# Lecture Notes
# MPRI 2 – 02
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# Denotational semantics of functional languages and linear logic

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# Table des matières

On conseille de faire les exercices au fur et à mesure de la lecture (parfois, la solution d'un exercice se trouve plus loin dans le texte). Les exercices avec un astérisque sont à faire en priorité (ils sont simples, et servent à vérifier que les notions ont été comprises).

Nos principaux ouvrages de référence, dont ce cours s'inspire d'ailleurs largement, sont [Kri90], [GLT89] (épuisé, mais accessible sur la page Web d'Yves Lafont) et [AC98].

## 0.1 Notations et terminologie générales

Si $I$ est un ensemble, $\mathcal{P}(I)$ est l'ensemble de ses parties et $\mathcal{P}_{\mathrm{fin}}(I)$ est l'ensemble de ses parties finies et $\mathcal{P}^*_{\mathrm{fin}}(I)$ est l'ensemble des éléments non vides de $\mathcal{P}_{\mathrm{fin}}(I)$. Si $I$ est un ensemble, on note $\#I$ son cardinal.

Soient $s \subseteq A \times B$ et $t \subseteq B \times C$ des relations binaires. On note $t\,s \subseteq A \times C$ leur composée relationnelle, donnée par

$$t\,s = \{(a, c) \in A \times C \mid \exists b \in B (a, b) \in s \text{ et } (b, c) \in t\}\,.$$

Si $R$ est une relation binaire sur un ensemble $S$, on note $R^*$ sa *clôture transitive et reflexive*, c'est-à-dire la plus petite relation binaire sur $S$ contenant $R$ qui est réflexive et transitive. On voit facilement que $a\ R^*\ b$ ssi on peut trouver une suite $a_1, \dots, a_n \in S$ telle que $a_1 = a$, $a_n = b$ et $a_i\ R\ a_{i+1}$ pour $i = 1, \dots, n-1$.

Un *ensemble filtrant* est un ensemble non vide partiellement ordonné $\Gamma$ tel que

$$\forall \gamma, \gamma' \in \Gamma \; \exists \delta \in \Gamma \quad \gamma, \gamma' \leq \delta \,.$$

Si $\Gamma$ et $\Delta$ sont des ensembles filtrants, alors $\Gamma \times \Delta$ (avec l'ordre produit) est aussi un ensemble filtrant. Une *famille filtrante* est une famille indexée par un ensemble filtrant.

On note $\mathbb{N}^+$ l'ensemble des entiers naturels non nuls.

If $(D, \leq_D)$ and $(E, \leq_E)$ are a partially ordered sets, a function $f : D \to E$ is non-decreasing (or monotone, or monotonic) if $\forall d, d' \in D \; d \leq_D d' \Rightarrow f(d) \leq_E f(d')$.

# Chapitre 1

# Syntaxe de PCF

On définit un lambda-calcul étendu, qui ressemble plus à un langage de programmation que le lambda-calcul pur : c'est le langage PCF (Programming with Computable Functions). Voir aussi [AC98], et [GLT89] qui parle du *système $T$* de Gödel, qui est similaire à PCF, avec un opérateur de "récursion primitive" moins expressif que l'opérateur de point fixe.

## 1.1 Syntaxe

On définit d'abord les types : $\iota$ est un type (le type des entiers), et si $A$ et $B$ sont des types, alors $A \to B$ est un type.

Les termes du langage PCF sont définis par la syntaxe suivante.

— Toute variable est un terme ;
— si $P$ et $Q$ sont des termes, $x$ une variable et $A$ un type, alors $\lambda x^A\, P$ et $(P)\, Q$ sont des termes ;
— si $P$ est un terme, alors $\mathsf{fix}(P)$ est un terme ;
— si $P$, $Q$ et $R$ sont des termes et si $z$ est une variable, alors $\mathsf{if}(P, Q, z \cdot R)$ est un terme ;
— si $n$ est un entier naturel, alors $\underline{n}$ est un terme ;
— si $P$ est un terme alors $\underline{\mathsf{succ}}(P)$ est un terme.

**Remarque 1.1.1** La conditionnelle appelle un commentaire car elle diffère de celles habituellement considérée dans PCF, comme par exemple dans [AC98]. Nous allons donner une sémantique opérationnelle en appel par nom à cette version de PCF. La conditionnelle est habitellement donnée par une construction plus simple $\mathsf{if}(M, P, Q)$ : on calcule $M$, si on obtient 0 on rend $P$ et sinon on rend $Q$. Mais en appel par nom, la valeur de $M$ est alors perdue. Bien sûr $P$ et $Q$ peuvent contenir d'autres copies de $M$ mais chacune de ces copies nécessitera un nouveau calcul (toujours le même en fait) de la valeur de $M$. Dans la construction $\mathsf{if}(M, P, z \cdot Q)$ on calcule $M$, si on obtient $\underline{0}$ on rend $P$ et si on obtient $\underline{n+1}$ on rend $Q\,[\underline{n}/z]$. Autrement dit, on a mis en mémoire la valeur de $M$ et on peut la réutiliser dans la suite sans avoir à réévaluer $M$ à chaque fois. Autrement dit encore, cette construction conditionnelle introduit une possibilité d'appel par valeur, restreinte au seul type de base. On verra que ce choix a une jutification sémantique très naturelle du point de vue de la logique linéaire, voir Section ??.

Dans le terme $\lambda x^A\, M$, la variable $x$ est liée. De même, dans $\mathsf{if}(M, P, z \cdot Q)$, la variable $z$ est liée dans $Q$. Les termes de PCF sont donc considérés à $\alpha$-conversion près, c'est-à-dire à renommage près des variables liées (attention, dans un tel renommage, il ne faut pas utiliser des variables libres dans le sous-terme associé au lieur considéré). Ce sont des difficultés superficielles que l'on peut régler en remplaçant les variables par des pointeurs ou des *indices de de Bruijn*.

Soit $M$ un terme, $x_1, \dots, x_l$ des variables deux à deux distinctes. Soient $P_1, \dots, P_l$ des termes. On définit la *substitution parallèle* $M\,[P_1/x_1, \dots, P_l/x_l]$ par récurrence sur $M$ :

— $x_i\,[P_1/x_1, \dots, P_l/x_l] = P_i$
— $x\,[P_1/x_1, \dots, P_l/x_l] = x$ si $x \notin \{x_1, \dots, x_n\}$
— $\underline{n}\,[P_1/x_1, \dots, P_l/x_l] = \underline{n}$
— $\underline{\mathsf{succ}}(M)\,[P_1/x_1, \dots, P_l/x_l] = \underline{\mathsf{succ}}(M\,[P_1/x_1, \dots, P_l/x_l])$

— the substituted term $\mathsf{if}(M, P, z \cdot Q)\,[P_1/x_1, \ldots, P_l/x_l]$ is defined as the term

$$\mathsf{if}(M\,[P_1/x_1, \ldots, P_l/x_l]\,, P\,[P_1/x_1, \ldots, P_l/x_l]\,, z \cdot Q\,[P_1/x_1, \ldots, P_l/x_l])$$

and one has to assume that $z$ is not free in the $P_i$'s (otherwise, apply an $\alpha$-conversion for $z$ in $\mathsf{if}(M, P, z \cdot Q)$

— $(\lambda x^A\, P)\,[P_1/x_1, \ldots, P_l/x_l] = \lambda x^A\,(P\,[P_1/x_1, \ldots, P_l/x_l])$ and one has to assume that $x$ is not free in the $P_i$'s (otherwise, perform an $\alpha$-conversion for the variable $x$)

— $\mathsf{fix}(P)\,[P_1/x_1, \ldots, P_l/x_l] = \mathsf{fix}((P\,[P_1/x_1, \ldots, P_l/x_l]))$.

## 1.2 Typage

Voici ensuite les règles de typage. Comme d'habitude, un contexte $\Gamma$ est une suite $(x_1 : A_1, \ldots, x_l : A_l)$ où les $x_i$ sont des variables deux à deux distinctes et où les $A_i$ sont des types.

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)\,N : B}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A\, M : A \to B} \qquad \frac{\Gamma \vdash M : A \to A}{\Gamma \vdash \mathsf{fix}(M) : A}$$

$$\frac{}{\Gamma \vdash \underline{n} : \iota} \qquad \frac{\Gamma \vdash M : \iota}{\Gamma \vdash \underline{\mathsf{succ}}(M) : \iota}$$

$$\frac{\Gamma \vdash M : \iota \qquad \Gamma \vdash P : A \qquad \Gamma, z : \iota \vdash Q : A}{\Gamma \vdash \mathsf{if}(M, P, z \cdot Q) : A}$$

**Remarque 1.2.1** Etant donnés un contexte de typage $\Gamma$ et un terme $M$, il existe au plus un type $A$ tel que $\Gamma \vdash M : A$. De plus la dérivation de typage qui mène à la conclusion $\Gamma \vdash M : A$ est unique et totalement déterminée par $M$. C'est pour obtenir cet effet que nous avons indiqué le type de la variable dans la construction $\lambda x^A\, M$.

1.2.1 USUAL CONDITIONAL. In the usual presentations of PCF, one uses another conditional $\mathsf{if}_{\mathrm{u}}(M, P, Q)$ with the following typing rule :

$$\frac{\Gamma \vdash M : \iota \qquad \Gamma \vdash P : A \qquad \Gamma : \iota \vdash Q : A}{\Gamma \vdash \mathsf{if}_{\mathrm{u}}(M, P, Q) : A}$$

Of course this conditional can be defined using ours as follows :

$$\mathsf{if}_{\mathrm{u}}(M, P, Q) = \mathsf{if}(M, P, z \cdot Q)$$

where $z$ does not occur free in $Q$.

## 1.3 Réduction.

On définit une relation de *réduction* $\beta$ sur les termes de PCF, suivant le même schéma que pour la beta-réduction du lambda-calcul. On définit donc, par récurrence sur $M$, l'ensemble des termes $M'$ tels que $M\ \beta\ M'$.

$$\frac{}{\big(\lambda x^A\, M\big)\, N\ \beta\ M\,[N/x]}$$

$$\frac{}{\mathsf{fix}(M)\ \beta\ (M)\,\mathsf{fix}(M)}$$

$$\frac{}{\mathsf{if}(\underline{0}, P, z \cdot Q)\ \beta\ P} \qquad \frac{}{\mathsf{if}(\underline{n+1}, P, z \cdot Q)\ \beta\ Q\,[\underline{n}/z]}$$

$$\frac{}{\underline{\mathsf{succ}}(\underline{n})\ \beta\ \underline{n+1}}$$

$$\frac{M\ \beta\ M'}{(M)\,N\ \beta\ (M')\,N} \qquad \frac{N\ \beta\ N'}{(M)\,N\ \beta\ (M)\,N'}$$

$$\frac{M\ \beta\ M'}{\mathsf{fix}(M)\ \beta\ \mathsf{fix}(M')}$$

$$\frac{M\ \beta\ M'}{\mathsf{if}(M, P, z\cdot Q)\ \beta\ \mathsf{if}(M', P, z\cdot Q)}$$

$$\frac{P\ \beta\ P'}{\mathsf{if}(M, P, z\cdot Q)\ \beta\ \mathsf{if}(M, P', z\cdot Q)} \qquad \frac{Q\ \beta\ Q'}{\mathsf{if}(M, P, z\cdot Q)\ \beta\ \mathsf{if}(M, P, z\cdot Q')}$$

$$\frac{M\ \beta\ M'}{\underline{\mathsf{succ}}(M)\ \beta\ \underline{\mathsf{succ}}(M')}$$

**Remarque 1.3.1** Autrement dit, dans PCF, un redex est un terme de l'une des formes suivantes :
- $(\lambda x^A\, R)\, Q$ qui se réduit en $R\,[Q/x]$,
- $\mathsf{fix}(P)$ qui se réduit en $(P)\,\mathsf{fix}(P)$,
- $\mathsf{if}(\underline{n}, P, z\cdot Q)$ avec $n \in \mathbb{N}$, qui se réduit en $P$ si $n = 0$ et en $Q\,[\underline{n-1}/z]$ si $n > 0$,
- $\underline{\mathsf{succ}}(\underline{n})$ qui se réduit en $\underline{n+1}$,

Et on a $M\ \beta\ M'$ si on obtient $M'$ en choisissant dans $M$ un redex (en une position quelconque) et en le réduisant.

### 1.3.1 Réduction du sujet et confluence.

**Lemme 1.3.2 (substitution)** *Soient $P, Q \in$ PCF. Si $\Gamma, x : A \vdash P : B$ et si $\Gamma \vdash Q : A$, alors $\Gamma \vdash P\,[Q/x] : B$.*

**Proposition 1.3.3 (réduction du sujet)** *Soit $M \in$ PCF. Si $\Gamma \vdash M : A$ et $M\ \beta\ M'$, alors $\Gamma \vdash M' : A$.*

**Exercice 1.3.1** Démontrer le lemme de substitution (par récurrence sur $P$) puis la réduction du sujet (par récurrence sur la dérivation du fait que le terme $M$ se réduit en $M'$).

It is useful to know that the reduction $\beta$ satisfies the *Church-Rosser property* :

**Théorème 1.3.4** *If $M\ \beta^*\ M_i$ for $i = 1, 2$, there exists $M'$ such that $M_i\ \beta^*\ M'$ for $i = 1, 2$. That is : the relation $\beta^*$ satisfies the* Diamond Property.

The Tait-Martin-Löf method consists in defining an auxiliary notion of *parallel reduction* $\rho$ by the following rules.

$$\frac{}{\underline{n}\ \rho\ \underline{n}} \qquad \frac{}{x\ \rho\ x}$$

$$\frac{M\ \rho\ M' \qquad N\ \rho\ N'}{(\lambda x^A\, M)\, N\ \rho\ M'\,[N'/x]} \qquad \frac{M\ \rho\ M'}{\lambda x^A\, M\ \rho\ \lambda x^A\, M'}$$

$$\frac{M\ \rho\ M' \qquad M\ \rho\ M''}{\mathsf{fix}(M)\ \rho\ (M')\,\mathsf{fix}(M'')}$$

$$\frac{P\ \rho\ P'}{\mathsf{if}(\underline{0}, P, z\cdot Q)\ \rho\ P'} \qquad \frac{Q\ \rho\ Q'}{\mathsf{if}(\underline{n+1}, P, z\cdot Q)\ \rho\ Q'\,[\underline{n}/z]}$$

$$\frac{M\ \rho\ M'}{\underline{\mathsf{succ}}(M)\ \rho\ \underline{\mathsf{succ}}(M')} \qquad \frac{}{\underline{\mathsf{succ}}(\underline{n})\ \rho\ \underline{n+1}}$$

$$\frac{M\ \rho\ M' \qquad N\ \rho\ N'}{(M)\,N\ \rho\ (M')\,N'}$$

$$\frac{M \;\rho\; M'}{\mathsf{fix}(M) \;\rho\; \mathsf{fix}(M')}$$

$$\frac{M \;\rho\; M' \qquad P \;\rho\; P' \qquad Q \;\rho\; Q'}{\mathsf{if}(M, P, z \cdot Q) \;\rho\; \mathsf{if}(M', P', z \cdot Q')}$$

If $M \;\rho\; M'$ then $M'$ is obtained from $M$ by reducing an arbitrary number of redexes *which are already present in* $M$. In other words, a single step of the $\rho$ reduction can contain several steps of $\beta$-reduction, but one is not allowed to reduce redexes which have been created by a $\beta$-reduction in the same $\rho$ reduction step.

**Exercice 1.3.2** Give examples of two terms $M$ and $M'$ such that $M \;\beta^*\; M'$ but not $M \rho M'$.

**Exercice 1.3.3** Check that $M \;\rho\; M$ for all term $M$. Prove that $\beta \subseteq \rho \subseteq \beta^*$. Conclude that $\rho^* = \beta^*$.

**Exercice 1.3.4** Prove that, if a relation $\gamma$ satisfies the Diamond Property, then its reflexive-transitive closure $\gamma^*$ satisfies also the Diamond Property.

So it suffices to prove that the relation $\rho$ satisfies the Diamond Property.

**Exercice 1.3.5** Assume that $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$ and assume that $M \;\rho\; M'$ and $N \;\rho\; N'$. Prove by induction on $M$ that $M\,[N/x] \;\rho\; M'\,[N'/x]$.

**Exercice 1.3.6** Prove that $\rho$ satisfies the Diamond Property. That is : Assume that $\Gamma \vdash M : A$ and that $M \;\rho\; M_i$ for $i = 1, 2$. Prove that there is a term $R$ such that $M_i \;\rho\; R$ for $i = 1, 2$. The proof is by induction on $M$. For each induction step, one has to consider all possible $\rho$-deduction rules which apply. For instance, if $M = (P)\,Q$, we have the following possibilities :

— $P \;\rho\; P_i$ and $Q \;\rho\; Q_i$ for $i = 1, 2$, with $M_i = (P_i)\,Q_i$.

— $P = \lambda x^B\, H$, $H \;\rho\; H_i$, $Q \;\rho\; Q_i$, $M_1 = H_1\,[Q_1/x]$ and $M_2 = \left(\lambda x^B\, H_2\right) Q_2$.

— A case symmetric to the previous one, swapping 1 and 2.

— $P = \lambda x^{B \to A}\, H$, $H \;\rho\; H_i$, $Q \;\rho\; Q_i$, $M_i = H_i\,[Q_i/x]$ for $i = 1, 2$.

In each case one applies the inductive hypothesis and the result of Exercise 1.3.5 in all cases but the first one.

The other inductive steps are dealt with similarly, the most important being the following ones :

— $M = \mathsf{if}(N, P, x \cdot Q)$

— $M = \mathsf{fix}(N)$.

The other ones are routine.

As usual, one of the main consequences of confluence is that, when a term normalizes, it has a unique normal form. Typically, if $\vdash M : \iota$, then either $M$ has no $\beta$-normal form, or there is a unique integer $n$ such that $M \;\beta^*\; \underline{n}$.

Also, defining $\sim_\beta$ as the least equivalence relation on terms which contains $\beta$, we can deduce from confluence that, if $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$, then $M \sim_\beta M'$ iff there exists a term $N$ such that $M \;\beta^*\; N$ and $M' \;\beta^*\; N$.

1.3.2 Réduction de tête (faible). On définit de même la *réduction de tête faible* $\beta_{\mathsf{wh}}$ (“faible” signifie qu'on ne réduit jamais sous un $\lambda$).

$$\frac{}{\left(\lambda x^A\, M\right) N \;\beta_{\mathsf{wh}}\; M\,[N/x]}$$

$$\frac{}{\mathsf{fix}(M) \;\beta_{\mathsf{wh}}\; (M)\,\mathsf{fix}(M)}$$

$$\frac{}{\mathsf{if}(\underline{0}, P, z \cdot Q) \;\beta_{\mathsf{wh}}\; P} \qquad \frac{}{\mathsf{if}(\underline{n+1}, P, z \cdot Q) \;\beta_{\mathsf{wh}}\; Q\,[\underline{n}/z]}$$

$$\frac{}{\mathsf{succ}(\underline{n}) \;\beta_{\mathsf{wh}}\; \underline{n+1}}$$

$$\frac{M \;\beta_{\mathsf{wh}}\; M'}{(M)\,N \;\beta_{\mathsf{wh}}\; (M')\,N}$$

$$\frac{M \; \beta_{\mathsf{wh}} \; M'}{\mathsf{if}(M, P, z \cdot Q) \; \beta_{\mathsf{wh}} \; \mathsf{if}(M', P, z \cdot Q)}$$

$$\frac{M \; \beta_{\mathsf{wh}} \; M'}{\underline{\mathsf{succ}}(M) \; \beta_{\mathsf{wh}} \; \underline{\mathsf{succ}}(M')}$$

**Remarque 1.3.5** On a $\beta_{\mathsf{wh}} \subseteq \beta$. Observer que $\beta_{\mathsf{wh}}$ est une *stratégie de réduction*, c'est-à-dire que, étant donné un terme de PCF $M$, ou bien $M$ est en “forme normale de tête” (non réductible au sens de $\beta_{\mathsf{wh}}$), ou bien $M$ contient exactement un redex tel que $M \; \beta_{\mathsf{wh}} \; M'$.

**Lemme 1.3.6** *Soit $M$ un terme de* PCF*. Il existe des variables $x_1, \ldots, x_n$, un terme $H$ et des termes $P_1, \ldots, P_l$ (avec $l \geq 0$) de* PCF *tels que*
- $M = \lambda {x_1}^{A_1} \ldots \lambda {x_n}^{A_n} \, (H) \, P_1 \cdots P_l$
- *$H$ n'est pas de la forme $\lambda x^A \, R$*
- *Si $H = (Q) \, P$, alors $Q = \lambda x^A \, R$*

*De plus, cette écriture de $M$ est unique, on l'appelle* forme canonique *de $M$. On appelle $H$ le* terme de tête *de $M$.*

*Démonstration.* Simple récurrence sur $M$. □

**Proposition 1.3.7** *Si $\vdash M : \iota$ et $M$ est $\beta_{\mathsf{wh}}$-normal, alors $M = \underline{k}$ pour un $k \in \mathbb{N}$.*

*Démonstration.* Par le lemme 1.3.6, on peut écrire $M = \lambda {x_1}^{A_1} \ldots \lambda {x_n}^{A_n} \, (H) \, P_1 \cdots P_l$, la forme canonique de $M$. Comme $\vdash M : \iota$, on a $n = 0$ et les termes $H, P_1, \ldots, P_l$ sont tous clos. De plus, on peut trouver des types $B_1, \ldots, B_l$ tels que

$$\vdash H : B_1 \to \cdots \to B_l \to \iota \quad \text{et} \quad \forall i \; \vdash P_i : B_i \, .$$

Or $H$ ne peut pas être une abstraction, puisque c'est un terme de tête. $H$ ne peut pas être de la forme $\mathsf{fix}(Q)$ ou $\left(\lambda x^A \, R\right) P$ puisque $M$ est $\beta_{\mathsf{wh}}$-normal. Et $H$ ne peut pas être une variable puisqu'il est clos. Donc $H$ est de l'une des formes suivantes :
- $H = \underline{k}$, et dans ce cas on a $l = 0$ et la preuve est terminée.
- $H = \mathsf{if}(P, Q, z \cdot R)$ et on a $\vdash P : \iota$ et $P$ est $\beta_{\mathsf{wh}}$-normal. Par hypothèse de récurrence, $P = \underline{k}$ pour un $k \in \mathbb{N}$, ce qui contredit l'hypothèse que $M$ est $\beta_{\mathsf{wh}}$-normal.
- $H = \underline{\mathsf{succ}}(P)$ et on a $\vdash P : \iota$ et $P$ est $\beta_{\mathsf{wh}}$-normal. Par hypothèse de récurrence, $P = \underline{k}$ pour un $k \in \mathbb{N}$, ce qui contredit l'hypothèse que $M$ est $\beta_{\mathsf{wh}}$-normal. □

**Exercice 1.3.7** Ecrire un terme clos $M$ tel que $\vdash M : \iota \to \iota$ tel que, pour tout $p \in \mathbb{N}$, on ait $(M) \, \underline{p} \; \beta^*_{\mathsf{wh}} \; \underline{2p}$.

Ecrire un terme clos $M$ tel que $\vdash M : \iota \to \iota \to \iota$ tel que $\left((M) \, \underline{p}\right) \underline{q} \; \beta^*_{\mathsf{wh}} \; \underline{p+q}$.

Ecrire un terme clos $M$ tel que $\vdash M : (\iota \to \iota) \to \iota$ qui (intuitivement) prend une fonction $f$ de type $\iota \to \iota$ et rend le plus petit entier $n$ tel que $f(n) = 0$, s'il en existe un (sinon, le programme peut boucler).

Soit $f : \mathbb{N} \to \mathbb{N}$ une fonction partielle et soit $D \subseteq \mathbb{N}$ son domaine de définition. Soit $M$ un terme clos tel que $\vdash M : \iota \to \iota$. On dit que $M$ représente $f$ si, pour tout $n \in \mathbb{N}$,
- si $n \in D$ alors $(M) \, \underline{n} \; \beta^*_{\mathsf{wh}} \; \underline{f(n)}$
- si $n \notin D$ alors la $\beta_{\mathsf{wh}}$-réduction ne termine pas sur $(M) \, \underline{n}$.

**Théorème 1.3.8** *Une fonction partielle $f : \mathbb{N} \to \mathbb{N}$ est représentable par un terme clos de* PCF *de type $\iota \to \iota$ si et seulement si $f$ est une fonction récursive partielle.*

La preuve n'est pa difficile mais supposerait d'introduire du matériel supplémentaire sur les fonctions récursives partielles. On peut donc voir PCF comme une extension de la notion de fonction récursive partielle à l'ordre supérieur.

1.3.3 Exemples de programmes La fonction prédécesseur :

$$\mathsf{pred} = \lambda x^\iota\, \mathsf{if}(x, \underline{0}, z \cdot z) \quad \text{et on a} \quad \vdash \mathsf{pred} : \iota \to \iota$$

L'addition :

$$\mathsf{add} = \lambda x^\iota\, \mathsf{fix}(\lambda a^{\iota\to\iota}\, \lambda y^\iota\, \mathsf{if}(y, x, z \cdot \underline{\mathsf{succ}}((a)\, z)))$$
$$\text{et on a} \quad \vdash \mathsf{add} : \iota \to (\iota \to \iota)$$

La fonction exponentielle $n \mapsto 2^n$ :

$$\mathsf{exp} = \mathsf{fix}(\lambda e^{\iota\to\iota}\, \lambda x^\iota\, \mathsf{if}(x, \underline{1}, z \cdot (\mathsf{add})\, (e)\, z\ (e)\, z))$$
$$\text{et on a} \quad \vdash \text{exp} : \iota \to \iota$$

Une fonction pour comparer des entiers :

$$\mathsf{cmp} = \mathsf{fix}(\lambda c^{\iota\to(\iota\to\iota)}\, \lambda x^\iota\, \lambda y^\iota\, \mathsf{if}(x, \underline{0}, z \cdot \mathsf{if}(y, \underline{1}, z' \cdot (c)\, z\, z')))$$
$$\text{et on a} \quad \vdash \mathsf{cmp} : \iota \to (\iota \to \iota)$$

Une fonction qui cherche le premier entier où une fonction de type $\iota \to \iota$, passée en argument, s'annule :

$$\lambda f^{\iota\to\iota}\, (\mathsf{fix}(\lambda g^{\iota\to\iota}\, \lambda x^\iota\, \mathsf{if}((f)\, x, x, z \cdot (g)\, \underline{\mathsf{succ}}(x))))\, \underline{0}$$

On peut introduire une construction "let" qui permet de mettre en mémoire des valeurs entières (mais pas des valeurs[1] quelconques, attention). Il fuffit de poser

$$\mathsf{let}(x, M \cdot N) = \mathsf{if}(M, N\, [\underline{0}/x]\, , z \cdot N\, [\underline{\mathsf{succ}}(z)/x])$$

et alors la règle de typage suivante est dérivable :

$$\frac{\Gamma \vdash M : \iota \qquad \Gamma, x : \iota \vdash N : A}{\Gamma \vdash \mathsf{let}(x, M \cdot N) : A}$$

1.3.4 Equivalences entre termes Une question naturelle est de savoir quand deux termes clos sont équivalents en tant que programmes. La $\beta$-réduction nous fournit une première réponse : on peut dire que deux termes clos $M$ et $M'$ sont équivalents si $\vdash M : A$ et $\vdash M' : A$ pour un type $A$ (ils ont le même type) et si $M \sim_\beta M'$ où $\sim_\beta$ est la clôture réflexive, symétrique et transitive de $\beta$ (modulo un théorème de confluence qu'il faudrait prouver, cela revient à dire que $M$ et $M'$ ont un réduit commun pour $\beta$).

Cependant, cette notion d'équivalence est beaucoup trop faible. Par exemple, les deux termes suivants sont manifestement identiques comme programmes, mais ne sont pas équivalent au sens de $\sim_\beta$ :

$$G = \lambda x^\iota\, \lambda y^\iota\, \mathsf{if}(x, \mathsf{if}(y, \underline{0}, z \cdot \underline{1}), z \cdot \underline{1}) \text{ et } D = \lambda y^\iota\, \lambda x^\iota\, \mathsf{if}(x, \mathsf{if}(y, \underline{0}, z \cdot \underline{1}), z \cdot \underline{1})\,.$$

Soient $M$ et $N$ deux termes de PCF clos de type $A$. On écrira $M \sqsubseteq_{\mathrm{obs}} N$ et on dira que $M$ et $N$ son*observationnellement équivalents* si, pour tout terme clos $C$ de type $A \to \iota$, on a

$$\forall k \in \mathbb{N} \quad (C)\, M\ \beta^*_{\mathsf{wh}}\ \underline{k} \Rightarrow (C)\, N\ \beta^*_{\mathsf{wh}}\ \underline{k}\,.$$

Clairement, $\sqsubseteq_{\mathrm{obs}}$ est une relation de préordre sur les termes clos de type $A$. On note $\simeq_{\mathrm{obs}}$ la relation d'équivalence associée. Autrement dit $M \simeq_{\mathrm{obs}} M'$ si $M \sqsubseteq_{\mathrm{obs}} M'$ et $M' \sqsubseteq_{\mathrm{obs}} M$, c'est-à-dire, pour tout terme clos $C$ de type $A \to \iota$ et tout entier $k$, on a $(C)\, M\ \beta^*_{\mathsf{wh}}\ \underline{k}$ si et seulement si $(C)\, M'\ \beta^*_{\mathsf{wh}}\ \underline{k}$. Intuitivement, cela signifie que $M$ et $M'$ sont interchangeables en tant que sous-programme de n'importe

1. En fait, dans notre calcul, les seules valeurs sont les valeurs entières $\underline{0}$, $\underline{1}$ etc ; les autres termes, y compris les variables et les abstractions, ne doivent pas être considérés comme des valeurs. C'est un choix très raisonnable car ce sont les seules entités du langage qu'il est vraiment sensé d'afficher sur un écran. Il n'y a pas de sens à afficher un terme représentant une fonction de type $\iota \to \iota$ par exemple car la même fonction (des entiers vers les entiers) pourrait être définie de nombreuses autres façons en général et il est impossible de décider si 2 termes représentent la même fonction puisque notre langage est Turing-complet.

quel programme $C$ qui produit un résultat observable (ici, un entier). Il n'est pas complètement trivial de prouver que $G \simeq_{\mathrm{obs}} D$, mais c'est néanmoins vrai.

De façon générale, il est très difficile de prouver que deux programmes sont observationnellement équivalents, à cause de la quantification universelle sur tous les "contextes" $C$. On va voir que la sémantique dénotationnelle donne un moyen puissant.

En utilisant la confluence de $\beta$ on montrerait facilement que si $\vdash M : A$ et $\vdash M' : A$ alors $M \sim_\beta M' \Rightarrow M \simeq_{\mathrm{obs}} M'$ mais la sémantique nous donnera aussi une preuve très simple de ce fait. Comme le montre l'exemple ci-dessus, la réciproque est très loind d'être vraie.

## 1.4 Probabilistic PCF

We extend our language PCF with a probabilistic choice primitive $\mathsf{rand}(r)$ where $r \in [0,1] \cap \mathbb{Q}$ and call pPCF this probabilistic PCF. The types of pPCF are the same as those of PCF, and we only add the new constants $\mathsf{rand}(r)$, with typing rule

$$\frac{r \in [0,1] \cap \mathbb{Q}}{\Gamma \vdash \mathsf{rand}(r) : \iota}$$

Intuitively, $\mathsf{rand}(r)$ has probbaility $r$ to reduce to $\underline{0}$ and $1-r$ to reduce to $\underline{1}$. We want now to give a precise content to this intuition. This requires some simple preliminary considerations.

1.4.1 Completed half real line and stochastic matrices. Let $\mathbb{R}_{\geq 0} = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$ and $\overline{\mathbb{R}_{\geq 0}} = \mathbb{R}_{\geq 0} \cup \{\infty\}$. These sets are naturally equipped with an addition and a multiplication, extended to the additional element $\infty$ by :

$$\lambda + \infty = \infty + \lambda = \infty$$
$$\lambda\infty = \infty\lambda = \begin{cases} 0 & \text{if } \lambda = 0 \\ \infty & \text{otherwise}. \end{cases}$$

Remember that any bounded subset of $\mathbb{R}_{\geq 0}$ has a least upper bound (lub) in $\mathbb{R}_{\geq 0}$ and that any subset of $\overline{\mathbb{R}_{\geq 0}}$ has a least upper bound in $\overline{\mathbb{R}_{\geq 0}}$. The lub of a set $A$ is denoted as $\sup A$ and the lub of an indexed family $(\lambda_i)_{i\in I}$ is denoted as $\sup_{i\in I} \lambda_i$.

Notice that addition and multiplication commute with these lubs.

**Lemme 1.4.1** *Given any $\lambda \in \overline{\mathbb{R}_{\geq 0}}$ and any subset $A$ of $\overline{\mathbb{R}_{\geq 0}}$, one has*

$$\lambda + \sup A = \sup \{\lambda + \lambda' \mid \lambda' \in A\}$$
$$\lambda \sup A = \sup \{\lambda\lambda' \mid \lambda' \in A\} .$$

This is why we have chosen $0\infty = 0$ (consider the case $\lambda = \infty$ and $A = \emptyset$ or $A = \{0\}$).

Let $I$ be a set. Equipped with the product order $\leq$ (that is, given $u, v \in \overline{\mathbb{R}_{\geq 0}}^I$, one has $u \leq v$ if $\forall i \in I\ u_i \leq v_i$), the set $\overline{\mathbb{R}_{\geq 0}}^I$ is a complete lattice. This means that any subset $A$ of $\overline{\mathbb{R}_{\geq 0}}^I$ has a least upper bound (lub for short) $\sup A \in \overline{\mathbb{R}_{\geq 0}}^I$. This lub is given by

$$\sup A = (\sup_{n\in\mathbb{N}} u(n)_i)_{i\in I} .$$

Given $u \in \overline{\mathbb{R}_{\geq 0}}^I$, we define $\sum_{i\in I} u_i \in \overline{\mathbb{R}_{\geq 0}}$ as the lub of the family $(\sum_{i\in I_0} u_i)_{I_0 \in \mathcal{P}_{\mathrm{fin}}(I)}$.

**Lemme 1.4.2** *If $\sum_{i\in I} u_i < \infty$ then the set $\{i \in I \mid u_i \neq 0\}$ is at most countable.*

*Démonstration.* Let $J = \{i \in I \mid u_i \neq 0\}$ and $J_n = \{i \in I \mid u_i \geq 1/2^n\}$ (we could take any non-increasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ instead of $1/2^n$) for $n \in \mathbb{N}$. We have $\frac{\# J_n}{2^n} \leq \sum_{i\in J_n} u_i \leq \sum_{i\in I} u_i < \infty$ and hence each $J_n$ is finite. Since $J = \cup_{n\in\mathbb{N}} J_n$ it follows that $J$ is at most countable. □

**Lemme 1.4.3** *Let $I$ and $J$ be sets and let $u \in \overline{\mathbb{R}_{\geq 0}}^{I\times J}$. Then*

$$\sum_{(i,j)\in I\times J} u_{i,j} = \sum_{i\in I}\sum_{j\in J} u_{i,j} = \sum_{j\in J}\sum_{i\in I} u_{i,j}\,.$$

*Démonstration.* We prove the first equation. For any finite subset $K$ of $I\times J$, with projections $K_1 \subseteq I$ and $K_2 \subseteq J$, we have

$$\begin{aligned}\sum_{(i,j)\in K} u_{i,j} &\leq \sum_{(i,j)\in K_1\times K_2} u_{i,j}\\ &= \sum_{i\in K_1}\sum_{j\in K_2} u_{i,j}\\ &\leq \sum_{i\in K_1}\sum_{j\in J} u_{i,j}\\ &\leq \sum_{j\in I}\sum_{i\in J} u_{i,j}\,.\end{aligned}$$

It follows that $\sum_{(i,j)\in I\times J} u_{i,j} \leq \sum_{i\in I}\sum_{j\in J} u_{i,j}$. We prove the converse inequation, so let $\lambda \in \mathbb{R}_{\geq 0}$, we assume that $\lambda < \sum_{i\in I}\sum_{j\in J} u_{i,j}$ and let us prove that $\lambda < \sum_{(i,j)\in I\times J} u_{i,j}$. Let $I_0$ be a finite subset of $I$ such that $\lambda < \sum_{i\in I_0}\sum_{j\in J} u_{i,j}$.

If $\sum_{i\in I_0}\sum_{j\in J} u_{i,j} = \infty$ there exists $i_0 \in I_0$ such that $\sum_{j\in J} u_{i_0,j} = \infty$ because $I_0$ is finite. Then we clearly have $\sum_{(i,j)\in I\times J} u_{i,j} = \infty$ and we are done. Assume now that $\sum_{i\in I_0}\sum_{j\in J} u_{i,j} < \infty$. For each $i \in I_0$ let $\lambda_i = \sum_{j\in J} u_{i,j} \in \mathbb{R}_{\geq 0}$, we have $\sum_{i\in I_0}\lambda_i > \lambda$ and hence $\lambda/(\sum_{i\in I_0}\lambda_i) < 1$. Let $\varepsilon = 1 - \lambda/(\sum_{i\in I_0}\lambda_i) > 0$. For each $i \in I_0$ there is a finite subset $J_0(i)$ such that $\sum_{j\in J_0(i)} u_{i,j} > (1-\varepsilon)\lambda_i$. Let $K_0 = \{(i,j) \in I\times J \mid i \in I_0 \text{ and } j \in J_0(i)\}$, this is a finite set. We have

$$\begin{aligned}\sum_{(i,j)\in I\times J} u_{i,j} &\geq \sum_{(i,j)\in K_0} u_{i,j}\\ &= \sum_{i\in I_0}\sum_{j\in J_0(i)} u_{i,j} \quad \text{since } K_0 \text{ is finite}\\ &> \sum_{i\in I_0}(1-\varepsilon)\lambda_i = \lambda\end{aligned}$$

by definition of $\varepsilon$. □

We will use quite often the following special case of the monotone convergence theorem.

