of all the elements of $R\langle X\rangle$ whose support does not meet u'. We prove then that $R\langle X \multimap Y\rangle$ is linearly isomorphic to the R-module of all linear and continuous functions from $R\langle X\rangle$ to $R\langle Y\rangle$ (in particular, $R\langle X^\perp\rangle$ is just the topological dual of $R\langle X\rangle$ equipped with the linear topology described above). We obtain in that way a model of multiplicative linear logic where morphisms are linear and continuous functions on these topological vector spaces. We also exhibit the additive structure: not surprisingly, $R\langle X \mathbin{\&} Y\rangle$ is both the direct product and the direct sum of X and Y.

We interpret then the exponential connectives of linear logic and retrieve the familiar structures: the exponential ! is an endofunctor on the category of finiteness spaces and linear continuous maps, and this functor has the canonical structure of comonad which allows to interpret full first order linear logic (in particular, there is a natural isomorphism between $!(X \mathbin{\&} Y)$ and $!X \otimes !Y$). We also show that the linear continuous functions from $R\langle !X\rangle$ to $R\langle Y\rangle$ can be seen as *entire* functions from $R\langle X\rangle$ to $R\langle Y\rangle$, that is, functions which are defined by a power series which converges on the whole space $R\langle X\rangle$. Last, we exhibit the categorical structure of the exponential which corresponds to the differential operations on these entire functions, showing that we have obtained in that way a model of the recently introduce *differential lambda-calculus* [ER01], categorically completely similar to the model of Köthe spaces presented in [Ehr02].

All these constructions can be seen as rephrasing Girard's *quantitative semantics* of the lambda-calculus presented in [Gir88] (see also [Has02]) where lambda-terms are interpreted as normal functors which are power series whose coefficients are (possibly infinite) sets; the role of our additional finiteness space structure is to keep these coefficients finite and recast the quantitative approach in a completely standard algebraic setting. The price to pay is the impossibility of interpreting fix-point operators, but we tend to consider this as a rather interesting feature: after all we do not have so many simple denotational models introducing natural divides between computational primitives.

[5] In the sense of [Lef42]. This is a notion of topology for vector spaces or modules where basic neighborhoods are *linear subspaces* and which is therefore quite different from the usual notions considered in functional analysis for instance, such as Banach spaces and their locally convex generalizations. Linear topologies have a more algebraic flavour and in particular make no topological assumptions on the underlying ring or field R which will always be endowed with the discrete topology.

1 The finitary relational model

Let I be a set and let $u, u' \subseteq I$. In this paper, we say that u and u' are *in duality*[6] if $u \cap u'$ is a *finite* set[7]. Let $\mathcal{F} \subseteq \mathcal{P}(I)$, we denote by $\mathcal{F}^\perp$ the set

$$\mathcal{F}^\perp = \{u' \subseteq I \mid \forall u \in \mathcal{F} \quad u \cap u' \text{ is finite}\} \subseteq \mathcal{P}(I)\,.$$

Obviously, if u' finite, then $u' \in \mathcal{F}^\perp$. It is also clear that if $u' \subseteq v' \in \mathcal{F}^\perp$, then $u' \in \mathcal{F}^\perp$, and that $\mathcal{F}^\perp$ is closed under finite unions.

Moreover, this duality operation on subsets of $\mathcal{P}(I)$ has the following immediate properties that we shall use implicitly. These are just "abstract non-sense" properties which do not use the particular definition of duality between elements of $\mathcal{P}(I)$ (here, having a finite intersection).

- $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{G}^\perp \subseteq \mathcal{F}^\perp$;
- $\mathcal{F} \subseteq \mathcal{F}^{\perp\perp}$;
- $\mathcal{F}^{\perp\perp\perp} = \mathcal{F}^\perp$.

The following statement is not really useful, but just shows that the closure $\mathcal{F}^{\perp\perp}$ of $\mathcal{F}$ admits a "direct" characterization.

Proposition 1 *Let $\mathcal{F} \subseteq \mathcal{P}(I)$ be downwards closed (that is, if $u \subseteq v \in \mathcal{F}$, then $u \in \mathcal{F}$). Let $u' \subseteq I$. One has $u' \in \mathcal{F}^\perp$ iff there is no infinite subset v of u' such that $v \in \mathcal{F}$ (that is $\mathcal{P}(u') \cap \mathcal{F} \subseteq \mathcal{P}_{\mathrm{fin}}(I)$). Let $u \subseteq I$. One has $u \in \mathcal{F}^{\perp\perp}$ iff, for any infinite subset v of u, there exists an infinite subset w of v such that $w \in \mathcal{F}$.*

The proof is straightforward.

A *finiteness space* is a pair $X = (|X|, \mathsf{F}(X))$ where $|X|$ is a set[8] and $\mathsf{F}(X)$ is a subset of $\mathcal{P}(|X|)$ satisfying $\mathsf{F}(X)^{\perp\perp} = \mathsf{F}(X)$. The elements of $\mathsf{F}(X)$ are called the *finitary* sets of X. Observe that if $|X|$ is finite then $\mathsf{F}(X)$ must be the powerset of $|X|$: in finite dimension, there is only one possible finiteness structure.

The finiteness semantics of linear logic associates with any formula G a finiteness space G^* and to any proof π of G, an element π^* of $\mathsf{F}(G^*)$. More precisely, $|G^*|$ will be the interpretation of G in the purely relational semantics of first order propositional linear logic (this interpretation is simply denoted by $|G|$), and π^* will be the interpretation of π in this purely relational semantics (see the appendix, Section 4).

1.1 Object constructions

As usual, we define various operations on finiteness spaces, which will interpret the corresponding logical operations on formulae. We first deal with the additive and multiplicative connectives. Let X and Y be finiteness spaces.

[6] One could say "orthogonal", according to the tradition of linear logic, but this is a misleading terminology when one deals with vector spaces so we shall avoid it.

[7] Other natural definitions of duality, giving rise to other models of linear logic, are:

- $u \cap u'$ has at most one element, which gives rise to the standard model of coherence spaces;
- $u \cap u'$ is not empty, which gives rise to a quite simple model of non uniform *totality*;
- $u \cap u'$ has exactly one element, which gives rise to Loader's totality spaces ([Loa94]);
- another natural choice, suggested by one of the referees of this paper, might be to require $u \cap u'$ to be cofinite. We have no idea about the resulting model, if any.

Due, maybe, to the logical complexity of the condition that the class of "good" sets should be equal to its bi-dual (see later the definition of finiteness spaces), it seems that these different choices lead to quite different interpretations. For instance, finiteness spaces are very different from coherence spaces. It is not clear whether these various cases can be handeled within a common framework. Interestingly enough, the hypercoherence model ([Ehr93]) does not seem to admit such a synthetic description.

[8] This set can be assumed to be countable, a property preserved by all the constructions we consider.

Orthogonal. The space $X^\perp$ is defined by $|X^\perp| = |X|$ and $\mathsf{F}(X^\perp) = \mathsf{F}(X)^\perp$.

Additives. The space 0 is defined by $|0| = \emptyset$ and $\mathsf{F}(0) = \{\emptyset\}$. The space $\top$ is its dual: $\top = 0^\perp = 0$, clearly.

The space $X \oplus Y$ is given by $|X \oplus Y| = |X| + |Y|$, and $\mathsf{F}(X \oplus Y) = \{u+v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}$ (where by "+" we denote the disjoint union of sets). Indeed, one checks easily that $\mathsf{F}(X \oplus Y) = \mathsf{F}(X^\perp \oplus Y^\perp)^\perp$ and therefore $\mathsf{F}(X \oplus Y)^{\perp\perp} = \mathsf{F}(X \oplus Y)$ and the operation $\oplus$ coincides with its De Morgan dual &, that is $X \,\&\, Y = X \oplus Y$.

Multiplicatives. One sets $|1| = \{*\}$ and $\mathsf{F}(1) = \{\emptyset, \{*\}\}$ (the only possible choice), and its dual $\perp$ satisfies $\perp = 1^\perp = 1$.

The space $X \otimes Y$ is defined by $|X \otimes Y| = |X| \times |Y|$ and

$$\mathsf{F}(X \otimes Y) = \{u \times v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}^{\perp\perp}.$$

The next lemma is important as it shows that the bi-duality closure used in the definition of $\mathsf{F}(X \otimes Y)$ is essentially useless: in this particular case, it boils down to the $\subseteq$-downwards closure of $\{u \times v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}$.

Lemma 2 *Let $w \subseteq |X \otimes Y|$. One has $w \in \mathsf{F}(X \otimes Y)$ iff $\pi_1(w) \in \mathsf{F}(X)$ and $\pi_2(w) \in \mathsf{F}(Y)$.*

Proof. Assume first that $\pi_1(w) \in \mathsf{F}(X)$ and $\pi_2(w) \in \mathsf{F}(Y)$. Then $w \subseteq \pi_1(w) \times \pi_2(w) \in \mathsf{F}(X \otimes Y)$, so that $w \in \mathsf{F}(X \otimes Y)$.

Conversely, assume that $w \in \mathsf{F}(X \otimes Y)$. Let $u' \in \mathsf{F}(X)^\perp$, it will be sufficient to show that $u_0 = \pi_1(w) \cap u'$ is finite. Let $f : u_0 \to \pi_2(w)$ be a function[9] such that for all $a \in u_0$, $(a, f(a)) \in w$ (that is $f \subseteq w$, identifying a function with its graph). We show that $f \in \mathsf{F}(X \otimes Y)^\perp = \{u \times v \mid u \in \mathsf{F}(X) \text{ and } v \in \mathsf{F}(Y)\}^\perp$. So let $u \in \mathsf{F}(X)$ and $v \in \mathsf{F}(Y)$. Since $u_0 \subseteq u' \in \mathsf{F}(X)^\perp$, the set $u_0 \cap u$ is finite and hence, since f is a function, $f \cap (u \times v)$ is finite. So $f \in \mathsf{F}(X \otimes Y)^\perp$ and therefore $f \cap w$ is finite, but $f \subseteq w$, so f is finite, and hence u_0 is finite, since $u_0 = \pi_1(f)$. □

The other multiplicative operations are defined as usual: the *par* is given by $X \parr Y = (X^\perp \otimes Y^\perp)^\perp$ and the *linear implication* by $X \multimap Y = (X \otimes Y^\perp)^\perp$. Tensor and par do not coincide in general (in contrast with what happened for the additives).

Lemma 3 *Let $t \subseteq |X \multimap Y| = |X| \times |Y|$. One has $t \in \mathsf{F}(X \multimap Y)$ iff the two following conditions hold:*

1) for any $u \in \mathsf{F}(X)$, one has $t(u) = \{b \in |Y| \mid \exists a \in u \ (a, b) \in t\} \in \mathsf{F}(Y)$

2) and for any $v' \in \mathsf{F}(Y)^\perp$, one has $t^\perp(v') \in \mathsf{F}(X)^\perp$ where $t^\perp = \{(b, a) \mid (a, b) \in t\} \subseteq |Y| \times |X| = |Y^\perp \multimap X^\perp|$ is the transpose of t.

Moreover, condition (2) can be weakened to:

2') for any $b \in |Y|$, one has $t^\perp(\{b\}) \in \mathsf{F}(X)^\perp$.

Proof. Assume first that $t \in \mathsf{F}(X \multimap Y)$. Let $u \in \mathsf{F}(X)$, we show that $t(u) \in \mathsf{F}(Y)$. So let $v' \in \mathsf{F}(Y)^\perp$. We have $u \times v' \in \mathsf{F}(X \otimes Y^\perp)$, so $t \cap (u \times v')$ is finite, so $t(u) \cap v' = \pi_2(t \cap (u \times v'))$ is finite. Hence $t(u) \in \mathsf{F}(Y)$. So condition (1) holds for t, and also condition (2) as clearly $t^\perp \in \mathsf{F}(Y^\perp \multimap X^\perp)$.

Conversely, assume that t satisfies conditions (1) and (2'). Let $u \in \mathsf{F}(X)$ and $v' \in \mathsf{F}(Y)^\perp$. The set $t_0 = t \cap (u \times v')$ is finite, since $\pi_2(t_0) = t(u) \cap v'$ is finite by (1), and $\pi_1(t_0) = \bigcup_{b \in \pi_2(t_0)} (t^\perp(\{b\}) \cap u)$ is finite, as a finite union of sets which are finite by (2'). □

[9] We use the Axiom of choice here. Can this be avoided?

Exponentials. These are the constructions which really introduce the infinite in logic, here they create infinite finitary sets. One takes for $|!X|$ the set $\mathcal{M}_{\mathrm{fin}}(|X|)$ of all finite multi-sets over $|X|$ and one sets

$$\mathsf{F}(!X) = \{u^! \mid u \in \mathsf{F}(X)\}^{\perp\perp} \tag{1}$$

where $u^! = \{\mu \in |!X| \mid |\mu| \subseteq u\} = \mathcal{M}_{\mathrm{fin}}(u)$ (a set which is infinite as soon as u is not empty). If $U \subseteq |!X|$, we define $|U| = \bigcup\{|\mu| \mid \mu \in U\}$ and call this set the *global support* of U.

The next lemma is an analogue of Lemma 2 and is therefore very important.

Lemma 4 *Let $U \subseteq |!X|$. One has $U \in \mathsf{F}(!X)$ iff $|U| \in \mathsf{F}(X)$.*

Proof. If $|U| \in \mathsf{F}(X)$, then $U \subseteq |U|^! \in \mathsf{F}(!X)$ and we conclude immediately.

Assume conversely that $U \in \mathsf{F}(!X)$. Let $u' \in \mathsf{F}(X)^\perp$, it will be sufficient to show that $u_0 = |U| \cap u'$ is finite. Let $f : u_0 \to U$ be a function such that, for each $a \in u_0$, one has $a \in |f(a)|$. Such a function exists[9], since $u_0 \subseteq |U|$. Let $U' = f(u_0)$. We contend that $U' \in \mathsf{F}(!X)^\perp$. Let $u \in \mathsf{F}(X)$, it will be sufficient to show that $U' \cap u^!$ is finite. But $f^{-1}(U' \cap u^!) = f^{-1}(U') \cap f^{-1}(u^!) = u_0 \cap f^{-1}(u^!) \subseteq u' \cap u$ and $u' \cap u$ is finite, so $U' \cap u^!$ is finite because f is surjective onto U'. Therefore $U' \in \mathsf{F}(!X)^\perp$. Hence $U \cap U' = U'$ is finite, and therefore, so is $|U'|$. But clearly $u_0 \subseteq |U'|$ and so u_0 is finite as announced. □

1.2 The category of finiteness spaces and finitary relations

We define the category of finiteness spaces and finitary relations **Fin**: its objects are the finiteness spaces and if X and Y are finiteness spaces, a morphism from X to Y in **Fin** is an element of $\mathsf{F}(X \multimap Y)$, a *finitary relation* from X to Y. Identity and composition of morphisms are defined as usual (as in the category of sets and relations). It results from Lemma 3 that we have defined a category in that way.

It is straightforward to check that **Fin** is cartesian, & being the cartesian product and $\top$ the terminal object (it is also co-cartesian, and finite sums coincide with finite products). The pairing operation on morphisms and the projections are defined exactly as in the category of sets and relations.

Monoidal structure. Let $t_1 \in \mathbf{Fin}(X_1, Y_1)$ and $t_2 \in \mathbf{Fin}(X_2, Y_2)$. We define as in the purely relational model

$$t_1 \otimes t_2 = \{((a_1, a_2), (b_1, b_2)) \mid (a_1, b_1) \in t_1 \text{ and } (a_2, b_2) \in t_2\}\,.$$

We prove that $t_1 \otimes t_2 \in \mathbf{Fin}(X_1 \otimes X_2, Y_1 \otimes Y_2)$. So let $u_1 \in \mathsf{F}(X_1)$, $u_2 \in \mathsf{F}(X_2)$ and $w' \in \mathsf{F}(Y_1 \otimes Y_2)^\perp$. It will be sufficient to show that $w_0 = (t_1 \otimes t_2) \cap ((u_1 \times u_2) \times w')$ is finite; indeed, by Lemma 2, any element of $\mathsf{F}(X_1 \otimes X_2)$ is included in a set of the shape $v_1 \times v_2$, with $v_1 \in \mathsf{F}(X_1)$ and $v_2 \in \mathsf{F}(X_2)$. But $\pi_2(w_0) = (t_1(u_1) \times t_2(u_2)) \cap w'$ and so $\pi_2(w_0)$ is finite. Let $(b_1, b_2) \in \pi_2(w_0)$. To conclude, it will be sufficient to show that the set $u = \{(a_1, a_2) \mid ((a_1, a_2), (b_1, b_2)) \in w_0\} \subseteq |X_1| \times |X_2|$ is finite. But $\pi_1(u) \subseteq u_1 \cap {t_1}^\perp(\{b_1\})$ and this latter intersection is finite, since $\{b_1\} \in \mathsf{F}({Y_1}^\perp)$. Similarly, $\pi_2(u)$ is finite, so u is finite.

The category **Fin**, equipped with the tensor product $\otimes$, is symmetric monoidal. The tensor product indeed is associative, as easily checked using again Lemma 2 (the isomorphism between $X \otimes (Y \otimes Z)$ and $(X \otimes Y) \otimes Z$ being the obvious bijection between the webs of these spaces). Symmetry of the tensor product is obvious. The tensor unit is 1 and the various required coherence diagrams (see e.g. [Bie95]) commute simply because they commute in the category of sets and relations. This category is monoidal closed, the objects of morphisms from X to Y being $X \multimap Y$; this is due to the associativity of the tensor product, to our definition of $X \multimap Y$ as $(X \otimes Y^\perp)^\perp$ and to the fact that the operation $X \mapsto X^\perp$ is an involutive (contravariant) endofunctor, defined on morphisms as the transpose operation $t \mapsto t^\perp$ mentioned in Lemma 3. Moreover, by Lemma 3, the obvious bijection between $|X^\perp|$ and $|X \multimap \perp|$ is an isomorphism between $X^\perp$ and $X \multimap \perp$ and so **Fin** is $\star$-autonomous.

We study now the exponentials. First, we define the functorial promotion of morphisms. Let $t \in \mathbf{Fin}(X,Y)$, we set as in the purely relational model

$$!t = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \mid (a_i,b_i) \in t \text{ for each } i = 1,\dots,n\}\,.$$

We prove that $!t \in \mathbf{Fin}(!X,!Y)$. Let $u \in \mathsf{F}(X)$ and let $V' \in \mathsf{F}(!Y)^\perp$, it will be sufficient to show that $W = {!t} \cap (u^! \times V')$ is finite; indeed, by Lemma 4, any element of $\mathsf{F}(!X)$ is included in a set of the shape $v^!$ with $v \in \mathsf{F}(X)$. But $\pi_2(W) = {!t}(u^!) \cap V' = t(u)^! \cap V'$ is finite by Lemma 3 applied to t. Now let $b_1,\dots,b_n \in |Y|$ be such that $\nu = [b_1,\dots,b_n] \in \pi_2(W)$. It will be sufficient to show that the set $U = \{\mu \mid (\mu,\nu) \in W\}$ is finite. But if $\mu \in U$, then μ can be written $\mu = [a_1,\dots,a_n]$ with $(a_i,b_i) \in t$ and $a_i \in u$ for $i = 1,\dots,n$. So U is contained in the image of the set $(t^\perp(\{b_1\}) \cap u) \times \cdots \times (t^\perp(\{b_n\}) \cap u)$ under the mapping $(a_1,\dots,a_n) \mapsto [a_1,\dots,a_n]$. But each of the sets $t^\perp(\{b_i\}) \cap u$ is finite by Lemma 3 (as $\{b_i\} \in \mathsf{F}(Y)^\perp$), and so U is finite.

There is a canonical bijection between $|!(X \mathbin{\&} Y)|$ and $|!X \otimes !Y|$: with each $\mu \in |!(X \mathbin{\&} Y)|$, associate the pair of multi-sets (λ,ρ), where λ is the restriction of μ to $|X|$ and ρ is the restriction of μ to $|Y|$ (multi-sets being considered as integer-valued functions). This bijection is an isomorphism of finiteness spaces from $!(X \mathbin{\&} Y)$ to $!X \otimes !Y$ (this is actually an immediate consequence of Lemmas 2 and 4):

$$!(X \mathbin{\&} Y) \simeq {!X} \otimes {!Y}\,. \tag{2}$$

It is also obvious that $!0$ and 1 are isomorphic.

It remains to check that the structure maps d^X (dereliction) and p^X (digging) of the purely relational model are morphisms in the category $\mathbf{Fin}$. Remember that

$$\mathrm{d}^X = \{([a],a) \mid a \in |X|\} \subseteq |!X \multimap X|\,.$$

Let $u \in \mathsf{F}(X)$ and $u' \in \mathsf{F}(X)^\perp$. Then $\mathrm{d}^X \cap (u^! \times u') = \{([a],a) \mid a \in u \cap u'\}$ is finite as $u \cap u'$ is finite. Therefore $\mathrm{d}^X \in \mathsf{F}(!X \multimap X)$.

Next, remember that

$$\mathrm{p}^X = \{(\mu_1 + \cdots + \mu_n, [\mu_1,\dots,\mu_n]) \mid \mu_1,\dots,\mu_n \in |!X|\} \subseteq |!X \multimap !!X|\,.$$

Let $u \in \mathsf{F}(X)$ and $\mathcal{U}' \in \mathsf{F}(!!X)^\perp$, we prove that $\mathrm{p}^X \cap (u^! \times \mathcal{U}')$ is finite. Observe that $(u^!)^! = \{[\mu_1,\dots,\mu_n] \mid \mu_1,\dots,\mu_n \in |!X| \text{ and } \mu_1 + \cdots + \mu_n \in u^!\}$, and therefore $\mathrm{p}^X \cap (u^! \times \mathcal{U}') = \{(\mu_1 + \cdots + \mu_n, [\mu_1,\dots,\mu_n]) \mid [\mu_1,\dots,\mu_n] \in (u^!)^! \cap \mathcal{U}'\}$, but $(u^!)^! \cap \mathcal{U}'$ is finite, and hence $\mathrm{p}^X \cap (u^! \times \mathcal{U}')$ is finite.

A finiteness property of the relational interpretation of linear logic. We have defined a semantics of linear logic, where formulae are interpreted as finiteness spaces and proofs as finitary subsets. But the operations on webs and morphisms used for defining this semantics are just the usual corresponding operations in the pure relational semantics. Therefore, we can derive from our model construction some properties of the purely relational semantics of linear logic. Typically, we have the following result, which was one of our initial motivations (the semantics is taken in the purely relational model of linear logic).

Theorem 5 *Let π be a proof of the sequent $\vdash \Gamma, G$ and let σ be a proof of the sequent $\vdash G^\perp, \Delta$. Let ρ be the proof of $\vdash \Gamma, \Delta$ obtained by applying a cut rule. Then, for any element (φ,ψ) of ρ^*, there is only a finite number of elements a of G^* such that $(\varphi,a) \in \pi^*$ and $(a,\psi) \in \sigma^*$.*

Having in mind the usual coherence space semantics, one might think that this result is completely trivial and that the announced finite set is always a singleton. But this would be forgetting about the essential difference between the purely relational model and the coherence space model concerning the issue of *uniformity*. The point is that, in coherence spaces, when one builds $!X$, one takes for $|!X|$ the collection of all finite multi-sets μ of elements of $|X|$ such that $|\mu|$ is a *clique* of X. It is only for that reason that, in the situation of the theorem, the "intermediate set" is a singleton. Without this uniformity restriction when building the exponential, the intermediate sets can have arbitrarily large cardinalities.

A typical example, formulated in a lambda-calculus style for notational convenience only, is as follows. Consider the term t, representing a normal proof of $\vdash (!\mathbf{Bool})^\perp, \mathbf{Bool}$ where $\mathbf{Bool} = 1 \oplus 1$ (with exactly two normal proofs denoted by $\mathbf{t}$ and $\mathbf{f}$),

$$t(x) = \mathtt{if}\, x\, \mathtt{then}\, (\mathtt{if}\, x\, \mathtt{then}\, \mathbf{t}\, \mathtt{else}\, \mathbf{f})\, \mathtt{else}\, (\mathtt{if}\, x\, \mathtt{then}\, \mathbf{t}\, \mathtt{else}\, \mathbf{f})\,.$$

Consider on the other hand the following term u, representing a normal proof of $\vdash \mathbf{Bool}^\perp, \mathbf{Bool}$,

$$u(y) = \mathtt{if}\, y\, \mathtt{then}\, \mathbf{t}\, \mathtt{else}\, \mathbf{t}\,.$$

Then, with the notations of the theorem, taking $\varphi = [\mathbf{t}, \mathbf{f}] \in |!\mathbf{Bool}|$ and $\psi = \mathbf{t} \in |\mathbf{Bool}|$, we get $\{\mathbf{t}, \mathbf{f}\}$ as set of intermediate points. In coherence spaces, this phenomenon does not occur simply because $[\mathbf{t}, \mathbf{f}]$ is not an element of the web of the coherence space associated with $!\mathbf{Bool}$ since $\mathbf{t}$ and $\mathbf{f}$ are not coherent, due to the definition of $\oplus$.

Infinitary additives. We have seen that $\oplus$ and $\&$ are interpreted in the same way in finiteness spaces. This is no more the case for the infinitary versions of these operations. Let $(X_l)_{l\in L}$ be a family of finiteness spaces, whose webs are assumed to be pairwise disjoint for notational convenience. Their direct product, denoted by $\&_{l\in L} X_l$, has $\sum_{l\in L} |X_l|$ as web, and a subset w of this web will belong to $\mathsf{F}(\&_{l\in L} X_l)$ iff each of its "projections" $w \cap |X_l|$ belongs to $\mathsf{F}(X_l)$. Indeed, it is easily seen that, with this definition of $\mathsf{F}(\&_{l\in L} X_l)$, a subset w' of $\sum_{l\in L} |X_l|$ belongs to $\mathsf{F}(\&_{l\in L} X_l)^\perp$ iff $w' \cap |X_l|$ is empty for almost all values of l, and $w' \cap |X_l| \in \mathsf{F}(X_l)^\perp$ for each $l \in L$. Therefore, $\mathsf{F}(\&_{l\in L} X_l)^{\perp\perp} = \mathsf{F}(\&_{l\in L} X_l)$. It results also from these considerations that the infinitary sum $\oplus_{l\in L} X_l$ is defined as follows: its web is $\sum_{l\in L} |X_l|$, and a subset w of this web belongs to $\mathsf{F}(\oplus_{l\in L} X_l)$ iff $w \cap |X_l|$ is empty for almost all values of l, and $w \cap |X_l|$ for each $l \in L$.