**Théorème 1.4.4** *Let $(u(n))_{n\in\mathbb{N}}$ be a family of elements of $\mathbb{R}^I_{\geq 0}$ which is non-decreasing for the product order (see Section 0.1, in this case, this condition means that $\forall n \in \mathbb{N}\, \forall i \in I\ u(n)_i \leq u(n+1)_i$). Then the family $(\sum_{i\in I} u(n)_i)_{n\in\mathbb{N}}$ is non-decreasing and satisfies*

$$\sup_{n\in\mathbb{N}}\sum_{i\in I} u(n)_i = \sum_{i\in I}\sup_{n\in\mathbb{N}} u(n)_i$$

*Démonstration.* For all $n \in \mathbb{N}$ we have $\forall i \in I\ u(n)_i \leq \sup_{q\in\mathbb{N}} u(q)_i$ and hence $\sum_{i\in I} u(n)_i \leq \sum_{i\in I}\sup_{q\in\mathbb{N}} u(q)_i$ so that $\sup_{n\in\mathbb{N}}\sum_{i\in I} u(n)_i \leq \sum_{i\in I}\sup_{n\in\mathbb{N}} u(n)_i$, we prove now the converse inequation. It suffices to prove that for any $\lambda \in \mathbb{R}_{\geq 0}$, if $\lambda < \sum_{i\in I}\sup_{n\in\mathbb{N}} u(n)_i$ then $\lambda < \sup_{n\in\mathbb{N}}\sum_{i\in I} u(n)_i$. So assume that $\lambda < \sum_{i\in I}\sup_{n\in\mathbb{N}} u(n)_i$. Then we can find a finite subset $I_0$ of $I$ such that $\lambda < \sum_{i\in I_0}\sup_{n\in\mathbb{N}} u(n)_i$. Let $i_1, \dots, i_k$ be an enumeration of $I_0$, by Lemma 1.4.1 we have

$$\sum_{i\in I_0}\sup_{n\in\mathbb{N}} u(n)_i = \sup_{(n_1,\dots,n_k)\in\mathbb{N}^k}\sum_{j=1}^{k} u(n_j)_{i_j} = \sup_{n\in\mathbb{N}}\sum_{i\in I_0} u(n)_i$$

the last equality results from the fact that for each $(n_1,\dots,n_k) \in \mathbb{N}^k$ we can find $n \in \mathbb{N}$ such that $n_j \leq n$ for each $j$, and from the fact that each sequence $(u(n)_i)_{n\in\mathbb{N}}$ is non-decreasing. So we have

$$\lambda < \sup_{n\in\mathbb{N}}\sum_{i\in I_0} u(n)_i \leq \sup_{n\in\mathbb{N}}\sum_{i\in I} u(n)_i$$

and this ends the proof. □

A *stochastic matrix* over $I$ is an $S \in [0,1]^{I\times I}$ such that

$$\forall i \in I \quad \sum_{j\in I} S_{i,j} = 1\,.$$

(if one has only $\sum_{j\in I} S_{i,j} \leq 1$ then one says that $S$ is *substochastic*). In this setting $I$ can be interpreted as a set of states and $S$ describes the dynamics of a system by stipulating the probability $S_{i,j}$ that it moves from state $i$ to state $j$.

There is an identity stochastic matrix $\mathsf{Id}$ given by $\mathsf{Id}_{i,j} = \delta_{i,j}$. If $S$ and $T$ are stochastic matrices we can define their product $S\,T \in \overline{\mathbb{R}_{\geq 0}}^{I\times I}$ given by

$$(S\,T)_{i,j} = \sum_{k\in I} S_{i,k} T_{k,j}\,.$$

Let $i \in I$, we have

$$\begin{aligned}\sum_{j\in I}(S\,T)_{i,j} &= \sum_{j\in I}\sum_{k\in I} S_{i,k}T_{k,j}\\ &= \sum_{k\in I}\sum_{j\in I} S_{i,k}T_{k,j} \quad \text{by Lemma 1.4.3}\\ &= \sum_{k\in I} S_{i,k}\sum_{j\in I} T_{k,j} \quad \text{by Lemma 1.4.1}\\ &= \sum_{k\in I} S_{i,k} = 1\end{aligned}$$

since $S$ and $T$ are stochastic matrices. Hence $S\,T$ is a stochastic matrix. So for each $n \in \mathbb{N}$ we have a stochastic matrix $S^n$ ($S^0 = \mathsf{Id}$ and $S^{n+1} = S\,S^n = S^n\,S$ since matrix product is an associative operation). Let $I^S_1$ be the set of all $i \in I$ such that $S_{i,i} = 1$, it is the set of all stationary states [2] of $S$; notice that $S_{i,j} = \delta_{i,j}$ if $i \in I^S_1$.

**Lemme 1.4.5** *For any $i \in I$ and $j \in I^S_1$ the sequence $(S^n_{i,j})_{n\in\mathbb{N}}$ is non-decreasing. We set $S^\infty_{i,j} = \sup_{n\in\mathbb{N}} S^n_{i,j} \in [0,1]$.*

*Démonstration.* We have

$$S^{n+1}_{i,j} = (S^n\,S)_{i,j} = \sum_{k\in I} S^n_{i,k}S_{k,j} \geq \sum_{k\in I^S_1} S^n_{i,k}S_{k,j} = S^n_{i,j}\,.$$

□

We call *path* a sequence $\pi = (i_1,\dots,i_r)$ of elements of $I$ such that $r \geq 1$, $i_r \in I^S_1$ and $\forall l \in \{1,\dots,r-1\}$ $i_l \neq i_k$ (but we can have $i_l = i_{l'}$ for $l < l' < k$). We write $\pi : i_1 \rightsquigarrow i_r$. The *length* of $\pi$ is $r-1 \geq 0$ and we write $\pi : i \rightsquigarrow^{\leq n} k$ if $\pi : i \rightsquigarrow k$ and the length of $\pi$ is $\leq n$. So $\pi : i \rightsquigarrow^{\leq 0} k$ means that $i = k$.

If $i \neq r$, we use $i \cdot \pi$ for the path $(i, i_1, \dots, j_r)$.

Then we set

$$\mathsf{Pr}(\pi) = S_{i_1,i_2}\cdots S_{i_{r-1},i_r}\,.$$

Notice that if $\pi = (j)$ (so that $j \in I^S_1$ and $\pi$ has length 0) we have $\mathsf{Pr}(\pi) = 1$.

**Théorème 1.4.6** *Let $i,j \in I$ with $j \in I^S_1$. For any $n \in \mathbb{N}$ one has $S^n_{i,j} = \sum_{\pi : i\rightsquigarrow^{\leq n} j} \mathsf{Pr}(\pi)$. Consequently*

$$S^\infty_{i,j} = \sum_{\pi : i\rightsquigarrow j} \mathsf{Pr}(\pi)\,.$$

2. They can be seen as very specific stationary distributions of the Markov chain $S$, namely those which are concentrated on a single element of $I$.

*Démonstration.* Given $n \in \mathbb{N}$ and $(i,j) \in I \times I_1^S$ we set $\Pi_n(i,j) = \{\pi \mid \pi : i \rightsquigarrow^{\leq n} j\}$. Then we have

$$\Pi_0(i,j) = \begin{cases} \{(j)\} & \text{if } i = j \\ \emptyset & \text{otherwise} \end{cases}$$

$$\Pi_{n+1}(i,j) = \begin{cases} \{(j)\} & \text{if } i = j \\ \{i \cdot \pi \mid k \in I \text{ and } \pi \in \Pi_n(k,j)\} & \text{otherwise.} \end{cases}$$

Therefore we have $\sum_{\pi : i \rightsquigarrow^{\leq 0} j} \mathsf{Pr}(\pi) = \delta_{i,j} = S^0_{i,j}$ so the property holds for $n = 0$. We deal now with the inductive step. Observe first that $S^{n+1}_{j,j} = 1$ because $S_{j,k} = \delta_{j,k}$ and hence $S^{n+1}_{j,j} = \sum_{\pi : j \rightsquigarrow^{\leq n+1} j} \mathsf{Pr}(\pi)$. Assume now that $i \neq j$, we have

$$\begin{aligned} \sum_{\pi : i \rightsquigarrow^{\leq n+1} j} \mathsf{Pr}(\pi) &= \sum_{k \in I} \sum_{\pi : k \rightsquigarrow^{\leq n} j} \mathsf{Pr}(i \cdot \pi) \quad \text{by the description above of } \Pi_{n+1}(j,j) \\ &= \sum_{k \in I} \sum_{\pi : k \rightsquigarrow^{\leq n} j} S_{i,k} \mathsf{Pr}(\pi) \\ &= \sum_{k \in I} S_{i,k} \sum_{\pi : k \rightsquigarrow^{\leq n} j} \mathsf{Pr}(\pi) \\ &= \sum_{k \in I} S_{i,k} S^n_{k,j} \quad \text{by inductive hypothesis} \\ &= S^{n+1}_{i,j} . \end{aligned}$$

□

1.4.2 The reduction stochastic matrix. We define a "probabilistic reduction strategy" by the following rules, which define two relations : $M\ \beta^{\mathsf{d}}_{\mathsf{wh}}\ M'$ means that $M$ reduced to $M'$ deterministically and $M\ \beta^{r}_{\mathsf{wh}}\ M'$ means that $M$ reduces to $M'$ with probability $r$ (for $r \in [0,1]$)

$$\frac{}{(\lambda x^A\, M)\, N\ \beta^{\mathsf{d}}_{\mathsf{wh}}\ M\,[N/x]}\ (\beta)$$

$$\frac{}{\mathsf{fix}(M)\ \beta^{\mathsf{d}}_{\mathsf{wh}}\ (M)\,\mathsf{fix}(M)}\ (\mathsf{fix})$$

$$\frac{}{\mathsf{if}(\underline{0}, P, z \cdot Q)\ \beta^{\mathsf{d}}_{\mathsf{wh}}\ P}\ (\mathsf{if}_0) \qquad \frac{}{\mathsf{if}(\underline{n+1}, P, z \cdot Q)\ \beta^{\mathsf{d}}_{\mathsf{wh}}\ Q\,[\underline{n}/z]}\ (\mathsf{if}_+)$$

$$\frac{}{\underline{\mathsf{succ}}(\underline{n})\ \beta^{\mathsf{d}}_{\mathsf{wh}}\ \underline{n+1}}\ (\mathsf{succ})$$

$$\frac{M\ \beta^{\mathsf{d}}_{\mathsf{wh}}\ M'}{M\ \beta^{1}_{\mathsf{wh}}\ M'}\ (\mathsf{d}) \qquad \frac{r \in [0,1] \cap \mathbb{Q}}{\mathsf{rand}(r)\ \beta^{r}_{\mathsf{wh}}\ \underline{0}}\ (\mathsf{rand}_0) \qquad \frac{r \in [0,1] \cap \mathbb{Q}}{\mathsf{rand}(r)\ \beta^{1-r}_{\mathsf{wh}}\ \underline{1}}\ (\mathsf{rand}_1)$$

$$\frac{M\ \beta^{r}_{\mathsf{wh}}\ M'}{(M)\, N\ \beta^{r}_{\mathsf{wh}}\ (M')\, N}\ (\mathsf{app})$$

$$\frac{M\ \beta^{r}_{\mathsf{wh}}\ M'}{\mathsf{if}(M, P, z \cdot Q)\ \beta^{r}_{\mathsf{wh}}\ \mathsf{if}(M', P, z \cdot Q)}\ (\mathsf{if}')$$

$$\frac{M\ \beta^{r}_{\mathsf{wh}}\ M'}{\underline{\mathsf{succ}}(M)\ \beta^{r}_{\mathsf{wh}}\ \underline{\mathsf{succ}}(M')}\ (\mathsf{succ}')$$

Notice that if $M \in \Lambda(\Gamma \vdash A)$ and $M'$ is a term, and if $\rho$ is a derivation of $M\ \beta^{r}_{\mathsf{wh}}\ M'$ for some $r$ in the deduction system above, then $r$ is uniquely determined by $\rho$, we use the notation $r(\rho)$ for this probability.

**Lemme 1.4.7** *Let $M \in \Lambda(\Gamma \vdash A)$. There are exactly three disjoint possibilities.*

— *Either there is no triple $(\rho, r, M')$ such that $\rho : M\ \beta^{r}_{\mathsf{wh}}\ M'$ and then $M \in \Lambda_0(\Gamma \vdash A)$*

— *or there is exactly one pair* $(\rho, M')$ *such that* $\rho : M \; \beta_{\mathsf{wh}}^{r(\rho)} \; M'$ *and then* $r(\rho) = 1$

— *or there are exactly two pairs* $(\rho_i, M_i)$ *(for* $i = 0, 1$*) such that* $\rho_i : M \; \beta_{\mathsf{wh}}^{r(\rho_i)} \; M_i$ *and then* $r(\rho_0) + r(\rho_1) = 1$ *and* $M_0 \neq M_1$*.*

*Démonstration.* It is clear that the three cases are pairwise disjoint.

The proof is by induction on $M$. If $M$ is a variable or a constant $\underline{n}$ we are in the first case. If $M = \mathsf{rand}(q)$ for some $q \in [0,1] \cap \mathbb{Q}$ we are in the third case with $M_i = \underline{i}$ for $i = 0, 1$ and $\rho_i$ consists of an instance of $(\mathsf{rand}_i)$.

For the inductive step, assume first that $M = \underline{\mathsf{succ}}(P)$ (so that $A = \iota$).

If $P = \underline{n}$ for some $n \in \mathbb{N}$ we are in the second case with $M' = \underline{n+1}$ and $\rho$ consists of an instance of $(\mathsf{succ})$. Otherwise, by inductive hypothesis, there are three disjoint possibilities.

— There is no triple $(\rho, r, P')$ such that $\rho : P \; \beta_{\mathsf{wh}}^{r} \; P'$ and then $P \in \Lambda_0(\Gamma \vdash A)$. Since $P$ is not of shape $\underline{n}$, no reduction applies to $M = \underline{\mathsf{succ}}(P)$ (such a reduction would end with a $(\mathsf{succ}')$ rule with a premise of shape $P \; \beta_{\mathsf{wh}}^{r} \; P'$) and hence $M \in \Lambda_0(\Gamma \vdash \iota)$ and we are in the first case also for $M$.

— There is exactly one pair $(\rho, P')$ such that $\rho : P \; \beta_{\mathsf{wh}}^{r(\rho)} \; P'$ and we know that $r(\rho) = 1$. In that case we have $M = \underline{\mathsf{succ}}(P) \; \beta_{\mathsf{wh}}^{1} \; \underline{\mathsf{succ}}(P')$ by a $(\mathsf{succ}')$ rule. Moreover, if $\theta : M = \underline{\mathsf{succ}}(P) \; \beta_{\mathsf{wh}}^{r(\theta)} \; M'$, then $\theta$ ends with the rule $(\mathsf{succ}')$ (this is the only possibility because $P$ is not of shape $\underline{n}$) and the premise is of shape $Q \; \beta_{\mathsf{wh}}^{r(\theta)} \; Q'$ with $\underline{\mathsf{succ}}(Q) = M$ and is obtained by a derivation $\rho'$. Hence $Q = P$. By our assumption about $P$ we must have $\rho' = \rho$ and $Q' = P'$. So there is exactly one $(\theta, M')$ such that $\theta : M \; \beta_{\mathsf{wh}}^{r(\theta)} \; M'$, namely : $\theta$ is obtained from $\rho$ by extending it with a $(\mathsf{succ}')$ rule, $r(\theta) = r(\rho) = 1$ and $M' = \underline{\mathsf{succ}}(P')$.

— There are exactly two pairs $(\rho_i, P_i)$ (for $i = 0, 1$) such that $\rho_i : P \; \beta_{\mathsf{wh}}^{r(\rho_i)} \; P_i$ for $i = 0, 1$ and moreover $r(\rho_0) + r(\rho_1) = 1$ and $P_0 \neq P_1$. In that case we have $M = \underline{\mathsf{succ}}(P) \; \beta_{\mathsf{wh}}^{r(\rho_i)} \; M_i = \underline{\mathsf{succ}}(P_i)$ by $(\mathsf{succ}')$ for $i = 0, 1$, notice that $M_0 \neq M_1$ since $P_0 \neq P_1$. Moreover, if $\theta : M = \underline{\mathsf{succ}}(P) \; \beta_{\mathsf{wh}}^{r(\theta)} \; M'$, then $\theta$ ends with the rule $(\mathsf{succ}')$ (this is the only possibility because $P$ is not of shape $\underline{n}$) and the premise is of shape $Q \; \beta_{\mathsf{wh}}^{r(\theta)} \; Q'$ with $\underline{\mathsf{succ}}(Q) = M$ and is obtained by a derivation $\rho'$. Hence $Q = P$. By our assumption about $P$ we must have $\rho' = \rho_0$ and $Q' = P_0$ or $\rho' = \rho_1$ and $Q' = P_1$. So there are exactly two pairs $(\theta_i, M_i)$ such that $\theta_i : M \; \beta_{\mathsf{wh}}^{r(\theta_i)} \; M_i$ namely : $\theta_i$ is obtained from $\rho_i$ by extending it with a $(\mathsf{succ}')$ rule, $r(\theta_i) = r(\rho_i)$ and $M_i = \underline{\mathsf{succ}}(P_i)$. So that $M_0 \neq M_1$ and $r(\theta_0) + r(\theta_1) = 1$.

The other cases of the inductive step are dealt with similarly. □

We define $\mathsf{Red}(\Gamma \vdash A) \in [0,1]^{\Lambda(\Gamma \vdash A) \times \Lambda(\Gamma \vdash A)}$ by

$$\mathsf{Red}(\Gamma \vdash A)_{M,M'} = \begin{cases} 1 & \text{if } M = M' \in \Lambda_0(\Gamma \vdash A) \\ r & \text{if } M \; \beta_{\mathsf{wh}}^{r} \; M' \\ 0 & \text{if there is no reduction } M \; \beta_{\mathsf{wh}}^{r} \; M' . \end{cases}$$

**Théorème 1.4.8** $\mathsf{Red}(\Gamma \vdash A)$ *is a stochastic matrix.*

*Démonstration.* This is an immediate consequence of Lemma 1.4.7. □

We are mainly interested by $\mathsf{Red}(\vdash \iota)$. Notice that $\Lambda_0(\vdash \iota) = \{\underline{n} \mid n \in \mathbb{N}\}$. Given $M \in \Lambda(\vdash \iota)$ (that is $M$ is closed of type $\iota$). Then $\mathsf{Red}(\vdash \iota)^{\infty}_{M,\underline{n}}$ is the probability that $M$ reduces to $M$ : this is precisely what Theorem 1.4.6 expresses.

### 1.4.3 Examples.

1.4.4 A simple choice sequence. The term $M_1 = \mathsf{if}(\mathsf{rand}(1/2), \underline{3}, d \cdot \mathsf{if}(\mathsf{rand}(1/3), \underline{0}, d \cdot \underline{5})) \in \Lambda(\vdash \iota)$ reduces to $\underline{3}$ with probability 1/2, to $\underline{0}$ with probability 1/6 and to $\underline{5}$ with probability 1/3.

1.4.5 The let construct. Assume that $M$ and $N$ are terms with $\Gamma \vdash M : \iota$ and $\Gamma, x : \iota \vdash N : A$, then we define

$$\mathsf{let}(M, x \cdot N) = \mathsf{if}(M, N, y \cdot N\,[\underline{\mathsf{succ}}(y)/x])$$

so that $\Gamma \vdash \mathsf{let}(M, x \cdot N) : A$. The fact that we can write such a macro is a crucial feature of our version of PCF, absolutely essential in probabilistic programming. It means that we can use *call by value* in PCF, but only for our unique data-type, which is the type of integers. Notice that $\mathsf{let}(M, x\cdot N)$ is not equivalent to $(\lambda x^\iota\, N)\, M$ as illustrated by the following example.

Take $M = \mathsf{rand}(\frac{1}{3})$ and $N = \mathsf{if}(x, \mathsf{if}(x, \underline{0}, d \cdot \underline{1}), d \cdot \underline{2})$. Then

$$
\begin{aligned}
P = (\lambda x^\iota\, N)\, M \;&\beta^1_{\mathsf{wh}}\; \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
&\beta^{\frac{1}{3}}_{\mathsf{wh}}\; \mathsf{if}(\underline{0}, \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
&\beta^1_{\mathsf{wh}}\; \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \underline{0}, d \cdot \underline{1}) \\
&\beta^{\frac{1}{3}}_{\mathsf{wh}}\; \underline{0} \\
P \;&\beta^1_{\mathsf{wh}}\; \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
&\beta^{\frac{1}{3}}_{\mathsf{wh}}\; \mathsf{if}(\underline{0}, \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
&\beta^1_{\mathsf{wh}}\; \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \underline{0}, d \cdot \underline{1}) \\
&\beta^{\frac{2}{3}}_{\mathsf{wh}}\; \underline{1} \\
P \;&\beta^1_{\mathsf{wh}}\; \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
&\beta^{\frac{2}{3}}_{\mathsf{wh}}\; \mathsf{if}(\underline{1}, \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
&\beta^1_{\mathsf{wh}}\; \underline{2}
\end{aligned}
$$

hence $\mathsf{Red}(\vdash \iota)^\infty_{P,\underline{0}} = \frac{1}{9}$, $\mathsf{Red}(\vdash \iota)^\infty_{P,\underline{1}} = \frac{2}{9}$, $\mathsf{Red}(\vdash \iota)^\infty_{P,\underline{2}} = \frac{2}{3}$ and $\mathsf{Red}(\vdash \iota)^\infty_{P,\underline{n}} = 0$ for $n \geq 3$. On the other hand we have

$$
\begin{aligned}
Q = \mathsf{let}(M, x \cdot N) = \mathsf{if}(\mathsf{rand}(\frac{1}{3}), \mathsf{if}(\underline{0}, \mathsf{if}(\underline{0}, \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}), z \cdot \mathsf{if}(\underline{\mathsf{succ}}(z), \mathsf{if}(\underline{\mathsf{succ}}(z), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2})) \\
\beta^{\frac{1}{3}}_{\mathsf{wh}}\; \mathsf{if}(\underline{0}, \mathsf{if}(\underline{0}, \mathsf{if}(\underline{0}, \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}), z \cdot \mathsf{if}(\underline{\mathsf{succ}}(z), \mathsf{if}(\underline{\mathsf{succ}}(z), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2})) \\
\beta^1_{\mathsf{wh}}\; \mathsf{if}(\underline{0}, \mathsf{if}(\underline{0}, \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
\beta^1_{\mathsf{wh}}\; \mathsf{if}(\underline{0}, \underline{0}, d \cdot \underline{1}) \\
\beta^1_{\mathsf{wh}}\; \underline{0} \\
Q\; \beta^{\frac{2}{3}}_{\mathsf{wh}}\; \mathsf{if}(\underline{1}, \mathsf{if}(\underline{0}, \mathsf{if}(\underline{0}, \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}), z \cdot \mathsf{if}(\underline{\mathsf{succ}}(z), \mathsf{if}(\underline{\mathsf{succ}}(z), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2})) \\
\beta^1_{\mathsf{wh}}\; \mathsf{if}(\underline{\mathsf{succ}}(\underline{0}), \mathsf{if}(\underline{\mathsf{succ}}(\underline{0}), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
\beta^1_{\mathsf{wh}}\; \mathsf{if}(\underline{1}, \mathsf{if}(\underline{\mathsf{succ}}(\underline{0}), \underline{0}, d \cdot \underline{1}), d \cdot \underline{2}) \\
\beta^1_{\mathsf{wh}}\; \underline{2}
\end{aligned}
$$

hence $\mathsf{Red}(\vdash \iota)^\infty_{Q,\underline{0}} = \frac{1}{3}$, $\mathsf{Red}(\vdash \iota)^\infty_{Q,\underline{1}} = 0$, $\mathsf{Red}(\vdash \iota)^\infty_{Q,\underline{2}} = \frac{2}{3}$ and $\mathsf{Red}(\vdash \iota)^\infty_{P,\underline{n}} = 0$ for $n \geq 3$. It appears that $P$ and $Q$ do not behave in the same way : in $Q$ the "dice" $\mathsf{rand}(\frac{1}{3})$ is tossed only once (and the result is stored) whereas it is tossed twice in two of the reduction pathes of $P$.

1.4.6 Tossing a coin untils we get 0. The term $M_2 = \mathsf{fix}(\lambda x^\iota\, \mathsf{if}(\mathsf{rand}(1/2), \underline{0}, d \cdot x)) \in \Lambda(\vdash \iota)$ tosses a fair $0, 1$-valued coin untils it gets 0. The probability of termination is 1 but there is no bound on the number of reduction steps : here our definition of $\mathsf{Red}(\Gamma \vdash A)^\infty$ as a sup is essential.

1.4.7 A random walk in $\mathbb{N}$. Here is a more involved example : $M_3 \in \Lambda(\vdash \iota \to \iota \to \iota)$ given by

$$M_3 = \mathsf{fix}(\lambda f^{\iota \to \iota \to \iota}\, \lambda x^\iota\, \lambda y^\iota\, \mathsf{if}(x, y, d \cdot \mathsf{if}(\mathsf{rand}(1/2), ((f)\, \underline{\mathsf{succ}}(x))\, \underline{\mathsf{succ}}(y), d \cdot ((f)\, (P)\, x)\, \underline{\mathsf{succ}}(y))))$$

where $P = \lambda z^\iota\, \mathsf{if}(z, \underline{0}, x \cdot x) \in \Lambda(\vdash \iota \to \iota)$ is the predecessor function. Then, given $n \in \mathbb{N}$, the execution of $((M)\, \underline{n})\, \underline{0}$ runs a "random walk" $n_0, n_1, \ldots$ on $\mathbb{N}$, starting with $n = n_0$ where $n_{i+1}$ is $n_i + 1$ or $n_i - 1$ (when $n_i > 0$) with probability $1/2$, the result of the function being the first integer $k$ such that $n_k = 0$.

1.4.8 A UNIFORM PROBABILITY DISTRIBUTION. We define closed terms $\mathsf{dbl}, \mathsf{pow} \in \Lambda(\vdash \iota \to \iota)$ which maps an integer $n$ to $2n$ and $2^n$ :

$$\begin{aligned}\mathsf{dbl} &= \lambda x^{\iota}\,(\mathsf{add})\,x\,x \\ \mathsf{pow} &= \mathsf{fix}(\lambda f^{\iota \multimap \iota}\,\lambda x^{\iota}\,\mathsf{if}(x, \underline{1}, y \cdot (\mathsf{dbl})\,(\mathsf{pow})\,y))\end{aligned}$$

Then we define $\mathsf{unif} \in \Lambda(\vdash \iota \to \iota)$ which maps $n$ to an integer chosen randomly uniformly in $\{0, \ldots, 2^n - 1\}$ :

$$\mathsf{unif} = \mathsf{fix}(\lambda u^{\iota \to \iota}\,\lambda x^{\iota}\,\mathsf{if}(x, \underline{0}, z \cdot \mathsf{if}(\mathsf{rand}(\frac{1}{2}), (u)\,z, d \cdot ((\mathsf{add})\,(u)\,z)\,(\mathsf{pow})\,z)))$$

1.4.9 A LAS VEGAS ALGORITHM. This function takes a function $f$, an integer $n$ and looks randomly for a $k \in \{0, \ldots, 2^n - 1\}$ such that $f(k) = 0$, it stops and returns such a $k$ as soon as it finds one. This function $\mathsf{rfind} \in \Lambda(\vdash (\iota \to \iota) \to \iota \to \iota)$ can be written as follows :

$$\mathsf{rfind} = \lambda f^{\iota \to \iota}\,\lambda x^{\iota}\,\mathsf{fix}(\lambda z^{\iota}\,\mathsf{let}((\mathsf{unif})\,x, y \cdot \mathsf{if}((f)\,y, y, d \cdot z)))$$

Notice the use of the let construct : it is required because if we find an integer $y$ such that "$(f)y = 0$", it is the very same value $y$ that we have to return as a result of the function. Without a let construct (or an if construct such as ours, which feeds the term of the "non-zero" branch of the conditional with the predecessor of the value of the tested term) it would be impossible to write such a (typical) probabilistic algorithm.

# Chapitre 2

# PCF with lazy integers

In PCF, integers are "strict" in the sense that they are either undefined or completely evaluated. It is however possible to deal with integers in a lazy way, that is, to deal with partially defined integers. The syntax of this variant LPCF of PCF is defined as follows.

$$M, N \ldots := x \mid \underline{0} \ \mid \underline{\mathsf{succ}}(M) \mid \mathsf{if}(M, N, x \cdot P) \mid (M)\, N \mid \lambda x^A\, M \mid \mathsf{fix}\, x^A \cdot M\,.$$

The typing rules are exactly the same as those of PCF, we record them for completeness.

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)\, N : B}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A\, M : A \to B} \qquad \frac{\Gamma, x : A \vdash M : A}{\Gamma \vdash \mathsf{fix}\, x^A \cdot M : A}$$

$$\frac{}{\Gamma \vdash \underline{0} : \iota} \qquad \frac{\Gamma \vdash M : \iota}{\Gamma \vdash \underline{\mathsf{succ}}(M) : \iota}$$

$$\frac{\Gamma \vdash M : \iota \qquad \Gamma \vdash P : A \qquad \Gamma, z : \iota \vdash Q : A}{\Gamma \vdash \mathsf{if}(M, P, z \cdot Q) : A}$$

## 2.1 Reduction

Just as for PCF, we define a reduction relation by means of "deduction rules"

$$\frac{}{\left(\lambda x^A\, M\right) N \;\beta\; M\,[N/x]}$$

$$\frac{}{\mathsf{fix}\, x^A \cdot M \;\beta\; M\left[\mathsf{fix}\, x^A \cdot M / x\right]}$$

$$\frac{}{\mathsf{if}(\underline{0}, P, z \cdot Q) \;\beta\; P} \qquad \frac{}{\mathsf{if}(\underline{\mathsf{succ}}(M), P, z \cdot Q) \;\beta\; Q\,[M/z]}$$

$$\frac{M \;\beta\; M'}{\underline{\mathsf{succ}}(M) \;\beta\; \underline{\mathsf{succ}}(M')}$$

$$\frac{M \;\beta\; M'}{(M)\, N \;\beta\; (M')\, N} \qquad \frac{N \;\beta\; N'}{(M)\, N \;\beta\; (M)\, N'}$$

$$\frac{M \;\beta\; M'}{\mathsf{fix}\, x^A \cdot M \;\beta\; \mathsf{fix}\, x^A \cdot M'}$$

$$\frac{M \;\beta\; M'}{\mathsf{if}(M, P, z \cdot Q) \;\beta\; \mathsf{if}(M', P, z \cdot Q)}$$

$$\frac{P\ \beta\ P'}{\mathsf{if}(M, P, z \cdot Q)\ \beta\ \mathsf{if}(M, P', z \cdot Q)} \qquad \frac{Q\ \beta\ Q'}{\mathsf{if}(M, P, z \cdot Q)\ \beta\ \mathsf{if}(M, P, z \cdot Q')}$$

**Remarque 2.1.1** In other words in LPCF, a redex si a term of one of the following shapes :
— $(\lambda x^A\, R)\, Q$ which reduces to $R\,[Q/x]$,
— $\mathsf{fix}\, x^A \cdot P$ which reduces to $P\,[\mathsf{fix}\, x^A \cdot P/x]$,
— $\mathsf{if}(\underline{0}, P, z \cdot Q)$ which reduces to $P$,
— $\mathsf{if}(\underline{\mathsf{succ}}(M), P, z \cdot Q)$ which reduces to $Q\,[M/z]$.
And $M\ \beta\ M'$ if one gets $M'$ by chosing anywhere in $M$ a redex and replacing it with its reduced form. The main difference wrt. PCF is that, in the conditional construct, the predecessor of an integer its passed to the term in the right branch without being evaluated.

As usual we set $\Omega^A = \mathsf{fix}\, x^A \cdot x$ which is a closed term such that $\vdash \Omega^A : A$, the ever-looping term of type $A$.

2.1.1 Weak head reduction. We define similarly a *weak head reduction* $\beta_{\mathsf{wh}}$ ("weak" means that one cannot reduce under $\lambda$s).

$$\frac{}{(\lambda x^A\, M)\, N\ \beta_{\mathsf{wh}}\ M\,[N/x]}$$

$$\frac{}{\mathsf{fix}\, x^A \cdot M\ \beta_{\mathsf{wh}}\ M\,[\mathsf{fix}\, x^A \cdot M/x]}$$

$$\frac{}{\mathsf{if}(\underline{0}, P, z \cdot Q)\ \beta_{\mathsf{wh}}\ P} \qquad \frac{}{\mathsf{if}(\underline{\mathsf{succ}}(M), P, z \cdot Q)\ \beta_{\mathsf{wh}}\ Q\,[M/z]}$$

$$\frac{M\ \beta_{\mathsf{wh}}\ M'}{(M)\, N\ \beta_{\mathsf{wh}}\ (M')\, N}$$

$$\frac{M\ \beta_{\mathsf{wh}}\ M'}{\mathsf{if}(M, P, z \cdot Q)\ \beta_{\mathsf{wh}}\ \mathsf{if}(M', P, z \cdot Q)}$$

**Remarque 2.1.2** One has $\beta_{\mathsf{wh}} \subseteq \beta$. Observe that $\beta_{\mathsf{wh}}$ is a *reduction strategy*, meaning that, given an LPCF term $M$, either $M$ is a "head normal form" (non reducible for $\beta_{\mathsf{wh}}$), or $M$ contains exactly one redex such that $M\ \beta_{\mathsf{wh}}\ M'$ for an LPCF term $M'$, which is uniquely determined.

In particular we cannot add the reduction rule

$$\frac{M\ \beta_{\mathsf{wh}}\ M'}{\underline{\mathsf{succ}}(M)\ \beta_{\mathsf{wh}}\ \underline{\mathsf{succ}}(M')}$$

as othewise a term like $M = \mathsf{if}(\underline{\mathsf{succ}}(\Omega^\iota), \underline{0}, z \cdot \underline{\mathsf{succ}}(\underline{0}))$ would be reducible in two different ways. With our definition of $\beta_{\mathsf{wh}}$ the only possible reduction is $M \beta_{\mathsf{wh}}\ \underline{\mathsf{succ}}(\underline{0})$. This example illustrates the lazyness of the language : we don't need to know anything about the term $P$ to say that the term $\underline{\mathsf{succ}}(P)$ represents a non-zero integer.

2.1.2 Subject reduction and confluence

**Proposition 2.1.3** *Assume that $\Gamma \vdash M : A$ and the $M\ \beta\ M'$. Then $\Gamma \vdash M' : A$.*

This is proven by induction on the derivation of the fact that $M\ \beta\ M'$, using the following substitution lemma.

**Lemme 2.1.4** *If $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash M : A$, then $\Gamma \vdash M\,[N/x] : B$.*

This is proven by induction on the typing derivation of $M$ (that is, on $M$).

It is useful to know that the reduction $\beta$ satisfies the Church-Rosser property :

**Théorème 2.1.5** *If $M\ \beta^*\ M_i$ for $i = 1, 2$, there exists $M'$ such that $M_i\ \beta^*\ M'$ for $i = 1, 2$. That is : the relation $\beta^*$ satisfies the* Diamond Property.

The Tait-Martin-Löf method consists in defining an auxiliary notion of *parallel reduction* $\rho$ by the following rules.

$$\frac{}{\underline{0}\;\rho\;\underline{0}} \qquad \frac{}{x\;\rho\;x}$$

$$\frac{M\;\rho\;M' \qquad N\;\rho\;N'}{\left(\lambda x^A\,M\right)N\;\rho\;M'\,[N'/x]} \qquad \frac{M\;\rho\;M'}{\lambda x^A\,M\;\rho\;\lambda x^A\,M'}$$

$$\frac{M\;\rho\;M'}{\mathsf{fix}\,x^A\cdot M\;\rho\;M'\left[\mathsf{fix}\,x^A\cdot M'/x\right]}$$

$$\frac{P\;\rho\;P'}{\mathsf{if}(\underline{0},P,z\cdot Q)\;\rho\;P'} \qquad \frac{Q\;\rho\;Q' \qquad M\;\rho\;M'}{\mathsf{if}(\underline{\mathsf{succ}}(M),P,z\cdot Q)\;\rho\;Q'\,[M'/z]}$$

$$\frac{M\;\rho\;M'}{\underline{\mathsf{succ}}(M)\;\rho\;\underline{\mathsf{succ}}(M')}$$

$$\frac{M\;\rho\;M' \qquad N\;\rho\;N'}{(M)\,N\;\rho\;(M')\,N'}$$

$$\frac{M\;\rho\;M'}{\mathsf{fix}\,x^A\cdot M\;\rho\;\mathsf{fix}\,x^A\cdot M'}$$

$$\frac{M\;\rho\;M' \qquad P\;\rho\;P' \qquad Q\;\rho\;Q'}{\mathsf{if}(M,P,z\cdot Q)\;\rho\;\mathsf{if}(M',P',z\cdot Q')}$$

If $M\;\rho\;M'$ then $M'$ is obtained from $M$ by reducing an arbitrary number of redexes *which are already present in* $M$. In other words, one is not allowed to reduce redexes which have have been created by a $\beta$-reduction in the same $\rho$ reduction step.