It is easily checked that the space $\&_{l\in L} X_l$ so defined is indeed the product of the spaces X_l in the category **Fin** (and therefore, $\oplus_{l\in L} X_l$ is their sum).

Typically, it is reasonable to consider the type of natural numbers as a solution to the fix-point equation $N = 1 \oplus N$, and for this reason, the finiteness space of natural numbers N is defined as follows: $|\mathsf{N}| = \mathbf{N}$, the set of natural numbers, and a subset u of $\mathbf{N}$ belongs to $\mathsf{F}(\mathsf{N})$ iff u is finite. The successor and predecessor functions (which define the natural bijection between the webs of the spaces $1 \oplus \mathsf{N}$ and N) are easily seen to be isomorphisms between these finiteness spaces, and so they are finitary. The dual is given by $|\mathsf{N}^\perp| = \mathbf{N}$ and $\mathsf{F}(\mathsf{N}^\perp) = \mathcal{P}(\mathbf{N})$.

2 Iteration and fix-points

One particularly interesting feature of this semantics is that it does not admit the usual fix-point operators as finitary objects. However, it is reasonably expressive in computational terms since there are finitary iteration operators as we shall see.

We extend linear logic with a type $\mathbf{N}$ of natural numbers, intuitively subject to the equation $\mathbf{N} = 1 \oplus \mathbf{N}$. This can be represented in sequent calculus by the following rules and axioms[10]:

$$\frac{}{\vdash \mathbf{N}}\ \textit{(Zero)}$$

is an axiom with denotation the finitary set $\{0\} \subseteq \mathbf{N}$.

$$\frac{\begin{array}{c}\vdots\,\pi\\ \vdash \mathbf{N}, \Gamma\end{array}}{\vdash \mathbf{N}, \Gamma}\ \textit{(Successor)}$$

[10]We do not claim that this particular presentation has any good proof-theoretic properties. We introduce it only for giving precise definitions of a possible extension of propositional linear logic with a basic type of natural numbers, and of the denotational semantics of such a system.

is a proof with denotation $\{(n+1,\gamma) \mid (n,\gamma) \in \pi^*\}$, which is finitary as soon as π^* is.

And last

$$\dfrac{\begin{array}{c}\vdots\,\zeta\\ \vdash \Gamma\end{array}\qquad\qquad \begin{array}{c}\vdots\,\sigma\\ \vdash \mathbf{N}^\perp,\Gamma\end{array}}{\vdash \mathbf{N}^\perp,\Gamma}\;\textit{(Case)}$$

is a proof, with denotation $\{(0,\gamma) \mid \gamma \in \zeta^*\} \cup \{(n+1,\gamma) \mid (n,\gamma) \in \sigma^*\}$, which is finitary as soon as ζ^* and σ^* are.

For each set U, we define an iteration operator $\mathsf{It}_U \subseteq U^+$ where

$$U^+ = \mathbf{N} \multimap\, !(!U \multimap U) \multimap\, !U \multimap U$$

(with connectives interpreted in the purely relational model of linear logic) as a union $\mathsf{It}_U = \bigcup_{n\in\mathbf{N}} \mathsf{It}_U^{(n)}$ of increasing approximations. We set $\mathsf{It}_U^{(0)} = \{(0,[],[a],a) \mid a \in U\}$ and

$$\mathsf{It}_U^{(n+1)} = \mathsf{It}_U^{(0)} \cup \{(m+1,\varphi+[(\mu_1,b_1),\dots,(\mu_k,b_k)],\sum_{i=1}^{k}\mu_i,b) \mid (m,\varphi,[b_1,\dots,b_k],b) \in \mathsf{It}_U^{(n)}\}\,.$$

Consider the operator $\mathsf{Case}_U \subseteq \mathbf{N} \multimap\, !(\mathbf{N} \multimap U) \multimap\, !U \multimap U$ given by

$$\mathsf{Case}_U = \{(0,[],[a],a) \mid a \in U\} \cup \{(m+1,[(m,b)],[],b)\}\,.$$

This case operator is definable in linear logic as follows:

$$\dfrac{\dfrac{\dfrac{\vdash U^\perp, U}{\vdash ?U^\perp, U}\;\textit{(Dereliction)}}{\vdash ?(\mathbf{N}\otimes U^\perp), ?U^\perp, U}\;\textit{(Weakening)}\qquad \dfrac{\dfrac{\dfrac{\vdash \mathbf{N}^\perp,\mathbf{N}\qquad \vdash U^\perp, U}{\vdash \mathbf{N}^\perp, \mathbf{N}\otimes U^\perp, U}\;\textit{(Tensor)}}{\vdash \mathbf{N}^\perp, ?(\mathbf{N}\otimes U^\perp), U}\;\textit{(Dereliction)}}{\vdash \mathbf{N}^\perp, ?(\mathbf{N}\otimes U^\perp), ?U^\perp, U}\;\textit{(Weakening)}}{\vdash \mathbf{N}^\perp, ?(\mathbf{N}\otimes U^\perp), ?U^\perp, U}\;\textit{(Case)}$$

Consider the definable operator $\Phi : !U^+ \multimap U^+$ given by

$$\Phi = \lambda\mathcal{I}\,\lambda n\,\lambda f\,\lambda x\,(((\mathsf{Case}_U)n)\lambda m\,(\mathcal{I})mf(f)x)x$$

(we adopt Krivine's notational conventions for the λ-calculus: the application of M to N is written $(M)N$ and $(\dots((M)N_1)\dots)N_k$ is written simply $(M)N_1\dots N_k$).

One can check that $\mathsf{It}_U^{(0)} = \Phi(\emptyset)$ and that $\mathsf{It}_U^{(n+1)} = \Phi(\mathsf{It}_U^{(n)})$ in the purely relational model. It follows that It_U is indeed a (tail-recursive) iteration operator.

Proposition 6 *Iteration is a finitary operation, that is: if X is a finiteness space, then*

$$\mathsf{It}_{|X|} \in \mathsf{F}(\mathsf{N} \multimap\, !(!X \multimap X) \multimap\, !X \multimap X)\,.$$

Proof. Observe first that if $(m,\varphi,\mu,a) \in \mathsf{It}_{|X|}^{(n)}$ then $m \le n$. Now let $u \in \mathsf{F}(\mathsf{N})$, that is: u is a finite subset of $\mathbf{N}$. Let $n \in \mathbf{N}$ be greater than all the elements of u. Then by the observation above, $\mathsf{It}_{|X|}(u) = \mathsf{It}_{|X|}^{(n)}(u)$. But $\mathsf{It}_{|X|}^{(n)} = \Phi^{n+1}(\emptyset)$ and Φ is finitary (since it is definable in linear logic), and therefore $\mathsf{It}_{|X|}^{(n)}$ is finitary. So, by Lemma 3, $\mathsf{It}_{|X|}(u) \in \mathsf{F}(!(!X \multimap X) \multimap\, !X \multimap X)$.

To conclude, it suffices to show that ${\mathsf{It}_{|X|}}^\perp(H) \in \mathsf{F}(\mathsf{N})^\perp$ for each $H \in \mathsf{F}(!(!X \multimap X) \multimap\, !X \multimap X)^\perp$. But this is trivial since $\mathsf{F}(\mathsf{N})^\perp = \mathcal{P}(\mathbf{N})$. $\square$

We have defined It_U as the least fix-point of the operator Φ and have obtained in that way a finitary operation. One might expect therefore that the least fix-point operator itself is finitary. We show now that this is not the case.

In the purely relational semantics, the fix-point operator at type U is the set $\mathsf{Y}_U \subseteq !(!U \multimap U) \multimap U$ given by $\mathsf{Y}_U = \bigcup_{n\in\mathbf{N}} \mathsf{Y}_U^{(n)}$ where $\mathsf{Y}_U^{(0)} = \emptyset$ and

$$\mathsf{Y}_U^{(n+1)} = \{([([a_1,\dots,a_k],a)] + \sum_{i=1}^{k} \varphi_i, a) \mid (\varphi_i, a_i) \in \mathsf{Y}_U^{(n)} \text{ for } i = 1,\dots,k\}\,.$$

The reader can check that this is just the usual definition of the least fix-point operator in the Kleisli category of the purely relational model of linear logic, which is a model of PCF (a cpo-enriched cartesian closed category, the order on morphisms being inclusion).

Our negative argument is based on a PCF term which has already been considered by Danos and Harmer in [DH00]. Using the least fix-point operator, we construct $f \subseteq !\mathbf{N} \multimap \mathbf{N}$ as follows:

$$f = (\mathsf{Y}_{!\mathbf{N}\multimap\mathbf{N}})\lambda g \lambda n\,(((\mathsf{Case}_{\mathbf{N}})n)\lambda m\,(1 + (g)(m+1)))0$$

in other words f is defined recursively as

$$f \;=\; \lambda n\, \mathtt{if}\, n = 0\, \mathtt{then}\, 0\, \mathtt{else}\, (1 + (f)n)\,.$$

Then $\{(k[1] + [0], k) \mid k \in \mathbf{N}\} \subseteq f$ and therefore $f(\{0,1\}^!) = \mathbf{N}$. But $\{0,1\}^! \in \mathsf{F}(!\mathsf{N})$ and $\mathbf{N} \notin \mathsf{F}(\mathsf{N})$. So $f \notin \mathsf{F}(!\mathsf{N} \multimap \mathsf{N})$. Therefore the fix-point operator $\mathsf{Y}_{!\mathbf{N}\multimap\mathbf{N}}$ is not finitary in the finiteness space $!(!X \multimap X) \multimap X$ where $X = !\mathsf{N} \multimap \mathsf{N}$.

Proposition 7 *In general, the fix-point operator $\mathsf{Y}_{|X|}$ is not finitary in $!(!X \multimap X) \multimap X$.*

3 Module associated with a finiteness space

We associate now a vector space, or more generally a module, with any finiteness space. The web of the finiteness space will be a kind of (generally infinite) "basis" for this vector space and the finitary structure will tell us which are the acceptable vectors (linear combinations of vectors taken in this basis).

Let R be a fixed ring (or even a semi-ring; in the case $R = \{0,1\}$ with the semi-ring structure defined by $1+1=1$, we retrieve the relational model presented in Section 1.2). When convenient, we shall assume that R is a field, but this is not necessary in general;

Given a finiteness space X, we define an R-module $R\langle X\rangle$ as follows. An element x of $R\langle X\rangle$ is an $|X|$-indexed family of elements of R whose support $|x| = \{a \in |X| \mid x_a \neq 0\}$ belongs to $\mathsf{F}(X)$. Module operations are defined componentwise. If $a \in |X|$ we denote by e_a the element of $R\langle X\rangle$ given by $(e_a)_b = \delta_{a,b}$. The family $(e_a)_{a\in|X|}$ is a kind of "canonical basis" of $R\langle X\rangle$ (although it is not a generating system in the standard algebraic sense, in general). If $\mathsf{F}(X) = \mathcal{P}_{\mathrm{fin}}(|X|)$ is the minimal finitary structure over $|X|$, then $R\langle X\rangle$ is just the free R-module generated by $|X|$.

We can endow the module $R\langle X\rangle$ with a *linear topology* in the sense of Lefschetz [Lef42, Bar76, Blu96]. This topology, that we denote by $\lambda(X)$, is defined as follows. For $u' \in \mathsf{F}(X^\perp)$, let us set

$$\mathrm{V}_X(u') = \{x \in R\langle X\rangle \mid |x| \cap u' = \emptyset\}$$

and observe that, for $u', v' \in \mathsf{F}(X^\perp)$, one has $\mathrm{V}_X(u') \cap \mathrm{V}_X(v') = \mathrm{V}_X(u' \cup v')$. We say that a subset U of $R\langle X\rangle$ is open if, for any $x \in U$, there exists $u' \in \mathsf{F}(X^\perp)$ such that $x + \mathrm{V}_X(u') \subseteq U$. In other words $(\mathrm{V}_X(u'))_{u'\in\mathsf{F}(X)^\perp}$ is a basis of neighborhood of 0 for this topology which is invariant by translation. Observe first the following easy facts (which by the way can be generalized to all linearly topologized vector spaces).

- If $\mathsf{F}(X) = \mathcal{P}_{\mathrm{fin}}(|X|)$ then $\lambda(X)$ is the discrete topology. This is in particular the case when $|X|$ is finite (all finite dimensional spaces have the discrete topology).

- If $\mathsf{F}(X) = \mathcal{P}(|X|)$ then $\lambda(X)$ is the product topology (R being endowed with the discrete topology and $R\langle X\rangle$ being considered as the product $R^{|X|}$). In general, the topology of $R\langle X\rangle$ will be finer than the product topology and coarser than the discrete topology on $R^{|X|}$.
- If a sequence $x(n)$ of elements of $R\langle X\rangle$ tends to 0 when $n \to \infty$, so does any sequence of the shape $\lambda_n x(n)$ where the $\lambda_n \in R$ are arbitrary scalars.
- Each basic neighborhood $\mathrm{V}_X(u')$ of 0 is obviously open, but also closed. Indeed, if $x \in R\langle X\rangle \setminus \mathrm{V}_X(u')$, then let $a \in |x| \cap u'$; we have $\mathrm{V}_X(u') \cap (x + \mathrm{V}_X(\{a\})) = \emptyset$. So the topology $\lambda(X)$ is always totally disconnected.

This linear topology $\lambda(X)$ is always Hausdorff, and also complete as we shall see immediately. Endowing R with the *discrete* topology, it is easy to see that addition and multiplication by a scalar are continuous operations.

A Cauchy sequence[11] in $R\langle X\rangle$ is a sequence $(x(n))_{n\in\mathbf{N}}$ of elements of $R\langle X\rangle$ such that, for each neighborhood U of 0, there exists $n \in \mathbf{N}$ such that, for all $p, q \in \mathbf{N}$, if $p, q \geq n$, then $x(p) - x(q) \in U$. In other words: for each $u' \in \mathsf{F}(X)^\perp$ there exists $n \in \mathbf{N}$ such that the restriction of $x(p)$ to u' does not depend on p for $p \geq n$, or: for each $u' \in \mathsf{F}(X)^\perp$, the sequence $(x(n)|_{u'})_{n\in\mathbf{N}}$ is ultimately constant.

Lemma 8 *The space $R\langle X\rangle$ is complete.*

Proof. Let $(x(n))_{n\in\mathbf{N}}$ be a Cauchy sequence in $R\langle X\rangle$. Let $u = \bigcup_{n\in\mathbf{N}} |x(n)|$, we show first that $u \in \mathsf{F}(X)$. Let $u' \in \mathsf{F}(X^\perp)$. Let n be such that $x(p)|_{u'} = x(n)|_{u'}$ for all $p \geq n$. Let $a \in u \cap u'$. Let $m \in \mathbf{N}$ be such that $a \in |x(m)|$. If $m \geq n$, since $x(m)|_{u'} = x(n)|_{u'}$, we have $a \in |x(n)|$, and therefore $a \in u' \cap \bigcup_{m=0}^{n} |x(m)|$ but $\bigcup_{m=0}^{n} |x(m)| \in \mathsf{F}(X)$ since $\mathsf{F}(X)$ is closed under finite unions. Hence $u \cap u'$ is finite, so $u \in \mathsf{F}(X)$. For each $a \in u$, $\{a\} \in \mathsf{F}(X^\perp)$, so there exists $n_a \in \mathbf{N}$ such that $x(n)_a = x(n_a)_a$ for all $n \geq n_a$. Let $x \in R^{|X|}$ be defined by $x_a = x(n_a)_a$. It is clear that $|x| \subseteq u$ and so that $x \in R\langle X\rangle$, and that $\lim_{n\to\infty} x(n) = x$. □

Due to completeness, we have a very simple criterion for the convergence of a series.

Lemma 9 *Let $(x(n))_{n\in\mathbf{N}}$ be a family of elements of $R\langle X\rangle$. The series $\sum_{n=0}^{\infty} x(n)$ converges in $R\langle X\rangle$ iff $\lim_{n\to\infty} x(n) = 0$.*

Proof. The condition is clearly necessary. It is sufficient because there is a basis of neighborhoods of 0 which consists of linear subspaces of $R\langle X\rangle$. Indeed, let V be such a neighborhood (typically, $V = \mathrm{V}_X(u')$ for some $u' \in \mathsf{F}(X^\perp)$). Let $n \in \mathbf{N}$ be such that $x(p) \in V$ whenever $p \geq n$. Then if $p, q \in \mathbf{N}$ are such that $n \leq p \leq q$, we have $\sum_{i=p+1}^{q} x(i) \in V$ since V is a linear subspace. Therefore, the considered series satisfies the Cauchy criterion and converges by completeness of $R\langle X\rangle$. □

It is clear that, when $(x(n))_{n\in\mathbf{N}}$ converges to 0, so does $(x(\sigma(n)))_{n\in\mathbf{N}}$, where σ is any permutation of $\mathbf{N}$, and that $\sum_{n=0}^{\infty} x(\sigma(n)) = \sum_{n=0}^{\infty} x(n)$ so that we can speak of summable families indexed over an arbitrary countable set. In particular, if $x \in R\langle X\rangle$, then the family $(x_a e_a)_{a\in|X|}$ is summable and its sum is equal to x.

Observe also that, if φ is an isomorphism from X to Y in the category **Fin**, then the function $\varphi_* : R\langle X\rangle \to R^{|Y|}$ defined by $\varphi_*(x)_b = x_{\varphi^{-1}(b)}$ takes its values in $R\langle Y\rangle$ and is a linear homeomorphism between $R\langle X\rangle$ and $R\langle Y\rangle$. Of course, not all linear homeomorphisms between the vector spaces associated to finiteness spaces are of this particular shape.

If X is a finiteness space, remember that we use $X^\perp$ for denoting the finiteness space $(|X|, \mathsf{F}(X)^\perp)$. Given $x \in R\langle X\rangle$ and $x' \in R\langle X^\perp\rangle$, the sum $\sum_{a\in|X|} x_a x'_a$ has only finitely many non-zero terms, and therefore defines an element of R that we denote by $\langle x, x'\rangle$.

Let X and Y be finiteness spaces. If $x \in R\langle X\rangle$ and $y \in R\langle Y\rangle$, we define $x \otimes y$ by $(x \otimes y)_{a,b} = x_a y_b$, and this is clearly an element of $R\langle X \otimes Y\rangle$.

[11] In full generality, we should consider nets, and not only sequences, but this would not change our reasonings.

Next remember that we have defined the finiteness space $X \multimap Y$ as $(X \otimes Y^\perp)^\perp$. A morphism from X to Y is an element A of $R\langle X \multimap Y\rangle$, to be considered as a matrix indexed over $|X| \times |Y|$, with coefficients in R. To such a matrix A (called a *finitary* matrix from X to Y), we can associate a linear map $\widehat{A} : R\langle X\rangle \to R^{|Y|}$ by setting $\widehat{A}(x)_b = \sum_{a\in|X|} A_{a,b}x_a \in R$ for each $b \in |Y|$. This sum indeed is finite by Lemma 3.

Conversely, let $f : R\langle X\rangle \to R\langle Y\rangle$ be a function. Define $\mathsf{M}(f) \in R^{|X|\times|Y|}$, the matrix of f, by $\mathsf{M}(f)_{a,b} = f(e_a)_b$.

Lemma 10 *The linear map $\widehat{A}$ takes its values in $R\langle Y\rangle$, is continuous and satisfies $\mathsf{M}(\widehat{A}) = A$.*

Proof. We have $|\widehat{A}(x)| \subseteq |A|(|x|) \in \mathsf{F}(Y)$ by Lemma 3 so $\widehat{A}$ takes its values in $R\langle Y\rangle$. For proving continuity we must show that for each $v' \in \mathsf{F}(Y^\perp)$, there exists $u' \in \mathsf{F}(X^\perp)$ such that $\widehat{A}(\mathrm{V}_X(u')) \subseteq \mathrm{V}_Y(v')$. Simply take $u' = \{a \in |X| \mid \exists b \in v'\ (a,b) \in |A|\}$. The last statement of the lemma is trivial. □

Lemma 11 *Let $f : R\langle X\rangle \to R\langle Y\rangle$ be a linear and continuous function. Then $\mathsf{M}(f) \in R\langle X \multimap Y\rangle$ and moreover, $\widehat{\mathsf{M}(f)} = f$.*

Proof. Let $u \in \mathsf{F}(X)$ and $v' \in \mathsf{F}(Y^\perp)$, we must show that $w = |\mathsf{M}(f)| \cap (u \times v')$ is finite. Since f is continuous, we know that there exists $u' \in \mathsf{F}(X^\perp)$ such that $f(\mathrm{V}_X(u')) \subseteq \mathrm{V}_Y(v')$. Let $a \in |X|$. If $a \notin u'$, then $e_a \in \mathrm{V}_X(u')$ and hence $|f(e_a)| \cap v' = \emptyset$. So, setting $w(a) = \{b \mid (a,b) \in w\}$, we have

$$w = \bigcup_{a\in u\cap u'} (\{a\} \times w(a))\,.$$

But for each a, $w(a)$ is a subset of $|f(e_a)| \cap v'$ and therefore is finite. So w itself is finite since $u \cap u'$ is finite. The last part of the lemma results from the continuity and linearity of f and from the fact that $x = \sum_{a\in|X|} x_a e_a$ for all $x \in R\langle X\rangle$. □

We summarize these simple observations.

Proposition 12 *There is a linear isomorphism between $R\langle X \multimap Y\rangle$ and the R-module of linear continuous functions from $R\langle X\rangle$ to $R\langle Y\rangle$.*

Although not technically essential, this proposition is important as it means that, in spite of the fact that the modules we consider are given together with a "basis" (the web of the underlying finiteness space), the notion of morphism between these modules is defined independently of these webs. This fact, together with the functoriality of the various operations on objects, shows that, at least in principle, the category of modules we consider could be presented in an intrinsic way. Mimicking what we did in [Ehr02], we could say for instance that a "finitary R-module" is an R-module M equipped with a linear topology (in the sense of [Lef42]) *such that there exists* (and *not* equipped with) a finiteness space X and a linear homeomorphism between M and the module $R\langle X\rangle$, equipped with the topology $\lambda(X)$; then all the constructions we perform on finiteness spaces can be transfered to finitary R-modules, that is, can be expressed in a web-independent way.

We want now to give a functional account[12] of the linear topology of $R\langle X \multimap Y\rangle$, in terms of the linear topologies of $R\langle X\rangle$ and $R\langle Y\rangle$. For this purpose, we shall first characterize the compact subsets of $R\langle X\rangle$ in terms of finitary sets.

Lemma 13 *A subset K of $R\langle X\rangle$ is compact for the topology $\lambda(X)$ iff*

1. *K is closed;*
2. *the set $|K| = \bigcup\{|x| \mid x \in K\}$ belongs to $\mathsf{F}(X)$;*
3. *for each $a \in |X|$, the set $\{x_a \mid x \in K\}$ is a finite subset of R.*

[12] Something that we have not been able to do for Köthe spaces in [Ehr02].

Proof. Assume first that K is compact. Condition (1) holds because $R\langle X\rangle$ is Hausdorff. Condition (3) holds because, for each $a \in |X|$, the projection $x \mapsto x_a$ from $R\langle X\rangle$ to R is continuous, and because the compact subsets for the discrete topology are just the finite ones.

We prove property (2), so let $u' \in \mathsf{F}(X^\perp)$. Since $\mathrm{V}_X(u')$ is a neighborhood of 0 and since K is compact, there is a finite subset M of K such that $K \subseteq M + \mathrm{V}_X(u')$. Let $a \in |K| \cap u'$. Let $x \in K$ be such that $a \in |x|$. Let $y \in M$ be such that $x \in y + \mathrm{V}_X(u')$. Since $a \in u'$, we have $x_a = y_a$ and hence $a \in |y|$. We have shown that $u' \cap |K| \subseteq u' \cap |M|$ and we conclude since $|M| \in \mathsf{F}(X)$ as M is finite.

Assume now that K satisfies the three conditions of the lemma and let us prove that K is compact. For $a \in |X|$, let $N_a = \{x_a \mid x \in K\}$. This set is finite, reduced to $\{0\}$ for $a \notin |K| \in \mathsf{F}(X)$, and will be considered as a discrete topological space. The topological product space $\prod_{a\in|X|} N_a$ is a topological subspace of $R\langle X\rangle$ (indeed, $\mathcal{P}_{\mathrm{fin}}(|K|) = \{|K| \cap u' \mid u' \in \mathsf{F}(X^\perp)\}$ and so the topology induced by $R\langle X\rangle$ on its subspace $\{x \in R\langle X\rangle \mid |x| \subseteq |K|\}$ is just the product topology, this subspace being identified with the product $R^{|K|}$) and is compact by Tychonov theorem. Therefore, K is compact as a closed subset of a compact space. □

Let us say that a subspace F of $R\langle X\rangle$ is *linearly compact* if F is the closure of the linear span of a compact subset of $R\langle X\rangle$. This notion coincides with the notion of linear compactness introduced in [Lef42]. More precisely, assuming that R is a field (what we do until the end of this paragraph),

Proposition 14 *Let F be a linear subspace of $R\langle X\rangle$. The following conditions are equivalent:*

1. *F is linearly compact;*
2. *F is closed and $|F| \in \mathsf{F}(X)$;*
3. *(the original definition of Lefschetz) for any filter $\mathcal{G}$ of closed affine subspaces of $R\langle X\rangle$ such that $G \cap F \neq \emptyset$ for each $G \in \mathcal{G}$, one has $\cap\mathcal{G} \cap F \neq \emptyset$.*

Proof. That (1) implies (2) is straightforward (observe that, given any $u \subseteq |X|$, the set $\{x \in R\langle X\rangle \mid |x| \subseteq u\}$ is closed). For proving the converse, one can proceed as follows (a kind of pivot method), assuming that $|X|$ is countable which, as already mentioned, is a reasonable restriction. First, enumerate $|F| = \{a_1, a_2, \dots\}$ (if this set is finite then F is finite-dimensional and one concludes trivially). Then choose one element $x(1) \in F$ such that $x(1)_{a_1} \neq 0$. Since R is a field, we can assume that $x(1)_{a_1} = 1$. Next we can linearly project F onto $R \cdot x(1)$ by the map $p : x \mapsto x_1 \cdot x(1)$ and we have $F = R \cdot x(1) \oplus F_1$ where $F_1 = (\mathrm{Id} - p)(F)$ is a closed subspace of F such that $a_1 \notin |F_1|$. We can iterate this process, producing a sequence $x(i)$ of elements of F such that $x(i)_{a_j} = 0$ for $j < i$ (this process can stop at some finite rank N, producing $F_N = 0$ and in that case again, F is finite-dimensional; in the sequel we use $N \in \mathbf{N} \cup \{\infty\}$ for dealing with both cases). Using Lemma 13, it is easy to check that the collection $\{x(i) \mid i < N\}$ is a compact subset of F whose linear span is dense in F (indeed, due to the fact that $|F| \in \mathsf{F}(X)$, the topology of F is simply the induced product topology of $R^{|F|}$). More precisely, any element x of F can be written exactly in one way as a converging sum $x = \sum_{i=1}^{\infty} \lambda_i x(i)$ with $\lambda_i \in R$ and conversely, each such sum converges to an element of F. This establishes a linear homeomorphism between F and R^N equipped with the product topology. We retrieve the fact, mentioned in [Lef42], that a linearly compact subspace is linearly homeomorphic to a power of R.