**Exercice 2.1.1** Give examples of two terms $M$ and $M'$ such that $M\;\beta^*\;M'$ but not $M\rho M'$.

**Exercice 2.1.2** Check that $M\;\rho\;M$ for all term $M$. Prove that $\beta\subseteq\rho\subseteq\beta^*$. Conclude that $\rho^*=\beta^*$.

**Exercice 2.1.3** Prove that, if a relation $\gamma$ satisfies the Diamond Property, then its reflexive-transitive closure $\gamma^*$ satisfies also the Diamond Property.

So it suffices to prove that the relation $\rho$ satisfies the Diamond Property.

**Exercice 2.1.4** Assume that $\Gamma,x:A\vdash M:B$ and $\Gamma\vdash N:A$ and assume that $M\;\rho\;M'$ and $N\;\rho\;N'$. Prove by induction on $M$ that $M\,[N/x]\;\rho\;M'\,[N'/x]$.

**Exercice 2.1.5** Prove that $\rho$ satisfies the Diamond Property. That is : Assume that $\Gamma\vdash M:A$ and that $M\;\rho\;M_i$ for $i=1,2$. Prove that there is a term $R$ such that $M_i\;\rho\;R$ for $i=1,2$. The proof is by induction on $M$. For each induction step, one has to consider all possible $\rho$-deduction rules which apply. For instance, if $M=(P)\,Q$, we have the following possibilities :
— $P\;\rho\;P_i$ and $Q\;\rho\;Q_i$ for $i=1,2$, with $M_i=(P_i)\,Q_i$.
— $P=\lambda x^B\,H$, $H\;\rho\;H_i$, $Q\;\rho\;Q_i$, $M_1=H_1\,[Q_1/x]$ and $M_2=\left(\lambda x^B\,H_2\right)Q_2$.
— A case symmetric to the previous one, swapping 1 and 2.
— $P=\lambda x^{B\to A}\,H$, $H\;\rho\;H_i$, $Q\;\rho\;Q_i$, $M_i=H_i\,[Q_i/x]$ for $i=1,2$.
In each case one applies the inductive hypothesis and the result of Exercise 2.1.4 in all cases but the first one.

The other inductive steps are dealt with similarly, the most important being the following ones :
— $M=\mathsf{if}(N,P,x\cdot Q)$ and then the main sub-case is when $N=\underline{\mathsf{succ}}(H)$
— $M=\mathsf{fix}\,x^A\cdot N$.
The other ones are routine.

As usual one of the main consequences of confluence is that, when a term normalizes, it has a unique normal form. Typically, if $\vdash M : \iota$, then either $M$ has no $\beta$-normal form, or there is a unique integer $n$ such that $M \ \beta^* \ \underline{n}$.

Also, defining $\sim_\beta$ as the least equivalence relation on terms which contains $\beta$, we can deduce from confluence that, if $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$, then $M \sim_\beta M'$ iff there exists a term $N$ such that $M \ \beta^* \ N$ and $M' \ \beta^* \ N$.

2.1.3 EXAMPLES For any type $A$, we have defined $\Omega^A = \mathsf{fix}\, x^A \cdot x$ which satisfies $\vdash \Omega^A : A$ and $\Omega^A \ \beta_{\mathsf{wh}} \ \Omega^A$ : it represents the ever-looping program of type $A$.

Given $n \in \mathbb{N}$ and a term $M$ such that $\Gamma \vdash M : \iota$, we define $\mathsf{succ}^n(M)$ such that $\Gamma \vdash \mathsf{succ}^n(M) : \iota$ by $\mathsf{succ}^0(M) = M$ and $\mathsf{succ}^{n+1}(M) = \underline{\mathsf{succ}}(\mathsf{succ}^n(M))$.

The integer $n$ is represented by $\underline{n} = \mathsf{succ}^n(\underline{0})$. We have various ways of defining an addition function. A first solution is

$$\mathsf{radd} = \lambda x^\iota \, \mathsf{fix}\, a^{\iota \to \iota} \cdot \lambda y^\iota \, \mathsf{if}(y, x, z \cdot \underline{\mathsf{succ}}((a)\, z))$$

then we have $(\mathsf{radd})\, M\, \underline{n} \ \beta^* \ \mathsf{succ}^n(M)$ for any term $M$ such that $\Gamma \vdash M : \iota$, so that in particular $(\mathsf{radd})\, \underline{m}\, \underline{n} \ \beta^* \ \underline{m+n}$. Observe that $(\mathsf{radd})\, M\, (\mathsf{succ}^n(\Omega^\iota)) \ \beta^* \ \mathsf{succ}^n(\Omega)$ where $\Omega$ is a term which does not $\beta_{\mathsf{wh}}$-normalize.

A different algorithm, which swaps its arguments at each recursive call is

$$\mathsf{add} = \mathsf{fix}\, a^{\iota \to \iota \to \iota} \cdot \lambda x^\iota \, \lambda y^\iota \, \mathsf{if}(y, x, z \cdot \underline{\mathsf{succ}}((a)\, z\, x))$$

Here is a function which computes the greatest lower bound of two integers

$$\mathsf{min} = \mathsf{fix}\, f^{\iota \Rightarrow \iota \Rightarrow \iota} \cdot \lambda x^\iota \, \lambda y^\iota \, \mathsf{if}(x, \underline{0}, x' \cdot \mathsf{if}(y, \underline{0}, y' \cdot \underline{\mathsf{succ}}((f)\, x'\, y')))$$

**Exercice 2.1.6** Prove that $(\mathsf{min})\, \underline{m}\, \mathsf{succ}^n(M) \ {\beta_{\mathsf{wh}}}^* \ \underline{m}$ and $(\mathsf{min})\, \mathsf{succ}^n(M)\, \underline{m} \ {\beta_{\mathsf{wh}}}^* \ \underline{m}$ as soon as $m \leq n$.

**Exercice 2.1.7** Consider the term :

$$\mathsf{it} = \lambda f^{A \to A} \, \lambda x^A \, \mathsf{fix}\, F^{\iota \to A} \cdot \lambda y^\iota \, \mathsf{if}(y, x, y' \cdot (f)\, ((F)\, y'))$$

Prove that $\vdash \mathsf{it} : (A \to A) \to A \to \iota \to A$ and explain the beavior of it.

Let $\underline{\infty} = \mathsf{fix}\, z^\iota \cdot \underline{\mathsf{succ}}(z)$. Let $M = \lambda f^{A \to A}\, (\mathsf{it})\, f\, \Omega^A\, \underline{\infty}$. Check that $\vdash M : (A \to A) \to A$ and that $M$ behaves exactly like $\lambda f^{A \to A} \, \mathsf{fix}\, x^A \cdot (f)\, x$.

# Chapitre 3

# A call-by-push-value programming language

We introduce now a programming language, more general than PCF, and similar to Paul Levy's call-by-push-value lambda-calculus. We call this language $\Lambda_{\mathsf{HP}}$ (for the time being) because it corresponds to a kind of "half-polarized" linear logic.

## 3.1 Syntax and typing

Types are given by the following BNF syntax. We define by mutual induction two kinds of types : *positive types* (denoted with letters $\varphi$, $\psi\ldots$) and *general types* (denoted with letters $\sigma$, $\tau\ldots$), given type variables $\zeta$, $\xi\ldots$ :

$$\varphi,\psi,\ldots := {!\sigma} \mid \varphi\otimes\psi \mid \varphi\oplus\psi \mid \zeta \mid \mathsf{Fix}\,\zeta\cdot\varphi \tag{3.1}$$

$$\sigma,\tau\ldots := \varphi \mid \varphi\multimap\sigma \mid \top \tag{3.2}$$

We consider the types up to the equation $\mathsf{Fix}\,\zeta\cdot\varphi = \varphi\,[\mathsf{Fix}\,\zeta\cdot\varphi/\zeta]$.

Terms are given by the following BNF syntax, given variables $x, y, \ldots$.

$$\begin{aligned} M, N\ldots := {} & x \mid M^! \mid \langle M, N\rangle \mid \mathsf{in}_1 M \mid \mathsf{in}_2 M \\ & \mid \lambda x^{\varphi}\, M \mid \langle M\rangle N \mid \mathsf{case}(M, x_1\cdot N_1, x_2\cdot N_2) \\ & \mid \mathsf{pr}_1 M \mid \mathsf{pr}_2 M \mid \mathsf{der}(M) \mid \mathsf{fix}\, x^{!\sigma}\, M \end{aligned}$$

This calculus can be seen as a special case of Levy's CBPV [Lev02] in which the type constructor $F$ is the identity (and $U$ is "!").

The notion of substitution is defined as usual. We give now the typing rules for these terms. A typing context is an expression $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_k : \varphi_k)$ where all types are positive and the $x_i$s are pairwise distinct variables.

$$\frac{\mathcal{P}\vdash M:\sigma}{\mathcal{P}\vdash M^!:{!\sigma}} \qquad \frac{\mathcal{P}\vdash M_1:\varphi_1 \qquad \mathcal{P}\vdash M_2:\varphi_2}{\mathcal{P}\vdash\langle M_1,M_2\rangle:\varphi_1\otimes\varphi_2} \qquad \frac{\mathcal{P}\vdash M:\varphi_i}{\mathcal{P}\vdash \mathsf{in}_i M:\varphi_1\oplus\varphi_2}$$

$$\frac{}{\mathcal{P},x:\varphi\vdash x:\varphi} \qquad \frac{\mathcal{P},x:\varphi\vdash M:\sigma}{\mathcal{P}\vdash\lambda x^{\varphi}\,M:\varphi\multimap\sigma} \qquad \frac{\mathcal{P}\vdash M:\varphi\multimap\sigma \qquad \mathcal{P}\vdash N:\varphi}{\mathcal{P}\vdash\langle M\rangle N:\sigma}$$

$$\frac{\mathcal{P}\vdash M:{!\sigma}}{\mathcal{P}\vdash\mathsf{der}(M):\sigma} \qquad \frac{\mathcal{P},x:{!\sigma}\vdash M:\sigma}{\mathcal{P}\vdash\mathsf{fix}\,x^{!\sigma}\,M:\sigma}$$

$$\frac{\mathcal{P}\vdash M:\varphi_1\oplus\varphi_2 \qquad \mathcal{P},x_1:\varphi_1\vdash M_1:\sigma \qquad \mathcal{P},x_2:\varphi_2\vdash M_2:\sigma}{\mathcal{P}\vdash\mathsf{case}(M,x_1\cdot M_1,x_2\cdot M_2):\sigma}$$

$$\frac{\mathcal{P}\vdash M:\varphi_1\otimes\varphi_2}{\mathcal{P}\vdash\mathsf{pr}_i M:\varphi_i}$$

**Remarque 3.1.1** It might seem strange to the reader acquainted with LL that the rules introducing the $\otimes$ connective and eliminating the $\multimap$ connective have an "additive" handling of typing contexts (the same typing context $\mathcal{P}$ occurs in both premises). The reason for this becomes clear in Section 4.7 where positive types are interpreted as !-coalgebras, which are equipped with morphisms allowing to interpret the structural rules of weakening and contraction. This is why typing contexts involve positive types only.

The next lemma is a simple observation.

**Lemme 3.1.2** *Let $\mathcal{P}$ be a typing context and $M$ be a term. There is at most one type $\sigma$ and one typing derivation of the judgment $\mathcal{P} \vdash M : \sigma$.*

One says that typing is *syntax driven*. Concretely this means that when a term $M$ is typeable, the corresponding typing derivation is isomorphic to $M$.

**Proposition 3.1.3 (Substitution Lemma for types)** *Assume that $\mathcal{P}, x : \varphi \vdash M : \sigma$ and $\mathcal{P} \vdash N : \varphi$. Then $\mathcal{P} \vdash M\,[N/x] : \sigma$.*

**Exercice 3.1.1** Prove this lemma (simple induction on $M$).

## 3.2 Weak reduction

We define now a *weak* reduction relation on terms, meaning that we never reduce within a "box" $M^!$ or under a $\lambda$. We first define the notion of *value* as follows :

— any variable $x$ is a value
— for any term $M$, the term $M^!$ is a value
— if $M$ is a value then $\mathsf{in}_i M$ is a value for $i = 1, 2$
— if $M_1$ and $M_2$ are values then $\langle M_1, M_2\rangle$ is a value.

**Remarque 3.2.1** A closed value is simply a tree whose leaves are "boxes" or "thunks" $M^!$ (where the $M$'s are arbitrary well typed closed terms) and whose internal nodes are either unary nodes bearing an index 1 or 2, or ordered binary nodes.

We use letters $V$, $W$... to denote values. Our reduction relation is defined as follows.

$$\frac{}{\mathsf{der}(M^!) \to_{\mathsf{w}} M} \qquad \frac{}{\langle \lambda x^{\varphi}\, M\rangle V \to_{\mathsf{w}} M\,[V/x]} \qquad \frac{}{\mathsf{pr}_i\langle V_1, V_2\rangle \to_{\mathsf{w}} V_i}$$

$$\frac{}{\mathsf{fix}\, x^{!\sigma}\, M \to_{\mathsf{w}} M\left[(\mathsf{fix}\, x^{!\sigma}\, M)^!/x\right]} \qquad \frac{M \to_{\mathsf{w}} M'}{\mathsf{der}(M) \to_{\mathsf{w}} \mathsf{der}(M')}$$

$$\frac{M \to_{\mathsf{w}} M'}{\langle M\rangle N \to_{\mathsf{w}} \langle M'\rangle N} \qquad \frac{N \to_{\mathsf{w}} N'}{\langle M\rangle N \to_{\mathsf{w}} \langle M\rangle N'}$$

$$\frac{M \to_{\mathsf{w}} M'}{\mathsf{pr}_i M \to_{\mathsf{w}} \mathsf{pr}_i M'} \qquad \frac{M_1 \to_{\mathsf{w}} M_1'}{\langle M_1, M_2\rangle \to_{\mathsf{w}} \langle M_1', M_2\rangle} \qquad \frac{M_2 \to_{\mathsf{w}} M_2'}{\langle M_1, M_2\rangle \to_{\mathsf{w}} \langle M_1, M_2'\rangle}$$

$$\frac{}{\mathsf{case}(\mathsf{in}_i V, x_1 \cdot M_1, x_2 \cdot M_2) \to_{\mathsf{w}} M_i\,[V/x_i]} \qquad \frac{M \to_{\mathsf{w}} M'}{\mathsf{in}_i M \to_{\mathsf{w}} \mathsf{in}_i M'}$$

$$\frac{M \to_{\mathsf{w}} M'}{\mathsf{case}(M, x_1 \cdot M_1, x_2 \cdot M_2) \to_{\mathsf{w}} \mathsf{case}(M', x_1 \cdot M_1, x_2 \cdot M_2)}$$

**Proposition 3.2.2 (Subject reduction)** *Assume that $\mathcal{P} \vdash M : \sigma$ and that $M \to_{\mathsf{w}} M'$. Then $\mathcal{P} \vdash M' : \sigma$.*

*Démonstration.* By induction on the deduction that $M \to_{\mathsf{w}} M'$.

Assume that $M = \mathsf{der}(N^!)$ and $M' = N$. Since $\mathcal{P} \vdash M : \sigma$ we must have $\mathcal{P} \vdash N^! : !\sigma$ and hence $\mathcal{P} \vdash N : \sigma$.

Assume that $M = \langle \lambda x^\varphi\, N\rangle V$ and $M' = N\,[x/V]$. Since $\mathcal{P} \vdash M : \sigma$ there must be a positive type $\psi$ such that $\mathcal{P} \vdash \lambda x^\varphi\, N : \psi \multimap \sigma$. Therefore we must have $\psi = \varphi$ and $\mathcal{P}, x : \varphi \vdash N : \sigma$. By Proposition 3.1.3 we have $\mathcal{P} \vdash N\,[V/x] : \sigma$ as required.

Assume that $M = \mathsf{pr}_i\langle V_1, V_2\rangle$ and $M' = V_i$. Since $\mathcal{P} \vdash M : \sigma$, there must be two positive types $\varphi_1$ and $\varphi_2$ such that $\mathcal{P} \vdash \langle V_1, V_2\rangle : \varphi_1 \otimes \varphi_2$, and we have $\sigma = \varphi_1$. For $j = 1, 2$ we must have $\mathcal{P} \vdash V_j : \varphi_j$ and therefore $\mathcal{P} \vdash N : \sigma$ (taking $j = i$).

Assume that $M = \mathsf{fix}\, x^{!\sigma}\, N$ and $M' = N\left[M^!/x\right]$. We know that $\mathcal{P} \vdash \mathsf{fix}\, x^{!\sigma}\, N : \sigma$ and therefore we must have $\mathcal{P}, x : !\sigma \vdash N : \sigma$. We also know that $\mathcal{P} \vdash M^! : !\sigma$ and therefore, by Proposition 3.1.3 we have $\mathcal{P} \vdash N\left[M^!/x\right] : \sigma$ as required.

Assume that $M = \mathsf{case}(\mathsf{in}_i N, x_1 \cdot R_1, x_2 \cdot R_2)$ and $M' = R_i\,[N/x_i]$. Since $\mathcal{P} \vdash M : \sigma$, there must be positive types $\varphi_1$ and $\varphi_2$ such that $\mathcal{P} \vdash \mathsf{in}_i N : \varphi_1 \oplus \varphi_2$ and also $\mathcal{P}, x_j : \varphi_j \vdash R_j : \sigma$ for $j = 1, 2$. We must have $\mathcal{P} \vdash N : \varphi_i$ and hence, by Proposition 3.1.3 we have $\mathcal{P} \vdash R_i\,[N/x] : \sigma$ as required.

Assume that $M = \mathsf{der}(N)$ and $M' = \mathsf{der}(N')$ with $N \to_{\mathsf{w}} N'$. We have $\mathcal{P} \vdash M : \sigma$ hence $\mathcal{P} \vdash N : !\sigma$. By inductive hypothesis we have $\mathcal{P} \vdash N' : !\sigma$ (remember that our proof is on the height of the derivation of the considered weak reduction step) and hence $\mathcal{P} \vdash M' : \sigma$ as required.

Assume that $M = \langle N\rangle R$ and $M' = \langle N'\rangle R$ with $N \to_{\mathsf{w}} N'$. Since $\mathcal{P} \vdash M : \sigma$ there is a positive type $\varphi$ such that $\mathcal{P} \vdash N : \varphi \multimap \sigma$ and $\mathcal{P} \vdash R : \varphi$. By inductive hypothesis $\mathcal{P} \vdash N' : \varphi \multimap \sigma$ and hence $\mathcal{P} \vdash M' : \sigma$ as required.

The remaining cases are similar and left to the reader. □

**Proposition 3.2.3** *Any value is $\to_{\mathsf{w}}$-normal. If $\varphi$ is a positive type, $\vdash M : \varphi$ and $M$ is $\to_{\mathsf{w}}$-normal, then $M$ is a value.*

This is easy. In the second statement $M$ has to be closed (the term $\langle \mathsf{der}(x)\rangle V$ is normal, is not a value and can be given a positive type).

**Proposition 3.2.4** *The relation $\to_{\mathsf{w}}$ has the diamond property : if $M \to_{\mathsf{w}} M_i$ for $i = 1, 2$, then there is a term $M'$ such that $M_i \to_{\mathsf{w}} M'$ for $i = 1, 2$.*

**Exercice 3.2.1** Prove this proposition by induction on the structure of $M$. Observe that, in the case where $M = \langle \lambda x^\varphi\, N\rangle V$, the only possible reduction from $M$ is $M \to_{\mathsf{w}} N\,[V/x]$ because $V$ and $\lambda x^\varphi\, N$ are $\to_{\mathsf{w}}$-normal.

3.2.1 Examples  Given any type $\sigma$, we define $\Omega^\sigma = \mathsf{fix}\, x^{!\sigma}\, \mathsf{der}(x)$ which satisfies $\vdash \Omega^\sigma : \sigma$. It is clear that $\Omega^\sigma \to_{\mathsf{w}} \mathsf{der}((\Omega^\sigma)^!) \to_{\mathsf{w}} \Omega^\sigma$ so that we can consider $\Omega^\sigma$ as the ever-looping program of type $\sigma$.

Unit type and natural numbers.  We define a unit type $1$ by $1 = !\top$, and we set $* = (\Omega^\top)^!$. We define the type $\iota$ of unary natural numbers by $\iota = 1 \oplus \iota$ (by this we mean that $\iota = \mathsf{Fix}\,\zeta \cdot (1 \oplus \zeta)$). We define $\underline{0} = \mathsf{in}_1 *$ and $\underline{n+1} = \mathsf{in}_2 \underline{n}$ so that we have $\mathcal{P} \vdash \underline{n} : \iota$ for each $n \in \mathbb{N}$.

Then, given a term $M$, we define the term $\mathsf{suc}(M) = \mathsf{in}_2 M$, so that we have

$$\frac{\mathcal{P} \vdash M : \iota}{\mathcal{P} \vdash \mathsf{suc}(M) : \iota}$$

Last, given terms $M$, $N_1$ and $N_2$ and a variable $x$, we define an "ifz" conditional by $\mathsf{if}(M, N_1, x \cdot N_2) = \mathsf{case}(M, z \cdot N_1, x \cdot N_2)$ where $z$ is not free in $N_1$, so that

$$\frac{\mathcal{P} \vdash M : \iota \qquad \mathcal{P} \vdash N_1 : \sigma \qquad \mathcal{P}, x : \iota \vdash N_2 : \sigma}{\mathcal{P} \vdash \mathsf{if}(M, N_1, x \cdot N_2) : \sigma}$$

Streams.  Let $\varphi$ be a positive type and $\mathsf{S}_\varphi$ be the positive type defined by $\mathsf{S}_\varphi = \varphi \otimes !\mathsf{S}_\varphi$, that is $\mathsf{S}_\varphi = \mathsf{Fix}\,\zeta \cdot (\varphi \otimes !\zeta)$. We can define a term $M$ such that $\vdash M : \mathsf{S}_\varphi \multimap \iota \multimap \varphi$ which computes the $n$th element of a stream :

$$M = \mathsf{fix}\, f^{!(\mathsf{S}_\varphi \multimap \iota \multimap \varphi)}\, \lambda x^{\mathsf{S}_\varphi}\, \lambda y^\iota\, \mathsf{if}(y, \mathsf{pr}_1 x, z \cdot \langle \mathsf{der}(f)\rangle \mathsf{der}(\mathsf{pr}_2 x)\, z)$$

Conversely, we can define a term $N$ such that $\vdash N : !(\iota \multimap \varphi) \multimap \mathsf{S}_\varphi$ which turns a function into a stream.

$$N = \mathsf{fix}\, F^{!(!(\iota \multimap \varphi) \multimap \mathsf{S}_\varphi)}\, \lambda f^{!(\iota \multimap \varphi)}\, \langle \langle \mathsf{der}(f) \rangle \underline{0}, (\langle \mathsf{der}(F) \rangle (\lambda x^\iota\, \langle \mathsf{der}(f) \rangle \mathsf{suc}(x))^!)^! \rangle$$

Observe that the recursive call of $F$ is encapsulated into a box, which makes the construction lazy.

LISTS. There are various possibilities for defining a type of lists of elements of a positive type $\varphi$. The simplest definition is $\lambda_0 = 1 \oplus (\varphi \otimes \lambda_0)$. This corresponds to the ordinary ML "strict" type of lists. But we can also define $\lambda_1 = 1 \oplus (\varphi \otimes !\lambda_1)$ and then we have a type of lazy lists where the tail of the list is computed only when required (this type contains also streams).

We could also consider $\lambda_2 = 1 \oplus (!\sigma \otimes \lambda_2)$ which allows to manipulate lists of objects of type $\sigma$ (which can be a general type) without accessing their elements.

# Chapitre 4

# Categories

## 4.1 Basic notions

A category $\mathcal{C}$ consists of :
— a class of objects $\mathsf{Obj}\,\mathcal{C}$
— for each $X, Y \in \mathsf{Obj}\,\mathcal{C}$, of a class of morphisms $\mathcal{C}(X,Y)$ from $X$ to $Y$,
— for each $X \in \mathsf{Obj}\,\mathcal{C}$, of a special element $\mathsf{Id}_X$ of $\mathcal{C}(X,X)$ called identity at $X$
— and, for each triple $(X,Y,Z) \in \mathcal{C}^3$, of a composition operation

$$\begin{array}{rcl} \circ : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) & \to & \mathcal{C}(X,Z) \\ (f,g) & \mapsto & g \circ f \end{array}$$

such that the following equations hold (for $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(Y,Z)$ and $h \in \mathcal{C}(Z,V)$) :

$$f \circ \mathsf{Id}_X = f \qquad \mathsf{Id}_Y \circ f = f \qquad h \circ (g \circ f) = (h \circ g) \circ f$$

We often denote composition as simple juxtaposition and $\mathsf{Id}_X$ as $\mathsf{id}_X$.

**Example 4.1.1** The category **Set** has sets as objects and functions as morphisms. It underlies most categories whose objects are sets endowed with a structure and morphisms are functions "preserving" this structure in some sense such as
— monoids and homomorphisms of monoids
— groups and homomorphisms of groups
— given a field, vector spaces on this field and linear functions
— topological spaces and continuous functions.

**Example 4.1.2** The category **Rel** is less usual but very important for us. Its objects are sets but now $\mathbf{Rel}(X,Y) = \mathcal{P}(X \times Y)$, whose elements are seen as relations from $X$ to $Y$, $\mathsf{Id}_X$ is the diagonal relation $\mathsf{Id}_X = \{(a,a) \mid a \in X\}$ and composition is the ordinary composition of relations : given $s \in \mathbf{Rel}(X,Y)$ and $t \in \mathbf{Rel}(Y,Z)$, then

$$t \circ s = \{(a,c) \mid \exists b \in Y \ (a,b) \in s \text{ et } (b,c) \in t\}\,.$$

We denote this composition by simple juxtaposition $t\,s$ as a product, and $\mathsf{Id}_X$ as $\mathsf{id}_X$. An example of categories built in that way is the category whose objects are finite sets and a morphism from $I$ to $J$ is an $I \times J$ matrix with coefficients in some (semi-)ring, composition being defined as the usual product of matrices.

An isomorphism is a morphism $f \in \mathcal{C}(X,Y)$ such that there is a morphism $g \in \mathcal{C}(Y,X)$ such that $g \circ f = \mathsf{Id}_X$ and $f \circ g = \mathsf{Id}_Y$. If $g$ and $g'$ satisfy these conditions then $g = g \circ \mathsf{Id}_Y = g \circ (f \circ g') = (g \circ f) \circ g' = \mathsf{Id}_X \circ g' = g'$ by the equations above and hence $g$ is fully determined by $f$ and is denoted as $f^{-1}$.

The opposite category of $\mathcal{C}$ is the category $\mathcal{C}^{\mathsf{op}}$ given by $\mathsf{Obj}\,\mathcal{C}^{\mathsf{op}} = \mathsf{Obj}\,\mathcal{C}$ and $\mathcal{C}^{\mathsf{op}}(X,Y) = \mathcal{C}(Y,X)$. The identities are the same and composition is defined in the obvious way (reversing the order of factors).

**Exercice 4.1.1** Prove that, in **Rel**, the isomorphisms are the relations which are (graphs) of bijections.

4.1.1 FUNCTORS Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ is an operation which
— maps any object $X$ of $\mathcal{C}$ to an object $F(X)$ of $\mathcal{D}$
— and any morphism $f \in \mathcal{C}(X, Y)$ to a morphism $F(f) \in \mathcal{C}(F(X), F(Y)$
such that, for any $X, Y, Z \in \mathsf{Obj}\,\mathcal{C}$ and $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ :

$$F(\mathsf{Id}_X) = \mathsf{Id}_{F(X)} \qquad F(g \circ f) = F(g) \circ F(f)$$

A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{D}$ (or, equivalently, from $\mathcal{C}$ to $\mathcal{D}^{\mathsf{op}}$).

A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* if, for any $X, Y \in \mathsf{Obj}\,\mathcal{C}$, the function $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ which maps $f$ to $F(f)$ is surjective. It is *faithful* if this function is injective.

For instance, the functor $P$ from **Rel** to **Set** which maps a set $X$ to $\mathcal{P}(X)$ and a relation $s \in \mathbf{Rel}(X, Y)$ to the function $P(s) : \mathcal{P}(X) \to \mathcal{P}(Y)$ given by $P(s)(u) = \{b \in Y \mid \exists a \in u\ (a, b) \in s\}$ is a functor from **Rel** to **Set**. This functor is faithful but not full.

**Exercice 4.1.2** Prove that this functor $P$ is faithful, but not full.

4.1.2 NATURAL TRANSFORMATIONS Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation from $F$ to $G$ is a family $T = (T_X)_{X \in \mathsf{Obj}\,\mathcal{C}}$ of morphisms such that, for each $X \in \mathsf{Obj}\,\mathcal{C}$ one has $T_X \in \mathcal{D}(F(X), G(X))$ and such that, for each $f \in \mathcal{C}(X, Y)$, one has $G(f) \circ T_X = T_Y \circ F(f)$. This is expressed by saying that the following fdiagram commutes :

$$\begin{array}{ccc} F(X) & \xrightarrow{T_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{T_Y} & G(Y) \end{array}$$

this means that the composition of morphisms on both sides coincide. One writes $S : F \stackrel{\bullet}{\to} G$. Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be three functors. If $S : F \stackrel{\bullet}{\to} G$ and $T : G \stackrel{\bullet}{\to} H$, one defines $T \circ S : F \stackrel{\bullet}{\to} H$ par $(T \circ S)_X = T_X \circ S_X$. In that way one defines the category $\mathcal{D}^{\mathcal{C}}$ of functors and natural transformations. This composition is often called the *horizontal composition* of natural transformations.

**Exercise 4.1.1** Let $F, F' : \mathcal{C} \to \mathcal{D}$ and $G, G' : \mathcal{D} \to \mathcal{E}$ be functors. Let $S : F \stackrel{\bullet}{\to} F'$ and $T : G \stackrel{\bullet}{\to} G'$ be natural transformations. Let $X \in \mathsf{Obj}\,\mathcal{C}$. Prove that $G'(S_X) \circ T_{F(X)} = T_{F'(X)} \circ G(S_X)$. One denote as $(T * S)_X \in \mathcal{E}(G(F(X)), G'(F'(X)))$ the morphism so defined. Prove that $T * S$ is a natural transformation $G \circ F \stackrel{\bullet}{\to} G' \circ F'$. Prove that this operation is associative and give its neutral element. It is called *vertical composition* of natural transformations. Let $F'' : \mathcal{C} \to \mathcal{D}$ and $G'' : \mathcal{D} \to \mathcal{E}$ be two other functors and $S' : F' \stackrel{\bullet}{\to} F''$ and $T' : G' \stackrel{\bullet}{\to} G''$ be two other natural transformations. Prove that $(T' \circ T) * (S' \circ S) = (T' * S') \circ (T * S)$. This property is called *exchange law*. The category of categories, with functors as morphisms and natural transformations as morphisms between morphisms, with these two laws of composition, is a 2-*category*.

## 4.2 Adjunctions

## 4.3 Monads and comonads

Let $\mathcal{C}$ be a category. A *monad* on $\mathcal{C}$ is a triple $(T, \varepsilon, \mu)$ where $T : \mathcal{C} \to \mathcal{C}$ is a functor, $\varepsilon : \mathsf{Id}_{\mathcal{C}} \stackrel{\bullet}{\to} T$ and $\mu : T^2 = T \circ T \stackrel{\bullet}{\to} T$ are natural transformations. One requires moreover the following commutations..

$$\begin{array}{ccc} T(X) & \xrightarrow{\varepsilon_{T(X)}} & T^2(X) \\ & \searrow^{\mathsf{Id}_{T(X)}} & \downarrow{\scriptstyle \mu_X} \\ & & t(X) \end{array} \qquad \begin{array}{ccc} T(X) & \xrightarrow{T(\varepsilon_X)} & T^2(X) \\ & \searrow^{\mathsf{Id}_{T(X)}} & \downarrow{\scriptstyle \mu_X} \\ & & T(X) \end{array} \qquad \begin{array}{ccc} T^3(X) & \xrightarrow{T(\mu_X)} & T^2(X) \\ \downarrow{\scriptstyle \mu_{T(X)}} & & \downarrow{\scriptstyle \mu_X} \\ T^2(X) & \xrightarrow{\mu_X} & T(X) \end{array}$$

One defines first the category of $T$-algebras $\mathcal{C}^T$, also called the *Eilenberg-Moore category of* $T$ : the objects of $\mathcal{C}^T$ are the pairs $(X, h)$ where $X \in \mathsf{Obj}\,\mathcal{C}$ and $h \in \mathcal{C}(T(X), X)$ such that the following diagrams commute.

$$
\begin{array}{ccc}
X & \xrightarrow{\varepsilon_X} & T(X) \\
 & {\scriptstyle \mathsf{Id}_X} \searrow & \downarrow{\scriptstyle h} \\
 & & X
\end{array}
\qquad
\begin{array}{ccc}
T^2(X) & \xrightarrow{T(h)} & T(X) \\
\downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle h} \\
T(X) & \xrightarrow{h} & X
\end{array}
$$

The elements of $\mathcal{C}^T((X,h),(Y,k))$ are the $f \in \mathcal{C}(X,Y)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
T(X) & \xrightarrow{h} & X \\
\downarrow{\scriptstyle T(f)} & & \downarrow{\scriptstyle f} \\
T(Y) & \xrightarrow{k} & Y
\end{array}
$$

**Exercise 4.3.1** Prove that one has defined a category $\mathcal{C}^T$.

One defines next the category of free $T$-algebras, or *Kleisli cetegory*, denoted as $\mathcal{C}_T$. First one sets $\mathsf{Obj}\,\mathcal{C}_T = \mathsf{Obj}\,\mathcal{C}$. Then $\mathcal{C}_T(X,Y) = \mathcal{C}(X,T(Y))$. In this category, the identity at $X$ is $\mathsf{Id}^{\mathsf{K}}_X = \varepsilon_X$ and composition is defined in the following way. Let $f \in \mathcal{C}_T(X,Y) = \mathcal{C}(X,T(Y))$ and $g \in \mathcal{C}_T(Y,Z) = \mathcal{C}(Y,T(Z))$. Then

$$g \circ^{\mathsf{K}} f = \mu_Z \circ T(g) \circ f\,.$$

**Exercise 4.3.2** Prove that we have defined a category $\mathcal{C}_T$.

**Exercise 4.3.3** If $X$ is an object of $\mathcal{C}$, check that $(T(X), \mu_X)$ is a $T$-algebra. It is called the *free $T$-algebra generated by* $X$ and denoted here as $F(X)$. Let $f \in \mathcal{C}_T(X,Y)$. We set $F(f) = \mu_Y \circ T(f)$. Prove that, in that way, one has defined a functor $F : \mathcal{C}_T \to \mathcal{C}^T$. Prove that this functor is full and faithful.

**Exercise 4.3.4** Let $M : \mathbf{Set} \to \mathbf{Set}$ the functor which, with any set $X$, associates the set $M(X)$ of all finite sequences $\langle a_1, \ldots, a_n\rangle$ of elements of $X$ and with any function $f : X \to Y$ associates the function $M(f) : M(X) \to M(Y)$ which maps $\langle a_1, \ldots, a_n\rangle$ to $\langle f(a_1), \ldots, f(a_n)\rangle$. If $X$ is a set, one defines $\varepsilon_X : X \to M(X)$ as the functions which maps $a$ to $\langle a\rangle$, and $\mu_X : M(M(X)) \to M(X)$ as the function which maps a sequence $\langle m_1, \ldots, m_n\rangle$ of finite sequences of elements of $X$ to their concatenation $m_1 \cdots m_n$.

— Prove that $\varepsilon$ and $\mu$ are natural transformations.
— Prove that $(M, \varepsilon, \mu)$ is a monad.
— Prove that $\mathbf{Set}^M$ is the category of monoids and morphisms of monoids.
— Explain why $\mathbf{Set}_M$ can be considered as the category of free monoids and morphisms of monoids.

**Exercise 4.3.5** Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the functor which maps a set $X$ to its powerset $\mathcal{P}(X)$ and $f \in \mathbf{Set}(X,Y)$ to the function $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ such that $\mathcal{P}(f)(x) = \{f(a) \mid a \in X\}$. Check that $\mathcal{P}$ is a functor. Find a structure of monad for this functor such that that category $\mathbf{Set}_{\mathcal{P}}$ is isomorphic to $\mathbf{Rel}$.

**Exercise 4.3.6** Reversing the direction of all arrows, explain what is a comonad and describe the Eilenberg-Moore and Kleisli categories of a comonad.

## 4.4 Limits and colimits

### 4.4.1 Projective limits (limits)

**Terminal objects.**

An object $T$ of a cetagory $\mathcal{C}$ is *terminal* if, for any object $X$ of $\mathcal{C}$, the set $\mathcal{C}(X,T)$ has exactly one element. Let $T$ and $T'$ be terminal objects of $\mathcal{C}$. Let $f$ be the unique element of $\mathcal{C}(T',T)$ and $f'$ the unique element of $\mathcal{C}(T,T')$. Since $\mathcal{C}(T,T)=\{\mathsf{Id}_T\}$, we must have $f\circ f'=\mathsf{Id}_T$ and also $f'\circ f=\mathsf{Id}_{T'}$. In other words, there is exactly one morphism from $T$ to $T'$, and this morphism is an iso. It is a very strong way to say that, if a category has a terminal object, this object is unique up to unique iso.

Terminal objects are a very special case of projective limit as we shall see, but, choosing the suitable category, any projective limit can be seen as a terminal object (this seems to be a general pattern of category theory : any notion is more general than any other notion).

**General case**

Let $\mathcal{C}$ be a category and $I$ be a small category (that is, such that $\mathsf{Obj}\, I$ is a set). There is an obvious functor $\Delta : \mathcal{C}\to\mathcal{C}^I$ which maps an object $X$ of $\mathcal{C}$ to the constant functor defined by $\Delta(X)(i)=X$ and $\Delta(X)(u)=\mathsf{Id}_X$. Let $D : I\to\mathcal{C}$ be a functor (such a "small" functor is sometimes called a *diagram*). A *projective cone* based on $D$ is a pair $(X,p)$ where $X\in\mathsf{Obj}\,\mathcal{C}$ and $p:\Delta(X)\overset{\bullet}{\to} D$. In other words it consists of the following data : the object $X$, and, for any $i\in\mathsf{Obj}\,I$, a mophism $p_i\in\mathcal{C}(X,D(i))$ such that, for each $\varphi\in I(i,j)$, one has $D(\varphi)\circ p_i=f_j$.

Let $(X,p)$ and $(Y,q)$ be projective cones based on $D$. A cone morphism from $(X,p)$ to $(Y,q)$ is an $h\in\mathcal{C}(X,Y)$ such that, for each $i\in I$, one has $q_i\circ h=p_i$. In that way we define a category $\mathcal{C}_D$. A *limiting projective cone* on $D$ is a terminal object of the category $\mathcal{C}_D$.

In other words, a limiting projective cone based on $D$ is a projective cone $(P,p)$ based on $D$ such that, for any other cone $(X,q)$ based on $D$, there is exactly one $h\in\mathcal{C}(P,X)$ such that $\forall i\in I\ p_i\circ h=q_i$. A limiting projective cone based on $D$ is also simply called a (projective, or inverse) limit of $D$.

**Proposition 4.4.1** *Let $(Y,(q_i)_{i\in I})$ and $(Y',(q'_i)_{i\in I})$ be projective limits of the diagram $D$. Then there is exactly one morphism $g\in\mathcal{C}(Y,Y')$ such that $\forall i\in I\ q'_i\circ g=q_i$. Moreover, $g$ is an iso.*

This is a direct consequence of a projective limit as terminal object in. Because of this sstrong uniqueness property one often uses the notation $\varprojlim D$ to denote this limit when it exists. Remember that, to be fully specified, a limit must be given as an object $P$ together with a family of morphisms $(p_i)_{i\in\mathsf{Obj}\,I}$ (the projective cone) which can be seen as some kind of "projections" from $P$ to the objects of the diagram $D$, whence the adjective "projective.

**Proposition 4.4.2** *Assume that all diagrams $D\in\mathsf{Obj}\,\mathcal{C}^I$ have a projective limit $(\varprojlim D,(p_i^D)_{i\in\mathsf{Obj}\,I})$. Then there is exactly one functor $L:\mathcal{C}^I\to\mathcal{C}$ such that $L(D)=\varprojlim D$ and, for each $T\in\mathcal{C}^I(D,E)$, the following triangle commute for each $i\in\mathsf{Obj}\,I$ :*

$$\begin{array}{ccc} \varprojlim D & \xrightarrow{L(f)} & \varprojlim E \\ {\scriptstyle p_i^D}\big\downarrow & & \big\downarrow{\scriptstyle p_i^E} \\ D(i) & \xrightarrow{T_i} & E(i) \end{array}$$

*Démonstration.* Observe that $(\varprojlim D,(T_i\circ p_i^D)_{i\in\mathsf{Obj}\,I})$ is a projective cone on $E$ and apply the universal property of the cone $(\varprojlim E,(p_i^E)_{i\in\mathsf{Obj}\,I})$. □

Here are a few examples of projective limits.

**Example 4.4.3** If $I$ is a *discrete category*, that is a category whose only morphisms are the identities (and therefore can be considered as a set since it is small), then $D$ is just an $I$-indexed family of objects of $\mathcal{C}$. In that case, when the projective limit $(P,(\pi_i)_{i\in I})$ of $D$ exists, it is called the *cartesian product* of the family $D$ and the morphisms $\pi_i\in\mathcal{C}(P,D_i)$ are called the *projections*. We will often use $\&_{i\in I} D_i$ to denote the object $P$. Special cases : if $I=\emptyset$, the projective limit consists simply of an object $\top$ charcterized by the fact that, for any object $X$ of $\mathcal{C}$, the set $\mathcal{C}(X,\top)$ is a singleton, whose unique element will be denoted $\mathrm{ast}_X$. In other words, $\top$ is a terminal object of $\mathcal{C}$. A category is cartesian if all finite families of objects have a cartesian product.

Assume that $\mathcal{C}$ is cartesian. The operation $(X_1, X_2) \mapsto X_1 \,\&\, X_2$ can be turned into a functor $\mathcal{C}^2 \to \mathcal{C}$ by Proposition 4.4.2 : let $f_i \in \mathcal{C}(X_i, Y_i)$ for $i = 1, 2$. We have $f_i \circ \pi_i \in \mathcal{C}(X_1 \,\&\, X_2, X_i)$ and hence there is a unique morphism $f_1 \,\&\, f_2 \in \mathcal{C}(X_1 \,\&\, X_2, Y_i)$ such that $\pi_i \circ (f_1 \,\&\, f_2) = f_i \circ \pi_i$ for $i = 1, 2$ and the operation which maps $(X_1, X_2)$ to $X_1 \,\&\, X_2$ and $(f_1, f_2) \in \mathcal{C}(X_1, Y_1) \times \mathcal{C}(X_2, Y_2)$ to $f_1 \,\&\, f_2$ is a functor.

**Example 4.4.4** Let $I$ be the category such that $\mathsf{Obj}\, I = \{1, 2\}$ and $I(1, 2) = \{\alpha, \beta\}$. A diagram is given by two objects $X$ and $Y$ of $\mathcal{C}$ and two morphisms $f, g \in \mathcal{C}(X, Y)$. A projective limit of this diagram consists of an object $E$ and a morphism $e \in \mathcal{C}(E, X)$ such that $f \circ e = g \circ e$ and, for any object $Z$ of $\mathcal{C}$ and any morphism $h \in \mathcal{C}(Z, X)$ such that $f \circ h = g \circ h$, there is exactly one morphism $h_0 \in \mathcal{C}(Z, E)$ such that $h = e \circ h_0$. Such a limit is called an *equalizer* of $f$ and $g$.

From now on, we drop the adjective "projective" and simply use the word *limit* and *cone* to refer to projective limits and cones.

**Exercice 4.4.1** Montrer que **Set** a tous les égaliseurs. Qu'en est-il de **Rel** ?

**Exercice 4.4.2** Examiner le cas où $I$ a $\{1, 2, 3\}$ pour ensembles d'objets, et $I(1, 3) = \{\alpha\}$, $I(2, 3) = \{\beta\}$ et $I(i, j) = \emptyset$ pour $i \neq j$ et $(i, j) \notin \{(1, 3), (2, 3)\}$. Une telle limite est appelée *produit fibré.*

**Exercice 4.4.3** En renversant le sens des flèches, expliciter les notions duales d'objet initial et de colimite (ou limite inductive).

4.4.2 Catégorie cartésienne. On donne une description directe de cette notion importante, indépendante de la notion générale de limite projective introduite ci-dessus.

Soit $\mathcal{C}$ une catégorie et $(X_i)_{i \in I}$ une famille d'objets de $\mathcal{C}$. Un *produit cartésien* de $(X_i)_{i \in I}$ est un couple $(Y, (\pi_i)_{i \in I})$ où $Y \in \mathsf{Obj}\, \mathcal{C}$ et $\pi_i \in \mathcal{C}(Y, X_i)$ pour $i \in I$ qui vérifie la *propriété universelle* suivante : pour tout $Z \in \mathsf{Obj}\, \mathcal{C}$ et toute famille $(f_i)_{i \in I}$ telle que $f_i \in \mathcal{C}(Z, X_i)$, il existe un unique $g \in \mathcal{C}(Z, Y)$ tel que, pour tout $i \in I$, on ait $\pi_i \circ g = f_i$.

**Théorème 4.4.5** *Soient $(Y, (\pi_i)_{i \in I})$ et $(Y', (\pi'_i)_{i \in I})$ deux produits cartésiens de la famille $(X_i)_{i \in I}$. Il existe un unique $h \in \mathcal{C}(Y, Y')$ tel que $\pi'_i \circ h = \pi_i$ pour tout $i \in I$, et $h$ est un isomorphisme.*

*Démonstration.* L'existence et l'unicité de $h$ résultent de la propriété universelle de $(Y', (\pi'_i)_{i \in I})$ et de l'existance des $\pi_i \in \mathcal{C}(Y, X_i)$. De la même façon, on définit un unique $h' \in \mathcal{C}(Y', Y)$ tel que $\pi_i \circ h' = \pi'_i$ pour tout $i \in I$. Alors $h' \circ h \in \mathcal{C}(Y, Y)$ vérifie $\pi_i \circ h' \circ h = \pi_i$ pour tout $i \in I$, or $\mathsf{Id}_Y$ vérifie également ces égalités, et donc $h' \circ h = \mathsf{Id}_Y$ par l'unicité dans la propriété universelle. De même on voit que $h \circ h' = \mathsf{Id}_{Y'}$. □

Donc le produit cartésien, quand il existe, est unique à unique isomorphisme près [1] : c'est typique des constructions définies par une propriété universelle. On peut à chaque fois démontrer une propriété similaire à celle énoncée par ce théorème (on ne les redémontrera pas par la suite pour les autres constructions universelles).

On dit que $\mathcal{C}$ est cartésienne si toutes les familles finies $(X_i)_{i \in I}$ d'objets de $\mathcal{C}$ ont un produit cartésien. On en choisit un, que l'on note $(\&_{i \in I} X_i, (\pi_i)_{i \in I})$. Soit $I$ un ensemble fini fixé. On peut définir un foncteur "produit cartésien" $P_I$ de $\mathcal{C}^I$ vers $\mathcal{C}$ en posant $P_I((X_i)_{i \in I}) = \&_{i \in I} X_i$, et, pour une famille $(f_i)_{i \in I}$ (avec $\forall i \in I\ f_i \in \mathcal{C}(X_i, Y_i)$), en définissant $g = P_I((f_i)_{i \in I})$ comme l'unique morphisme $g \in \mathcal{C}(\&_{i \in I} X_i, \&_{i \in I} Y_i)$ tel que $\forall i \in I\ \pi_i \circ g = f_i \circ \pi_i$ pour $i = 1, 2$. La fonctorialité dit que $\mathsf{Id}_{X_1} \,\&\, \mathsf{Id}_{X_2} = \mathsf{Id}_{X_1 \& X_2}$ et que, si $f_i \in \mathcal{C}(X_i, Y_i)$ et $g_i \in \mathcal{C}(Y_i, Z_i)$ pour $i = 1, 2$, alors

$$(g_1 \,\&\, g_2) \circ (f_1 \,\&\, f_2) = (g_1 \circ f_1) \,\&\, (g_2 \circ f_2)\,.$$

Elle se démontre en utilisant la propriété universelle du produit cartésien : par exemple, les morphismes ci-dessus sont tous les deux des morphismes $h \in \mathcal{C}(X_1 \,\&\, X_2, Z_1 \,\&\, Z_2)$ tels que $h \circ \pi_i = g_i \circ f_i \circ \pi_i$ pour $i = 1, 2$, ils sont donc égaux puisqu'il n'y a qu'un seul tel morphisme.

On suppose toujours $\mathcal{C}$ cartésienne. On démontre de la même façon que le produit cartésien admet $\top$ comme "élément neutre" à gauche et à droite, c'est-à-dire qu'il existe des isomorphismes naturels

1. Plus précisément : à un unique morphisme près, et en plus ce morphisme est un isomorphisme.

$\lambda_X : \top \ \& \ X \to X$ et $\rho_X : X \ \& \ \top \to X$. Il est associatif au sens où il existe des isomorphismes naturels $\alpha_{X_1,X_2,X_3} : (X_1 \ \& \ X_2) \ \& \ X_3 \to X_1 \ \& \ (X_2 \ \& \ X_3)$ et il est symétrique au sens où il existe des isomorphismes naturels $\sigma_{X_1,X_2} : X_1 \ \& \ X_2 \to X_2 \ \& \ X_1$. Ces morphismes satisfont des commutations sur lesquelles on reviendra quand on parlera de catégories monoïdales.

On peut encore axiomatiser les catégories cartésiennes de façon équationnelles : la catégorie $\mathcal{C}$ est cartésienne si elle a un objet terminal et les opérations suivantes :

— une opération qui à $X_1, X_2 \in \mathsf{Obj}\,\mathcal{C}$ associe $X_1 \ \& \ X_2$ et $\pi_i \in \mathcal{C}(X_1 \ \& \ X_2, X_i)$ pour $i = 1, 2$ ;

— et une opération qui $(f_i \in \mathcal{C}(Y, X_i))_{i=1,2}$ associe $\langle f_1, f_2 \rangle \in \mathcal{C}(Y, X_1 \ \& \ X_2$

qui vérifient les équations suivantes :

$$\pi_i \circ \langle f_1, f_2 \rangle = f_i \quad \text{pour } i = 1, 2$$
$$\langle f_1, f_2 \rangle \circ g = \langle f_1 \circ g, f_2 \circ g \rangle$$
$$\langle \pi_1, \pi_2 \rangle = \mathsf{Id}_{X_1 \& X_2}$$

Pour prouver qu'une catégorie est cartésienne, il peut être plus facile de démontrer ces équations plutôt que de prouver directement la propriété universelle. On laisse la vérification de l'équivalence entre ces deux présentations comme un exercice.

**Example 4.4.6** Les produits dans **Set** et dans **Rel**. Dans **Set**, l'objet terminal est n'importe quel ensemble à 1 élément. Le produit cartésien de $X_1$ et $X_2$ est le produit habituel $X_1 \times X_2$ et les projections sont, elles aussi, les fonctions de projection usuelle. Si $f_i \in \mathbf{Set}(Y, X_i)$, la fonction $\langle f_1, f_2 \rangle$ est donnée par $\langle f_1, f_2 \rangle(a) = (f_1(a), f_2(a))$.

Dans **Rel**, l'objet terminal est l'ensemble vide puisque $\mathbf{Rel}(X, \emptyset) = \mathcal{P}(X \times \emptyset) = \mathcal{P}(\emptyset) = \{\emptyset\}$ est bien un singleton. Le produit cartésien d'une famille d'objets $(X_i)_{i \in I}$ est l'ensemble $\&_{i \in I} X_i = \bigcup_{i \in I} \{i\} \times X_i$ (c'est une union disjointe). Les projections sont données par

$$\pi_i = \{((i, a), a) \mid i \in I \text{ et } a \in X_i\} .$$

Soit $s_i \in \mathbf{Rel}(Y, X_i)$, pour chaque $i \in I$. On définit $\langle s_i \rangle_{i \in I} \in \mathbf{Rel}(Y, \&_{i \in I} X_i)$ par

$$\langle s_i \rangle_{i \in I} = \{(b, (i, a)) \mid i \in I \text{ et } (b, a) \in s_i\}$$

et on vérifie facilement que $\langle s_i \rangle_{i \in I}$ est l'unique élément de $\pi_i$ tel que $\pi_i \ \langle s_j \rangle_{j \in I} = s_i$ pour chaque $i \in I$.

## 4.5 Catégorie cartésiennes fermées

Soit $\mathcal{C}$ une catégorie cartésienne. Soient $X, Y \in \mathsf{Obj}\,\mathcal{C}$. Un *objet des morphismes* de $X$ vers $Y$ est la donnée d'un couple $(E, e)$ où $E$ est un objet de $\mathcal{C}$ et $e \in \mathcal{C}(E \ \& \ X, Y)$ sont tels que, pour tout $f \in \mathcal{C}(Z \ \& \ X, Y)$ il existe un unique $f' \in \mathcal{C}(Z, E)$ tel que $e \circ (f' \ \& \ \mathsf{Id}_X) = f$. Cette notion est de nouveau définie par une propriété universelle, et donc de façon unique à unique isomorphisme près comme on va le voir.

Soit $(E', e')$ un autre objet des morphismes de $X$ vers $Y$. Comme $e' \in \mathcal{C}(E' \ \& \ X, Y)$ et donc il existe un unique $h' \in \mathcal{C}(E', E)$ tel que $e \circ (h \ \& \ \mathsf{Id}_X) = e'$ et de même il existe un unique $h' \in \mathcal{C}(E, E')$ tel que $e' \circ (h' \ \& \ \mathsf{Id}_X) = e$. On a donc $e \circ ((h \circ h') \ \& \ \mathsf{Id}_X) = e$, et comme $e \circ (\mathsf{Id}_E \ \& \ \mathsf{Id}_X) = e$, on a $h \circ h' = \mathsf{Id}_E$ et de même $h' \circ h = \mathsf{Id}_{E'}$, donc $h$ est un isomorphisme dont $h'$ est l'inverse.

Il a donc un sens d'introduire des notations : l'objet des morphismes $E$ sera noté $X \Rightarrow Y$, le morphisme $e$ dit d'*évaluation* sera noté $\mathsf{Ev}_{X,Y}$ ou simplement $\mathsf{Ev}$ et si $f \in \mathcal{C}(Z \ \& \ X, Y)$, l'unique morphisme $h : \mathcal{C}(Z, X \Rightarrow Y)$ tel que $\mathsf{Ev} \circ (h \ \& \ \mathsf{Id}_X)$ sera noté $\mathsf{Cur}(f)$ (*curryfication* de $f$, en l'honneur de Haskell Curry, père du $\lambda$-calcul).

Tout comme le produit cartésien, ces constructions peuvent être caractérisées par un système de trois équations :

$$\mathsf{Ev} \circ (\mathsf{Cur}(f) \ \& \ \mathsf{Id}_X) = f$$
$$\mathsf{Cur}(f) \circ g = \mathsf{Cur}(f \circ (g \ \& \ \mathsf{Id}_X)) \quad \text{où } g \in \mathcal{C}(Z', Z)$$
$$\mathsf{Cur}(\mathsf{Ev}) = \mathsf{Id}_{X \Rightarrow Y} \ .$$

Encore une fois, il peut être plus facile de vérifier ces équations que de prouver directement la propriété universelle.

**Exercice 4.5.1** Soient $X, Y \in \mathsf{Obj}\,\mathcal{C}$. Soit $\mathcal{C}_{X,Y}$ la catégorie suivante : un objet de $\mathcal{C}_{X,Y}$ est un couple $(Z, f)$ où $Z \in \mathsf{Obj}\,\mathcal{C}$ et $f \in \mathcal{C}(Z \,\&\, X, Y)$. L'ensemble $\mathcal{C}_{X,Y}((Z, f), (Z', f'))$ est l'ensemble des $g \in \mathcal{C}(Z, Z')$ tels que $f' \circ (g \,\&\, \mathsf{Id}_X) = f$. Vérifier qu'on a bien défini ainsi une catégorie, et qu'un objet des morphismes de $X$ vers $Y$ est exactement un objet terminal dans cette catégorie.

Enoncer la propriété d'unicité des objets des morphismes qui en résulte.

**Exercice 4.5.2** Vérifier que **Set** est cartésienne fermée et que **Rel** ne l'est pas.

## 4.6 Qu'est-ce qu'un modèle de la logique linéaire ?

Il y a plusieurs façons différentes de présenter catégoriquement les modèles de la logique linéaire. Nous suivons l'approche proposée par Seely et légèrement corrigée par Bierman, nous parlerons simplement de catégories de Seely. Notre principale référence est le long et très détaillé article de Melliès [Mel09].

4.6.1 Catégories monoïdales. Alors qu'une catégorie cartésienne est une catégorie ayant une certaine propriété (existence de limites sur les diagrammes discrets finis), une catégorie monoïdale n'est pas qu'une catégorie, c'est une structure $(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha)$ où $\mathcal{L}$ est une catégorie, 1 est un objet de $\mathcal{L}$, $\otimes$ est un foncteur $\mathcal{L}^2 \to \mathcal{L}$ (noté de façon infixe) et les autres éléments du sextuplet sont des isomorphismes naturels.

Dans une telle catégorie $\mathcal{L}$ (dont les morphismes sont intuitivement des fonctions linéaires), la composition est notée par simple juxtaposition. Si $s \in \mathcal{L}(X, Y)$ et $t \in \mathcal{L}(Y, Z)$, la composition de ces deux morphismes est notée $t\,s \in \mathcal{L}(X, Z)$.

- $\lambda_X : 1 \otimes X \to X$ est un iso naturel exprimant que 1 est élément neutre à gauche pour $\otimes$, et de même $\rho_X : X \otimes 1 \to X$ est un iso naturel exprimant que 1 est neutre à droite.
- $\alpha_{X_1,X_2,X_3} : (X_1 \otimes X_2) \otimes X_3 \to X_1 \otimes (X_2 \otimes X_3)$ est un isomorphisme naturel exprimant que le produit tensoriel est une opération associative.

Ces isomorphismes doivent satisfaire entre eux des *conditions de cohérence* qui permettent de donner une valeur unique à tous les isomorphismes que l'on peut avoir envie d'écrire de façon générique. Par exemple, on doit avoir la commutation suivante :

$$\begin{array}{ccc} (1 \otimes X_1) \otimes X_2 & \xrightarrow{\lambda_{X_2 \otimes X_3}} & X_1 \otimes X_2 \\ {\scriptstyle \alpha_{1,X_1,X_2}} \big\downarrow & \nearrow {\scriptstyle \lambda_{X_1 \otimes X_2}} & \\ 1 \otimes (X_1 \otimes X_2) & & \end{array}$$

qui expriment que les deux façons canoniques de passer de $(1 \otimes X_1) \otimes X_2$ à $X_1 \otimes X_2$ sont équivalentes. On a un diagramme similaire pour $\rho$, et le fameux pentagone de McLane :

$$\begin{array}{ccc} ((X_1 \otimes X_2) \otimes X_3) \otimes X_4 & \xrightarrow{\alpha_{X_1 \otimes X_2, X_3, X_4}} & (X_1 \otimes X_2) \otimes (X_3 \otimes X_4) \\ {\scriptstyle \alpha_{X_1,X_2,X_3 \otimes X_4}} \big\downarrow & & \\ (X_1 \otimes (X_2 \otimes X_3)) \otimes X_4 & & \big\downarrow {\scriptstyle \alpha_{X_1,X_2,X_3 \otimes X_4}} \\ {\scriptstyle \alpha_{X_1,X_2 \otimes X_3,X_4}} \big\downarrow & & \\ X_1 \otimes ((X_2 \otimes X_3) \otimes X_4) & \xrightarrow{X_1 \otimes \alpha_{X_2,X_3,X_4}} & X_1 \otimes (X_2 \otimes (X_3 \otimes X_4)) \end{array}$$

Une catégorie monoïdale symétrique est une structure $(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \sigma)$ telle que $(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha)$ soit une catégorie symétrique, et $\sigma_{X_1}X_2 : X_1 \otimes X_2 \to X_2 \otimes X_1$ soit un isomorphisme naturel ; il exprime que le produit tensoriel est “commutatif”. Cet isomorphisme doit satisfaire la propriété fondamentale suivante :

$$\sigma_{X_2,X_1}\, \sigma_{X_1,X_2} = \mathsf{Id}_{X_1 \otimes X_2} \;.$$

Il existe une notion de *catégorie monoïdale tressée* dans laquelle il y a un isomorphisme naturel $\sigma$, qui ne satisfait pas cette condition, mais une condition plus faible liée aux groupes de tresses.

Dans une catégorie monoïdale symétrique, il faut aussi que d'autres diagrammes de cohérence commutent, à savoir

$$\begin{array}{ccccc}
(X_1 \otimes X_2) \otimes X_3 & \xrightarrow{\alpha_{X_1,X_2,X_3}} & X_1 \otimes (X_2 \otimes X_3) & \xrightarrow{\sigma_{X_1,X_2 \otimes X_3}} & (X_2 \otimes X_3) \otimes X_1 \\
\downarrow{\scriptstyle \sigma_{X_1,X_2} \otimes X_3} & & & & \downarrow{\scriptstyle \alpha_{X_2,X_3,X_1}} \\
(X_2 \otimes X_1) \otimes X_3 & \xrightarrow{\alpha_{X_2,X_1,X_3}} & X_2 \otimes (X_1 \otimes X_3) & \xrightarrow{X_2 \otimes \sigma_{X_1,X_3}} & X_2 \otimes (X_3 \otimes X_1)
\end{array}$$

ainsi que

$$\begin{array}{ccc}
1 \otimes X & \xrightarrow{\lambda_X} & X \\
\downarrow{\scriptstyle \sigma_{1,X}} & \nearrow{\scriptstyle \rho_X} & \\
X \otimes 1 & &
\end{array}$$

4.6.2 MONOIDAL CLOSENESS. A symmetric monoidal category (SMC) $(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \sigma)$ is *closed* if, for any object $X$ of $\mathcal{L}$, the functor $\mathcal{L} \to \mathcal{L}$ which maps $Z$ to $Z \otimes X$ ha a right adjoint. This adjoint functor is denoted as $X \multimap \_$.

In other words and without using the notion of adjunction, for any objects $X$ and $Y$ of $\mathcal{L}$, there is a pair $(X \multimap Y, \mathsf{ev})$ called *linear hom object* ) where $X \multimap Y$ is an object of $\mathcal{L}$ and $\mathsf{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$ and, for each morphism $f \in \mathcal{L}(Z \otimes X, Y)$, there is a morphism $\mathsf{cur}(f) \in \mathcal{L}(Z, X \multimap Y)$ such that the following conditions hold.

— $\mathsf{ev}\,(\mathsf{cur}\,(f) \otimes X) = f$

— si $g \in \mathcal{L}(Z', Z)$, alors $\mathsf{cur}\,(f)\ g = \mathsf{cur}\,(f\,(g \otimes X))$

— et $\mathsf{cur}\,(\mathsf{ev}) = \mathsf{Id}_{X \multimap Y}$.

This pair $(X \multimap Y, \mathsf{ev})$ can also be charaterized by a universal property : for any object $Z$ of $\mathcal{L}$ and any morphism $f \in \mathcal{L}(Z \otimes X, Y)$ there is exactly one morphism $g \in \mathcal{L}(Z, X \multimap Y)$ such as $\mathsf{ev}\,(g \otimes X) = f$. This morphism $g$ is denoted as $\mathsf{cur}\,(f)$ and is called the *linear curryfication* of $f$.

Since these linear hom objects are characterized by a universal property, their existence is a property of the SMC $\mathcal{L}$ (together with its monoidal structure) and not a further structure. An SMC where all pair of objects has an hom object is called a *symmetric monoidal closed category (SMCC).*

**Lemme 4.6.1** *The map* $\mathsf{cur} : \mathcal{L}(Z \otimes X, Y) \to \mathcal{L}(Z, X \multimap Y)$ *is a bijection.*

**Exercise 4.6.1** Prove this lemma, and that the inverse of $\mathsf{cur}$ is given by $\mathsf{cur}^{-1}(g) = \mathsf{ev}\,(g \otimes X)$ for $g \in \mathcal{L}(Z, X \multimap Y)$.

Let $f \in \mathcal{L}(X', X)$ and $g \in \mathcal{L}(Y, Y')$. Then by this universal property there is exactly one morphism $f \multimap g \in \mathcal{L}(X \multimap Y, X' \multimap Y')$ such that the following diagram commutes :

$$\begin{array}{ccccc}
(X \multimap Y) \otimes X' & & \xrightarrow{(f \multimap g) \otimes X'} & & (X' \multimap Y') \otimes X' \\
\downarrow{\scriptstyle (X \multimap Y) \otimes f} & & & & \downarrow{\scriptstyle \mathsf{ev}} \\
(X \multimap Y) \otimes X & \xrightarrow{\mathsf{ev}} & Y & \xrightarrow{g} & Y'
\end{array}$$

In other words we have $(f \multimap g) = \mathsf{cur}\,(g\ \mathsf{ev}\,((X \multimap Y) \otimes f))$.

**Exercise 4.6.2** Prove that $\_ \multimap \_$ is a functor $\mathcal{L}^{\mathsf{op}} \times \mathcal{L} \to \mathcal{L}$.

Given a functor $F : \mathcal{C} \to \mathcal{L}$ and $X \in \mathsf{Obj}\,\mathcal{L}$ we use $F \otimes X : \mathcal{C} \to \mathcal{L}$ for the composition of the functors $F$ and $\_ \otimes X$. It is a general fact that any functor which is a left adjoint of another functors commutes with arbitrary existing colimits (and dually any functor which is a right adjoint commutes with arbitrary existing limits).

**Proposition 4.6.2** *Assume that $\mathcal{L}$ is an SMCC and let $I$ be a small category. Assume that any diagram $D : I \to \mathcal{L}$ has a colimit $(\varinjlim D, (e_i)_{i \in \mathsf{Obj}\, I})$ in $\mathcal{L}$. Then for any object $X$ and any diagram $D : I \to \mathcal{L}$ there is an isomorphism $(\varinjlim D) \otimes X \to \varinjlim (D \otimes X)$. This isomorphism is natural in $D$ and $X$ and is symmetric monoidal.*

**Exercise 4.6.3** Prove this proposition.

Soit $Z$ un objet de $\mathcal{L}$. On peut construire un morphisme naturel

$$\eta_X \in \mathcal{L}(X, (X \multimap Z) \multimap Z)$$

On a en effet $\mathsf{ev} : (X \multimap Z) \otimes X \to Z$ et donc $\mathsf{ev}\,\sigma : X \otimes (X \multimap Z) \to Z$, et on pose $\eta_X = \mathsf{cur}\,(\mathsf{ev}\,\sigma) : X \to (X \multimap Z) \multimap Z$.

4.6.3 $*$-AUTONOMIE. On appelle *catégorie $*$-autonome* une structure

$$(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \sigma, \bot)$$

où $(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \sigma)$ est une catégorie monoïdale symétrique fermée (pour laquelle on utilise les notations introduites ci-dessus), et $\bot$ est un objet de $\mathcal{L}$, dit *objet dualisant*, qui possède la propriété suivante : le morphisme naturel associé $\eta_X \in \mathcal{L}(X, (X \multimap \bot) \multimap \bot)$ est un isomorphisme, pour tout objet $X$.

L'$*$-autonomie d'une catégorie monoïdale symétrique fermée n'est pas une propriété, mais une structure (à savoir la donnée de $\bot$).

On note $X^\bot = (X \multimap \bot)$ en sorte que les objets $X$ et $X^{\bot\bot}$ sont canoniquement isomorphes par l'isomorphisme $\eta_X$ associé à $\bot$.

On peut alors définir une nouvelle opération : $X \parr Y = (X^\bot \otimes Y^\bot)^\bot$.

**Exercice 4.6.1** Exhiber des isomorphismes naturels $\lambda', \rho', \alpha', \sigma'$ tels que la structure $(\mathcal{L}, \parr, \bot, \lambda', \rho', \alpha', \sigma')$ soit une catégorie monoïdale symétrique. Exhiber un isomorphisme naturel entre $X \multimap Y$ et $X^\bot \parr Y$.

4.6.4 PRODUIT CARTÉSIEN. Ce que nous avons présenté correspond au fragment multiplicatif de la logique linéaire. Pour interpréter le fragment additif, il suffit de demander que la catégorie $*$-autonome $(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \sigma, \bot)$ soit telle que $\mathcal{L}$ soit cartésienne. Quand c'est le cas, on note $\top$ l'objet terminal et $\&$ le produit cartésien binaire.

**Exercice 4.6.2** Exhiber des isomorphismes naturels entre $X \parr \top$ et $\top$ et entre $(X_1 \& X_2) \parr Y$ et $(X_1 \parr Y) \& (X_2 \parr Y)$.

**Exercice 4.6.3** Montrer que $\mathcal{L}$ est co-cartesienne, avec $0 = \top^\bot$ comme objet initial, et $X \oplus Y = (X^\bot \& Y^\bot)^\bot$ comme co-produit binaire.

If $(X_i)_{i \in I}$ is a family of objects of $\mathcal{L}$ which has a coproduct in $\mathcal{L}$, this coproduct is denoted $\bigoplus_{i \in I} X_i$ and the injections are denoted as $\overline{\pi}_j \in \mathcal{L}(X_j, \bigoplus_{i \in I} X_i)$ for each $j \in I$. If $(f_i)_{i \in I}$ is a family of morphisms $f_i \in \mathcal{L}(X_i, Y)$ then we denote as $[f_i]_{i \in I}$ the unique morphism in $f \in \mathcal{L}(\bigoplus_{i \in I} X_i, Y)$ such that $f\,\overline{\pi}_i = f_i$ for each $i \in I$.

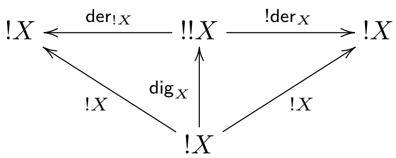
Given $g \in \mathcal{L}(Y, Z)$, we have $g\,[f_i]_{i \in I} = [g\,f_i]_{i \in I}$. In the special case where $Y = \bigoplus_{i \in I} X_i$ and $f_i = \overline{\pi}_i$, we have $[f_i]_{i \in I} = \mathsf{Id}_Y$.

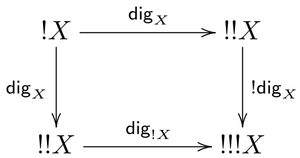
More generally, let $X$ be an object of $\mathcal{L}$ and assume now that $(f_i)_{i \in I}$ is a family of morphisms such that $f_i \in \mathcal{L}(X_i \otimes X, Y)$. Using monoidal closedness, we can similarly show that there is an unique morphism $f \in \mathcal{L}((\bigoplus_{i \in I} X_i) \otimes X, Y)$ such that $f\,(\overline{\pi}_i \otimes X) = f_i$ for each $i \in I$.

**Exercice 4.6.4** Complete the proof of this last fact.

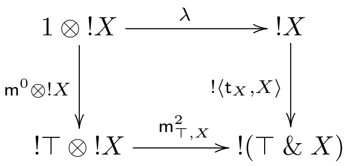
4.6.5 Exponentielles. Une *exponentielle* sur une catégorie $*$-autonome cartésienne $(\mathcal{L}, 1, \otimes, \lambda, \rho, \alpha, \sigma, \perp)$ est la donnée d'un foncteur $!\_ : \mathcal{L} \to \mathcal{L}$, d'une structure de comonade sur ce foncteur et d'un isomorphisme naturel dit isomorphisme de Seely.

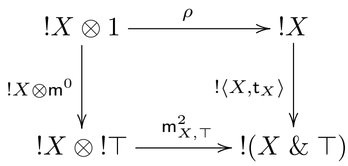
Plus précisément, et pour fixer les notations, une exponentielle est une structure $(!\_, \mathsf{der}, \mathsf{dig}, \mathsf{m}^2, \mathsf{m}^0)$ où $!\_ : \mathcal{L} \to \mathcal{L}$ est un foncteur, $\mathsf{der}_X \in \mathcal{L}(!X, X)$ et $\mathsf{dig}_X \in \mathcal{L}(!X, !!X)$ sont des transformations naturelles, $\mathsf{m}^0 : 1 \to !\top$ est un isomorphisme et $\mathsf{m}^2_{X,Y} : !X \otimes !Y \to !(X \,\&\, T)$ est un isomorphisme naturel (ces deux derniers isomorphismes sont appelés *isomorphismes de Seely* alors qu'ils sont plutôt dûs à Girard qui n'en a toutefois pas donné les propriétés catégoriques générales). Ces morphismes doivent satisfaire les diagrammes suivants. Les 3 premiers expriment que $(!\_, \mathsf{der}, \mathsf{dig})$ est une comonade.



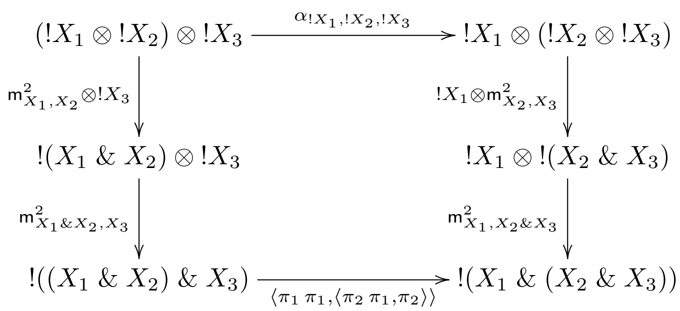



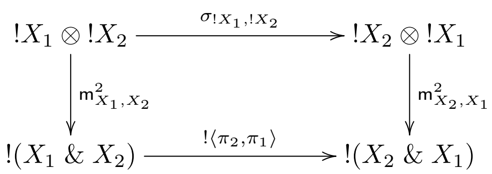

Les deux suivants concernent les isomorphismes de Seely.




où on rappelle que $\mathsf{t}_X$ est l'unique élément de $\mathcal{L}(X, \top)$ ; par conséquent $\langle \mathsf{t}_X, X\rangle$ est un isomorphisme (qui dit que $\top$ est neutre à gauche pour &).