We leave the equivalences concerning (3) to the reader, as they are not essential to our purpose. □

As in [Lef42], let us say that a linearly topologized vector space is *locally linearly compact* if its topology admits a sub-basis of neighborhoods of 0 which consists of linearly compact subspaces only. We have a straightforward characterization of the finiteness spaces which give rise to locally linearly compact spaces $R\langle X\rangle$.

Proposition 15 *The space $R\langle X\rangle$ is locally linearly compact if and only if there exist $u \in \mathsf{F}(X)$ and $u' \in \mathsf{F}(X^\perp)$ such that $u \cup u' = |X|$. In that case we shall simply say that X is locally linearly compact.*

Proof. If F is locally linearly compact, let F be a linearly compact neighborhood of 0 and set $u = |F| \in \mathsf{F}(X)$. There must exist $u' \in \mathsf{F}(X^\perp)$ such that $\mathrm{V}_X(u') \subseteq F$, and for such an u' we have $u \cup u' = |X|$.

Conversely, if we have two such subsets u and u' of $|X|$, then for any $v' \in \mathsf{F}(X^\perp)$, the 0-neighborhood $\mathrm{V}_X(v' \cup u') = \{x \in R\langle X\rangle \mid |x| \subseteq u \setminus (u' \cup v')\}$ is a linearly compact subspace of $R\langle X\rangle$, and since these neighborhoods generate the topology $\lambda(X)$, the space $R\langle X\rangle$ is locally linearly compact. □

Saying that X is locally linearly compact means intuitively that $\mathsf{F}(X)$ has a greatest element. More precisely, it means that the quotient order associated to the following preorder $\sqsubseteq$ on $\mathsf{F}(X)$

$$u \sqsubseteq v \quad \text{if} \quad u \setminus v \text{ is finite}$$

has a greatest element. This property is easily seen to be preserved by all the space constructions we consider (including linear negation) *apart from the exponentials* as we shall see soon.

Let X and Y be finiteness spaces. If F is a linearly compact subspace of $R\langle X\rangle$ and V is a neighborhood of 0 in $R\langle Y\rangle$ (we can of course assume that V is also a linear subspace of $R\langle Y\rangle$), we define

$$\mathcal{W}(F, V) = \{A \in R\langle X \multimap Y\rangle \mid \widehat{A}(F) \subseteq V\}\,.$$

As an immediate corollary of Proposition 14, we obtain the following characterization of the linear topology of $R\langle X \multimap Y\rangle$.

Proposition 16 *The subsets $\mathcal{W}(F, V)$ constitute a basis of neighborhoods of* 0 *for the topology* $\lambda(X \multimap Y)$.

When restricted to the case where $|Y|$ is a singleton, this is exactly the topology prescribed by Lefschetz for the topological dual of a linearly topologized vector space.

Monoidal structure. Given two matrices $A \in R\langle X \multimap Y\rangle$ and $B \in R\langle Y \multimap Z\rangle$, we define their product $C = BA$ indexed over $|X| \times |Z|$ by $C_{a,c} = \sum_{b\in|Y|} B_{b,c}A_{a,b}$. It is easy to check that this sum is finite, and that the resulting matrix belongs to $R\langle X \multimap Z\rangle$. Moreover, one checks that $\widehat{BA} = \widehat{B} \circ \widehat{A}$. The identity matrix $\mathrm{I} \in R\langle X \multimap X\rangle$ is defined by $\mathrm{I}_{a,b} = \delta_{a,b}$. In that way, we have defined a category whose objects are the finiteness spaces and whose morphisms are the finitary matrices (or equivalently, the linear continuous functions).

We denote by $\mathbf{Fin}\,(R)$ this category.

In $\mathbf{Fin}\,(R)$, the operation $\otimes$ defines a tensor product, whose object part has been defined above. Given $A \in R\langle X \multimap Y\rangle$ and $A' \in R\langle X' \multimap Y'\rangle$, we define $A \otimes A' \in R^{(|X|\times|X'|)\times(|Y|\times|Y'|)}$ by $(A \otimes A')_{(a,a'),(b,b')} = A_{a,b}A'_{a',b'}$. Then one checks easily that $A \otimes A' \in R\langle(X \otimes X') \multimap (Y \otimes Y')\rangle$, and that this operation $\otimes$ on morphisms is functorial. If $x \in R\langle X\rangle$ and $x' \in R\langle X'\rangle$, we have in particular $\widehat{A \otimes A'}(x \otimes x') = \widehat{A}(x) \otimes \widehat{A'}(x')$.

It is routine then to check that $(\mathbf{Fin}\,(R), \otimes)$ is a symmetric monoidal category, the unit of the tensor being the finiteness space 1 given by $|1| = \{\star\}$ (so that $R\langle 1\rangle = R$). This symmetric monoidal category is closed (with $X \multimap Y$ as objects of morphisms from X to Y), and is actually a $\star$-autonomous category, $\perp = 1$ being the dualizing object. If $f : R\langle X \otimes Y\rangle \to R\langle Z\rangle$ is linear and continuous, then the corresponding linear and continuous function $f' : R\langle X\rangle \to R\langle Y \multimap Z\rangle$ is given by $f'(x)(y) = f(x\otimes y)$ (considering $f'(x)$ as a continuous linear function). The evaluation function $\mathrm{ev} : R\langle(Y \multimap Z) \otimes Y\rangle \to R\langle Z\rangle$ is given by $\mathrm{ev}(f \otimes x) = f(x)$ (again, identifying $R\langle Y \multimap Z\rangle$ with the space of continuous and linear functions from $R\langle Y\rangle$ to $R\langle Z\rangle$).

Universal property of the tensor product. We assume in this paragraph again that R is a field (because under this hypothesis we have a simple characterization of linearly compact subspaces: see Proposition 14). Of course, the function $\tau : R\langle X\rangle \times R\langle Y\rangle \to R\langle X \otimes Y\rangle$ defined by $\tau(x, y) = x \otimes y$ is bilinear, and so any continuous linear function $g : R\langle X \otimes Y\rangle \to R\langle Z\rangle$ determines a bilinear function $f = g \circ \tau : R\langle X\rangle \times R\langle Y\rangle \to R\langle Z\rangle$. These bilinear mappings can be characterized as the *hypocontinuous* ones, a mild adaptation[13] of a standard notion which is strictly weaker in general than continuity with respect to the product topology.

[13] Standard hypocontinuity involves *boundedness*, a notion which does not really make sense here.

Let us first argue that such a bilinear map cannot be required to be continuous. For this, it will be clearly enough to show that the bilinear map $e : R\langle X\rangle \times R\langle X^\perp\rangle \to R$ given by $e(x, x') = \langle x, x'\rangle$ is not continuous in general. One checks easily that this function is continuous if and only if there exists $u' \in \mathsf{F}(X^\perp)$ and $u \in \mathsf{F}(X)$ such that $u \cup u' = |X|$, that is, if and only if X is locally linearly compact.

We show that some of our space constructions give rise to non locally linearly compact spaces. Let $X = !\mathsf{N}$ (the space N has been defined at the end of Section 1.2; its web is $\mathbf{N}$ and a subset of $\mathbf{N}$ is finitary if it is finite). Let $U \in \mathsf{F}(X)$ and $U' \in \mathsf{F}(X^\perp)$, we want to show that $U \cup U'$ cannot be equal to $|X|$. Let $u = |U| = \bigcup\{|\mu| \mid \mu \in U\} \in \mathsf{F}(N) = \mathcal{P}_{\mathrm{fin}}(\mathbf{N})$. Let $a \in \mathbf{N} \setminus u$. Then for each non-zero integer n we have $n[a] \notin U$. On the other hand, the set $\{n[a] \in U' \mid n > 0\}$ must be finite. So there is a non-zero integer n such that $n[a] \notin U \cup U'$ and therefore $U \cup U' \neq |X|$. So $!N$ is not locally linearly compact and the corresponding bilinear evaluation map is not continuous.

Let X, Y and Z be finiteness spaces. A bilinear function $f : R\langle X\rangle \times R\langle Y\rangle \to R\langle Z\rangle$ is hypocontinuous if, for any linear neighborhood W of 0 in $R\langle Z\rangle$:

- for any linearly compact subspace F of $R\langle X\rangle$ there is a linear neighborhood V of 0 in $R\langle Y\rangle$ such that $f(F \times V) \subseteq W$
- and for any linearly compact subspace G of $R\langle Y\rangle$ there is a linear neighborhood U of 0 in $R\langle X\rangle$ such that $f(U \times G) \subseteq W$.

Of course, if both spaces X and Y are locally linearly compact, then a bilinear map on $R\langle X\rangle \times R\langle Y\rangle$ is hypocontinuous if and only if it is continuous.

The tensor product we have defined has the standard universal property with respect to this notion of bilinear mappings.

Proposition 17 *The bilinear function $\tau : R\langle X\rangle \times R\langle Y\rangle \to R\langle X \otimes Y\rangle$ defined by $\tau(x, y) = x \otimes y$ is hypocontinuous, and moreover, for any hypocontinuous bilinear function $f : R\langle X\rangle \times R\langle Y\rangle \to R\langle Z\rangle$, there is a unique continuous linear function $g : R\langle X \otimes Y\rangle \to R\langle Z\rangle$ such that $f = g \circ \tau$.*

This is a direct corollary of Proposition 12, Proposition 16 and of the monoidal closeness of the category $\mathbf{Fin}\,(R)$.

We can also characterize $R\langle X \otimes Y\rangle$ as the topological dual of the space of all hypocontinuous bilinear functions from $R\langle X\rangle \times R\langle Y\rangle$ to R equipped with the topology of uniform convergence on all linearly compact subspaces of $R\langle X\rangle \times R\langle Y\rangle$ (a basis of neighborhood for this space is given by the sets $\{g \mid g(F \times G) = \{0\}\}$ for F and G linearly compact subspaces of $R\langle X\rangle$ and $R\langle Y\rangle$ respectively.).

Metrizability. We have seen that the modules associated to finiteness spaces cannot be assumed to be locally linearly compact. The next natural question to ask is whether they can be assumed to be metrizable. The answer again is no and can be obtained as follows: show first that $R\langle X\rangle$ is metrizable iff there is a non decreasing sequence $(u_n)_{n\in\mathbf{N}}$ of elements of $\mathsf{F}(X)$ such that for each $u \in \mathsf{F}(X)$ there exists n such that $u \subseteq u_n$. Then, show that the finiteness space $(!N)^\perp$ has not this property (this boils down to a Cantor diagonal argument). So logical constructs obliges us to consider a fairly general class of finiteness spaces.

Products and coproducts. This category has all denumerable products and coproducts. Let indeed $(X_i)_{i\in I}$ be an at most countable family of finiteness spaces. Then $\&_{i\in I} X_i$ is the product of the spaces X_i in the category $\mathbf{Fin}\,(R)$. It is clear that $R\langle \&_{i\in I} X_i\rangle$ is canonically isomorphic to the product module $\prod_{i\in I} R\langle X_i\rangle$ and that its topology is the product of the topologies $\lambda(X_i)$. The finiteness space $\oplus_{i\in I} X_i$ is the sum of the spaces X_i in this category and $R\langle \oplus_{i\in I} X_i\rangle$ is the sub-module of $\prod_{i\in I} R\langle X_i\rangle$ whose elements are the families which vanish in almost all components of the product. Of course, when I is finite, one has $\oplus_{i\in I} X_i = \&_{i\in I} X_i$, a property which is completely standard in this kind of categories where morphisms can be added and where composition commutes to these sums (a category enriched over commutative monoids, see [Mac71]).

Exponentials. Let us first introduce some additional notations concerning finite multi-sets. If μ is an element of $\mathcal{M}_{\mathrm{fin}}(I)$, we define its *size* (or cardinality) as $\#\mu = \sum_{i\in I}\mu(i) \in \mathbf{N}$. We also define its *factorial* as $\mu! = \prod_{i\in I}\mu(i)! \in \mathbf{N}$. If $\mu,\nu \in \mathcal{M}_{\mathrm{fin}}(I)$ are such that $\nu \leq \mu$, we define the *binomial coefficient*

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu-\nu)!} = \prod_{i\in I}\binom{\mu(i)}{\nu(i)}\,.$$

For $x \in R^I$ and $\mu \in \mathcal{M}_{\mathrm{fin}}(I)$, we define $x^\mu \in R$ as $x^\mu = \prod_{i\in I} x_i^{\mu(i)}$. Since the multi-set μ is finite, this product makes sense (we adopt the usual convention that $0^0 = 1$). It is essential to observe that

$$x^\mu \neq 0 \Rightarrow |\mu| \subseteq |x|\,. \tag{3}$$

With these notations, the usual binomial equation immediately generalizes as follows: for $x, y \in R^I$ and $\mu \in \mathcal{M}_{\mathrm{fin}}(I)$, one has

$$(x+y)^\mu = \sum_{\nu\leq\mu}\binom{\mu}{\nu} x^\nu y^{\mu-\nu}\,. \tag{4}$$

Let now S be a commutative monoid (with additive notations for the operations). If $\mu \in \mathcal{M}_{\mathrm{fin}}(S)$, we denote by $\Sigma(\mu)$ the element of S given by $\Sigma(\mu) = \sum_{s\in S}\mu(s)s$.

We introduce next *multinomial coefficients* for multi-sets. Let J be another index set. Let $\mu \in \mathcal{M}_{\mathrm{fin}}(I)$ and let $\sigma \in \mathcal{M}_{\mathrm{fin}}(I\times J)$ (to be considered here as a J-indexed collection of multi-sets over I). If the following property holds:

$$\forall i \in I \quad \sum_{j\in J}\sigma(i,j) = \mu(i)$$

then we define the multinomial coefficient

$$\begin{bmatrix}\mu\\ \sigma\end{bmatrix} = \frac{\mu!}{\sigma!} \in \mathbf{N}\,.$$

The binomial coefficient $\binom{\mu}{\nu}$ corresponds to the particular case $J = \{1,2\}$, $\sigma(i,1) = \nu(i)$ and $\sigma(i,2) = \mu(i) - \nu(i)$.

Let $x \in R\langle X\rangle$. We define $x^! \in R^{|!X|}$ by $x^!_\mu = x^\mu$. It results from property (3) that $|x^!| \subseteq |x|^! \in \mathsf{F}(!X)$, so that $x^! \in R\langle !X\rangle$. We describe now the action of ! on morphisms. Let $A \in R\langle X \multimap Y\rangle$, we define $!A \in R^{|!X|\times|!Y|}$ by setting, for $\mu \in |!X|$ and $\nu \in |!Y|$:

$$(!A)_{\mu,\nu} = \sum_{\sigma\in L(\mu,\nu)}\begin{bmatrix}\nu\\ \sigma\end{bmatrix} A^\sigma\,, \tag{5}$$

where $L(\mu,\nu)$ is the (finite) set of all multi-sets σ over $|X|\times|Y|$ such that $\sum_{b\in|Y|}\sigma(a,b) = \mu(a)$ for each $a \in |X|$ and $\sum_{a\in|X|}\sigma(a,b) = \nu(b)$ for each $b \in |Y|$.

The first thing to observe is that $!A \in R\langle !X \multimap !Y\rangle$ as, if $(\mu,\nu) \in |!A|$, then μ and ν must have same cardinality n, and must be of the shape $[a_1,\dots,a_n]$ and $[b_1,\dots,b_n]$ respectively, with $(a_i,b_i) \in |A|$ for each i. In other words $|!A| \subseteq !|A|$ where the exponential in the right-hand side of this equation is taken in the relational category **Fin** of Section 1.2. But then since $|A| \in \mathsf{F}(X \multimap Y)$, it follows that $|!A| \in \mathsf{F}(!X \multimap !Y)$.

Next we claim that this operation is functorial. That $!\mathrm{Id} = \mathrm{Id}$ is fairly clear. Let $A \in R\langle X \multimap Y\rangle$ and $B \in R\langle Y \multimap Z\rangle$, we must check now that $(!B)(!A) = !(BA)$. The proof is based on the following simple identity.

Lemma 18 *Let I and J be sets and let $\alpha \in \mathcal{M}_{\mathrm{fin}}(I)$ and $\beta \in \mathcal{M}_{\mathrm{fin}}(J)$ be such that $\#\alpha = \#\beta = n$. Then*

$$\begin{bmatrix}n\\ \alpha\end{bmatrix}\begin{bmatrix}n\\ \beta\end{bmatrix} = \sum_{\gamma\in L(\alpha,\beta)}\begin{bmatrix}n\\ \gamma\end{bmatrix}\,.$$

Proof. We can assume without loss of generality that I and J are finite. Let U_i $(i \in I)$ and V_j $(j \in J)$ be pairwise distinct formal indeterminates. In the algebra $\mathcal{P}$ of polynomials of indeterminates (U_i) and (V_j) (over any field with characteristic 0), we compute the expression $(\sum_{i\in I} U_i)^n(\sum_{j\in J} V_j)^n$ in two different ways. First, we have

$$\begin{aligned}\left(\sum_{i\in I} U_i\right)^n \left(\sum_{j\in J} V_j\right)^n &= \left(\sum_{\alpha\in\mathcal{M}_{\mathrm{fin}}(I)} \begin{bmatrix} n \\ \alpha \end{bmatrix} U^\alpha\right)\left(\sum_{\beta\in\mathcal{M}_{\mathrm{fin}}(J)} \begin{bmatrix} n \\ \beta \end{bmatrix} V^\beta\right) \\ &= \sum_{\substack{\alpha\in\mathcal{M}_{\mathrm{fin}}(I)\\ \beta\in\mathcal{M}_{\mathrm{fin}}(J)}} \begin{bmatrix} n \\ \alpha \end{bmatrix}\begin{bmatrix} n \\ \beta \end{bmatrix} U^\alpha V^\beta .\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(\sum_{i\in I} U_i\right)^n \left(\sum_{j\in J} V_j\right)^n &= \left(\sum_{(i,j)\in I\times J} U_i V_j\right)^n \\ &= \sum_{\substack{\gamma\in\mathcal{M}_{\mathrm{fin}}(I\times J)\\ \#\gamma=n}} \begin{bmatrix} n \\ \gamma \end{bmatrix} \prod_{(i,j)\in I\times J} (U_i V_j)^{\gamma(i,j)} \\ &= \sum_{\substack{\gamma\in\mathcal{M}_{\mathrm{fin}}(I\times J)\\ \#\gamma=n}} \begin{bmatrix} n \\ \gamma \end{bmatrix} \prod_{i\in I} U_i^{\sum_{j\in J}\gamma(i,j)} \prod_{j\in J} V_j^{\sum_{i\in I}\gamma(i,j)} \\ &= \sum_{\substack{\alpha\in\mathcal{M}_{\mathrm{fin}}(I)\\ \beta\in\mathcal{M}_{\mathrm{fin}}(J)}} \left(\sum_{\gamma\in L(\alpha,\beta)} \begin{bmatrix} n \\ \gamma \end{bmatrix}\right) U^\alpha V^\beta\end{aligned}$$

and we conclude. $\square$

Let us prove as announced that $(!B)(!A) = !(BA)$. Given $\mu \in |!X|$ and $\rho \in |!Z|$, we have

$$\begin{aligned} !(BA)_{\mu,\rho} &= \sum_{\varphi\in L(\mu,\rho)} \begin{bmatrix} \rho \\ \varphi \end{bmatrix} \prod_{(a,c)\in|X|\times|Z|} \left(\sum_{b\in|Y|} B_{b,c}A_{a,b}\right)^{\varphi(a,c)} \\ &= \sum_{\varphi\in L(\mu,\rho)} \begin{bmatrix} \rho \\ \varphi \end{bmatrix} \prod_{(a,c)\in|X|\times|Z|} \left(\sum_{\substack{\nu\in|!Y|\\ \#\nu=\varphi(a,c)}} \begin{bmatrix} \varphi(a,c) \\ \nu \end{bmatrix} \prod_{b\in|Y|} (B_{b,c}A_{a,b})^{\nu(b)}\right) \\ &= \sum_{\varphi\in L(\mu,\rho)} \begin{bmatrix} \rho \\ \varphi \end{bmatrix} \sum_{\psi\in L'(\varphi)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)} \\ &= \sum_{\substack{\varphi\in L(\mu,\rho)\\ \psi\in L'(\varphi)}} \begin{bmatrix} \rho \\ \psi \end{bmatrix} \prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)}\end{aligned}$$

where $L'(\varphi)$ is the set of all $\psi \in \mathcal{M}_{\mathrm{fin}}(|X|\times|Y|\times|Z|)$ such that $\sum_{b\in|Y|}\psi(a,b,c) = \varphi(a,c)$, for all $(a,c) \in$

$|X| \times |Y|$. Given $\nu \in |!Y|$, let $L(\mu,\nu,\rho)$ be the set of all $\psi \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y| \times |Z|)$ such that

$$\begin{aligned}
\forall a \in |X| \quad & \sum_{\substack{b\in|Y|\\c\in|Z|}} \psi(a,b,c) = \mu(a)\,,\\
\forall b \in |Y| \quad & \sum_{\substack{a\in|X|\\c\in|Z|}} \psi(a,b,c) = \nu(b) \quad \text{and}\\
\forall c \in |Z| \quad & \sum_{\substack{a\in|X|\\b\in|Y|}} \psi(a,b,c) = \rho(c)\,.
\end{aligned}$$

Using this notation, we get

$$!(BA)_{\mu,\rho} = \sum_{\nu\in|!Y|} \sum_{\psi\in L(\mu,\nu,\rho)} \begin{bmatrix}\rho\\ \psi\end{bmatrix} \prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)}$$

and so it will be sufficient to show that, for all $\nu \in |!Y|$, one has

$$\sum_{\psi\in L(\mu,\nu,\rho)} \begin{bmatrix}\rho\\ \psi\end{bmatrix} \prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)} = \sum_{\substack{\sigma\in L(\mu,\nu)\\ \tau\in L(\nu,\rho)}} \begin{bmatrix}\rho\\ \tau\end{bmatrix} \begin{bmatrix}\nu\\ \sigma\end{bmatrix} B^{\tau} A^{\sigma}\,.$$

But given $\psi \in L(\mu,\nu,\rho)$, we can define $\psi_1 \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y|)$ and $\psi_2 \in \mathcal{M}_{\mathrm{fin}}(|Y| \times |Z|)$ by $\psi_1(a,b) = \sum_{c\in|Z|} \psi(a,b,c)$ and $\psi_2(b,c) = \sum_{a\in|X|} \psi(a,b,c)$. It is quite clear that $\psi_1 \in L(\mu,\nu)$ and that $\psi_2 \in L(\nu,\rho)$, and also that

$$\prod_{(a,b,c)\in|X|\times|Y|\times|Z|} (B_{b,c}A_{a,b})^{\psi(a,b,c)} = B^{\psi_2} A^{\psi_1}\,.$$

Therefore, we are reduced to showing that, for any $\sigma \in L(\mu,\nu)$ and $\tau \in L(\nu,\rho)$,

$$\begin{bmatrix}\rho\\ \tau\end{bmatrix} \begin{bmatrix}\nu\\ \sigma\end{bmatrix} = \sum_{\substack{\psi\in L(\mu,\nu,\rho)\\ \psi_1=\sigma,\ \psi_2=\tau}} \begin{bmatrix}\rho\\ \psi\end{bmatrix}\,.$$

Let L be the set of all $\psi \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y| \times |Z|)$ such that $\psi_1 = \sigma$ and $\psi_2 = \tau$. It is clear that $L \subseteq L(\mu,\nu,\rho)$. Multiplying both sides of the equation above by $\nu!/\rho!$, we are left with showing that

$$\begin{bmatrix}\nu\\ \tau\end{bmatrix} \begin{bmatrix}\nu\\ \sigma\end{bmatrix} = \sum_{\psi\in L} \begin{bmatrix}\nu\\ \psi\end{bmatrix}\,,$$

which results from Lemma 18 and from the fact that L can be considered as the set of all families $(\psi_b)_{b\in|Y|}$ such that, for each $b \in |Y|$, $\psi_b \in L(\sigma_b, \tau^b)$, where $\sigma_b \in |!X|$ is defined by $\sigma_b(a) = \sigma(a,b)$ and $\tau^b \in |!Z|$ is defined by $\tau^b(c) = \tau(b,c)$. □

From the functoriality of !, we can deduce that, for all $A \in R\langle X \multimap Y\rangle$ and $x \in R\langle X\rangle$, the following essential equation holds:

$$(A \cdot x)^! = !A \cdot x^!\,. \tag{6}$$

Indeed, we can define a linear morphism $\tilde{x} \in R\langle 1 \multimap X\rangle$ by $\tilde{x}_{*,a} = x_a$ (for each $a \in |X|$) and it is clear that $x^! = !\tilde{x} \cdot 1^!$ where $1 \in R\langle 1\rangle$ is the unit of R. Equation (6) was by the way our starting point for arriving to expression (5).

Comonadic structure. This functor ! has a structure of comonad given by two natural transformations $\mathrm{d}^X \in R\langle !X \multimap X\rangle$ (dereliction) and $\mathrm{p}^X \in R\langle !X \multimap !!X\rangle$ (digging). The matrices of these natural transformations have only 0 and 1 coefficients and are given by

$$\mathrm{d}^X_{\mu,a} = \delta_{\mu,[a]} \quad \text{and} \quad \mathrm{p}^X_{\mu,M} = \delta_{\mu,\Sigma(M)}\,.$$

We have already checked in Section 1 that $|\mathrm{d}^X| \in \mathsf{F}(!X \multimap X)$ and $|\mathrm{p}^X| \in \mathsf{F}(!X \multimap !!X)$ (these supports are exactly the dereliction and digging morphisms of the finiteness spaces relational model presented in that section).