On dit que $(!\_, \mathsf{m}^0, \mathsf{m}^2)$ est une *foncteur monoïdal symétrique* de la catégorie monoïdale symétrique $(\mathcal{L}, \&, \top)$ vers la catégorie monoïdale symétrique $(\mathcal{L}, \otimes, 1)$ (on omet les isomorphismes naturels de structure monoïdale symétrique). C'est la bonne façon catégorique de dire que "!_ envoie le produit cartésien sur le produit tensoriel", ce qui justifie d'ailleurs la terminologie "exponentielle" : le produit cartésien est additif, le produit tensoriel est multiplicatif, à rapprocher de $e^{x+y} = e^x e^y$.

Finalement, il faut encore qu'une condition technique soit satisfaite, qui se traduit par la commutation du diagramme suivant. Ces conditions relient le digging aux isomorphismes de Seely.

$$\begin{array}{ccc}
!X \otimes !Y & \xrightarrow{\mathsf{m}^2_{X,Y}} & !(X \,\&\, Y) \\
 & & \downarrow{\scriptstyle \mathsf{dig}_{X\&Y}} \\
\downarrow{\scriptstyle \mathsf{dig}_X \otimes \mathsf{dig}_Y} & & !!(X \,\&\, Y) \\
 & & \downarrow{\scriptstyle !\langle !\pi_1, !\pi_2\rangle} \\
!!X \otimes !!Y & \xrightarrow{\mathsf{m}^2_{!X,!Y}} & !(!X \,\&\, !Y)
\end{array}$$

4.6.6 DERIVED STRUCTURES Given $f \in \mathcal{L}(!X, Y)$, we can define $f^! \in \mathcal{L}(!X, !Y)$ by $f^! = !f\ \mathsf{dig}_X$. This is the *unary promotion* of $f$.

Given more generally $f \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, Y)$ we want now to define an $n$-ary promotion $f^! \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, !Y)$.

For this we define $\mu^0 \in \mathcal{L}(1, !1)$ and $\mu^2_{X,Y} \in \mathcal{L}(!X \otimes !Y, !(X \otimes Y))$. The first of these morphisms is defined as the following composition of morphisms in $\mathcal{L}$

$$1 \xrightarrow{\mathsf{m}^0} !\top \xrightarrow{\mathsf{dig}_\top} !!\top \xrightarrow{!(\mathsf{m}^0)^{-1}} !1$$

The second morphism is defined as the following composition in $\mathcal{L}$

$$\begin{array}{ccccc}
!X \otimes !Y & & & & !(X \otimes Y) \\
\downarrow{\scriptstyle \mathsf{m}^2_{X,Y}} & & & & \uparrow{\scriptstyle !(\mathsf{der}_X \otimes \mathsf{der}_Y)} \\
!(X \,\&\, Y) & \xrightarrow{\mathsf{dig}_{X\&Y}} & !!(X \,\&\, Y) & \xrightarrow{!(\mathsf{m}^2_{X,Y})^{-1}} & !(!X \otimes !Y)
\end{array}$$

It results straightforwardly from the definition that $\mu^2_{X,Y}$ is natural in $X$ and $Y$. These two morphisms equip the functor ! with a lax[2] symmetric monoidal structure, from the monoidal category $(\mathcal{L}, 1, \otimes)$ to itself. This means that the following diagrams commute

$$\begin{array}{ccc}
1 \otimes !X & \xrightarrow{\mu^0 \otimes !X} & !1 \otimes !X \\
 & & \downarrow{\scriptstyle \mu^2_{1,X}} \\
 & \searrow{\scriptstyle \lambda_{!X}} & !(1 \otimes X) \\
 & & \downarrow{\scriptstyle !\lambda_X} \\
 & & !X
\end{array}
\qquad
\begin{array}{ccc}
!X \otimes 1 & \xrightarrow{!X \otimes \mu^0} & !X \otimes !1 \\
 & & \downarrow{\scriptstyle \mu^2_{X,1}} \\
 & \searrow{\scriptstyle \rho_{!X}} & !(X \otimes 1) \\
 & & \downarrow{\scriptstyle !\rho_X} \\
 & & !X
\end{array}$$

$$\begin{array}{ccccc}
(!X \otimes !Y) \otimes !Z & \xrightarrow{\mu^2_{X,Y} \otimes !Z} & !(X \otimes Y) \otimes !Z & \xrightarrow{\mu^2_{X\otimes Y,Z}} & !((X \otimes Y) \otimes Z) \\
\downarrow{\scriptstyle \alpha_{!X,!Y,!Z}} & & & & \downarrow{\scriptstyle !\alpha_{X,Y,Z}} \\
!X \otimes (!Y \otimes !Z) & \xrightarrow{!X \otimes \mu^2_{Y,Z}} & !X \otimes !(Y \otimes Z) & \xrightarrow{\mu^2_{X,Y\otimes Z}} & !(X \otimes (Y \otimes Z))
\end{array}$$

$$\begin{array}{ccc}
!X \otimes !Y & \xrightarrow{\mu^2_{X,Y}} & !(X \otimes Y) \\
\downarrow{\scriptstyle \sigma_{!X,!Y}} & & \downarrow{\scriptstyle !\sigma_{Y,X}} \\
!Y \otimes !X & \xrightarrow{\mu^2_{Y,X}} & !(Y \otimes X)
\end{array}$$

**Exercice 4.6.5** Prove that the three diagrams above commute.

2. "lax" means that the associated natural transformations are not isos in general.

If we consider the isomorphisms $\alpha$ as identities (that is, if we identify the objects $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$), then it makes sense to write an $n$-ary tensor as $X_1 \otimes \cdots \otimes X_n$, without parentheses. This is of course an abuse of notation which can be suitably corrected by inserting parentheses and explicit isomorphisms. The property above of $\mu^2_{,}$ means precisely that, independently of these choices of representations of $n$-ary tensors, we can canonical morphism

$$\mu^n \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_1, !(X_1 \otimes \cdots \otimes X_n))$$

by combining freely occurrences of $\mu^2_{,}$, the order in which we use them does not matter thanks to the coherence diagrams commutations satisfied by the monoidality isomorphisms associated with $\otimes$ (we can actually even insert 1's and permute factors).

Thanks to these morphisms, we can generalize promotion as follows.

Let $f \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, Y)$, we define $f^! \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, !Y)$ as the following composition of morphisms in $\mathcal{L}$

$$\begin{array}{c} !X_1 \otimes \cdots \otimes !X_n \\ \Big\downarrow \mathsf{dig}_{X_1} \otimes \cdots \otimes \mathsf{dig}_{X_n} \\ !!X_1 \otimes \cdots \otimes !!X_n \\ \Big\downarrow \mu^n_{!X_1,\ldots,!X_n} \\ !(!X_1 \otimes \cdots \otimes !X_n) \\ \Big\downarrow !f \\ !Y \end{array}$$

We simply denote this morphism again as $f^!$ because the only case where this choice can introduce an ambiguity is $n = 1$, and in that case, both notions coincide. Observe that this definition also makes sense when $n = 0$, in which case we have $f \in \mathcal{L}(1, Y)$ and $f^! \in \mathcal{L}(1, !Y)$.

Let $f \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, Y)$. The two following diagrams commute.

$$\begin{array}{ccc} !X_1 \otimes \cdots \otimes !X_n & \xrightarrow{f^!} & !Y \\ & \searrow^{f} & \Big\downarrow \mathsf{der}_Y \\ & & Y \end{array} \qquad \begin{array}{ccc} !X_1 \otimes \cdots \otimes !X_n & \xrightarrow{f^!} & !Y \\ & \searrow^{(f^!)^!} & \Big\downarrow \mathsf{dig}_Y \\ & & !!Y \end{array}$$

**Exercice 4.6.6** Prove these commutations.

**Exercice 4.6.7** Let $g \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n \otimes !Y, Z)$ and $f \in \mathcal{L}(!X_{n+1} \otimes \cdots \otimes !X_p, Y)$. Prove that the following diagram commutes

$$\begin{array}{ccc} !X_1 \otimes \cdots \otimes !X_p & \xrightarrow{!X_1 \otimes \cdots \otimes !X_n \otimes f^!} & !X_1 \otimes \cdots \otimes !X_n \otimes !Y \\ & \searrow^{(g\,(!X_1 \otimes \cdots \otimes !X_n \otimes f^!))^!} & \Big\downarrow g^! \\ & & !Z \end{array}$$

We define now morphisms corresponding to the structural rules of Linear Logic, weakening and contraction. Let $X$ be an object of $\mathcal{L}$. We define $\mathsf{wf}_X \in \mathcal{L}(!X, 1)$ as the following composition of morphisms in $\mathcal{L}$ :

$$!X \xrightarrow{!\mathsf{t}_X} !\top \xrightarrow{(\mathsf{m}^0)^{-1}} 1$$

where $\mathsf{t}_X$ is the unique element of $\mathcal{L}(X, \top)$ (since $\top$ is the terminal object of $\mathcal{L}$). Similarly, we define $\mathsf{cf}_X \in \mathcal{L}(!X, !X \otimes !X)$ as the following composition of morphisms

$$!X \xrightarrow{!\langle \pi_1, \pi_2 \rangle} !(X \,\&\, X) \xrightarrow{(\mathsf{m}^2_{X,X})^{-1}} !X \otimes !X$$

Then one proves easily (exercise) that $(!X, \mathsf{wf}_X, \mathsf{cf}_X)$ is a symmetric comonoid in the SMC $(\mathcal{L}, 1, \otimes)$ meaning that the following diagrams commute. The first two commutations mean that $\mathsf{wf}_X$ is the "neutral element" of this comonoid.

$$
\begin{array}{ccc}
!X & \xrightarrow{\mathsf{cf}_X} & !X \otimes !X \\
 & \searrow^{!X} & \downarrow{\scriptstyle \mathsf{wf}_X \otimes !X} \\
 & & 1 \otimes !X \\
 & & \downarrow{\scriptstyle \lambda_X} \\
 & & !X
\end{array}
\qquad
\begin{array}{ccc}
!X & \xrightarrow{\mathsf{cf}_X} & !X \otimes !X \\
 & \searrow^{!X} & \downarrow{\scriptstyle !X \otimes \mathsf{wf}_X} \\
 & & !X \otimes 1 \\
 & & \downarrow{\scriptstyle \rho_X} \\
 & & !X
\end{array}
$$

The next diagram expresses the associativity of the comultiplication $\mathsf{cf}_X$.

$$
\begin{array}{ccccc}
!X & \xrightarrow{\mathsf{cf}_X} & !X \otimes !X & \xrightarrow{\mathsf{cf}_X \otimes !X} & (!X \otimes !X) \otimes !X \\
{\scriptstyle !X}\downarrow & & & & \downarrow{\scriptstyle \alpha_{!X,!X,!X}} \\
!X \otimes !X & & \xrightarrow{\quad !X \otimes (\mathsf{cf}_X) \quad} & & !X \otimes (!X \otimes !X)
\end{array}
$$

The last diagram expresses commutativity of comultiplication.

$$
\begin{array}{ccc}
!X & \xrightarrow{\mathsf{cf}_X} & !X \otimes !X \\
 & \searrow_{\mathsf{cf}_X} & \downarrow{\scriptstyle \sigma_{!X,!X}} \\
 & & !X \otimes !X
\end{array}
$$

**Exercice 4.6.8** Prove these commutations. Given $f \in \mathcal{L}(!X_1 \otimes \cdots \otimes !X_n, Y)$, prove that the following diagrams commute, where $\varphi$ and $\psi$ are isos that you will give explicitly.

$$
\begin{array}{ccc}
!X_1 \otimes \cdots \otimes !X_n & \xrightarrow{f^!} & !Y \\
{\scriptstyle \mathsf{wf}_{X_1} \otimes \cdots \otimes \mathsf{wf}_{X_n}}\downarrow & & \downarrow{\scriptstyle \mathsf{wf}_Y} \\
1 \otimes \cdots \otimes 1 & \xrightarrow{\varphi} & 1
\end{array}
$$

$$
\begin{array}{ccc}
!X_1 \otimes \cdots \otimes !X_n & \xrightarrow{\qquad f^! \qquad} & !Y \\
{\scriptstyle \mathsf{cf}_{X_1} \otimes \cdots \otimes \mathsf{cf}_{X_n}}\downarrow & & \downarrow{\scriptstyle \mathsf{cf}_Y} \\
(X_1 \otimes X_1) \otimes \cdots \otimes (X_n \otimes X_n) & \xrightarrow{\psi} & !Y \otimes !Y
\end{array}
$$

4.6.7 Catégorie de Kleisli. Il est alors possible de définir une catégorie cartésienne fermée. C'est la catégorie de Kleisli associée à la comonade $!\_$ que l'on note $\mathcal{L}_!$ (la construction de cette catégorie n'utilise que le fait que $!\_$ est une comonade). Les objets de $\mathcal{L}_!$ sont ceux de $\mathcal{L}$, et on a $\mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$. L'identité est $\mathsf{der}_X \in \mathcal{L}_!(X, X)$. Etant donnés $f \in \mathcal{L}_!(X, Y)$ et $g \in \mathcal{L}_!(Y, Z)$, la composition est donnée par

$$g \circ f = g\,(!f)\ \mathsf{dig}_X = g\, f^!\,.$$

**Exercice 4.6.9** Vérifier que $\mathcal{L}_!$ est bien une catégorie : associativité de la composition, neutralité de $\mathsf{der}_X$ pour la composition.

Le reste de la structure permet d'obtenir le résultat suivant.

**Proposition 4.6.3** *La catégorie $\mathcal{L}_!$ est cartésienne fermée.*

*Démonstration.* L'objet terminal est $\top$ et le produit cartésien de $X_1$ et $X_2$ est $(X_1 \mathbin{\&} X_2, \pi_1 \ \mathsf{der}_{!(X_1 \& X_2)}, \pi_2 \ \mathsf{der}_{!(X_1 \& X_2)})$. En effet, si on se donne $f_i \in \mathcal{L}_!(Z, X_i)$ on a $\langle f_1, f_2 \rangle \in \mathcal{L}_!(Z, X_1 \mathbin{\&} X_2)$ et on peut vérifier que les équations voulues sont satisfaites.

L'objet des morphismes de $X$ vers $Y$ est $(!X \multimap Y, \mathsf{Ev})$ où $\mathsf{Ev}$ est défini comme la composition suivante

$$
\begin{array}{ccccc}
!((!X \multimap Y) \mathbin{\&} X) & \xrightarrow{(\mathsf{m}^2_{!X \multimap Y, X})^{-1}} & !(!X \multimap Y) \otimes !X & \xrightarrow{\mathsf{der}_{!X \multimap Y} \otimes \mathsf{Id}} & (!X \multimap Y) \otimes !X \\
 & & & & \Big\downarrow \mathsf{ev} \\
 & & & & Y
\end{array}
$$

Soit $f \in \mathcal{L}_!(Z \mathbin{\&} X, Y)$, alors $f \ \mathsf{m}^2_{Z,X} \in \mathcal{L}(!Z \otimes !X, Y)$ et donc

$$\mathsf{Cur}\,(f) = \mathsf{cur}\left(f \ \mathsf{m}^2_{Z,X}\right) \in \mathcal{L}_!(Z, !X \multimap Y)\,.$$

On laisse au lecteur le soin de vérifier que les équations voulues sont satisfaites. □

4.6.8 Catégories des coalgèbres. Une !-colagèbre est un couple $P = (\underline{P}, \mathsf{h}_P)$ où $\underline{P}$ est un objet de $\mathcal{L}$ et $\mathsf{h}_P \in \mathcal{L}(\underline{P}, !\underline{P})$ vérifie les commutations suivantes

$$
\begin{array}{ccc}
\underline{P} & \xrightarrow{\mathsf{h}_P} & !\underline{P} \\
 & {\scriptstyle \mathsf{Id}_{\underline{P}}} \searrow & \Big\downarrow \mathsf{der}_{\underline{P}} \\
 & & \underline{P}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\underline{P} & \xrightarrow{\mathsf{h}_P} & !\underline{P} \\
\mathsf{h}_P \Big\downarrow & & \Big\downarrow \mathsf{dig}_{\underline{P}} \\
!\underline{P} & \xrightarrow{!\mathsf{h}_P} & !!\underline{P}
\end{array}
$$

Etant donnés deux coalgèbres $P$ et $Q$, un morphisme de $P$ vers $Q$ est un morphisme $f \in \mathcal{L}(\underline{P}, h_Q)$ tel que le diagramme suivant commute

$$
\begin{array}{ccc}
\underline{P} & \xrightarrow{f} & \underline{Q} \\
\mathsf{h}_P \Big\downarrow & & \Big\downarrow \mathsf{h}_Q \\
!\underline{P} & \xrightarrow{!f} & !\underline{Q}
\end{array}
$$

**Exercice 4.6.10** Vérifier qu'on a ainsi défini une catégorie (l'identité et la composition sont celles de $\mathcal{L}$).

This category is denoted as $\mathcal{L}^!$ and is called the Eilenberg-Moore category of ! (or, simply, the category of coalgebras of !).

L'objet 1 de $\mathcal{L}$ a une structure de coalgèbre canonique qui est donnée par $\mu^0 \in \mathcal{L}(1, !1)$, see Section 4.6.6. On notera encore 1 cette !-coalgèbre.

**Exercice 4.6.11** Vérifier qu'il s'agit bien d'une structure de colagèbre. Pour le second diagramme, il vaut mieux commencer en calculant la composée $!\mathsf{h}_1 \ \mathsf{h}_1$.

4.6.9 Coproducts and products of coalgebras. Soit $(P_i)_{i \in I}$ une famille dénombrable de colagèbres. Sur $X = \bigoplus_{i \in I} \underline{P_i}$, on définit une structure de colagèbre en définissant un morphisme $h \in \mathcal{L}(X, !X)$. Pour chaque $i \in I$, on observe d'abord que $!\overline{\pi}_i \ \mathsf{h}_{P_i} \in \mathcal{L}(\underline{P_i}, !X)$ et donc on pose $h = [!\overline{\pi}_i \ \mathsf{h}_{P_i}]_{i \in I}$. Autrement dit, $h$ est l'unique morphisme de $\mathcal{L}(X, !X)$ tel que $h \, \overline{\pi}_i = !\overline{\pi}_i \ \mathsf{h}_{P_i}$ pour tout $i \in I$.

Let us check that we have defined a coalgebra. We have

$$\begin{aligned}
\mathsf{der}_X\ h &= [\mathsf{der}_X\ !\overline{\pi}_i\,\mathsf{h}_{P_i}]_{i\in I}\\
&= [\overline{\pi}_i\ \mathsf{der}_{h_{P_i}}\ \mathsf{h}_{P_i}]_{i\in I}\\
&= [\overline{\pi}_i]_{i\in I} = \mathsf{Id}_X
\end{aligned}$$

$$\begin{aligned}
\text{and } !h\,h &= [!h\,!\overline{\pi}_i\,\mathsf{h}_{P_i}]_{i\in I}\\
&= [!(h\,\overline{\pi}_i)\,\mathsf{h}_{P_i}]_{i\in I}\\
&= [!(!\overline{\pi}_i\,\mathsf{h}_{P_i})\,\mathsf{h}_{P_i}]_{i\in I}\\
&= [!!\overline{\pi}_i\,!\mathsf{h}_{P_i}\,\mathsf{h}_{P_i}]_{i\in I}\\
&= [!!\overline{\pi}_i\ \mathsf{dig}_{!h_{P_i}}\ \mathsf{h}_{P_i}]_{i\in I}\\
&= [\mathsf{dig}_X\ !\overline{\pi}_i\,\mathsf{h}_{P_i}]_{i\in I}\\
&= \mathsf{dig}_X\ h
\end{aligned}$$

We use $\bigoplus_{i\in I} P_i$ to denote this coalgebra.

Let $P$ and $Q$ be coalgebras. We endow $X = \underline{P}\otimes\underline{Q}$ with a structure of caolgebra which is defined as the following composition of morphisms in $\mathcal{L}$, that we denote as $\mathsf{h}_{P\otimes Q}$ :

$$\underline{P}\otimes\underline{Q} \xrightarrow{\mathsf{h}_P\otimes\mathsf{h}_Q} !\underline{P}\otimes !\underline{Q} \xrightarrow{\mu^2_{\underline{P},\underline{Q}}} !(\underline{P}\otimes\underline{Q})$$

Given any object $X$ of $\mathcal{L}$, we can define an object $\mathsf{E}(X)$ of $\mathcal{L}^!$ as follows : $\mathsf{E}(X) = (!X, \mathsf{dig}_X)$. This is the *free* !-coalgebra generated by $X$. This operation can be extended into a functor : given $f \in \mathcal{L}(X,Y)$ then it results from the naturality of $\mathsf{dig}$ that $\mathsf{E}(f) = !f \in \mathcal{L}^!(\mathsf{E}(X),\mathsf{E}(Y))$.

**Exercice 4.6.12** Let $X$ and $Y$ be objects of $\mathcal{L}$. Prove that the coalgebras $\mathsf{E}(X \,\&\, Y)$ and $\mathsf{E}(X)\otimes\mathsf{E}(Y)$ are isomorphic in the category $\mathcal{L}^!$.

**Exercice 4.6.13** Given objects $X$ and $Y$ of $\mathcal{L}$, prove that there is a (natural) bijective correspondence between $\mathcal{L}_!(X,Y)$ and $\mathcal{L}^!(\mathsf{E}(X),\mathsf{E}(Y))$. In other words $\mathcal{L}_!$ can be considered as the category of free coalgebras (this is its intended meaning).

**Lemme 4.6.4** *Let $R$ and $R'$ be coalgebras. Let $f \in \mathcal{L}^!(R,R')$ be such that $f$ is an iso in $\mathcal{L}$. Then $f$ is an iso in $\mathcal{L}^!$.*

**Exercice 4.6.14** Prove this lemma.

So to prove that a morphism of coalgebras is an iso of coalgebras, it suffices to prove that it is an iso in $\mathcal{L}$.

**Proposition 4.6.5** *Let $(P_i)_{i\in I}$ be a countable family of coalgebras and $Q$ be a coalgebra. Then the coalgebras $(\bigoplus_{i\in I} P_i)\otimes Q$ and $\bigoplus_{i\in I}(P_i\otimes Q)$ are isomorphic (naturally) in $\mathcal{L}^!$.*

*Démonstration.* To define a (natural) morphism $f \in \mathcal{L}^!(\bigoplus_{i\in I}(P_i\otimes Q), (\bigoplus_{i\in I} P_i)\otimes Q)$, it suffices to define $f_i \in \mathcal{L}^!(P_i\otimes Q, (\bigoplus_{j\in I} P_j)\otimes Q)$ for each $i\in I$. We set $f_i = \overline{\pi}_i\otimes Q$. To prove that $f$ is an iso in $\mathcal{L}^!$ it suffices to prove that it has an inverse in $\mathcal{L}$ by Lemma 4.6.4. So we define a morphism $g \in \mathcal{L}((\bigoplus_{i\in I}\underline{P_i})\otimes\underline{Q}, Y)$ where $Y = \bigoplus_{i\in I}(\underline{P_i}\otimes\underline{Q})$. For this, by monoidal closeness, it suffices to define a morphism $g' \in \mathcal{L}(\bigoplus_{i\in I}\underline{P_i}, \underline{Q}\multimap Y)$. By the universal property of the coproduct, it suffices to define, for each $i\in I$, a morphism $g'_i \in \mathcal{L}(\underline{P_i}, \underline{Q}\multimap Y)$, that is, by monoidal closeness again, a morphism $f'_i \in \mathcal{L}^!(\underline{P_i}\otimes\underline{Q}, Y)$, we take of course $f'_i = \overline{\pi}_i$ (the $i$-th injection into the coproduct $\bigoplus_{i\in I}(\underline{P_i}\otimes\underline{Q})$). □

**Exercice 4.6.15** Complete the proof by showing that $g$ is the (left and right) inverse of $f$.

4.6.10 Structural structure of coalgebras Let $P$ be a coalgebra. We can endow $P$ with a structure of commutative $\otimes$-comonoid in $\mathcal{L}^!$. This means that we can define a morphism $\mathsf{w}_P \in \mathcal{L}^!(P, 1)$ and a morphism $\mathsf{c}_P \in \mathcal{L}^!(P, P \otimes P)$. These morphisms are defined as the following composition of morphisms in $\mathcal{L}$ :

$$\underline{P} \xrightarrow{\mathsf{h}_P} !\underline{P} \xrightarrow{\mathsf{wf}_{!\underline{P}}} 1$$

$$\underline{P} \xrightarrow{\mathsf{h}_P} !\underline{P} \xrightarrow{\mathsf{cf}_{!\underline{P}}} !\underline{P} \otimes !\underline{P} \xrightarrow{\mathsf{der}_{\underline{P}} \otimes \mathsf{der}_{\underline{P}}} \underline{P} \otimes \underline{P}$$

where the morphisms $\mathsf{wf}_{!\underline{P}}$ and $\mathsf{cf}_{!\underline{P}}$ are those defined in Section 4.6.6.

**Exercice 4.6.16** Prove that $\mathsf{w}_P$ and $\mathsf{c}_P$ are morphisms of $\mathcal{L}^!$, and that the following diagrams commute, meaning that $(P, \mathsf{w}_P, \mathsf{c}_P)$ is a commutative $\otimes$-comonoid, meaning that the following diagrams commute :

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\mathsf{c}_P} & \underline{P} \otimes \underline{P} \\ & & \downarrow \mathsf{w}_P \otimes \underline{P} \\ & \searrow^{\underline{P}} & 1 \otimes \underline{P} \\ & & \downarrow \lambda^{-1} \\ & & \underline{P} \end{array} \qquad \begin{array}{ccc} \underline{P} & \xrightarrow{\mathsf{c}_P} & \underline{P} \otimes \underline{P} \\ & & \downarrow \underline{P} \otimes \mathsf{w}_P \\ & \searrow^{\underline{P}} & \underline{P} \otimes 1 \\ & & \downarrow \rho^{-1} \\ & & \underline{P} \end{array}$$

$$\begin{array}{ccccc} \underline{P} & \xrightarrow{\mathsf{c}_P} & \underline{P} \otimes \underline{P} & \xrightarrow{\mathsf{c}_P \otimes \underline{P}} & (\underline{P} \otimes \underline{P}) \otimes \underline{P} \\ \mathsf{c}_P \downarrow & & & & \downarrow \alpha \\ \underline{P} \otimes \underline{P} & & \xrightarrow{\underline{P} \otimes \mathsf{c}_P} & & \underline{P} \otimes (\underline{P} \otimes \underline{P}) \end{array} \qquad \begin{array}{ccc} \underline{P} & \xrightarrow{\mathsf{c}_P} & \underline{P} \otimes \underline{P} \\ & \searrow^{\mathsf{c}_P} & \downarrow \sigma \\ & & \underline{P} \otimes \underline{P} \end{array}$$

**Proposition 4.6.6** *Let $f \in \mathcal{L}^!(P, Q)$ be a coalgebra morphism. Then $f$ is a morphism from the comonoid associated with $P$ to the comonoid associated with $Q$.*

This means that the following two diagrams commute :

$$\begin{array}{ccc} \underline{P} & \xrightarrow{f} & \underline{Q} \\ & \searrow^{\mathsf{w}_P} & \downarrow \mathsf{w}_Q \\ & & 1 \end{array} \qquad \begin{array}{ccc} \underline{P} & \xrightarrow{f} & \underline{Q} \\ \mathsf{c}_P \downarrow & & \downarrow \mathsf{c}_Q \\ \underline{P} \otimes \underline{P} & \xrightarrow[f \otimes f]{} & \underline{Q} \otimes \underline{Q} \end{array}$$

**Exercice 4.6.17** Prove the two commutations above.

**Exercice 4.6.18** Prove that, for any object $X$ of $\mathcal{L}$, one has $\mathsf{w}_{\mathsf{E}(X)} = \mathsf{wf}_{!X}$ and $\mathsf{c}_{\mathsf{E}(X)} = \mathsf{cf}_{!X}$.

**Exercice 4.6.19** Prove that $P \otimes Q$ is the cartesian product of $P$ and $Q$ in the category $\mathcal{L}^!$ (warning : the proof of this statement is rather painful). Prove also that $1$ (equiped with $\mathsf{h}_1$ defined above) is the terminal object of $\mathcal{L}^!$, which is therefore a cartesian category.

Given an object $P$ of $\mathcal{L}^!$ and an object $X$ of $\mathcal{L}$, we define now the *generalized promotion* of a morphism $f \in \mathcal{L}(\underline{P}, X)$ : it is the morphism $f^! = !f\, \mathsf{h}_P \in \mathcal{L}(\underline{P}, !X)$.

**Exercice 4.6.20** Prove that $f^! \in \mathcal{L}^!(P, \mathsf{E}(X))$ and that the operation $f \mapsto f^!$ is a bijection $\mathcal{L}(\underline{P}, X) \to \mathcal{L}^!(P, \mathsf{E}(X))$ (define its inverse).

Let $\mathsf{U} : \mathcal{L}^! \to \mathcal{L}$ be the forgetful functor defined on objects by $\mathsf{U}(P) = \underline{P}$ and $\mathsf{U}(f) = f$. We have seen essentially that the functor $\mathsf{E}$ is *right adjoint* to the functor $\mathsf{U}$. Observe that the functor ! is $\mathsf{U} \circ \mathsf{E}$ : this a general phenomenon that any comonad can be described by means of an adjunction (actually by means of many adjunctions : such a decomposition of ! is called a *factorization*). Some people prefer to present models of LL as such adjunctions. The "linear/non linear" presentation of LL categorical semantics is a typical example of this. We prefer the comonadic presentation because it is agnostic on this choice of factorization which reflects a particular veiwpoint on LL.

4.6.11 Subobjects and embedding retraction pairs Remember that a partially ordered set $\Gamma$ is directed if $\Gamma \neq \emptyset$ and $\forall \gamma_1, \gamma_2 \in \Gamma\, \exists \gamma \in \Gamma\ \gamma_1 \leq \gamma$ and $\gamma_2 \leq \gamma$.

**Exercice 4.6.21** Prove that a partially ordered set is directed iff for any $n \in \mathbb{N}$ and any family $\gamma_1, \dots, \gamma_n$ of elements of $\Gamma$, there is an element $\gamma$ of $\Gamma$ such that $\gamma_i \leq \gamma$ for $i = 1, \dots, n$.

Given a class $\mathcal{M}$ equiped with a partial order $\sqsubseteq$, a directed family of elements of $\mathcal{M}$ is an indexed family $(X_\gamma)_{\gamma\in\Gamma}$ where $\Gamma$ is a directed set, $X_\gamma \in \mathcal{M}$ for each $\gamma \in \Gamma$ and $\gamma \leq \gamma' \Rightarrow X_\gamma \sqsubseteq X_{\gamma'}$.

We describe now the categorical structure required for an LL categorical model $\mathcal{L}$ to allow the interpretation of *recursive types*.

We assume to be given an order relation $\sqsubseteq$ on the class of objects of $\mathcal{L}$; we denote as $\mathcal{L}_\sqsubseteq$ the corresponding partially ordered class. This class is also considered as a category with $\mathcal{L}_\sqsubseteq(X, Y) = \{*_{X,Y}\}$ (a singleton set whose unique elements is denoted as $*_{X,Y}$) if $X \sqsubseteq Y$ and $\mathcal{L}_\sqsubseteq(X, Y) = \emptyset$ otherwise, identities and composition defined in the obvious way. We also make the following assumptions :

— 0 (the initial object of $\mathcal{L}$) is also the initial object of $\mathcal{L}_\sqsubseteq$, that is, its least element for the partial order $\sqsubseteq$.

— Any countable directed family $(X_\gamma)_{\gamma\in\Gamma}$ of $\mathcal{L}_\sqsubseteq$ has a least upper bound denoted as $\bigsqcup_{\gamma\in\Gamma} X_\gamma$.

We say that the partially ordered class $\mathcal{L}_\sqsubseteq$ is *directed-cocomplete* or is a *complete partially ordered class (cpoc)*.

To make the connection between the partially ordered class $\mathcal{L}_\sqsubseteq$ and the category $\mathcal{L}$, we assume to be given a functor $\mathcal{I} : \mathcal{L}_\sqsubseteq \to \mathcal{L}^{\mathsf{op}} \times \mathcal{L}$ such that $\mathcal{I}(X) = (X, X)$ for each object $X$ of $\mathcal{L}$. Given objects $X$ and $Y$ such that $X \sqsubseteq Y$ we use the notations $\mathsf{i}^-_{X,Y} \in \mathcal{L}(Y, X)$ and $\mathsf{i}^+_{X,Y} \in \mathcal{L}(X, Y)$ for the first and second component of the pair of morphisms $\mathcal{I}(*_{X,Y})$.

So we know that

$$\begin{aligned}
\mathsf{i}^+_{X,X} &= \mathsf{Id}_X \\
\mathsf{i}^-_{X,X} &= \mathsf{Id}_X \\
X \sqsubseteq Y \sqsubseteq Z \Rightarrow \mathsf{i}^+_{X,Z} &= \mathsf{i}^+_{Y,Z}\, \mathsf{i}^+_{X,Y} \\
X \sqsubseteq Y \sqsubseteq Z \Rightarrow \mathsf{i}^-_{X,Z} &= \mathsf{i}^-_{X,Y}\, \mathsf{i}^-_{Y,Z}
\end{aligned}$$

We assume moreover that

$$X \sqsubseteq Y \Rightarrow \mathsf{i}^-_{X,Y}\, \mathsf{i}^+_{X,Y} = \mathsf{Id}_X$$

This latter property reflects the intuition that $X$ is a subobject of $Y$, that $\mathsf{i}^+_{X,Y}$ is an embedding morphism of $X$ into $Y$ and that $\mathsf{i}^-_{X,Y}$ is a kind of retraction of $Y$ onto $X$.

We assume moreover that $\mathcal{I}$ preserves the "directed colimits" that we have assumed to exist in $\mathcal{L}_\sqsubseteq$. This means that the following property holds. Let $(X_\gamma)_{\gamma\in\Gamma}$ be a countable directed family of $\mathcal{L}_\sqsubseteq$. For each $\gamma \in \Gamma$, assume that we are given a morphism $f_\gamma \in \mathcal{L}(X_\gamma, Y)$ such that, for each $\gamma, \delta \in \Gamma$, if $\gamma \leq \delta$ then $f_\delta\, \mathsf{i}^+_{X_\gamma, X_\delta} = f_\gamma$ (such a family $(f_\gamma)_{\gamma\in\Gamma}$ is called a *cocone* to $Y$ based on $(X_\gamma)_{\gamma\in\Gamma}$). Then there is exactly one $f \in \mathcal{L}(\bigsqcup_{\gamma\in\Gamma} X_\gamma, Y)$ such that

$$\forall \gamma \in \Gamma \quad f\, \mathsf{i}^+_{X_\gamma, \bigsqcup_{\delta\in\Gamma} X_\delta} = f_\gamma\,.$$

Observe that, conversely, if we are given $g \in \mathcal{L}(\bigsqcup_{\gamma\in\Gamma} X_\gamma, Y)$, then we can define $g_\gamma = g\, \mathsf{i}^-_{X_\gamma, \bigsqcup_{\delta\in\Gamma} X_\delta}$ for each $\gamma \in \Gamma$. Then, assuming that $\gamma \leq \gamma'$, we have

$$\begin{aligned}
g_{\gamma'}\, \mathsf{i}^+_{X_\gamma, X_{\gamma'}} &= g\, \mathsf{i}^-_{Y, \bigsqcup \mathcal{D}}\, \mathsf{i}^+_{X,Y} \\
&= g\, \mathsf{i}^-_{X, \bigsqcup \mathcal{D}}\, \mathsf{i}^-_{X,Y}\, \mathsf{i}^+_{X,Y} \\
&= g_X\,.
\end{aligned}$$

So our assumption means that there is a bijection between $\mathcal{L}(\bigsqcup_{\gamma\in\Gamma} X_\gamma, Y)$ and the sets on all cocones to $Y$ based on $(X_\gamma)_{\gamma\in\Gamma}$.

**Exercice 4.6.22** Let $\Gamma_i$ be directed partially ordered sets for $i = 1,2$. Prove that $\Gamma_1 \times \Gamma_2$, equipped with the product order, is directed. Let $(X_{i,\gamma_i})_{\gamma_i \in \Gamma_i}$ be directed families of $\mathcal{L}_\sqsubseteq$ for $i = 1,2$. Prove that $(X_{1,\gamma_1} \times X_{2,\gamma_2})_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2}$ is directed in $\mathcal{L}_\sqsubseteq^2$ and that

$$\bigsqcup_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} (X_{1,\gamma_1} \times X_{2,\gamma_2}) = (\bigsqcup_{\gamma_1\in\Gamma_1} X_{\gamma_1}, \bigsqcup_{\gamma_2\in\Gamma_2} X_{\gamma_2}).$$

We assume next that all the operations of linear logic are Scott-continuous in the following sense. The next exercise is a useful preliminary observation.

We make the following additional assumptions on $\mathcal{L}$.

— If $X_i \sqsubseteq Y_i$ for $i = 1,2$ then $X_1 \otimes X_2 \sqsubseteq Y_1 \otimes Y_2$, $\mathsf{i}^+_{X_1\otimes X_2,Y_1\otimes Y_2} = \mathsf{i}^+_{X_1,Y_1} \otimes \mathsf{i}^+_{X_2,Y_2}$ and $\mathsf{i}^-_{X_1\otimes X_2,Y_1\otimes Y_2} = \mathsf{i}^-_{X_1,Y_1} \otimes \mathsf{i}^-_{X_2,Y_2}$.
Moreover, given directed families $(X_{i,\gamma_i})_{\gamma_i\in\Gamma_i}$ of $\mathcal{L}_\sqsubseteq$ for $i = 1,2$, one has

$$\bigsqcup_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} X_{\gamma_1} \otimes X_{\gamma_2} = \bigsqcup_{\gamma_1\in\Gamma_1} X_{\gamma_1} \otimes \bigsqcup_{\gamma_2\in\Gamma_1} X_{\gamma_2}$$

— If $X_i \sqsubseteq Y_i$ for $i = 1,2$ then $X_1 \oplus X_2 \sqsubseteq Y_1 \oplus Y_2$, $\mathsf{i}^+_{X_1\otimes X_2,Y_1\oplus Y_2} = \mathsf{i}^+_{X_1,Y_1} \oplus \mathsf{i}^+_{X_2,Y_2}$ and $\mathsf{i}^-_{X_1\otimes X_2,Y_1\oplus Y_2} = \mathsf{i}^-_{X_1,Y_1} \oplus \mathsf{i}^-_{X_2,Y_2}$.
Moreover, given directed families $(X_{i,\gamma_i})_{\gamma_i\in\Gamma_i}$ of $\mathcal{L}_\sqsubseteq$ for $i = 1,2$, one has

$$\bigsqcup_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} X_{\gamma_1} \oplus X_{\gamma_2} = \bigsqcup_{\gamma_1\in\Gamma_1} X_{\gamma_1} \oplus \bigsqcup_{\gamma_2\in\Gamma_1} X_{\gamma_2}$$

— If $X \sqsubseteq Y$ then $X^\perp \sqsubseteq Y^\perp$ (warning : linear negation is covariant), $\mathsf{i}^+_{X^\perp,Y^\perp} = (\mathsf{i}^-_{X,Y})^\perp$ and $\mathsf{i}^-_{X^\perp,Y^\perp} = (\mathsf{i}^+_{X,Y})^\perp$. Moreover, given a directed family $(X_\gamma)_{\gamma\in\Gamma}$, we have $\bigsqcup({X_\gamma}^\perp) = (\bigsqcup_{\gamma\in\Gamma} X_\gamma)^\perp$.

— If $X \sqsubseteq Y$ then $!X \sqsubseteq !Y$, $\mathsf{i}^+_{!X,!Y} = !(\mathsf{i}^+_{X,Y})$ and $\mathsf{i}^-_{!X,!Y} = !(\mathsf{i}^-_{X,Y})$. Moreover, given a directed family $(X_\gamma)_{\gamma\in\Gamma}$, we have $\bigsqcup(!X_\gamma) = !(\bigsqcup_{\gamma\in\Gamma} X_\gamma)$.

It follows If $X_i \sqsubseteq Y_i$ for $i = 1,2$ then $X_1 \multimap X_2 \sqsubseteq Y_1 \multimap Y_2$, $\mathsf{i}^+_{X_1\multimap X_2,Y_1\multimap Y_2} = \mathsf{i}^-_{X_1,Y_1} \multimap \mathsf{i}^+_{X_2,Y_2}$ and $\mathsf{i}^-_{X_1\multimap X_2,Y_1\multimap Y_2} = \mathsf{i}^+_{X_1,Y_1} \multimap \mathsf{i}^-_{X_2,Y_2}$.
Moreover, given directed families $(X_{i,\gamma_i})_{\gamma_i\in\Gamma_i}$ of $\mathcal{L}_\sqsubseteq$ for $i = 1,2$, one has

$$\bigsqcup_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} (X_{\gamma_1} \multimap X_{\gamma_2}) = \bigsqcup_{\gamma_1\in\Gamma_1} X_{\gamma_1} \multimap \bigsqcup_{\gamma_2\in\Gamma_1} X_{\gamma_2}$$

We need to extend this notion of embedding-retraction pair to !-coalgebras because we want to define fix-points of positive types. Let ${\mathcal{L}^!}_\sqsubseteq$ be the partially ordered class whose elements are those of $\mathcal{L}^!$ and where $P \sqsubseteq Q$ if $\underline{P} \sqsubseteq \underline{Q}$ (in $\mathcal{L}_\sqsubseteq$ of course) and $\mathsf{i}^+_{\underline{P},\underline{Q}} \in \mathcal{L}^!(P,Q)$. In this definition, it is important *not to require* $\mathsf{i}^-_{\underline{P},\underline{Q}}$ to be a coalgebra morphism. We still use $\mathsf{U}$ for the obvious forgetful functor ${\mathcal{L}^!}_\sqsubseteq \to \mathcal{L}_\sqsubseteq$. Observe that $\otimes$ and $\oplus$ define functors $({\mathcal{L}^!}_\sqsubseteq)^2 \to {\mathcal{L}^!}_\sqsubseteq$ and that $!\_$ defines a functor $\mathcal{L}_\sqsubseteq \to {\mathcal{L}^!}_\sqsubseteq$.

Let $(P_\gamma)_{\gamma\in\Gamma}$ be a directed family in ${\mathcal{L}^!}_\sqsubseteq$. Let $X = \bigsqcup_{\gamma\in\Gamma} \underline{P_\gamma}$. We want to equip $X$ with a structure of !-coalgebra, so we need to define a morphism $h \in \mathcal{L}(X, !X)$. For this, it suffices to define, for each $\gamma \in \Gamma$, a morphism $h(\gamma) \in \mathcal{L}(\underline{P_\gamma}, !X)$ in such a way that

$$\gamma \leq \gamma' \Rightarrow h(\gamma')\, \mathsf{i}^+_{\underline{P_\gamma},\underline{P_{\gamma'}}} = h(\gamma). \tag{4.1}$$

For $\gamma \in \Gamma$, we set $h(\gamma) = !\mathsf{i}^+_{\underline{P_\gamma},X}\, \mathsf{h}_{P_\gamma}$. Condition (4.1) holds thanks to to the fact that if $P \sqsubseteq Q$ in ${\mathcal{L}^!}_\sqsubseteq$ then $\mathsf{i}^+_{\underline{P},\underline{Q}}$ is a coalgebra morphism. We know that there is exactly one morphism $h \in \mathcal{L}(X, !X)$ such that $\forall\gamma \in \Gamma\ h\, \mathsf{i}^+_{\underline{P_\gamma},X} = !(\mathsf{i}^+_{\underline{P_\gamma},})\, \mathsf{h}_{P_\gamma}$.

To prove that $\mathsf{der}_X\ h = \mathsf{Id}_X$, it suffices to prove that $\forall\gamma \in \Gamma\ \mathsf{der}_X\ h\, \mathsf{i}^+_{\underline{P_\gamma},X} = \mathsf{i}^+_{\underline{P_\gamma},X}$. We have $\forall\gamma \in \Gamma\ \mathsf{der}_X\ h\, \mathsf{i}^+_{\underline{P_\gamma},X} = \mathsf{der}_X\ !(\mathsf{i}^+_{\underline{P_\gamma},X})\, \mathsf{h}_{P_\gamma} = \mathsf{i}^+_{\underline{P_\gamma},X}\, \mathsf{der}_{\underline{P_\gamma}}\ \mathsf{h}_{P_\gamma} = \mathsf{i}^+_{\underline{P_\gamma},X}$.

**Exercice 4.6.23** Prove similarly that $\mathsf{dig}_X\ h = !h\, h$.

So $(X, h)$ is a !-coalgebra, that we denote as $\bigsqcup_{\gamma \in \Gamma} P_\gamma$. Therefore $\mathcal{L}^{!}{}_{\sqsubseteq}$ is a cpoc, with least element 0.

Given cpoc's $\mathcal{M}$ and $\mathcal{M}'$, a functional (this means "function", but acting on a class rather than on a set) is *continuous* if its is monotone and commute with countable directed lubs.

**Proposition 4.6.7** *$\otimes$ and $\oplus$ are continuous functionals from $(\mathcal{L}^{!}{}_{\sqsubseteq})^2$ to $\mathcal{L}^{!}{}_{\sqsubseteq}$. "!" is a continuous functional $!\_ : \mathcal{L}_{\sqsubseteq} \to \mathcal{L}^{!}{}_{\sqsubseteq}$. The functional $\multimap : (\mathcal{L}_{\sqsubseteq})^2 \to \mathcal{L}_{\sqsubseteq}$ is continuous.*

**Exercice 4.6.24** Prove this proposition.

**Théorème 4.6.8** *Let $\Phi : (\mathcal{L}^{!}{}_{\sqsubseteq})^{n+1} \to \mathcal{L}^{!}{}_{\sqsubseteq}$ be a continuous functional. There is a continuous functional $\overline{\mathsf{Fix}}(\Phi) : (\mathcal{L}^{!}{}_{\sqsubseteq})^n \to \mathcal{L}^{!}{}_{\sqsubseteq}$ which is equal to the functional $\Psi : (\mathcal{L}^{!}{}_{\sqsubseteq})^n \to \mathcal{L}^{!}{}_{\sqsubseteq}$ defined by $\Psi(P_1, \dots, P_n) = \Phi(P_1, \dots, P_n, \overline{\mathsf{Fix}}(\Phi)(P_1, \dots, P_n))$.*

*Démonstration.* Let $\vec{P} = (P_1, \dots, P_n)$ be a tuple of objects of $\mathcal{L}^{!}$. Consider the funciolan $\Phi_{\vec{P}} : \mathcal{L}^{!}{}_{\sqsubseteq} \to \mathcal{L}^{!}{}_{\sqsubseteq}$ defined by $\Phi_{\vec{P}}(P) = \Phi(\vec{P}, P)$. Consider the set of natural numbers equipped with the usual order relation as a directed set. We define a directed family $(P_i)_{i \in \mathbb{N}}$ in $\mathcal{L}^{!}{}_{\sqsubseteq}$ as follows. First, we set $P_i = \Phi^i_{\vec{P}}(0)$. Since $\Phi$ is monotone, we have $P_i \sqsubseteq P_{i+1}$ for all $i \in \mathbb{N}$. We set $\overline{\mathsf{Fix}}(\Phi)(\vec{P}) = \bigsqcup_{i \in \mathbb{N}} P_i$. Then the announced equality of objects holds by continuity of $\Phi$. □

**Exercice 4.6.25** Complete the proof above by showing that the functional $\overline{\mathsf{Fix}}(\Phi)$ is monotone and continuous.

4.6.12 Fixpoints. Let $X$ be an object of $\mathcal{L}$. We want to define a morphism $\overline{\mathsf{fix}} \in \mathcal{L}_!(X \Rightarrow X, X) = \mathcal{L}(!(!X \multimap X), X)$ which will be used for interpreting the fixpoint operator of PCF. This means that, given $f \in \mathcal{L}_!(Y, X \Rightarrow X)$, the following diagram should commute in $\mathcal{L}_!$ :

$$\begin{array}{ccc} Y & \xrightarrow{\quad f \quad} & X \to X \\ {\scriptstyle \langle f, \overline{\mathsf{fix}} \circ f \rangle} \downarrow & & \downarrow {\scriptstyle \overline{\mathsf{fix}}} \\ (X \to X) \mathbin{\&} X & \xrightarrow{\quad \mathsf{Ev} \quad} & X \end{array}$$

For this purpose, we define an operator

$$\mathcal{Y} : \mathcal{L}_!(X \Rightarrow X, X) \to \mathcal{L}_!(X \Rightarrow X, X)$$

as follows : given $F \in \mathcal{L}(X \Rightarrow X, X)$ we define $\mathcal{Y}(F) \in \mathcal{L}(X \Rightarrow X, X)$ as the following composition in $\mathcal{L}_!$ (not in $\mathcal{L}$, take care !) :

$$X \Rightarrow X \xrightarrow{\langle \mathsf{Id}, F \rangle} (X \Rightarrow X) \mathbin{\&} X \xrightarrow{\mathsf{Ev}} X$$

In $\mathcal{L}$ now, this morphism $\mathcal{Y}$ can be written equivalently as the following composition of morphisms

$$!(!X \multimap X) \xrightarrow{\mathsf{c}_{!X \multimap X}} !(!X \multimap X) \otimes !(!X \multimap X) \xrightarrow{\mathsf{der}_{!X \multimap X} \otimes F^!} (!X \multimap X) \otimes !X \xrightarrow{\mathsf{ev}} X$$

Then $\overline{\mathsf{fix}}$ will be chosen as a morphism such that $\mathcal{Y}(\overline{\mathsf{fix}}) = \overline{\mathsf{fix}}$, if available (often, there is an order relation around and $\overline{\mathsf{fix}}$ can be defined as least fixpoint). Indeed, if this equation hold, we have

$$\begin{aligned} \overline{\mathsf{fix}} \circ f &= \mathcal{Y}(\overline{\mathsf{fix}}) \circ f \\ &= \mathsf{Ev} \circ \langle \mathsf{Id}, \overline{\mathsf{fix}} \rangle \circ f \\ &= \mathsf{Ev} \circ \langle f, \overline{\mathsf{fix}} \circ f \rangle \end{aligned}$$

which is exactly the expected commutation.

The trick is to define $\overline{\mathsf{fix}}$ as a fixpoint of a higher type operator : this garantees that $\overline{\mathsf{fix}}$ is a morphism in the category $\mathcal{L}_!$.

This ends the presentation of the categorical material needed to interpret the CBPV language of Chapter 3.

## 4.7 Semantics of CBPV in a model of LL

We explain now how types and programs of the programming language presented in Chapter 3 can be interpreted in any model of LL.

4.7.1 INTERPRETING TYPES. First, we interpret types. With any positive type $\varphi$ and any list $\vec{\zeta} = (\zeta_1, \ldots, \zeta_k)$ of type variables containing all the free variables of $\varphi$, we associate a continuous functional $[\varphi]^!_{\vec{\zeta}} : {\mathcal{L}^!}^k_{\sqsubseteq} \to \mathcal{L}^!_{\sqsubseteq}$. Simultaneously, with any general type $\sigma$ and any list $\vec{\zeta} = (\zeta_1, \ldots, \zeta_k)$ of type variables containing all the free variables of $\sigma$, we associate a continuous functional $[\sigma]_{\vec{\zeta}} : {\mathcal{L}^!}^k_{\sqsubseteq} \to \mathcal{L}_{\sqsubseteq}$.

We set

$$
\begin{aligned}
[\zeta_i]^!_{\vec{\zeta}}(\vec{P}) &= P_i \\
[!\sigma]^!_{\vec{\zeta}}(\vec{P}) &= !([\sigma]_{\vec{\zeta}}(\vec{P})) \\
[\varphi_1 \otimes \varphi_2]^!_{\vec{\zeta}}(\vec{P}) &= [\varphi_1]^!_{\vec{\zeta}}(\vec{P}) \otimes [\varphi_2]^!_{\vec{\zeta}}(\vec{P}) \\
[\varphi_1 \oplus \varphi_2]^!_{\vec{\zeta}}(\vec{P}) &= [\varphi_1]^!_{\vec{\zeta}}(\vec{P}) \oplus [\varphi_2]^!_{\vec{\zeta}}(\vec{P})
\end{aligned}
$$

Let $\varphi$ be a positive type and $\zeta$ not occurring in $\vec{\zeta}$, then $[\varphi]^!_{\vec{\zeta},\zeta}$ is a continuous functional ${\mathcal{L}^!}^k_{\sqsubseteq} \times \mathcal{L}^!_{\sqsubseteq} \to \mathcal{L}^!_{\sqsubseteq}$. So by Theorem 4.6.8 we have a continuous functional $\overline{\mathsf{Fix}}([\varphi]^!_{\vec{\zeta},\zeta}) : {\mathcal{L}^!}^k_{\sqsubseteq} \to \mathcal{L}^!_{\sqsubseteq}$ and we set $[\mathsf{Fix}\,\zeta \cdot \varphi]^!_{\vec{\zeta}} = \overline{\mathsf{Fix}}([\varphi]^!_{\vec{\zeta},\zeta})$.

We define now the interpretation of general types :

$$
\begin{aligned}
[\varphi]_{\vec{\zeta}}(\vec{P}) &= \underline{[\varphi]^!_{\vec{\zeta}}(\vec{P})} \\
[\psi \multimap \tau]_{\vec{\zeta}}(\vec{P}) &= \underline{[\varphi]^!_{\vec{\zeta}}(\vec{P})} \multimap [\tau]_{\vec{\zeta}}(\vec{P})
\end{aligned}
$$

**Lemme 4.7.1** *With the notations above,* $[\mathsf{Fix}\,\zeta \cdot \varphi]^!_{\vec{\zeta}} = [\varphi\,[\mathsf{Fix}\,\zeta \cdot \varphi/\zeta]]^!_{\vec{\zeta}}$.

*Démonstration.* One applies Theorem 4.6.8, showing that if $\varphi$ and $\psi$ are positive types and $\vec{\zeta}$ is repetition-free and contains all the free variables of $\psi$, and if $\zeta$ is not in $\vec{\zeta}$ and is such that $\vec{\zeta},\zeta$ contains all the free variables of $\varphi$, then one has $[\varphi\,[\psi/\zeta]]^!(\vec{P}) = [\varphi]^!_{\vec{\zeta},\zeta}(\vec{P}, [\psi]^!_{\vec{\zeta}}(\vec{P}))$. This is obtained by a simple induction on $\varphi$. □

If $\varphi$ is closed, then $[\varphi]^!$ is an object of $\mathcal{L}^!$ and if $\sigma$ is closed then $[\sigma]$ is an object of $\mathcal{L}$. Notice that $[\varphi] = \underline{[\varphi]^!}$.

Let $\mathcal{P} = (x_1 : \varphi_1, \ldots, x_k : \varphi_k)$ be a typing context. Then we set $[\mathcal{P}]^! = [\varphi_1]^! \otimes \cdots \otimes [\varphi_k]^!$, which is an object of $\mathcal{L}^!$.

4.7.2 INTERPRETING TERMS. Given a term $M$ and a typing context $\mathcal{P}$ such that there is a type $\sigma$ with $\mathcal{P} \vdash M : \sigma$, observe that there is only one such type $\sigma$ and that the typinge derivation of $\mathcal{P} \vdash M : \sigma$ is uniquely determined : on says that typing is *syntax-driven*.

If $\mathcal{P} \vdash M : \sigma$, we define $[M]_{\mathcal{P}} \in \mathcal{L}([\mathcal{P}], [\sigma])$. The definition is by induction on the typing derivation, that is, by induction on $M$.

If $M = x_i$ then $\sigma = \varphi_i$ and $[M]_{\mathcal{P}}$ is defined as the following composition of morphisms in $\mathcal{L}$ :

$$
\begin{array}{ccc}
[\varphi_1] \otimes \cdots \otimes [\varphi_{i-1}] \otimes [\varphi_i] \otimes [\varphi_{i+1}] \otimes \cdots \otimes [\varphi_k] & \longrightarrow & 1 \otimes \cdots \otimes 1 \otimes [\varphi_i] \otimes 1 \otimes \cdots \otimes 1 \\
 & & \downarrow \\
 & & [\varphi_i]
\end{array}
$$

where the first morphism is obtained by tensoring weakening morphisms $\mathsf{w}_{P_j}$ for $j \neq i$ with the identity at $[\varphi_i]$ and a the second one is a monoidal isomorphism. So $[M]_{\mathcal{P}}$ is the $i$-th projection in the cartesian category $\mathcal{L}^!$ (whose cartesian product is $\otimes$). Observe that $[M]_{\mathcal{P}} \in \mathcal{L}^!([\mathcal{P}]^!, [\varphi_i]^!)$.

We denote this projection $\mathsf{pr}^{\otimes}_i$.

If $M = N^!$ with $\mathcal{P} \vdash N : \tau$ and $\sigma = !\tau$ then we know by inductive hypothesis that $[N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\tau])$ and therefore $[N]_\mathcal{P}{}^! \in \mathcal{L}^!([\mathcal{P}]^!, ![\tau])$ (we use the fact that $[\mathcal{P}] = \underline{[\mathcal{P}]^!}$ and the generalized promotion operation defined in Section 4.6.10). So we set $[M]_\mathcal{P} = [N]_\mathcal{P}{}^! \in \mathcal{L}^!([\mathcal{P}]^!, [!\tau]^!)$.

If $M = \langle M_1, M_2 \rangle$ with $\mathcal{P} \vdash M_i : \varphi_i$ for $i = 1,2$ and $\sigma = \varphi_1 \otimes \varphi_2$, then we have $[M_i]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_i])$. Then we define $[M]_\mathcal{P}$ as the following composition of morphisms :

$$[\mathcal{P}] \xrightarrow{\mathsf{c}_{[\mathcal{P}]^!}} [\mathcal{P}] \otimes [\mathcal{P}] \xrightarrow{[M_1]_\mathcal{P} \otimes [M_2]_\mathcal{P}} [\varphi_1] \otimes [\varphi_2]$$

So that $[M]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\sigma])$. Observe that if $[M_i]_\mathcal{P}$ are !-coalgebra morphisms, that is, if $[M_i]_\mathcal{P} \in \mathcal{L}^!([\mathcal{P}]^!, [\varphi_i]^!)$ for $i = 1,2$, then that $[M]_\mathcal{P} \in \mathcal{L}^!([\mathcal{P}]^!, [\varphi_1]^! \otimes [\varphi_2]^!)$.

If $M = \mathsf{in}_i N$ with $\mathcal{P} \vdash N : \varphi_i$ for $i = 1$ or $i = 2$, and $\sigma = \varphi_1 \oplus \varphi_2$ then by inductive hypothesis we have $[M]_\mathcal{P} = \overline{\pi}_i\, [N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_1] \oplus [\varphi_2])$. Observe that, if $[N]_\mathcal{P} \in \mathcal{L}^!([\mathcal{P}]^!, [\varphi_i]^!)$, then $[M]_\mathcal{P} \in \mathcal{L}^!([\mathcal{P}]^!, [\varphi_1]^! \oplus [\varphi_2]^!)$.

If $M = \lambda x^\psi\, N$ with $\mathcal{P}, x : \psi \vdash N : \tau$ and $\sigma = \varphi \multimap \tau$ so that, we have $[N]_{\mathcal{P},x:\psi} \in \mathcal{L}([\mathcal{P}] \otimes [\psi], [\tau])$ and we set $[M]_\mathcal{P} = \mathsf{cur}\,([N]_{\mathcal{P},x:\psi}) \in \mathcal{L}([\mathcal{P}], [\psi] \multimap [\tau])$.

If $M = \langle N \rangle R$ with $\mathcal{P} \vdash N : \psi \multimap \sigma$ and $\mathcal{P} \vdash R : \psi$ then we have $[N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\psi] \multimap [\tau])$ and $[R]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\psi])$ then we set $[M]_\mathcal{P} = \mathsf{ev}\,([N]_\mathcal{P} \otimes [R]_\mathcal{P})\ \mathsf{c}_{[\mathcal{P}]^!} \in \mathcal{L}([\mathcal{P}], [\sigma])$.

If $M = \mathsf{der}(N)$ with $\mathcal{P} \vdash N : !\sigma$ then we have $[N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], ![\sigma])$ and we set $[M]_\mathcal{P} = \mathsf{der}_{[\sigma]}\, [N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\sigma])$.

If $M = \mathsf{pr}_i N$ with $i = 1$ or $i = 2$, $\mathcal{P} \vdash N : \varphi_1 \otimes \varphi_2$ and $\sigma = \varphi_i$ so that $[N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_1] \otimes [\varphi_2])$ and we set $[M]_\mathcal{P} = \mathsf{pr}_i^\otimes\, [N]_\mathcal{P}$ (remember that $\mathsf{pr}_i^\otimes$ is the $i$-th projection of the cartesian product $\otimes$ of $\mathcal{L}^!$, the definition is recorded above).

If $M = \mathsf{case}(N, x_1 \cdot R_1, x_2 \cdot R_2)$ with $\mathcal{P} \vdash N : \varphi_1 \oplus \varphi_2$ and $\mathcal{P}, x_i : \varphi_i \vdash R_i : \sigma$ for $i = 1,2$, then we have $[N]_\mathcal{P} \in \mathcal{L}([\mathcal{P}], [\varphi_1] \oplus [\varphi_2])$ and $[R_i]_\mathcal{P} \in \mathcal{L}([\mathcal{P}] \otimes [\varphi_i], [\sigma])$ for $i = 1,2$. By Proposition 4.6.5, we have a canonical and natural iso

$$\chi \in \mathcal{L}^!([\mathcal{P}] \otimes ([\varphi_1] \oplus [\varphi_2]), ([\mathcal{P}] \otimes [\varphi_1]) \oplus ([\mathcal{P}] \otimes [\varphi_2]))\,.$$

We set $[M]_\mathcal{P} = [[R_1]_\mathcal{P}, [R_2]_\mathcal{P}]\, \chi(\mathcal{P} \otimes [N]_\mathcal{P})\ \mathsf{c}_{[\mathcal{P}]^!}$ (remember that $[[R_1]_\mathcal{P}, [R_2]_\mathcal{P}]$ is the "cotupling" of $[R_1]_\mathcal{P}$ and $[R_2]_\mathcal{P}$, see Section 4.6.9).

Last, if $M = \mathsf{fix}\, x^{!\sigma}\, N$ with $\mathcal{P}, x : !\sigma \vdash N : \sigma$ then we have $[N]_{\mathcal{P},x:!\sigma} \in \mathcal{L}([\mathcal{P}] \otimes ![\sigma], [\sigma])$ so that $\mathsf{cur}\,([N]_{\mathcal{P},x:!\sigma}) \in \mathcal{L}([\mathcal{P}], ![\sigma] \multimap [\sigma])$ and hence

$$(\mathsf{cur}\,([N]_{\mathcal{P},x:!\sigma}))^! \in \mathcal{L}([\mathcal{P}], !(![\sigma] \multimap [\sigma]))$$

using that fact that $[\mathcal{P}] = \underline{[\mathcal{P}]^!}$. We set $[M]_\mathcal{P} = \overline{\mathsf{fix}}\,(\mathsf{cur}\,([N]_{\mathcal{P},x:!\sigma}))^!$.

This ends the definition of the interpretation of types and terms.

The first crucial observation is the following.

**Proposition 4.7.2** *If $\mathcal{P} \vdash V : \varphi$ and $V$ is a value, then $[V]_\mathcal{P} \in \mathcal{L}^!([\mathcal{P}]^!, [\varphi]^!)$.*

The proof is by simple inspection of the definition of the interpretation of terms above, four first cases which corresponds to the value constructors.

**Proposition 4.7.3 (Substitution Lemma)** *If $\mathcal{P}, x : \varphi \vdash M : \sigma$ and $\mathcal{P} \vdash V : \varphi$, then*

$$[M\,[V/x]]_\mathcal{P} = [M]_{\mathcal{P},x:\varphi}\,(\mathcal{P} \otimes [V]_\mathcal{P})\ \mathsf{c}_{[\mathcal{P}]^!}\ .$$

**Exercice 4.7.1** Prove this proposition, by induction on $M$. The proof uses in an essential way Proposition 4.7.2 which allow $[V]_\mathcal{P}$ to commute with the structural rules and promotion boxes which appear in the interpretation of $M$.

An easy consequence of this lemma is the invariance of this interpretation under reduction.

**Théorème 4.7.4** *If $\mathcal{P} \vdash M : \sigma$ and $M \to_\mathsf{w} M'$ the $[M]_\mathcal{P} = [M']_\mathcal{P}$.*

**Exercice 4.7.2** Prove the theorem, by induction on the derivation of the reduction $M \to_\mathsf{w} M'$ in the definition of weak reduction in Section 3.2.

# Chapitre 5

# Un peu de théorie des domaines

Ce chapitre n'est pas essentiel pour la suite ; il est surtout là pour la culture générale. Il présente des notions plus générales que celles sur lesquelles nous concentrerons notre attention à partir de la section 5.3, à laquelle on peut sauter directement. On trouvera plus de détails sur ce sujet dans [AC98].

## 5.1 Domaines de Scott

Si $X$ est un ensemble partiellement ordonné (la relation d'ordre sera toujours notée $\leq$), un sous-ensemble $D$ de $X$ est dit *filtrant* s'il est filtrant pour l'ordre de $X$ restreint à $D$. Autrement dit : $D$ est non vide, et $\forall x, y \in D\, \exists z \in D\; x \leq z$ et $y \leq z$.

Si une partie $C$ de $X$ a un sup, ce sup sera toujours noté $\bigvee C$. Le sup de deux éléments $x, y \in X$, s'il existe, sera noté $x \vee y$.

Un *cpo* (*ordre partiel complet*) est un ordre partiel dans lequel toutes les parties filtrantes ont un sup. Autrement dit, toute famille croissante $(x_\gamma)_{\gamma\in\Gamma}$ d'éléments de $X$ indexée par un ensemble filtrant $\Gamma$ doit avoir un sup $\bigvee_{\gamma\in G} x_\gamma \in X$.

Un élément $x_0$ d'un cpo $X$ est dit *isolé* (on dit souvent *compact* mais ce n'est pas une très bonne terminologie) si, pour toute partie filtrante $D$ de $X$, on a

$$x_0 \leq \bigvee D \Rightarrow \exists x \in D\; x_0 \leq x$$

(la réciproque étant toujours vraie). L'intuition est qu'un élément isolé est «fini». Par exemple, si $X = \mathcal{P}(\mathbb{N})$, ordonné par l'inclusion, les isolés de $D$ sont les parties finies de $\mathbb{N}$.

**Lemme 5.1.1** *Soit $X$ un cpo et $B \subseteq X$ un ensemble fini d'éléments isolés de $X$. Si $B$ a un sup, alors $\bigvee B$ est isolé.*

*Démonstration.* Soient $x_1, \ldots, x_n$ les éléments de $B$. Soit $D \subseteq X$ une partie filtrante telle que $\bigvee B \leq \bigvee D$, c'est-à-dire que pour tout $i \in \{1, \ldots, n\}$, $x_i \leq \bigvee D$. Comme $x_i$ est isolé et $D$ est filtrant, il existe $y_i \in D$ tel que $x_i \leq y_i$. Comme $D$ est filtrant, il existe $y \in D$ tel que $y_i \leq y$ pour tout $i$. On a $\bigvee B \leq y$. □

Un *domaine de Scott* est un cpo $X$ tel que

1. toute partie bornée de $X$ a un sup (en particulier la partie vide a un sup, donc $X$ a un élément minimal, qu'on note $\perp$) ;
2. l'ensemble des éléments isolés de $X$ est dénombrable ;
3. tout élément de $x$ est le sup de l'ensemble de ses minorants isolés (on dit que $X$ est *algébrique*).

La condition correspondant à la conjonction des deux dernières conditions est souvent appelée *$\omega$-algébricité*. Observer que, par la condition (1), et par le lemme 5.1.1, l'ensemble des minorants isolés d'un élément est filtrant.

**Lemme 5.1.2** *Soit $X$ un domaine de Scott et soient $x, y \in X$. On a $x \leq y$ si et seulement si, pour tout élément isolé $x_0$ de $X$, si $x_0 \leq x$, alors $x_0 \leq y$.*

**Exercice 5.1.1** Prouver ce lemme.

**Exercice 5.1.2** Soit $X$ l'ensemble des fonctions partielles de $\mathbb{N}$ vers $\mathbb{N}$, ordonnées par l'inclusion des graphes (autrement dit, $f \leq g$ si le domaine de $f$ est contenu dans celui de $g$ et $f$ et $g$ coïncident sur le domaine de $f$). Montrer que $X$ est un domaine de Scott.

Soient $X$ et $Y$ deux ensembles partiellement ordonnés. Soit $f : X \to Y$ une fonction croissante. Si $D \subseteq X$ est filtrant dans $X$, alors $f(X)$ est filtrant dans $Y$. Supposons que $X$ et $Y$ sont des cpo. Alors $f : X \to Y$ est *continue* si

— $f$ est croissante (et donc l'image par $f$ de toute partie filtrante de $X$ est filtrante dans $Y$)

— et pour tout $D \subseteq X$ filtrant, on a $f(\bigvee D) = \bigvee f(D)$.

Observer que, dès que $f$ est croissante, on a $\bigvee f(D) \leq f(\bigvee D)$. Donc, pour montrer qu'une fonction croissante $f$ est continue, il suffit de montrer que, pour toute partie filtrante $D$, on a $f(\bigvee D) \leq \bigvee f(D)$.

L'exercice suivant est essentiel pour comprendre l'intuition derrière cette définition, dans le cas des domaines de Scott : une fonction est continue si, pour obtenir une information finie sur le résultat, il suffit d'une information finie sur l'argument.

**Exercice 5.1.3** Soient $X$ et $Y$ des domaines de Scott et soit $f : X \to Y$ une fonction. Alors $f$ est continue si et seulement si

— $f$ est croissante

— et, pour tout $x \in X$ et tout élément isolé $y_0$ de $Y$, si $y_0 \leq f(x)$, il existe un élément isolé $x_0$ de $X$ tel que $x_0 \leq x$ et $y_0 \leq f(x_0)$.

**Exercice 5.1.4** Soient $X$ et $Y$ des domaines de Scott et soit $f : X \to Y$ une fonction croissante. Montrer que $f$ est continue si et seulement si, pour toute suite croissante $(x_n)_{n\in\mathbb{N}}$ d'éléments de $X$, on a $f(\bigvee_{n=0}^{\infty} x_n) = \bigvee_{n=0}^{\infty} f(x_n)$. [ *Indication:* utiliser le fait que $X$ a un nombre au plus dénombrable d'élements isolés. ]

Soit $\mathbb{O}$ le domaine de Scott $\mathbb{O} = \{\bot < \top\}$. Soit $X$ un domaine de Scott. On dit que $U \subseteq X$ est un *ouvert de Scott* si la «fonction caractéristique» de $U$, de $X$ vers $\mathbb{O}$, qui envoie $x$ sur $\top$ si $x \in U$ et sur $\bot$ sinon, est continue. Autrement dit, $U \subseteq X$ est un ouvert de Scott si les deux conditions suivantes sont vérifiées :

— si $x \in U$ et $x \leq y$, alors $y \in U$ ;

— pour toute partie filtrante $D$ de $X$, si $\bigvee D \in U$, alors $U \cap D \neq \emptyset$.

**Exercice 5.1.5** Soit $X$ un domaine de Scott. Montrer que les ouverts de Scott définissent une topologie sur $X$, et que cette topologie est *séparée au sens* $T_0$ : si $x, y \in X$ sont distincts, alors il existe un ouvert $U$ tel que $x \in U$ et $y \notin U$, ou un ouvert $U$ tel que $y \in U$ et $x \notin U$. C'est la notion la plus faible de séparation on topologie.

**Exercice 5.1.6** Soient $X$ et $Y$ des domaines de Scott. Montrer qu'une fonction $f : X \to Y$ est continue (au sens ci-dessus) si et seulement si $f$ est continue au sens de la *topologie de Scott* qu'on vient de définir sur $X$ et sur $Y$.

## 5.2 La catégorie des domaines de Scott.

Soit **Scott** la catégorie dont les objets sont les domaines de Scott et les morphismes sont les fonctions continues (il est clair que l'identité est continue et que la composée de deux fonctions continues et continue, donc il s'agit bien d'une catégorie).

**Proposition 5.2.1** *La catégorie* **Scott** *est cartésienne.*

*Démonstration.* L'objet terminal est $\{\bot\}$. Le produit cartésien de deux domaines de Scott $X_1$ et $X_2$ est l'ensemble $X_1 \times X_2$, avec l'ordre produit ($(x_1, x_2) \leq (y_1, y_2)$ si $x_1 \leq y_1$ et $x_2 \leq y_2$). On notera aussi ce produit $X_1$ & $X_2$. Les projections sont définies de façon standard.

**Exercice 5.2.1** Vérifier que $X_1 \mathbin{\&} X_2$ est bien un domaine de Scott. Il faudra vérifier en particulier que $(x, y)$ est isolé si et seulement si $x$ et $y$ le sont.

□

La remarque suivante nous sera utile dans la suite.

**Lemme 5.2.2** *Soient $X$, $Y$ et $Z$ des domaines de Scott et soit $f : X \mathbin{\&} Y \to Z$ une fonction croissante. Alors $f$ est continue si et seulement si, pour toutes $D \subseteq X$ et $E \subseteq Y$ filtrantes, on a*

$$f(\bigvee D, \bigvee E) = \bigvee f(D \times E)\,.$$

*Démonstration.* On suppose d'abord $f$ continue est on prend $D$ et $E$ comme dans l'énoncé du lemme. Alors $D \times E$ est une partie filtrante de $X \mathbin{\&} Y$, dont le sup est clairement $(\bigvee D, \bigvee E)$. Donc on a bien l'égalité voulue.

Pour la réciproque, soit $F \subseteq X \mathbin{\&} Y$ une partie filtrante. Soit $D = \pi_1(F)$ et $E = \pi_2(F)$, ce sont des parties filtrantes de $X$ et $Y$ respectivement. On a $\bigvee F = (\bigvee D, \bigvee E)$, car les projections sont continues. On doit montrer que $f(\bigvee F) \leq \bigvee f(F)$. Or, par hypothèse,

$$f(\bigvee F) = f(\bigvee D, \bigvee E) = \bigvee f(D \times E)$$

et il s'agit donc de vérifier que $\bigvee f(D \times E) \leq f(F)$. Pour cela, il suffit de montrer que tout élément de $D \times E$ est majoré par un élément de $F$ dans $X \mathbin{\&} Y$. Si $(x, y) \in D \times E$, on peut trouver $x' \in X$ et $y' \in Y$ tels que $(x, y'), (x', y) \in F$, et comme $F$ est filtrant, on peut trouver $(x'', y'') \in F$ tel que $(x'', y'') \geq (x, y'), (x', y)$, et donc $(x'', y'') \geq (x, y)$. □

**Exercice 5.2.2** Déduire de ce lemme qu'une fonction $f : X \mathbin{\&} Y \to Z$ est continue si et seulement si elle est séparément continue (*i.e.* pour chaque $x \in X$, la fonction $y \mapsto f(x, y)$ et continue, et pour tout $y \in Y$, la fonction $x \mapsto f(x, y)$ est continue). Cette propriété est-elle vraie des fonctions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ ?