The naturality of these morphisms can be checked by simple computations. The following equations express that these natural transformations endow the functor ! with a comonad structure, they are checked similarly.

$$\mathrm{d}^{!X} \circ \mathrm{p}^X = \mathrm{Id}_{!X}\,, \quad !\mathrm{d}^X \circ \mathrm{p}^X = \mathrm{Id}_{!X}$$

and

$$\mathrm{p}^{!X} \circ \mathrm{p}^X = !\mathrm{p}^X \circ \mathrm{p}^X\,.$$

Let us just check that p^X is natural, so let $A \in R\langle X \multimap Y\rangle$, and let us check that, for $\mu \in |!X|$ and $N \in |!!Y|$, we have

$$(\mathrm{p}^Y \circ !A)_{\mu,N} = (!!A \circ \mathrm{p}^X)_{\mu,N}\,,$$

that is

$$(!A)_{\mu,\Sigma(N)} = \sum_{\substack{M\in|!!X|\\ \Sigma(M)=\mu}} (!!A)_{M,N}\,. \tag{7}$$

Let $x \in R\langle X\rangle$ and let $N \in |!!Y|$. Applying twice equation (6), we have $(A\cdot x)^{!!} = !!A \cdot x^{!!}$. We obtain therefore

$$\begin{aligned}
\sum_{\mu\in|!X|} (!A)_{\mu,\Sigma(N)} x^\mu &= \sum_{\mu\in|!X|} (!A)_{\mu,\Sigma(N)} x^!_\mu \\
&= (!A\cdot x^!)_{\Sigma(N)} \\
&= (Ax)^!_{\Sigma(N)} \quad \text{by equation (6)} \\
&= (Ax)^{!!}_N \\
&= (!!A\cdot x^{!!})_N \quad \text{by equation (6) again} \\
&= \sum_{M\in|!!X|} (!!A)_{M,N} x^{!!}_M \\
&= \sum_{\mu\in|!X|} \left(\sum_{\substack{M\in|!!X|\\ \Sigma(M)=\mu}} (!!A)_{M,N} \right) x^\mu\,.
\end{aligned}$$

To summarize, the following equation holds in R:

$$\sum_{\mu\in|!X|} (!A)_{\mu,\Sigma(N)} x^\mu = \sum_{\mu\in|!X|} \left(\sum_{\substack{M\in|!!X|\\ \Sigma(M)=\mu}} (!!A)_{M,N} \right) x^\mu\,, \tag{8}$$

from which we deduce equation (7) by the following simple argument. Let $I \subseteq |X|$ be finite. Let $U = (U_a)_{a\in I}$ be a family of pairwise distinct formal indeterminates. Let $R' = R[U]$ be the ring of polynomials with coefficients in R and indeterminates U_a. Using the canonical embedding of R into R', we can consider A as an element of $R'\langle X \multimap Y\rangle$. Therefore, equation (8) holds, with scalars now taken in R' (we have only used the ring structure of R for proving this equation). Let $x \in R'\langle X\rangle$ be defined by: $x_a = U_a$ if $a \in I$ and $x_a = 0$ otherwise. In that particular case, equation (8) gives

$$\sum_{\mu\in\mathcal{M}_{\mathrm{fin}}(I)} (!A)_{\mu,\Sigma(N)} U^\mu = \sum_{\mu\in\mathcal{M}_{\mathrm{fin}}(I)} \left(\sum_{\substack{M\in|!!X| \\ \Sigma(M)=\mu}} (!!A)_{M,N} \right) U^\mu \,.$$

We conclude, since the monomials U^μ are linearly independent in R' (considered as an R-module).

Fundamental isomorphism and the co-algebraic structure. Given two finiteness spaces X and Y, remember from Section 1 that there is a canonical isomorphism of finiteness spaces (2)

$$!(X \mathbin{\&} Y) \simeq {!X} \otimes {!Y} \,.$$

Given $x \in R\langle X\rangle$ and $y \in R\langle Y\rangle$, we have $x^! \otimes y^! \in R\langle !X \otimes !Y\rangle$. The corresponding element of $R\langle !(X \mathbin{\&} Y)\rangle$ is easily seen to be $(x \oplus y)^!$.

Since & is the cartesian product in $\mathbf{Fin}\,(R)$, there is a diagonal linear map $\Delta^X : X \to X \mathbin{\&} X$ whose matrix is given by $\Delta^X_{a,(i,b)} = \delta_{a,b}$. Let

$$\mathrm{contr}^X : {!X} \to {!X} \otimes {!X}$$

be obtained by composing $!\Delta^X$ with the isomorphism (2). Similarly, let

$$\mathrm{weak}^X : {!X} \to 1$$

be obtained by composing $!0$ (where 0 is the unique morphism $X \to 0$) with the isomorphism $!0 \simeq 1$. Using (5), the matrices of these operators are easily seen to be given by

$$\mathrm{weak}^X_{\mu,*} = \delta_{\mu,[]} \quad \text{and} \quad \mathrm{contr}^X_{\mu,(\lambda,\rho)} = \delta_{\mu,\lambda+\rho} \,.$$

As it is standard, these two morphisms define a structure of co-algebra on $!X$ (we are actually in a "new-Seely" situation, following the terminology of [Bie95]). The first one is used for interpreting the weakening rule of linear logic and the second one is used for the contraction rule.

Morphisms as power series. The category whose objects are finiteness spaces, where a morphism from X to Y is a linear and continuous function from $R\langle !X\rangle$ to $R\langle Y\rangle$ (that is, a matrix in $R\langle !X \multimap Y\rangle$), with d^X as identity at X and composition of $\varphi \in R\langle !X \multimap Y\rangle$ with $\psi \in R\langle !Y \multimap Z\rangle$ defined as the product of matrices

$$\psi \, !\varphi \, \mathrm{p}^X \tag{9}$$

is the *Kleisli category* of the comonad ! (the category of co-free co-algebras of the co-monad), and is cartesian closed, essentially because of the fundamental isomorphism presented above. If we consider $|X|$ as a set of formal indeterminates, then morphisms in the Kleisli category can be considered as power series (this is the basic idea of Girard's quantitative semantics [Gir88]):

- An element of $R\langle X\rangle$ is a *valuation* in R for these indeterminates (subject to the restriction that its domain must belong to $\mathsf{F}(X)$).
- An element μ of $|!X|$ is a *multi-exponent* (or equivalently a primitive monomial, that is a pure monomial without coefficient) on the indeterminates of $|X|$: if $\mu = [\xi_1, \dots, \xi_n]$, the corresponding primitive monomial is the formal product of indeterminates $\xi_1 \dots \xi_n$ (let us denote here by ξ^μ this formal product). The value of this monomial for a valuation $x \in R\langle X\rangle$ is just $x_{\xi_1} \times \cdots \times x_{\xi_n}$ (product computed in R), a value that we have already decided to denote as x^μ.

- An element φ of $R\langle !X \multimap Y\rangle$ is seen as the following power series, with coefficients in $R\langle Y\rangle$:

$$\sum_{\mu\in|!X|} \xi^\mu \Big(\sum_{b\in|Y|} \varphi_{\mu,b} e_b \Big) .$$

  This viewpoint is sensible, since the application of φ to $x \in R\langle X\rangle$ in the Kleisli category under consideration is just $\varphi \cdot x^! = \sum_{\mu\in|!X|} x^\mu \big(\sum_{b\in|Y|} \varphi_{\mu,b} e_b\big) \in R\langle Y\rangle$ (this sum being always finite by definition of $!X$).

One can check just as in [Ehr02] that the Kleisli composition (9) corresponds to the usual composition of power series (substitution), so that this Kleisli category is a cartesian closed category of power series. A matrix $\varphi \in R\langle !X \multimap Y\rangle$ will therefore be called a power series from X to Y. If $x \in R\langle X\rangle$, we denote by $\varphi(x)$ the value $\varphi \cdot x^!$ of this power series at x.

Let us give two concrete examples. In both cases we shall take a singleton for $|Y|$ so that $R\langle Y\rangle = R$ and our power series will be scalar-valued.

- If $|X|$ is also a singleton $\{\xi\}$, then $|!X| = \{\xi^n \mid n \in \mathbf{N}\}$ and $\mathsf{F}(!X) = \mathcal{P}(|!X|)$. In that case, $R\langle !X \multimap Y\rangle$ is the space $R[\xi]$ of all polynomials of the indeterminate ξ. Similarly, when $|X|$ is finite (and then we have seen that there is only one finiteness structure for X), $R\langle !X \multimap Y\rangle$ is the space of polynomials of n indeterminates, where n is the cardinality of $|X|$. It is only when $|X|$ become infinite that true "power series" come in, as shown by our second example.

- Consider now the case where $X = \mathsf{N}$, whose web will be considered as an infinite set of indeterminates $\{\xi_i \mid i \in \mathbf{N}\}$ rather than as $\mathbf{N}$. An element of $R\langle X\rangle$ is an R-valuation of these indeterminates whose support must be finite, that is, which must take the value 0 for all but a finite number of indeterminates, so that $R\langle \mathsf{N}\rangle$ can again be assimilated to the space $R[\chi]$ of all polynomials of the indeterminate χ: the element $\sum a_n \xi_n$ of $R\langle \mathsf{N}\rangle$ corresponds to the polynomial $\sum a_n \chi^n$. An element ξ^μ of $|!X|$ is a primitive monomial on these indeterminates and a collection of such monomials is finitary (in $!X$) if it mentions only a finite number of indeterminates. A power series $\varphi = \sum_{\mu\in|!X|} \varphi_\mu \xi^\mu \in R\langle !X \multimap Y\rangle$ must therefore have only finitely many non-zero coefficients of monomials mentioning the elements of any given finite set of indeterminates. For that reason, the sum $\sum_{\mu\in|!X|} \varphi_\mu x^\mu \in R\langle !X \multimap Y\rangle$ will have only finitely non-zero terms, for any $x \in R\langle X\rangle$. Such a power series can perfectly be infinite, and can even have an unbounded degree[14]; consider for instance the series $\sum_{n=0}^{\infty} \xi_0^n \xi_n$ which represents the map $P \mapsto P(P(0))$ from $R[\chi]$ to R (identifying $R\langle \mathsf{N}\rangle$ with $R[\chi]$). So these power series cannot be considered as polynomials, though, when computing their values on actual valuations of their indeterminates, finite computations only are required, in sharp contrast with usual power series. They are infinite objects only when they mention infinitely many indeterminates.

Algebraic structure. Due to the fact that (finite) products and co-products coincide in $\mathbf{Fin}\,(R)$, there is also a co-diagonal linear morphism $\mathrm{a}^X : X \,\&\, X \to X$ (as a function $R\langle X\rangle \times R\langle X\rangle \to R\langle X\rangle$, this is simply addition: $\mathrm{a}^X(x,y) = x + y$), whose matrix is given by $\mathrm{a}^X_{(i,a),b} = \delta_{a,b}$. Similarly, there is a zero-map $0 \to X$. The map $!\mathrm{a}^X$, composed with the isomorphism (2) gives rise to a linear morphism

$$\mathrm{c}^X : !X \otimes !X \to !X$$

whose matrix is given by $\mathrm{c}^X_{(\lambda,\rho),\mu} = \binom{\mu}{\lambda} \delta_{\lambda+\rho,\mu}$. The value of this coefficient can be obtained by applying Formula (5) in the particular case where $A = \mathrm{a}^X$, or more simply by observing that we must have

$$\mathrm{c}^X(x^! \otimes y^!) = (x + y)^!$$

for each $x, y \in R\langle X\rangle$, and by applying the generalized binomial equation (4).

[14]Total degree, or degree in a given indeterminate, as the next example shows.

Applying the functor ! to the zero-map mentioned above, we obtain similarly a linear map $\mathrm{u}^X : 1 \to\, !X$, that is an element u^X of $R\langle X\rangle$ which is defined by $\mathrm{u}^X_\mu = \delta_{\mu,[]}$.

The linear map c^X can be considered as defining a binary, bilinear and hypocontinuous commutative and associative multiplication on $R\langle !X\rangle$. Given $S, T \in R\langle !X\rangle$, we write $S * T$ or simply ST for $\mathrm{c}^X(S,T)$. This multiplication admits u^X as neutral element. So we have endowed $!X$ with a structure of commutative algebra whose multiplication can be interpreted as a kind of "convolution product", if we remember that $R\langle !X\rangle$ is the topological dual of $R\langle (!X)^\perp\rangle$, which itself can be seen as a space of power series from $R\langle X\rangle$ to R (which play the rôle of test functions in the theory of distributions). From this viewpoint, the element $x^!$ of $R\langle !X\rangle$ (when $x \in R\langle X\rangle$) corresponds to the "Dirac mass at x" which maps a test function φ to its value at x, that is $\varphi(x)$. The unit u^X corresponds to the Dirac mass at 0, and the convolution product is given by $(S * T)(\varphi) = S(\lambda x\, T(\lambda y\, \varphi(x+y)) = T(\lambda y\, S(\lambda x\, \varphi(x+y)))$ (using notations from the lambda-calculus). See [Ehr02] for more details (the setting is different but the analogy with the convolution product of distributions is preserved).

To summarize, $!X$ has a structure of co-algebra and of algebra, and is indeed a commutative and co-commutative Hopf algebra (an antipode can be defined, by applying the functor ! to the linear map $x \mapsto -x$, from $R\langle X\rangle$ to itself, one obtains in that way the matrix $S \in R\langle !X \multimap\, !X\rangle$ given by $S_{\mu,\nu} = (-1)^{\#\mu}\delta_{\mu,\nu}$). This kind of Hopf algebra seems to be known as a "divided power algebra".