**Exercice 5.2.3** Soient $X$ et $Y$ deux domaines de Scott et $f, g : X \to Y$ deux fonctions continues. L'ensemble $Z = \{x \in X \mid f(x) = g(x)\}$, muni de la relation d'ordre induite par celle de $X$, est-il un domaine de Scott ?

**Théorème 5.2.3** *La catégorie* **Scott** *est cartésienne fermée.*

*Démonstration.* Soient $X$ et $Y$ des domaines de Scott. Soit $X \Rightarrow Y$ l'ensemble des fonctions continues de $X$ vers $Y$, muni de la relation d'ordre dite *ordre extensionnel* :

$$f \leq g \quad \text{si} \quad \forall x \in X\ f(x) \leq g(x)\,.$$

On va vérifier que $X \Rightarrow Y$, avec cette relation d'ordre, est un domaine de Scott.

Soit $\mathcal{D} \subseteq X \Rightarrow Y$ un ensemble filtrant. Pour tout $x \in X$, l'ensemble $\{f(x) \mid f \in \mathcal{D}\} \subseteq Y$ est filtrant. Soit $h : X \to Y$ définie par $h(x) = \bigvee_{f \in \mathcal{D}} f(x)$. Alors $h$ est croissante (le vérifier). Soit $D$ une partie filtrante de $X$. On doit montrer que $h(\bigvee D) \leq \bigvee_{x \in D} h(x)$.

Comme chaque $f \in \mathcal{D}$ est continue, on a

$$h(\bigvee D) = \bigvee_{f \in \mathcal{D}} \bigvee_{x \in D} f(x)\,.$$

Il suffit donc de voir que, pour tout $f \in \mathcal{D}$, on a $\bigvee_{x \in D} f(x) \leq \bigvee_{x \in D} h(x)$, ce qui est clair car $\forall x \in X\ f(x) \leq h(x)$. Par suite, $h$ est continue. D'autre part, soit $g \in X \Rightarrow Y$ tel que $g \geq f$ pour tout $f \in D$. Soit $x \in X$. On a $g(x) \geq f(x)$ pour tout $f \in D$, et donc $g(x) \geq h(x)$. Donc $g \geq h$. Par suite $h$ est le sup de $D$ dans $X \Rightarrow Y$.

Donc $X \Rightarrow Y$ est un cpo.

**Exercice 5.2.4** Montrer que $X \Rightarrow Y$ est *borné-complet* (toute partie bornée a un sup) et que le sup d'une partie bornée $\mathcal{B}$ de $X \Rightarrow Y$ est donné par

$$(\bigvee \mathcal{B})(x) = \bigvee_{f\in\mathcal{B}} f(x)\,.$$

Le raisonnement est similaire à celui qu'on vient de faire.

Soient $x_0 \in X$ et $y_0 \in Y$ isolés. Soit $[x_0, y_0]$ la *fonction de seuil* $[x_0, y_0] : X \to Y$ définie par

$$[x_0, y_0](x) = \begin{cases} y_0 & \text{si } x \geq x_0 \\ \bot & \text{sinon.} \end{cases}$$

Comme $x_0$ est isolé, cette fonction est continue. Soit $f \in X \Rightarrow Y$. On a

$$[x_0, y_0] \leq f \Leftrightarrow y_0 \leq f(x_0)\,.$$

Soit $\mathcal{D} \subseteq X \Rightarrow Y$ un ensemble filtrant, et supposons que $[x_0, y_0] \leq \bigvee \mathcal{D}$. Cela signifie que

$$y_0 \leq \bigvee_{f\in\mathcal{D}} f(x_0)$$

et donc, comme $y_0$ est isolé, il existe $f \in \mathcal{D}$ tel que $y_0 \leq f(x_0)$, c'est-à-dire $[x_0, y_0] \leq f$. Donc les fonctions de seuil sont isolées dans $X \Rightarrow Y$.

Si $f \in X \Rightarrow Y$, soit $s(f)$ l'ensemble des fonctions de seuil majorées par $f$, autrement dit

$$s(f) = \{[x_0, y_0] \mid x_0 \in \mathsf{i}(X),\ y_0 \in \mathsf{i}(Y) \text{ et } y_0 \leq f(x_0)\}\,,$$

où $\mathsf{i}(X)$ est l'ensemble des éléments isolés de $X$.

On montre que $f = \bigvee s(f)$ (ce sup existe, puisque $X \Rightarrow Y$ est borné-complet). Il suffit de montrer que $f \leq \bigvee s(f)$, l'autre inégalité étant évidente. Soit $x \in X$, on montre que $f(x) \leq \bigvee_{f_0\in s(f)} f_0(x)$. Soit $y_1$ un élément isolé de $Y$ ; par le lemme 5.1.2, il suffit de montrer que, si $y_1 \leq f(x)$, alors $y_1 \leq \bigvee_{f_0\in s(f)} f_0(x)$. On suppose donc que $y_1 \leq f(x)$. Comme $f$ est continue, il existe $x_1$, isolé dans $X$, tel que $x_1 \leq x$ et $y_1 \leq f(x_1)$. Donc $f_1 = [x_1, y_1] \in s(f)$ et comme $f_1(x) = y_1$, on en déduit que $y_1 \leq \bigvee_{f_0\in s(f)} f_0(x)$.

Soit alors $f \in X \Rightarrow Y$ et soit $\mathcal{B}$ l'ensemble des minorants de $f$ qui sont isolés dans $X \Rightarrow Y$. On a bien sûr $\bigvee \mathcal{B} \leq f$. Mais $s(f) \subseteq \mathcal{B}$ (toute fonction de seuil est isolée) et on vient de voir que $\bigvee s(f) = f$, donc $\bigvee \mathcal{B} = f$. On a ainsi montré que $X \Rightarrow Y$ est algébrique. Pour conclure que c'est un domaine de Scott, il suffit de montrer qu'il n'a qu'un nombre dénombrable d'éléments isolés.

Soit donc $f_0 \in X \Rightarrow Y$ isolé. Pour toute partie finie $\mathcal{B}$ de $s(f)$, $\bigvee \mathcal{B}$ est un élément isolé de $X \Rightarrow Y$ qui est majoré par $f_0$. De plus, si $\mathcal{B}, \mathcal{B}' \in \mathcal{P}_{\text{fin}}(s(f))$, on a $\mathcal{B} \subseteq \mathcal{B}' \Rightarrow \bigvee \mathcal{B} \leq \bigvee \mathcal{B}'$. Donc l'ensemble

$$\mathcal{D} = \{\bigvee \mathcal{B} \mid \mathcal{B} \in \mathcal{P}_{\text{fin}}(s(f_0))\}$$

est une partie filtrante de $X \Rightarrow Y$, bornée par $f_0$ et contenant $s(f)$, ces deux dernières propriétés impliquant que $f_0 = \bigvee \mathcal{D}$. Donc comme $f_0$ est isolé, il existe une partie finie $\mathcal{B}$ de $s(f_0)$ telle que $f_0 = \bigvee \mathcal{B}$.

On a ainsi montré que tout élément isolé de $X \Rightarrow Y$ est le sup d'une famille finie de fonctions de seuil. Or, comme $X$ et $Y$ sont $\omega$-algébriques, l'ensemble des fonctions de seuil est au plus dénombrable, et par conséquent, l'ensemble des fonctions isolées est au plus dénombrable (l'ensemble des parties finies d'un ensemble dénombrable est dénombrable).

Soit $\mathsf{Ev} : (X \Rightarrow Y) \,\&\, X \to Y$ définie par $\mathsf{Ev}(f, x) = f(x)$. Cette fonction est clairement croissante. Pour montrer qu'elle est continue, on applique le lemme 5.2.2. Soient $\mathcal{D} \subseteq X \Rightarrow Y$ et $D \subseteq X$ filtrantes. On a

$$\begin{aligned}
(\bigvee \mathcal{D})(\bigvee D) &= \bigvee_{f\in\mathcal{D}} f(\bigvee D) \\
&= \bigvee_{f\in\mathcal{D}} \bigvee_{x\in D} f(x) \\
&= \bigvee \mathsf{Ev}(\mathcal{D} \times D)
\end{aligned}$$

et on conclut.

Soient $X$, $Y$ et $Z$ des domaines de Scott et soit $f : Z \,\&\, X \to Y$ une fonction continue. Pour chaque $z \in Z$, la fonction $f_z : X \to Y$ définie par $f_z(x) = f(z, x)$ est continue, et on définit $\mathsf{Cur}(f) : Z \to (X \Rightarrow Y)$ par $\mathsf{Cur}(f)(z) = f_z$.

**Exercice 5.2.5** Vérifier que $\mathsf{Cur}(f)$ est continue. Vérifier que les trois équations de clôture cartésienne sont satisfaites.

On a ainsi prouvé que **Scott** est une catégorie cartésienne fermée. □

## 5.3 Treillis complets premier-algébriques

On va travailler avec des domaines de Scott très particuliers, qui sont les treillis complets premier-algébriques. Ils ont d'excellentes propriétés de symétrie (un peu comme les espaces de cohérence), que les domaines de Scott généraux ne possèdent pas.

Un *treillis complet* est un ensemble partiellement ordonné dont tout sous-ensemble a un sup. Soit $E$ un treillis complet. Si $A \subseteq E$, on note $\bigvee A$ le sup de $A$ dans $E$. On note $\bot_E$ le plus petit élément de $E$ (qui existe, c'est $\bigvee \emptyset$) et $\top_E$ le plus grand, qui est $\bigvee E$.

Un élément $p$ de $E$ est *premier* si, pour tout $A \subseteq E$, si $p \leq \bigvee A$, alors il existe $x \in A$ tel que $p \leq x$. On dit que $E$ est *premier-algébrique* si tout élément de $E$ est le sup de ses minorants premiers. On dénote par $\mathsf{Pr}\, E$ l'ensemble des éléménts premiers de $E$, considéré comme ordre partiel (avec la restriction de l'ordre de $E$).

5.3.1 Représentation des treillis complets premier-algébriques. Soit $S = (|S|, \leq_S)$ un ensemble préordonné ($|S|$ est la trame de $S$ et $\leq_S$ est une relation binaire, transitive et réflexive sur $|S|$). La relation sur $|S|$ définie par $a \sim_S b$ si $a \leq_S b$ et $b \leq_S a$ est alors une relation d'équivalence, sur les classes d'équivalence de laquelle $\leq_S$ induit une relation d'ordre.

On note $\mathcal{I}(S)$ l'ensemble des *segments initiaux* de $|S|$, c'est-à-dire, des sous-ensembles $u$ de $|S|$ tels que $\forall a \in u, \forall b \in |S|\ b \leq a \Rightarrow b \in u$. Noter que si $a \in u \in \mathcal{I}(S)$ et si $b \sim_S a$ ($\sim_S$ étant la relation d'équivalence définie ci-dessus), alors $b \in u$ : les éléments de $\mathcal{I}(S)$ sont clos pour la relation d'équivalence associée au préordre de $S$.

Muni de l'inclusion comme relation d'ordre, $\mathcal{I}(S)$ est un treillis complet : une réunion quelconque d'éléments de $\mathcal{I}(S)$ est un élément de $\mathcal{I}(S)$. Si $u \subseteq |S|$, on note $\downarrow u$ le plus petit élément de $\mathcal{I}(S)$ contenant $u$, on a donc

$$\downarrow u = \{b \in |S| \mid \exists a \in u\ b \leq a\}\,.$$

Si $a \in |S|$, on note $\downarrow a = \downarrow \{a\}$.

**Lemme 5.3.1** $(\mathcal{I}(S), \subseteq)$ *est un treillis complet premier-algébrique dont les éléments premiers sont les* $\downarrow a$*, pour* $a \in |S|$.

**Exercice 5.3.1** Prouver ce lemme.

**Proposition 5.3.2** *Soit $E$ un treillis complet premier-algébrique. La fonction $\varphi$ qui envoie $x \in E$ sur $\{p \in \mathsf{Pr}\, E \mid p \leq x\}$ est un isomorphisme entre $E$ et $\mathcal{I}(\mathsf{Pr}\, E)$.*

*Démonstration.* Il suffit de montrer que $\varphi$ est une bijection croissante dont la récipoque est aussi croissante. La fonction $\varphi$ est croissante car si $p \leq x \leq y$, alors $p \leq y$ et donc $\varphi(x) \subseteq \varphi(y)$.

Soit $\psi : \mathcal{I}(\mathsf{Pr}\, E) \to E$ définie par $\psi(u) = \bigvee u$. Si $x \in E$, alors $\psi(\varphi(x)) = \bigvee\{p \in \mathsf{Pr}\, E \mid p \leq x\} = x$ car $E$ est premier-algébrique. Soit $u \in \mathcal{I}(\mathsf{Pr}\, E)$. Si $p \in u$, on a $p \leq \bigvee u = \psi(u)$, donc $p \in \varphi(\psi(u))$, et donc $u \subseteq \varphi(\psi(u))$. Réciproquement, soit $p \in \mathsf{Pr}\, E$ tel que $p \in \varphi(\psi(u))$, c'est-à-dire $p \leq \bigvee u$. Comme $p$ est premier, il existe $q \in u$ tel que $p \leq q$, et comme $u \in \mathcal{I}(\mathsf{Pr}\, E)$ ($u$ est un segment initial), on a $p \in u$.

Donc $\psi$ est l'inverse de $\varphi$, ce qui montre que $\varphi$ est une bijection, et comme $\psi$ est clairement croissante, $\varphi$ est un isomorphisme d'ordres partiels entre $E$ et $\mathcal{I}(\mathsf{Pr}\, E)$. □

Donc tout treillis complet premier-algébrique $E$ est de la forme $\mathcal{I}(S)$, pour un préordre $S$. La proposition ci-dessus montre qu'on peut prendre pour $S$ un ordre partiel (l'ordre induit sur les premiers de

$E$). Mais la construction des exponentielles qu'on va voir en section 6.4 montre qu'il est plus commode de considérer des préordres quelconques.

**Exercice 5.3.2** Décrire $\mathcal{I}(S)$ dans les cas suivants :

1. $|S| = \mathbb{N}$ avec l'ordre discret ($n \leq_S m$ si $n = m$)
2. $|S| = \mathbb{N}$ avec l'ordre usuel sur les entiers ($n \leq_S m$ si $n \leq m$)
3. $|S| = \mathbb{N}$ avec l'opposé de l'ordre usuel sur les entiers ($n \leq_S m$ si $m \leq n$.)

5.3.2 Morphismes linéaires. Soient $E$ et $F$ deux treillis complets. Une fonction $f : E \to F$ est linéaire si, pour toute partie $A$ de $E$, on a $f(\bigvee A) = \bigvee f(A)$ (où, comme d'habitude, $f(A)$ est l'image directe de $A$ par $f$, c'est-à-dire l'ensemble $\{f(x) \mid x \in A\}$). Il est clair que la composée de deux fonctions linéaires est linéaire et que l'identité est une fonction linéaire, donc les treillis complets et les fonctions linéaires forment une catégorie.

Remarquer que si $f : E \to F$ est linéaire, alors $f(\perp_E) = \perp_F$, mais par contre on n'a pas forcément $f(\top_E) = \top_F$. Remarquer aussi que $f$ est croissante, car si $x \leq y$ dans $E$, on a $y = x \vee y$ et donc $f(y) = f(x \vee y) = f(x) \vee f(y)$.

# Chapitre 6

# Modèle de Scott et modèle relationnel

Dans ce chapitre, on présente deux modèles fondamentaux de la logique linéaire : le modèle de Scott et le modèle relationnel. On les décrit dans un cadre commun : la catégorie des ordres partiels **PoLR**. La catégorie des ensembles et des relations est juste la sous-catégorie de **PoLR** formée des préordres "discrets" (la relation de préordre est juste la diagonale). Si on n'est intéressé que par cette catégorie, on peut sauter directement à a Section 6.7.

Un *ensemble préordonné* (ou préordre) est une structure $S = (|S|, \leq_S)$ où $|S|$ est un ensemble (qu'il est raisonnable de supposer au plus dénombrable, même si ce n'est pas techniquement indispensable) et $\leq_S$ est une relation binaire, réflexive et transitive sur $|S|$. On définit alors

$$\mathcal{I}(S) = \{u \subseteq |S| \mid \forall a \in u \, \forall a' \in |S| \; a' \leq_S a \Rightarrow a' \in u\}\,.$$

Autrement dit, $\mathcal{I}(S)$ est l'ensemble des sous-ensembles de $|S|$ qui sont "clos vers le bas" pour la relation $\leq_S$. Ces sous-ensembles de $|S|$ sont parfois appelés *segments initiaux*.

Si $u \subseteq |S|$, on définit

$$\downarrow u = \{a \in |S| \mid \exists a' \in u \; a \leq_S a'\}$$

et on a bien sûr $\downarrow u \in \mathcal{I}(S)$ (c'est le plus petit élément de $\mathcal{I}(S)$ qui contient $u$). Si $a \in |S|$, on écrire souvent $\downarrow a$ au lieu de $\downarrow \{a\}$. Donc $\downarrow a = \{a' \in |S| \mid a' \leq_S a\}$.

Muni de l'inclusion, $\mathcal{I}(S)$ est un treillis complet, ce qui signifie simplement que si $(u_i)_{i\in I}$ est une famille quelconque d'éléments de $\mathcal{I}(S)$, on a $\bigcup_{i\in I} u_i \in \mathcal{I}(S)$. En particulier, il y a un plus petit éléments qui est $\emptyset$ et un plus grand élément qui est $|S|$.

Soient $S$ et $T$ des ensembles préordonnés. Une fonction $f : \mathcal{I}(S) \to \mathcal{I}(T)$ est *linéaire* si elle commute à toutes les unions, autrement dit, pour toute famille $(u_i)_{i\in I}$ déléments de $\mathcal{I}(S)$, on a

$$f(\bigcup_{i\in I} u_i) = \bigcup_{i\in I} f(u_i)\,.$$

Une telle fonction est forcément croissante par rapport à $\subseteq$. En effet, si $u_1, u_2 \in \mathcal{I}(S)$ vérifient $u_1 \subseteq u_2$, on a $u_1 \cup u_2 = u_2$ et donc $f(u_2) = f(u_1 \cup u_2) = f(u_1) \cup f(u_2)$ et donc $f(u_1) \subseteq f(u_2)$. Observe also that $f(\emptyset) = \emptyset$, but of course it is not true that necessarily $f(|S|) = |T|$ !

## 6.1 La catégorie PoL

C'est la catégorie dont les objets sont les ensembles préordonnés, et les morphismes sont les fonctions linéaires entre treillis complets associés. Plus précisément, un objet de **PoL** est un ensemble préordonné $S$ et un morphisme $f \in \mathbf{PoL}(S, T)$ est une fonction linéaire $\mathcal{I}(S) \to \mathcal{I}(T)$.

Si $S$ et $T$ sont des préordres, $S^{\mathsf{op}}$ désigne le préordre opposé (autrement dit $|S^{\mathsf{op}}| = |S|$, $a \leq_{S^{\mathrm{op}}} a'$ si $a' \leq_S a$) et $S \times T$ désigne le préordre produit, avec $|S \times T| = |S| \times |T|$, avec $(a, b) \leq_{S\times T} (a', b')$ si $a \leq_S a'$ et $b \leq_T b'$.

6.1.1 Trace linéaire associée à une fonction linéaire. Soient $S$ et $T$ des péordres et soit $f \in \mathbf{PoL}(S,T)$. On définit la *trace linéaire* de $f$

$$\mathsf{tr}(f) = \{(a,b) \in |S| \times |T| \mid b \in f(\downarrow a)\}\,.$$

On vérifie que $\mathsf{tr}(f) \in \mathcal{I}(S^{\mathrm{op}} \times T)$. Soit $(a,b) \in \mathsf{tr}(f)$ et soit $(a',b') \in S \times T$ tel que $(a',b') \leq_{S^{\mathrm{op}} \times T} (a,b)$, c'est-à-dire $a \leq_S a'$ et $b' \leq_T b$. On a $\downarrow a \subseteq \downarrow a'$ donc $f(\downarrow a) \subseteq f(\downarrow a')$ et par suite $b \in f(\downarrow a')$, donc $(a',b') \in \mathsf{tr}(f)$ puisque $f(\downarrow a')$ est un segment initial.

On a en particulier $\mathsf{tr}(\mathsf{Id}_S) = \{(a,a') \in |S| \times |S| \mid a' \leq_S a\}$.

Let $s \subseteq E \times F$ and $t \subseteq F \times G$ be relations. Their composition $t\,s \subseteq E \times G$ is defined as

$$t\,s = \{(a,c) \in E \times G \mid \exists b \in F\ (a,b) \in s \text{ and } (b,c) \in t\}\,.$$

**Lemme 6.1.1** *Soient $f \in \mathbf{PoL}(S,T)$ et $g \in \mathbf{PoL}(T,U)$. On a $\mathsf{tr}(g \circ f) = \mathsf{tr}(g)\,\mathsf{tr}(f)$.*

*Démonstration.* Soit $(a,c) \in \mathsf{tr}(g \circ f)$, c'est-à-dire $c \in g(y)$ où $y = f(\downarrow a)$. Comme $g$ est linéaire et comme $y = \bigcup_{b \in y} \downarrow b$, on a $g(y) = \bigcup_{b \in y} g(\downarrow b)$, donc il existe $b \in y$ tel que $c \in g(\downarrow b)$, c'est-à-dire $(b,c) \in \mathsf{tr}(g)$. Comme $b \in f(\downarrow a)$, on a $(a,b) \in \mathsf{tr}(f)$. Donc $(a,c) \in \mathsf{tr}(g)\,\mathsf{tr}(f)$. Réciproquement, soit $(a,c) \in \mathsf{tr}(g)\,\mathsf{tr}(f)$. Soit $b \in |T|$ tel que $(a,b) \in \mathsf{tr}(f)$ et $(b,c) \in \mathsf{tr}(g)$. On a $b \in f(\downarrow a)$, donc $\downarrow b \subseteq f(\downarrow a)$ et donc $g(\downarrow b) \subseteq g(f(\downarrow a))$ par croissance de $g$. Or $c \in g(\downarrow b)$, ce qui montre que $c \in g(f(\downarrow a))$, c'est-à-dire $c \in \mathsf{tr}(g \circ f)$. □

6.1.2 Fonction linéaire associée à une relation. Soit $t \in \mathcal{I}(S^{\mathrm{op}} \times T)$. On définit la fonction

$$\begin{aligned} \mathsf{fun}(t) : \mathcal{I}(S) &\to \mathcal{P}(T) \\ u &\mapsto \{b \mid \exists a \in u\ (a,b) \in t\}\,. \end{aligned}$$

Soit $b \in \mathsf{fun}(t)(x)$ et soit $b' \in |T|$ tel que $b' \leq b$. Soit $a \in |S|$ tel que $(a,b) \in t$. On a $(a,b') \leq_{S^{\mathrm{op}} \times T} (a,b)$, donc $(a,b') \in t$. Par suite $b' \in \mathsf{fun}(t)(x)$, et donc $\mathsf{fun}(t)$ prend ses valeurs dans $\mathcal{I}(T)$. De plus, cette fonction est linéaire, car si $A \subseteq \mathcal{I}(S)$ et si $b \in |T|$, on a $b \in \mathsf{fun}(t)(\bigcup A)$ si et seulement s'il existe $a \in \bigcup A$ tel que $(a,b) \in t$, ce qui est équivalent à dire qu'il existe $u \in A$ tel que $a \in u$ et $(a,b) \in t$, et cette dernière propriété est équivalente à dire que $b \in \bigcup\{\mathsf{fun}(t)(u) \mid u \in A\}$.

6.1.3 Un isomorphisme d'ordre. Donc $\mathsf{fun}(t) \in \mathbf{PoL}(S,T)$. On considère $\mathbf{PoL}(S,T)$ comme un ensemble partiellement ordonné, par l'ordre ponctuel (dit parfois aussi ordre extensionnel) : $f \leq g$ si $\forall u \in \mathcal{I}(S)\ f(u) \subseteq g(u)$.

**Proposition 6.1.2** *Les fonctions $\mathsf{tr}$ et $\mathsf{fun}$ sont inverses l'une de l'autre et définissent un isomorphisme d'ordre entre $\mathbf{PoL}(S,T)$ et $\mathcal{I}(S^{\mathrm{op}} \times T)$.*

*Démonstration.* Soit $f \in \mathbf{PoL}(S,T)$. On montre que $\mathsf{fun}(\mathsf{tr}(f)) = f$. Soit donc $u \in \mathcal{I}(S)$. Soit $b \in f(u)$. Comme $u = \bigcup_{a \in u} \downarrow a$ et comme $f$ est linéaire, on a $f(u) = \bigcup_{a \in x} f(\downarrow a)$. Donc, il existe $a \in u$ tel que $b \in f(\downarrow a)$, c'est-à-dire $(a,b) \in \mathsf{tr}(f)$. Par suite $b \in \mathsf{fun}(\mathsf{tr}(f))(u)$. Réciproquement, soit $b \in \mathsf{fun}(\mathsf{tr}(f))(u)$. Soit donc $a \in u$ tel que $(a,b) \in \mathsf{tr}(f)$. Cela signifie que $b \in f(\downarrow a)$, et comme $\downarrow a \subseteq u$ et comme $f$ est croissante, on a $b \in f(u)$.

Soit maintenant $t \in \mathcal{I}(S^{\mathrm{op}} \times T)$, montrons que $\mathsf{tr}(\mathsf{fun}(t)) = t$. Soit donc $(a,b) \in |S| \times |T|$. Supposons que $(a,b) \in t$ et montrons que $(a,b) \in \mathsf{tr}(\mathsf{fun}(t))$. Comme $a \in \downarrow a$, on a $b \in \mathsf{fun}(t)(\downarrow a)$ et donc $(a,b) \in \mathsf{tr}(\mathsf{fun}(t))$. Réciproquement, supposons $(a,b) \in \mathsf{tr}(\mathsf{fun}(t))$. Cela signifie que $b \in \mathsf{fun}(t)(\downarrow a)$, c'est-à-dire qu'il existe $a' \in \downarrow a$ tel que $(a',b) \in t$. On a $a' \leq a$ et donc $(a,b) \leq_{S^{\mathrm{op}} \times T} (a',b)$ et comme $t \in \mathcal{I}(S^{\mathrm{op}} \times T)$, on a $(a,b) \in t$.

Les fonctions $\mathsf{fun}$ et $\mathsf{tr}$ sont donc inverses l'une de l'autre. Montrons que $\mathsf{fun}$ est croissante. Soient donc $s,t \in \mathcal{I}(S^{\mathrm{op}} \times T)$ tels que $s \subseteq t$, on montre que $\mathsf{fun}(s) \leq \mathsf{fun}(t)$. Soit donc $u \in \mathcal{I}(S)$ et soit $b \in \mathsf{fun}(s)(u)$. Cela signifie qu'il existe $a \in u$ tel que $(a,b) \in s$ tel que $a \in u$. Comme $s \subseteq t$, on a $(a,b) \in t$ et par suite $b \in \mathsf{fun}(t)(u)$, ce qui montre que $\mathsf{fun}(s) \leq \mathsf{fun}(t)$. Réciproquement, soient $f,g \in \mathbf{PoL}(S,T)$ tels que $f \leq g$ et montrons que $\mathsf{tr}(f) \subseteq \mathsf{tr}(g)$. Soit $(a,b) \in \mathsf{tr}(f)$. Cela signifie que $b \in f(\downarrow a)$. Comme $f \leq g$, on a $f(\downarrow a) \subseteq g(\downarrow a)$ et donc $b \in g(\downarrow a)$, c'est-à-dire que $(a,b) \in \mathsf{tr}(g)$. Donc $\mathsf{tr}(f) \subseteq \mathsf{tr}(g)$.

On a montré que $\mathsf{fun}$ et $\mathsf{tr}$ définissent un isomorphisme entre ensembles partiellement ordonnés. □

Il découle de ce qui précède que $\mathbf{PoL}(S,T)$ est un treillis complet.

**Exercise 6.1.1** Montrer que, si $\mathcal{F} \subseteq \mathbf{PoL}(S,T)$, alors $f = \bigvee \mathcal{F}$ (le sup dans ce treillis complet) est donné par $f(u) = \bigcup_{g\in\mathcal{F}} g(u)$.

**Remarque 6.1.3** On peut donc, au choix, considérer un morphisme de $S$ vers $T$ dans **PoL** soit comme une fonction linéaire de $\mathcal{I}(S)$ vers $\mathcal{I}(T)$, soit comme un élément de $\mathcal{I}(S^{\mathrm{op}} \times T)$. On passera d'un point de vue à l'autre selon le contexte. On pose $S \multimap T = S^{\mathrm{op}} \times T$.

6.1.4 Isomorphismes forts. Par définition, un isomorphisme de $S$ vers $T$ est une fonction linéaire $f : \mathcal{I}(S) \to \mathcal{I}(T)$ qui est bijective, croissante, et dont la réciproque est aussi croissante. Deux ensembles préordonnés peuvent être isomorphes tout en étant très différents. Par exemple, si $S$ a un seul élément, et si $|T| = \mathbb{N}$ (l'ensemble des entiers naturels), avec le préordre tel que $n \leq m$ pour tous $n, m \in \mathbb{N}$, alors $\mathcal{I}(S)$ et $\mathcal{I}(T)$ sont isomorphes (et isomorphes à l'ordre partiel $\{\bot, \top\}$ avec $\bot < \top$). Toutefois, dans la suite, on aura souvent affaire à des isomorphismes beaucoup plus restrictifs.

On appelle *isomorphisme fort* de $S$ vers $T$ une application $\theta : |S| \to |T|$ qui est une bijection telle que, pour tous $a, a' \in |S|$, on ait $a \leq a'$ si et seulement si $\theta(a) \leq \theta(a')$. Autrement dit, un isomorphisme fort est un isomorphisme de préordres. On écrit $S \simeq T$ s'il existe un isomorphisme fort de $S$ vers $T$.

Un tel isomorphisme fort $\theta : S \to T$ induit un isomorphisme de $S$ vers $T$ dans la catégorie **PoL**, à savoir la fonction $\widehat{\theta} : \mathcal{I}(S) \to \mathcal{I}(T)$ défine par $\widehat{\theta}(x) = \{\theta(a) \mid a \in x\}$.

**Exercise 6.1.2** Vérifier que, si $\theta : S \to T$ est un isomorphisme fort, alors $\widehat{\theta}$ est bien à valeur dans $\mathcal{I}(T)$, que c'est un isomorphisme d'ordres partiels et que $\widehat{\theta^{-1}} = \widehat{\theta}^{-1}$. Vérifier également que $\mathsf{tr}(\widehat{\theta}) = \downarrow_{S^{\mathrm{op}}\times T} \mathsf{Gr}(\theta)$ (où $\mathsf{Gr}(\theta) \subseteq |S| \times |T|$ est le graphe de la fonction $\theta$), c'est-à-dire, $\mathsf{tr}(\widehat{\theta}) = \{(a,b) \in |S| \times |T| \mid b \leq \theta(a)\}$.

## 6.2 Structure monoïdale

A partir de maintenant, nous verrons les morphismes de $S$ vers $T$ comme les éléments de $\mathcal{I}(S \multimap T)$. Pour éviter les confusions, nous noterons **PoLR** la catégorie dont les objets sont les préordres et telle que $\mathbf{PoLR}(S,T) = \mathcal{I}(S \multimap T)$, l'identité $\mathsf{Id}_S$ sur $S$ étant donné par $\mathsf{Id}_S = \{(a,a') \in S \multimap S \mid a' \leq a\}$ et la composition étant définie de façon relationnelle : si $s \in \mathcal{I}(S \multimap T)$ et $t \in \mathcal{I}(T \multimap U)$, la composition de $s$ et $t$ est

$$t\,s = \{(a,c) \in S \multimap U \mid \exists b \in |T|\ (a,b) \in s \text{ et } (b,c) \in t\}\,.$$

Si $s \in \mathbf{PoLR}(S,T)$ et $u \in \mathcal{I}(S)$, l'application de $s$ (vue comme fonction linéaire) à $x$ est notée $s\,u$. On a donc

$$s\,u = \mathsf{fun}(s)(u) = \{b \in |T| \mid \exists a \in u\ (a,b) \in |S|\}\,.$$

**Exercise 6.2.1** Vérifier directement que $\mathsf{Id}$ ainsi défini est bien l'élément neutre de la composition, à gauche et à droite. Vérifier aussi que $(t\,s)\,u = t\,(s\,u)$.

On a montré à la section 6.1 que les catégories **PoL** et **PoLR** sont isomorphes.

6.2.1 Le foncteur produit tensoriel. On note 1 le préordre qui n'a qu'un élément (que l'on notera $*$). Si $S_1$ et $S_2$ sont des préordres, on définit $S_1 \otimes S_2 = S_1 \times S_2$, muni du préordre produit. Si $u_i \in \mathcal{I}(S_i)$, on pose $u_1 \otimes u_2 = u_1 \times u_2$, et il est clair que $u_1 \otimes u_2 \in S_1 \otimes S_2$.

Soient $s_i \in \mathbf{PoLR}(S_i, T_i)$ pour $i = 1,2$. On définit

$$s_1 \otimes s_2 = \{((a_1,a_2),(b_1,b_2)) \mid (a_i,b_i) \in s_i \text{ pour } i = 1,2\} \subseteq (S_1 \otimes S_2) \multimap (T_1 \otimes T_2)\,.$$

**Exercise 6.2.2** Montrer que $s_1 \otimes s_2 \in \mathbf{PoLR}(S_1 \otimes S_2, T_1 \otimes T_2)$.

**Proposition 6.2.1** *Soient $s_i \in \mathbf{PoLR}(S_i,T_i)$ et $t_i \in \mathbf{PoLR}(T_i,U_i)$ pour $i = 1,2$. On a*

$$(t_1 \otimes t_2)\,(s_1 \otimes s_2) = (t_1\,s_1) \otimes (t_2\,s_2)\,.$$

*On a aussi $\mathsf{Id}_{S_1} \otimes \mathsf{Id}_{S_2} = \mathsf{Id}_{S_1 \otimes S_2}$. Si $u_i \in \mathcal{I}(S_i)$ (pour $i = 1,2$), on a $(s_1 \otimes s_2)\,(u_1 \otimes u_2) = (s_1\,u_1) \otimes (s_2\,u_2)$.*

*Démonstration.* On montre la dernière équation, les deux autres sont laissées au lecteur. Soit $(b_1, b_2) \in (s_1 \otimes s_2)\,(u_1 \otimes u_2)$. Il existe $(a_1, a_2) \in u_1 \otimes u_2$ tel que $((a_1, a_2), (b_1, b_2)) \in s_1 \otimes s_2$. Cela signifie que $a_i \in u_i$ et $(a_i, b_i) \in s_i$ pour $i = 1, 2$. On a donc $b_i \in s_i\, u_i$ pour $i = 1, 2$, et donc $(b_1, b_2) \in (s_1\, u_1) \otimes (s_2\, u_2)$.

Réciproquement, si $(b_1, b_2) \in (s_1\, u_1) \otimes (s_2\, u_2)$, il existe $a_i \in u_i$ tel que $(a_i, b_i) \in s_i$ pour $i = 1, 2$. On a $(a_1, a_2) \in u_1 \otimes u_2$ et $((a_1, a_2), (b_1, b_2)) \in s_1 \otimes s_2$. Donc $(b_1, b_2) \in (s_1 \otimes s_2)\,(u_1 \otimes u_2)$. □

Donc $\otimes$ est un foncteur $\mathbf{PoLR}^2 \to \mathbf{PoLR}$. On définit facilement des isomorphismes forts $\lambda : 1 \otimes S \to S$, $\rho : S \otimes 1 \to S$, $\alpha : (S_1 \otimes S_2) \otimes S_3 \to S_1 \otimes (S_2 \otimes S_3)$ et $\sigma : S_1 \otimes S_2 \to S_2 \otimes S_1$, et on vérifie facilement que ces isomorphismes forts satisfont les commutations de diagrammes voulues pour faire de $(\mathbf{PoLR}, \otimes, 1, \lambda, \rho, \alpha, \sigma)$ une catégorie monoïdale symétrique.

Comme tout foncteur, $\otimes$ envoie les isomorphismes sur des isomorphismes. On a mieux.

**Lemme 6.2.2** *Si $S \simeq S'$ et $T \simeq T'$, alors $S \otimes T \simeq S' \otimes T'$.*

La vérification est immédiate. En toute rigueur, si $\eta : S \to S'$ et $\theta : T \to T'$ sont des isomorphismes fort, il faut vérifier que l'isomorphisme fort $S \otimes T \to S' \otimes T'$ associés (notons-le $\eta \otimes \theta$) vérifie $\widehat{\eta \otimes \theta} = \widehat{\eta} \otimes \widehat{\theta}$. Là aussi, la vérification est immédiate.