Derivatives. There is moreover a linear "anti-dereliction" map $\partial^X_0 : X \to\, !X$, simply given by the matrix $(\partial^X_0)_{a,\mu} = \delta_{[a],\mu}$, so that $\mathrm{d}^X \circ \partial^X_0 = \mathrm{Id}_X$. Let φ be a power series from X to Y, that is, a linear map from $!X$ to Y. Then $A = \varphi\partial^X_0 \in R\langle X \multimap Y\rangle$ is given by $A_{a,b} = \varphi_{[a],b}$ and so is the "linear part" of φ, which is precisely what a derivative at 0 of φ should be. Remember indeed that, when $f : E \to F$ is a function between two Banach spaces (for instance), the derivative of f at 0 (when it exists) is the (necessarily unique) linear continuous function $h : E \to F$ such that $(f(x) - f(0) - h(x))/\|x\| \to 0$ when $x \to 0$, meaning that $h(x)$ is the best possible linear approximation of $f(x) - f(0)$. Similarly, in the present setting, we have

$$\varphi(x) - \varphi(0) - A \cdot x = \sum_{b\in|Y|} \left(\sum_{\substack{\mu\in|!X|\\ \#\mu\geq 2}} \varphi_\mu x^\mu \right) e_b$$

which means that all the terms in $\varphi(x) - \varphi(0) - A\cdot x$ have a total degree ≥ 2. Let us denote by $\varphi'_0 = A = \varphi\partial^X_0$ this derivative.

Let $x \in R\langle X\rangle$. The derivative of φ at x is the derivative at 0 of the power series $\psi : R\langle X\rangle \to R\langle Y\rangle$ defined by $\psi(u) = \varphi(x+u)$: $\varphi'_x = \psi'_0$. The map $\varphi' : R\langle X\rangle \to R\langle (X \multimap Y)\rangle$ defined by $\varphi'(x) = \varphi'_x$ is itself "analytic", i.e. can be defined by a power series as follows: composing $\mathrm{Id}_{!X} \otimes \partial^X_0 : !X \otimes X \to\, !X \otimes !X$ and $\mathrm{c}^X : !X \otimes !X \to\, !X$, we obtain a linear map

$$\partial^X : !X \otimes X \to\, !X\,,$$

therefore $\varphi\partial^X$ is a linear map $!X \otimes X \to\, !Y$ which can be transposed (using monoidal closeness) into a map $!X \to (X \multimap Y)$ which turns out to be φ'.

This derivation process can therefore be iterated: to $\varphi \in R\langle !X \multimap Y\rangle$, we can associate $\varphi^{(n)} \in\, !X \multimap (X^{\otimes n} \multimap Y)$, the n-the derivative of φ. This derivative can also be obtained by precomposing φ with a morphism

$$\partial^X_n : !X \otimes X^{\otimes n} \to\, !X$$

defined by induction over n as follows: $\partial^X_0 = \mathrm{Id}_{!X}$ and $\partial^X_{n+1} = \partial^X_n(\partial^X \otimes \mathrm{Id}_{X^{\otimes n}})$. The matrix of this operator is given by

$$(\partial^X_n)_{\mu,(a_1,\dots,a_n),\nu} = \frac{\nu!}{\mu!}\delta_{\mu+[a_1,\dots,a_n],\nu}\,.$$

So $\varphi^{(n)}(x)$, seen as an n-linear map from $R\langle X\rangle^n$ to $R\langle X\rangle$, is symmetrical, as it is standard.

The Taylor formula and the exponential. Let X be a finiteness space. The linear and continuous map ∂_0^X defines an embedding of $R\langle X\rangle$ into $R\langle !X\rangle$, with retraction d^X, so that $R\langle X\rangle$ can canonically be considered as a subspace of $R\langle !X\rangle$, what we do now. Given $x \in R\langle X\rangle$, the corresponding element of $R\langle !X\rangle$, still denoted by x, is $\sum_{a\in|X|} x_a e_{[a]}$. If $n \in \mathbf{N}$, we write x^n for the n-th power of x (multiplication being the convolution product $*$ on $R\langle !X\rangle$). By definition of this product, for each $\mu \in |!X|$, we have

$$x^n_\mu = \delta_{n,\#\mu} \sum_{\substack{(a_1,\dots,a_n)\in|X|^n \\ [a_1,\dots,a_n]=\mu}} \mu! \prod_{i=1}^n x_{a_i} = \delta_{n,\#\mu} n! x^\mu$$

since there are exactly $n!/\mu!$ tuples $(a_1,\dots,a_n) \in |X|^n$ such that $[a_1,\dots,a_n] = \mu$ (when $n = \#\mu$). Observe in particular that $|x^n| = \{\mu \in |x|^! \mid \#\mu = n\}$.

Lemma 19 *The series* $\sum_{n=0}^\infty \frac{x^n}{n!}$ *converges to* $x^!$ *in* $R\langle !X\rangle$*.*

Proof. As to convergence, it suffices to prove that $\lim_{n\to\infty} x^n = 0$ in $R\langle !X\rangle$. Let $U' \in \mathsf{F}((!X)^\perp)$. Since $|x| \in \mathsf{F}(X)$, the set $U' \cap |x|^!$ is finite. Then if $n \in \mathbf{N}$ is such that $n > \#\mu$ for all $\mu \in U' \cap |x|^!$, we have $x^n \in \mathrm{V}_{!X}(U')$ and so $\lim_{n\to\infty} x^n = 0$.

Let $\mu \in |!X|$ and let $m = \#\mu$, we have

$$\begin{aligned}\left(\sum_{n=0}^\infty \frac{x^n}{n!}\right)_\mu &= \sum_{n=0}^\infty \frac{x^n_\mu}{n!} \\ &= \frac{1}{m!} m! x^\mu = x^!_\mu\end{aligned}$$

and hence $\sum_{n=0}^\infty \frac{x^n}{n!} = x^!$. $\square$

Let φ be a power series from X to Y, so that φ can be seen as a linear function from $!X$ to Y: $\varphi \in R\langle !X \multimap Y\rangle$. Then we have, by continuity of φ considered as a linear map (using a "dot notation" for linear application):

$$\begin{aligned}\varphi(x) &= \varphi \cdot x^! \\ &= \sum_{n=0}^\infty \frac{1}{n!} \varphi \cdot x^n\end{aligned}$$

for all $x \in R\langle X\rangle$. For all such x, we have

$$\varphi^{(n)}(0) \cdot x^{\otimes n} = (\varphi \partial_n^X) \cdot (0^! \otimes x^{\otimes n}) = \varphi \cdot x^n$$

where $x^{\otimes n} = \underbrace{x \otimes \cdots \otimes x}_{n\times} \in X^{\otimes n}$, and hence the Taylor formula holds

$$\varphi(x) = \sum_{n=0}^\infty \frac{1}{n!} \varphi^{(n)}(0) \cdot x^{\otimes n}\,.$$

Remark: There is a canonical notion of polynomial in this setting: let us say that a power series φ from X to Y is a polynomial if there is an integer n such that the n-th derivative of φ (which is a power series from X to $X^{\otimes n} \multimap Y$) is 0, and in that case let us call total degree of φ the value $n-1$, where n is the least such integer (taking the $-\infty$ as value of this degree when $\varphi = 0$). Such a polynomial φ has a finite Taylor expansion and can therefore be written $\varphi(x) = \sum_{k=0}^{n-1} A_k \cdot x^{\otimes k}$ where $A_k \in R\langle X^{\otimes k} \multimap Y\rangle$ for $k = 0,\dots,n-1$ (A_k can be seen as k-linear hypocontinuous map which can obviously be assumed to be symmetrical). Conversely, any power series of that shape is a polynomial.

Concluding remarks

Among several problems raised by this interpretation, let us point out the impossibility of defining *in a standard way* a finitary model of the pure lambda-calculus. Such a model would be a finiteness space U together with an embedding retraction pair from $!U \multimap U$ into U. But (under mild hypotheses, e.g. the assumption that the space N is a retract of U) this would induce fix-point operators whose existence has been disproved in section 2.

This situation is embarrassing for two reasons. First we know by the work of Ryu Haswgawa ([Has97]) that all coefficients in the quantitative interpretation of pure lambda-terms are finite, and the purpose of finiteness spaces being to keep all coefficients finite, pure lambda-terms should admit a finitary interpretation. Second, the finiteness space model presented here is perfectly adapted for interpreting the simply typed differential lambda-calculus[15] of [ER03], but this calculus admits a natural untyped version whose denotational semantics would require something like a finitary model of the pure lambda-calculus.

The solution might be a non standard interpretation which would not use a directed limit or co-limit (like the construction of D_∞ in Scott domains or coherence spaces) because these infinitary constructions are not available in the category of finiteness spaces and finitary relations (or in the category of finiteness spaces and linear continuous functions, a semi-ring of coefficients being given); this is due to the fact that finitary sets are not closed under directed unions in general.

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I want to thank François Lamarche who suggested to me that Lefschetz linear topologies should be the right framework for describing the topological structure of a module built over a finiteness space.

4 Appendix: the interpretation of proofs in the category of sets and relations

To each formula G of first order propositional linear logic (without atoms, only logical constants), we associate a set $|G|$ as follows.

- $|0| = |\top| = \emptyset$ and $|G \oplus H| = |G \mathbin{\&} H| = |G| + |H|$ (where $+$ denotes the disjoint union on sets which can be defined for instance by $S + T = \{1\} \times S \cup \{2\} \times T$);
- $|1| = |\bot| = \{*\}$ and $|F \otimes G| = |F \parr G| = |F| \times |G|$ (where $*$ is a distinguished element);
- $!F = ?F = \mathcal{M}_{\mathrm{fin}}(|F|)$ where $\mathcal{M}_{\mathrm{fin}}(S)$ is the set of finite multi-sets over S.

If $\Gamma = G_1, \dots, G_n$ is a list of formulae, then $|\Gamma| = |G_1 \parr \cdots \parr G_n| = |G_1| \times \cdots \times |G_n|$. Given a formula G, the formula $G^\perp$ is defined by induction using the usual De Morgan identities of linear logic. It is clear then that $|G^\perp| = |G|$.

To each proof π of a sequent in first order propositional linear logic $\vdash \Gamma$, we associate a subset of π^* of the set $|\Gamma|$ by induction on π.

Tensor unit: if the proof π is

$$\frac{}{\vdash 1}$$

then $\pi^* = \{*\}$.

With unit: if the proof π is

$$\frac{}{\vdash \Gamma, \top}$$

then $\pi^* = \emptyset$.

[15] We did not give the boring details of this interpretation, but we hope to have presented our category in a sufficiently detailed way for making this interpretation straightforward to readers acquainted with denotational models of the lambda-calculus.

With: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, G\end{array}\qquad\qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Gamma, H\end{array}}{\vdash \Gamma, G \,\&\, H}$$

then $\pi^* = \{(c,(1,a)) \mid (c,a) \in {\pi_1}^*)\} \cup \{(c,(2,b)) \mid (c,b) \in {\pi_2}^*)\}$.

Left plus: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, S\end{array}}{\vdash \Gamma, G \oplus H}$$

then $\pi^* = \{(c,(1,a)) \mid (c,a) \in {\pi_1}^*)\}$. And similarly if π ends with a right plus rule.

Par unit: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, \bot}$$

then $\pi^* = \{(c,*) \mid c \in {\pi_1}^*\}$.

Par: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, G, H\end{array}}{\vdash \Gamma, G \parr H}$$

then $\pi^* = \{(c,(a,b)) \mid (c,a,b) \in {\pi_1}^*\}$.

Tensor: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, G\end{array}\qquad\qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Delta, H\end{array}}{\vdash \Gamma, \Delta, G \otimes H}$$

then $\pi^* = \{(c,d,(a,b)) \mid (c,a) \in {\pi_1}^*) \text{ and } (d,b) \in {\pi_2}^*\}$.

Weakening: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?G}$$

then $\pi^* = \{(c,[]) \mid c \in {\pi_1}^*\}$.

Contraction: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, ?G, ?G\end{array}}{\vdash \Gamma, ?G}$$

then $\pi^* = \{(c,x+y) \mid (c,x,y) \in {\pi_1}^*\}$ where $x+y$ denotes the sum of the multi-sets x and y.

Dereliction: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, G\end{array}}{\vdash \Gamma, ?G}$$

then $\pi^* = \{(c,[a]) \mid (c,a) \in {\pi_1}^*\}$.

Promotion: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash ?G^1,\dots,?G^k,G\end{array}}{\vdash ?G^1,\dots,?G^k,!G}$$

then π^* is the set of all $k+1$-tuples of the shape $(\sum_{j=1}^n x_j^1,\dots,\sum_{j=1}^n x_j^k,[a_1,\dots,a_n])$ where $((x_j^1,\dots,x_j^k,a_j))_{j=1,\dots,n}$ is any finite family of elements of ${\pi_1}^*$.

The exchange rule does not deserve particular mention.

Cut: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash \Gamma,G\end{array}\qquad\qquad \begin{array}{c}\vdots\ \pi_2\\ \vdash \Delta,G^\perp\end{array}}{\vdash \Gamma,\Delta}$$

then $\pi^* = \{(c,d) \mid \exists a\,(c,a) \in {\pi_1}^* \text{ and } (d,a) \in {\pi_2}^*\}$.

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