6.2.2 Propriété universelle du produit tensoriel. Soit $S$, $T$ et $U$ des préordres. Une fonction $f : \mathcal{I}(S) \times \mathcal{I}(T) \to \mathcal{I}(U)$ est dite *bilinéaire* si elle est croissante et vérifie

$$\forall A \subseteq \mathcal{I}(X)\, \forall v \in \mathcal{I}(Y) \quad f(\bigcup A, v) = \bigcup_{u \in A} f(u, v)$$

et

$$\forall u \in \mathcal{I}(X)\, \forall B \subseteq \mathcal{I}(Y) \quad f(u, \bigcup B) = \bigcup_{v \in B} f(u, v)\,.$$

Comme d'habitude, à cause de la croissance de $f$, ces égalités sont équivalentes à des inclusions, à savoir $f(\bigcup A, v) \subseteq \bigcup_{u \in A} f(u, v)$ et $f(u, \bigcup B) \subseteq \bigcup_{v \in B} f(u, v)$.

Soit $\tau : \mathcal{I}(S) \times \mathcal{I}(T) \to \mathcal{I}(S \otimes T)$ la fonction définie par $\tau(u, v) = u \times v = u \otimes v$.

**Lemme 6.2.3** *La fonction $\tau$ est bilinéaire.*

*Démonstration.* On a $(a, b) \in (\bigcup A) \times v$ si et seulement s'il existe $u \in A$ tel que $(a, b) \in u \times v$. □

On peut maintenant exprimer la propriété universelle du produit tensoriel.

**Proposition 6.2.4** *Soit $f : \mathcal{I}(S) \times \mathcal{I}(T) \to \mathcal{I}(U)$ une fonction bilinéaire. Il existe une et une seule fonction linéaire $g : \mathcal{I}(S \otimes T) \to \mathcal{I}(U)$ telle que $f = g \circ \tau$.*

*Démonstration.* On définit une notion de *trace* pour une fonction bilinéaire :

$$\mathsf{tr_b}(f) = \{((a, b), c) \in (|S| \times |T|) \times |U| \mid c \in f(\downarrow_S a, \downarrow_T b)\}\,.$$

Par croissance de $f$, on a $\mathsf{tr_b}(f) \in \mathcal{I}((S^{\mathrm{op}} \times T^{\mathrm{op}}) \times U) = \mathcal{I}((S \otimes T) \multimap U)$. Soit $g = \mathsf{fun}(\mathsf{tr_b}(f)) : \mathcal{I}(S \otimes T) \to \mathcal{I}(U)$, c'est une fonction linéaire.

Pour toute fonction bilinéaire $h : \mathcal{I}(S) \times \mathcal{I}(T) \to \mathcal{I}(U)$, on a

$$h(u, v) = \bigcup_{(a,b) \in u \times v} h(\downarrow_S a, \downarrow_T b)$$

car $u = \bigcup_{a \in u} \downarrow_S a$ et $v = \bigcup_{b \in v} \downarrow_T b$ pour tous $u \in \mathcal{I}(S)$ et $v \in \mathcal{I}(T)$. Donc la fonction $h$ est connue dès qu'on connaît ses valeurs sur les éléments $(\downarrow_S a, \downarrow_T b)$ de $\mathcal{I}(S) \times \mathcal{I}(T)$.

Or $g \circ \tau$ est bilinéaire car $g$ est linéaire. On a

$$\begin{aligned}
(g \circ \tau)(\downarrow_S a, \downarrow_T b) &= g(\downarrow_S a \times \downarrow_T b) \\
&= g(\downarrow_{S \otimes T} (a, b)) \\
&= \{c \mid \exists (a', b') \leq_{S \times T} (a, b) \ \ ((a', b'), c) \in \mathsf{tr_b}(f)\} \\
&= \{c \mid \exists (a', b') \leq (a, b) \ \ c \in f(\downarrow_S a', \downarrow_T b')\} \\
&= f(\downarrow_S a, \downarrow_T b)
\end{aligned}$$

donc $g \circ \tau = f$. Il reste à voir que $g$ est caractérisée de façon unique par cette propriété. Or $g$ est linéaire sur $S \otimes T$ et est donc caractérisée par sa valeur sur les $\downarrow_{S\otimes T}(a, b)$, et on a $g(\downarrow_{S\otimes T}(a, b)) = f(\downarrow_S a, \downarrow_T b)$ comme on vient de le voir. □

**Remarque 6.2.5** Même si on a caractérisé $\otimes$ par une propriété universelle, on n'a pas pour autant ramené son existence à une simple propriété de **PoL** (comme pour le produit cartésien). En effet, cette propriété universelle fait référence à la notion de fonction bilinéaire qui ne fait pas partie de la définition de **PoL**. On pourrait la prendre comme définition des morphismes de cette catégorie, mais il faudrait alors quitter le monde des catégories pour passer dans celui des multicatégories (où les morphismes ont plusieurs sources et un but) qui ont une axiomatique plus compliquée dans laquelle les isos de cohérence apparaissent sous une autre forme. Il n'y a pas de miracle !

**Exercise 6.2.3** Une fonction bilinéaire $f : \mathcal{I}(S) \times \mathcal{I}(S) \to \mathcal{I}(T)$ est symétrique si $f(x_1, x_2) = f(x_2, x_1)$ pour tous $x_1, x_2 \in \mathcal{I}(S)$. Trouver un préordre $U$ et une fonction bilinéaire symétrique $\gamma : \mathcal{I}(S) \times \mathcal{I}(S) \to \mathcal{I}(U)$ avec la propriété universelle suivante : pour tout préordre $T$ et toute fonction bilinéaire symétrique $f : \mathcal{I}(S) \times \mathcal{I}(S) \to \mathcal{I}(T)$, il existe exactement une fonction linéaire $h : \mathcal{I}(U) \to \mathcal{I}(T)$ telle que $f = h \circ \gamma$.

6.2.3 Clôture monoïdale. Soient $S, T$ et $U$ des préordres.

Soit $\mathsf{ev} \subseteq |((S \multimap T) \otimes S) \multimap T|$ défini par

$$\mathsf{ev} = \{(((a', b), a), b') \mid a' \leq_S a \text{ et } b' \leq_T b\}.$$

On vérifie que $\mathsf{ev} \in \mathbf{PoLR}((S \multimap T) \otimes S, T)$. Soit donc $(((a', b), a), b') \in \mathsf{ev}$ et soit $(((a'_1, b_1), a_1), b'_1) \in ((S \multimap T) \otimes S) \multimap T$ tel que

$$(((a'_1, b_1), a_1), b'_1) \leq_{((S\multimap T)\otimes S)\multimap T} (((a', b), a), b').$$

Cela signifie que $b'_1 \leq b'$, $a \leq a_1$, $b \leq b_1$ et $a'_1 \leq a'$. Comme $a' \leq a$ et $b' \leq b$, on a $a'_1 \leq a_1$ et $b'_1 \leq b_1$. Par suite, $(((a'_1, b_1), a_1), b'_1) \in \mathsf{ev}$, et donc $\mathsf{ev} \in \mathcal{I}(((S \multimap T) \otimes S) \multimap T) = \mathbf{PoLR}((S \multimap T) \otimes S, T)$.

**Lemme 6.2.6** *Soient $s \in \mathcal{I}(S \multimap T)$ et $u \in \mathcal{I}(S)$. On a $\mathsf{ev}\,(s \otimes u) = s\,u$.*

*Démonstration.* Soit $b' \in \mathsf{ev}\,(s \otimes u)$. On peut trouver $(a', b) \in s$ et $a \in u$ tels que $a' \leq a$ et $b' \leq b$. Comme $x \in \mathcal{I}(S)$, on a $a' \in x$ et comme $s \in \mathcal{I}(S \multimap T)$, on a $(a, b') \in s$, donc $b' \in s\,u$. Réciproquement, si $b \in s\,u$, on peut trouver $a \in u$ tel que $(a, b) \in s$. On a $(((a, b), a), b) \in \mathsf{ev}$ et $((a, b), a) \in s \otimes u$. Donc $b \in \mathsf{ev}\,(s \otimes u)$. □

Soit $s \in \mathbf{PoLR}(U \otimes S, T)$. Soit

$$\mathsf{cur}\, s = \{((c, (a, b)) \in U \multimap (S \multimap T) \mid ((c, a), b) \in s\}.$$

**Exercise 6.2.4** Montrer que $\mathsf{cur}\, s \in \mathbf{PoLR}(U, S \multimap T)$, que $\mathsf{ev}\,(\mathsf{cur}\, s \otimes \mathsf{Id}_S) = s$ et que $\mathsf{cur}\, s$ est le seul élément de $\mathbf{PoLR}(U, S \multimap T)$ qui vérifie cette équation.

On a montré que **PoLR**, avec la structure monoïdale symétrique définie ci-dessus, est monoïdale fermée.

Soit $\bot = 1$ (on utilise deux noms différents pour le même objet car, en logique linéaire, il représente deux formules distinctes).

Soit $S$ un préordre. La bijection $|S| \to |S \multimap \bot|$ qui envoie $a$ sur $(a, *)$ définit un isomorphisme fort de $S^{\mathrm{op}}$ vers $S \multimap \bot$.

On a $\mathsf{ev} \in \mathbf{PoLR}((S \multimap \bot) \otimes S, \bot)$ et donc $\mathsf{ev}\,\sigma \in \mathbf{PoLR}(S \otimes (S \multimap \bot), \bot)$, et donc

$$\mathsf{cur}\,\mathsf{ev}\,\widehat{\sigma} \in \mathbf{PoLR}(S, (S \multimap \bot) \multimap \bot).$$

On a $\mathsf{ev} = \{(((a', *), a), *) \mid a' \leq_S a\}$, et donc $\mathsf{cur}\,\mathsf{ev}\,\widehat{\sigma} = \{(a, ((a', *), *)) \mid a' \leq a\}$. Donc $\mathsf{cur}\,\mathsf{ev}\,\widehat{\sigma} = \widehat{\eta}$ où $\eta : S \to ((S \multimap \bot) \multimap \bot)$ est l'isomorphisme fort évident.

On a vu que la catégorie monoïdale **PoLR** est $\star$-*autonome*, avec $\bot$ comme objet dualisant. Le dual $S^\bot = S \multimap \bot$ sera assimilé à $S^{\mathrm{op}}$. En particulier, on peut définir un cotenseur (appelé *par*), en posant $S \parr T = (S^\bot \otimes T^\bot)^\bot$ ; il se trouve qu'ici, on a $S \parr T = S \otimes T$ (mais on rappelle que ce n'est pas le cas dans la catégorie des espaces cohérents, par exemple).

**Lemme 6.2.7** *Si $S \simeq S'$, alors $S^\perp \simeq (S')^\perp$.*

Vérification immédiate.

## 6.3 Structure additive

Soit $(S_i)_{i\in I}$ une famille de préordres. On définit $S = \&_{i\in I} S_i$, le *avec* des $S_i$, de la façon suivante : $|S| = \bigcup_{i\in I}\{i\} \times |S_i|$, avec la relation de préordre selon laquelle $(i,a) \leq (j,b)$ si $i = j$ et $a \leq_{S_i} b$. On définit également, pour chaque $i \in I$ :

$$\pi_i = \{((i,a),a') \mid a' \leq_{S_i} a\}\,.$$

On voit facilement que $\pi_i \in \mathbf{PoLR}(S,S_i)$. Soit $T$ un préordre et soient $s_i \in \mathbf{PoLR}(T,S_i)$. Soit

$$s = \{(b,(i,a)) \mid i \in I \text{ et } (b,a) \in s_i\}\,.$$

On vérifie que $s \in \mathbf{PoLR}(T, \&_{i\in I} S_i)$. Soit $i \in I$ et $(b,a) \in s_i$. Soit $(b',(j,a')) \in T \multimap \&_{k\in I} S_k$ tel que $(b',(j,a')) \leq_{T\multimap\&_{k\in I} S_k} (b,(i,a))$. Cela signifie que $b \leq_T b'$, $i = j$ et $a' \leq_{S_i} a$, et donc $(b',a') \in s_i$. Par suite, on a bien $(b',(j,a')) \in s$ et donc $s \in \mathbf{PoLR}(T, \&_{i\in I} S_i)$. De plus, on vérifie facilement que $\pi_i\, s = s_i$ pour tout $i \in I$. On note $s = \langle s_i\rangle_{i\in I}$.

Soit maintenant $s \in \mathbf{PoLR}(T, \&_{i\in I} S_i)$ tel qu'on ait $\pi_i\, s = s_i$ pour tout $i \in I$. Soit $(b,(i,a)) \in T \multimap \&_{k\in I} S_k$ tel que $(b,(i,a)) \in s$. Comme $((i,a),a) \in \pi_i$, on a $(a,b) \in \pi_i\, s = s_i$. Cela montre que $s \subseteq \langle s_i\rangle_{i\in I}$. Réciproquement, soit $i \in I$ et soit $(b,a) \in s_i$. Comme $s_i = \pi_i\, s$, il existe $a' \in |S|_i$ tel que $((i,a'),a) \in \pi_i$ et $(b,(i,a')) \in s$. On a donc $a \leq a'$, et par suite $(b,(i,a)) \in s$, ce qui montre que $\langle s_i\rangle_{i\in I} \subseteq s$.

On a donc montré que $\&_{i\in I} S_i$, avec les projections $\pi_i$, est le produit cartésien de la famille $(S_i)_{i\in I}$. Dans le cas particulier où $I = \emptyset$, le produit cartésien est l'objet terminal de **PoLR**, qui est le préordre $\emptyset$, que l'on note $\top$.

**Lemme 6.3.1** *Si $S_i \simeq S'_i$ pour tout $i \in I$, alors $\&_{i\in I} S_i \simeq \&_{i\in I} S'_i$*

Vérification immédiate.

Noter qu'on a un isomorphisme d'ordre entre $\prod_{i\in I} \mathcal{I}(S_i)$ (muni de l'ordre produit) et $\mathcal{I}(\&_{i\in I} S_i)$, qui envoie la famille $\langle u_i\rangle_{i\in I}$ sur $\cup_{i\in I}\{i\} \times u_i$. On peut voir cela comme le cas particulier de la construction ci-dessus où $T$ est le préordre à un point.

6.3.1 Foncteur produit cartésien. Si $S$ est un préordre, on note $S^I$ le produit cartésien $\&_{i\in I} S_i$ dans lequel $S_i = S$ pour chaque $i \in I$. Autrement dit, $S^I = I \times S$, en mettant sur $I$ le préordre *discret* ($i \leq j$ iff $i = j$). Cette opération est fonctorielle : si $s \in \mathbf{PoLR}(S,T)$, alors $s^I = \langle s\,\pi_i\rangle_{i\in I} \in \mathbf{PoLR}(S^I, T^I)$ est donné par

$$s^I = \{((i,a),(i,b)) \mid i \in I \text{ et } (a,b) \in |S|\}\,.$$

6.3.2 Coproduit On a aussi un coproduit, défini par $\bigoplus_{i\in I} S_i = \left(\&_{i\in I} {S_i}^\perp\right)^\perp$. On voit facilement que $\bigoplus_{i\in I} S_i = \&_{i\in I} S_i$.

## 6.4 Exponentielles de Scott

On définit $|!_{\mathsf{s}}S| = \mathcal{P}_{\mathrm{fin}}(|S|)$, l'ensemble des parties finies de $|S|$. On munit cet ensemble du préordre suivant : si $u^1, u^2 \in |!_{\mathsf{s}}S|$, on dira que $u^1 \leq u^2$ si, pour tout $a \in u^1$, il existe $a' \in u^2$ tel que $a \leq a'$. Observer que cette relation est transitive et réflexive (c'est un préordre), et qu'elle n'est pas antisymétrique en général, même quand $\leq_S$ l'est. Observer aussi que $u^1 \leq_{!_{\mathsf{s}}S} u^2$ si et seulement si $\downarrow u^1 \subseteq \downarrow u^2$.

**Exercise 6.4.1** Trouver un ensemble partiellement ordonné $S$ tel que $!_{\mathsf{s}}S$ ne soit pas un ordre partiel.

Le treillis complet $\mathcal{I}(S)$ est en particulier un cpo.

**Lemme 6.4.1** *Un élément $u$ de $\mathcal{I}(S)$ est isolé (voir Section 5.1) si et seulement si $u = \downarrow u^0$ où $u^0 \in\, !_{\mathsf{s}}S$.*

*Démonstration.* Soit $u \in \mathcal{I}(S)$ isolé. Soit $D = \{\downarrow u^0 \mid u^0 \in \mathcal{P}_{\mathrm{fin}}(u)\}$. L'ensemble $D$ est filtrant et on a $u = \bigcup D$. Comme $u$ est isolé, il existe $u^0 \in \mathcal{P}_{\mathrm{fin}}(u)$ tel que $u \subseteq \downarrow u^0$, et donc $u = \downarrow u^0$ (puisque $u^0 \subseteq u$).

Réciproquement, soit $u = \downarrow u^0$ avec $u^0 \in |!_{\mathsf{s}}S|$. Soit $D$ une partie filtrante de $\mathcal{I}(S)$ et supposons que $u \subseteq \bigcup D$. Comme $u^0 \subseteq \bigcup D$ et comme $D$ est filtrant, il existe $v \in D$ tel que $u^0 \subseteq v$. Mais comme $v \in \mathcal{I}(S)$, on a aussi $u \subseteq v$, ce qui prove que $u$ est isolé. □

6.4.1 TRACE D'UNE FONCTION CONTINUE. Soit $f : \mathcal{I}(S) \to \mathcal{I}(T)$ une fonction Scott-continue. Cela signifie que $f$ est croissante et commute aux unions des familles filtrantes. Autrement dit, pour toute partie filtrante $D$ de $\mathcal{I}(S)$, on a $f(\bigcup D) \subseteq \bigcup f(D)$ (l'autre inclusion résulte simplement du fait que $f$ est croissante).

On définit

$$\mathsf{Tr}(f) = \{(u^0, b) \in |!_{\mathsf{s}}S| \times |T| \mid b \in f(\downarrow u^0)\}\,.$$

On voit facilement que $\mathsf{Tr}(f) \in \mathcal{I}(!_{\mathsf{s}}S \multimap T)$. Réciproquement, soit $t \in \mathcal{I}(!_{\mathsf{s}}S \multimap T)$. On définit une fonction

$$\begin{aligned} \mathsf{Fun}(t) : \mathcal{I}(S) &\to \mathcal{P}(T) \\ u &\mapsto \{b \mid \exists u^0 \subseteq u\ (u^0, b) \in t\} \end{aligned}$$

On vérifie facilement que $\mathsf{Fun}(t)$ prend ses valeurs dans $\mathcal{I}(T)$ et est croissante. On montre que $f = \mathsf{Fun}(t)$ est continue. Soit donc $D \subseteq \mathcal{I}(S)$ un ensemble filtrant, il faut montrer que $f(\bigcup D) \subseteq \bigcup f(D)$. Soit $b \in f(\bigcup D)$, il existe $u^0 \in |!_{\mathsf{s}}S|$ tel que $u^0 \subseteq \bigcup D$ et $(u^0, b) \in t$. Comme $u^0$ est fini et $D$ filtrant, il existe $u \in D$ tel que $u^0 \subseteq u$. Donc $b \in f(u) \subseteq \bigcup f(D)$ (vu que $u \in D$).

Soit $\mathbf{PoC}$ la catégorie dont les objets sont les préordres, et telle que $\mathbf{PoC}(S, T)$ soit l'ensemble des fonctions continues de $\mathcal{I}(S)$ vers $\mathcal{I}(T)$. On munit cet ensemble de morphismes de l'ordre ponctuel déjà vu : $f \leq g$ si et seulement si $f(u) \subseteq g(u)$ pour tout $u \in \mathcal{I}(S)$.

**Exercise 6.4.2** Montrer que $\mathbf{PoL}(S, T) \subseteq \mathbf{PoC}(S, T)$, et que cette inclusion est stricte en général (donner un contre-exemple).

**Proposition 6.4.2** *Les fonctions* $\mathsf{Tr}$ *et* $\mathsf{Fun}$ *définissent un isomorphisme d'ordre entre* $\mathbf{PoC}(S, T)$ *et* $\mathcal{I}(!_{\mathsf{s}}S \multimap T)$.

*Démonstration.* Soit $f \in \mathbf{PoC}(S, T)$ et montrons que $\mathsf{Fun}(\mathsf{Tr}(f)) = f$. Soit donc $u \in \mathcal{I}(S)$, on montre que $\mathsf{Fun}(\mathsf{Tr}(f))(u) = f(u)$. Soit $b \in |T|$. Supposons d'abord que $b \in f(u)$. Soit $D = \{\downarrow u^0 \mid u^0 \in \mathcal{P}_{\mathrm{fin}}(u)\}$. On a $\bigcup D = u$, et $D$ est filtrant. Donc $f(u) = f(\bigcup D) \subseteq \bigcup f(D)$, et par suite il existe $u^0 \in \mathcal{P}_{\mathrm{fin}}(u)$ tel que $b \in f(\downarrow u^0)$, c'est-à-dire $(u^0, b) \in \mathsf{Tr}(f)$. Comme $u^0 \subseteq u$, on a $b \in \mathsf{Fun}(\mathsf{Tr}(f))(u)$. Supposons maintenant que $b \in \mathsf{Fun}(\mathsf{Tr}(f))(u)$. Soit donc $u^0 \in |!_{\mathsf{s}}S|$ tel que $(u^0, b) \in \mathsf{Tr}(f)$ et $u^0 \subseteq u$. On a $b \in f(\downarrow u^0)$, et comme $\downarrow u^0 \subseteq u$ et comme $f$ est croissante, on a $b \in f(u)$.

Soit $t \in \mathcal{I}(!_{\mathsf{s}}S \multimap T)$. On montre que $t = \mathsf{Tr}(\mathsf{Fun}(t))$. Soient $u^0 \in |!_{\mathsf{s}}S|$ et $b \in |T|$. On suppose d'abord $(u^0, b) \in t$. On a $b \in \mathsf{Fun}(t)(\downarrow u^0)$, puisque $u^0 \subseteq \downarrow u^0$, et donc $(u^0, b) \in \mathsf{Tr}(\mathsf{Fun}(t))$. Réciproquement, supposons $(u^0, b) \in \mathsf{Tr}(\mathsf{Fun}(t))$. Cela signifie que $b \in \mathsf{Fun}(t)(\downarrow u^0)$, et donc il existe $u^1 \in |!_{\mathsf{s}}S|$ tel que $u^1 \subseteq \downarrow u^0$ et $(u^1, b) \in t$. On a $u^1 \leq_{!_{\mathsf{s}}S} u^0$ et donc $(u^0, b) \in t$ puisque $t \in \mathcal{I}(!_{\mathsf{s}}S \multimap T)$.

On montre ensuite que $\mathsf{Tr}$ est croissante. Soient donc $f, f' \in \mathbf{PoC}(S, T)$ telles que $f \leq f'$ et soit $(u^0, b) \in \mathsf{Tr}(f)$. C'est que $b \in f(\downarrow u^0)$, or $f(\downarrow u^0) \subseteq f'(\downarrow {}^0 u)$ et donc $(u^0, b) \in \mathsf{Tr}(f')$, ce qui montre que $\mathsf{Tr}(f) \subseteq \mathsf{Tr}(f')$.

Soient $t, t' \in \mathbf{PoC}(S, T)$ tels que $t \subseteq t'$ et soit $u \in \mathcal{I}(S)$. Soit $b \in \mathsf{Fun}(t)(u)$. Soit $u^0$ tel que $(u^0, b) \in t$ et $u^0 \subseteq u$. Comme $t \subseteq t'$, on a $(u^0, b) \in t'$ et donc $b \in \mathsf{Fun}(t')(u)$, ce qui montre que $\mathsf{Fun}(t) \leq \mathsf{Fun}(t')$. □

6.4.2 L'EXPONENTIELLE COMME FONCTEUR. Si $u \in \mathcal{I}(S)$, on définit $u^{!_{\mathsf{s}}} \in \mathcal{I}(!_{\mathsf{s}}S)$ en posant

$$u^{!_{\mathsf{s}}} = \mathcal{P}_{\mathrm{fin}}(u)\,.$$

Il faut remarquer qu'on a bien $u^{!_{\mathsf{s}}} \in \mathcal{I}(!_{\mathsf{s}}S)$. Soit en effet $u^0 \in u^{!_{\mathsf{s}}}$ et $u^1 \in |!_{\mathsf{s}}S|$ tel que $u^1 \leq_{!_{\mathsf{s}}S} u^0$. Soit $a' \in u^1$, il existe $a \in u^0$ tel que $a' \leq_S a$, or $a \in u$ et donc $a' \in u$ puisque $u \in \mathcal{I}(S)$. Il en résulte que $u^1 \subseteq u$ et donc que $u^1 \in u^{!_{\mathsf{s}}}$.

**Lemme 6.4.3** *Soit $t \in \mathbf{PoL}(!S,T)$. On a $\mathsf{Fun}(t)(u) = t\,u^{!_\mathsf{s}}$.*

*Démonstration.* Soit $b \in \mathsf{Fun}(t)(u)$. Soit $u^0 \in |!_\mathsf{s}S|$ tel que $u^0 \subseteq u$ et $(u^0, b) \in t$. On a $u^0 \in u^{!_\mathsf{s}}$ et donc $b \in t\,u^{!_\mathsf{s}}$. Réciproquement, soit $b \in t\,u^{!_\mathsf{s}}$. Soit $u^0 \in u^{!_\mathsf{s}}$ tel que $(u^0, b) \in t$. On a $u^0 \subseteq u$ et donc $b \in \mathsf{Fun}(t)(u)$. □

Ce lemme très simple est important car il a la conséquence suivante, qui donne un critère suffisant pour que deux éléments de $\mathbf{PoL}(!S,T)$ soient égaux.

**Lemme 6.4.4** *Soient $t, t' \in \mathbf{PoL}(!S,T)$. Si on a $t\,u^{!_\mathsf{s}} = t'\,u^{!_\mathsf{s}}$ pour tout $u \in \mathcal{I}(S)$, alors $t = t'$.*

*Démonstration.* En effet, on a alors $\mathsf{Fun}(t) = \mathsf{Fun}(t')$, et donc $t = t'$ par la proposition 6.4.2. □

Soit $t \in \mathbf{PoL}(S,T)$. Soit

$$!_\mathsf{s}t = \{(u^0, v^0) \in |!S| \times |!T| \mid \forall b \in v^0\, \exists a \in u^0\ (a, b) \in t\}\,.$$

**Lemme 6.4.5** *Si $t \in \mathbf{PoL}(S,T)$, alors $!_\mathsf{s}t \in \mathbf{PoL}(!_\mathsf{s}S, !_\mathsf{s}T)$.*

*Démonstration.* Soit $(u^0, v^0) \in !_\mathsf{s}t$ est soit $(u^1, v^1) \in !_\mathsf{s}S \times !_\mathsf{s}T$ tel que $(u^1, v^1) \leq_{!_\mathsf{s}S \multimap !_\mathsf{s}T} (u^0, v^0)$, il faut voir que $(u^1, v^1) \in !_\mathsf{s}t$. Soit $b' \in v^1$. Comme $v^1 \leq_{!_\mathsf{s}T} v^0$, il existe $b \in v^0$ tel que $b' \leq_T b$. Comme $(u^0, v^0) \in !_\mathsf{s}t$, il existe $a \in u^0$ tel que $(a, b) \in t$. Comme $u^0 \leq_{!_\mathsf{s}S} u^1$, il existe $a' \in u^1$ tel que $a \leq_S a'$. On a $(a', b') \leq_{S \multimap T} (a, b)$, et donc $(a', b') \in t$ puisque $t \in \mathcal{I}(S \multimap T)$. Puisque $a' \in u^1$, on a montré que $(u^1, v^1) \in !_\mathsf{s}t$, et donc $!_\mathsf{s}t \in \mathbf{PoL}(!_\mathsf{s}S, !_\mathsf{s}T)$. □

**Lemme 6.4.6** *Si $t \in \mathbf{PoL}(S,T)$ et $u \in \mathcal{I}(S)$, alors $!_\mathsf{s}t\,u^{!_\mathsf{s}} = (t\,u)^{!_\mathsf{s}}$.*

*Démonstration.* Soit $v^0 \in !_\mathsf{s}t\,u^{!_\mathsf{s}}$, montrons que $v^0 \in (t\,u)^{!_\mathsf{s}}$. Il existe $u^0 \in u^{!_\mathsf{s}}$ tel que $(u^0, v^0) \in !_\mathsf{s}t$. Soit $b \in v^0$, il existe $a \in u^0$ tel que $(a, b) \in t$. Comme $u^0 \subseteq u$, on a $a \in u$ et donc $b \in t\,u$, et donc $v^0 \in (t\,u)^{!_\mathsf{s}}$.

Réciproquement, soit $v^0 \in (t\,u)^{!_\mathsf{s}}$, c'est-à-dire que $v^0 \in |!_\mathsf{s}S|$ et $v^0 \subseteq t\,u$. Pour chaque $b \in v^0$, on choisit $a_b \in u$ tel que $(a_b, b) \in t$. Soit $u^0 \in !_\mathsf{s}S$ qui contient tous les $a_b$ (pour $b \in v^0$). On a $(u^0, v^0) \in !_\mathsf{s}t$ et $u^0 \in u^{!_\mathsf{s}}$ et donc $v^0 \in !_\mathsf{s}t\,u^{!_\mathsf{s}}$. □

**Proposition 6.4.7** *L'opération $t \mapsto !_\mathsf{s}t$ est fonctorielle. Autrement dit, $!_\mathsf{s}\mathsf{Id}_S = \mathsf{Id}_{!_\mathsf{s}S}$ et, si $s \in \mathbf{PoL}(S,T)$ et $t \in \mathbf{PoL}(T,U)$, on a $!_\mathsf{s}(t\,s) = !_\mathsf{s}t\,!_\mathsf{s}s$.*

*Démonstration.* Soit $u \in \mathcal{I}(S)$. On a $!_\mathsf{s}\mathsf{Id}_S\,u^{!_\mathsf{s}} = u^{!_\mathsf{s}}$ par le lemme 6.4.6, donc $!_\mathsf{s}\mathsf{Id}_S = \mathsf{Id}_{!_\mathsf{s}S}$ par le lemme 6.4.4. Et on a $!_\mathsf{s}(t\,s)\,u^{!_\mathsf{s}} = ((t\,s)\,u)^{!_\mathsf{s}}$ par le lemme 6.4.6, donc $!_\mathsf{s}(t\,s)\,u^{!_\mathsf{s}} = (t\,(s\,u))^{!_\mathsf{s}} = !_\mathsf{s}t\,(s\,u)^{!_\mathsf{s}} = (!_\mathsf{s}t\,!_\mathsf{s}s)\,u^{!_\mathsf{s}}$ en appliquant deux fois le lemme 6.4.6, et on conclut grâce au lemme 6.4.4. □

**Lemme 6.4.8** *Si $S \simeq S'$, alors $!_\mathsf{s}S \simeq !_\mathsf{s}S'$.*

La vérification est immédiate.

### 6.4.3 Structure de comonade de l'exponentielle.

Soit $\mathsf{der}^\mathsf{s}_S \subseteq |!S \multimap S|$ donné par

$$\mathsf{der}^\mathsf{s}_S = \{(u^0, a) \mid \exists a' \in u^0\ a \leq_S a'\}\,.$$

**Lemme 6.4.9** *On a $\mathsf{der}^\mathsf{s}_S \in \mathbf{PoL}(!S,S)$ et, pour tout $u \in \mathcal{I}(S)$, on a $\mathsf{der}^\mathsf{s}_S\,u^{!_\mathsf{s}} = u$. De plus $\mathsf{der}^\mathsf{s}_S$ est naturel en $S$ ; autrement dit, si $s \in \mathbf{PoLR}(S,T)$, on a $\mathsf{der}^\mathsf{s}_T\,!_\mathsf{s}s = s\,\mathsf{der}^\mathsf{s}_S$.*

*Démonstration.* Soit $(u^0, a) \in \mathsf{der}^\mathsf{s}_S$ et soit $(u^1, a^1) \in |!_\mathsf{s}S \multimap S|$ tel que $(u^1, a^1) \leq_{!_\mathsf{s}S \multimap S} (u^0, a)$. Soit $a^2 \in u^0$ tel que $a \leq a^2$. Comme $u^0 \leq_{!_\mathsf{s}S} u^1$, il existe $a^3 \in u^1$ tel que $a^2 \leq a^3$. Comme $a^1 \leq a$, on a $a^1 \leq a^3 \in u^1$, ce qui montre que $(u^1, a^1) \in \mathsf{der}_S$, et donc $\mathsf{der}^\mathsf{s}_S \in \mathbf{PoL}(!S,S)$.

Soit $u \in \mathcal{I}(S)$. Soit $a \in u$. On a $(\{a\}, a) \in \mathsf{der}_S$, donc $a \in \mathsf{der}^\mathsf{s}_S\,u^{!_\mathsf{s}}$ puisque $\{a\} \in u^{!_\mathsf{s}}$. Si $a \in \mathsf{der}^\mathsf{s}_S\,u^{!_\mathsf{s}}$, soit $u^0 \in u^{!_\mathsf{s}}$ tel que $(u^0, a) \in \mathsf{der}^\mathsf{s}_S$. Soit $a^1 \in u^0$ tel que $a \leq a^1$. Comme $a^1 \in u^0 \subseteq u$ et comme $u \in \mathcal{I}(S)$, on a $a \in u$. Donc $\mathsf{der}^\mathsf{s}_S\,u^{!_\mathsf{s}} = u$.

La naturalité en résulte : on a $(\mathsf{der}^{\mathsf{s}}_T\, !_{\mathsf{s}}s)\, u^{!_{\mathsf{s}}} = \mathsf{der}^{\mathsf{s}}_T\, (s\, u)^{!_{\mathsf{s}}} = s\, u = (s\, \mathsf{der}^{\mathsf{s}}_S)\, u^{!_{\mathsf{s}}}$. On conclut par le lemme 6.4.4. □

On définit ensuite $\mathsf{dig}^{\mathsf{s}}_S \subseteq |!_{\mathsf{s}}S \multimap !_{\mathsf{s}}!_{\mathsf{s}}S|$ par

$$\mathsf{dig}^{\mathsf{s}}_S = \{(u^0, U^0) \mid \forall u^1 \in U^0 \quad u^1 \leq_{!_{\mathsf{s}}S} u^0\}\,.$$

**Lemme 6.4.10** *On a* $\mathsf{dig}^{\mathsf{s}}_S \in \mathbf{PoL}(!_{\mathsf{s}}S, !_{\mathsf{s}}!_{\mathsf{s}}S)$ *et, pour tout* $u \in \mathcal{I}(S)$*, on a* $\mathsf{dig}^{\mathsf{s}}_S\, u^{!_{\mathsf{s}}} = u^{!_{\mathsf{s}}!_{\mathsf{s}}}$*. De plus,* $\mathsf{dig}^{\mathsf{s}}_S$ *est naturel en* $S$ *; autrement dit, si* $s \in \mathbf{PoL}(S,T)$*, on a* $\mathsf{dig}^{\mathsf{s}}_T\, !_{\mathsf{s}}s = !_{\mathsf{s}}!_{\mathsf{s}}s\, \mathsf{dig}^{\mathsf{s}}_S$*.*

*Démonstration.* Soit $(u^0, U^0) \in \mathsf{dig}^{\mathsf{s}}_S$ et soit

$$(u^1, U^1) \in |!_{\mathsf{s}}S \multimap !_{\mathsf{s}}!_{\mathsf{s}}S|$$

tel que $(u^1, U^1) \leq_{!_{\mathsf{s}}S \multimap !_{\mathsf{s}}!_{\mathsf{s}}S} (u^0, U^0)$, on montre que $(u^1, U^1) \in \mathsf{dig}^{\mathsf{s}}_S$. Soit $v^1 \in U^1$ on doit montrer que $v^1 \leq_{!_{\mathsf{s}}S} u^1$. Soit $a^1 \in v^1$. Comme $U^1 \leq_{!_{\mathsf{s}}!_{\mathsf{s}}S} U^0$, on peut trouver $v^0 \in U^0$ tel que $v^1 \leq_{!_{\mathsf{s}}S} v^0$. Comme $a^1 \in v^1$, il existe $a \in v^0$ tel que $a^1 \leq a$. Or $v^0 \leq_{!_{\mathsf{s}}S} u^0$ puisque $(u^0, U^0) \in \mathsf{dig}^{\mathsf{s}}_S$. Comme $u^0 \leq_{!_{\mathsf{s}}S} u^1$ on peut trouver $a^2 \in u^1$ tel que $a^1 \leq a^2$. On a montré que $v^1 \leq_{!_{\mathsf{s}}S} u^1$, ce qu'on voulait.

Soit $u \in \mathcal{I}(S)$. On montre que $\mathsf{dig}^{\mathsf{s}}_S\, u^{!_{\mathsf{s}}} = u^{!_{\mathsf{s}}!_{\mathsf{s}}}$. Soit donc $U^0 \in !_{\mathsf{s}}!_{\mathsf{s}}S$. On suppose d'abord que $U^0 \in \mathsf{dig}^{\mathsf{s}}_S\, u^{!_{\mathsf{s}}}$, et donc qu'il existe $u^0 \in u^{!_{\mathsf{s}}}$ tel que $(u^0, U^0) \in \mathsf{dig}^{\mathsf{s}}_X$. Soit $u^1 \in U^0$. Par definition de $\mathsf{dig}^{\mathsf{s}}_S$, on a $u^1 \leq_{!_{\mathsf{s}}S} u^0$. Comme $u^0 \in u^{!_{\mathsf{s}}}$ et $u \in \mathcal{I}(S)$, on a $u^1 \in u^{!_{\mathsf{s}}}$ et donc $U^0 \in u^{!_{\mathsf{s}}!_{\mathsf{s}}}$. Supposons réciproquement que $U^0 \in u^{!_{\mathsf{s}}!_{\mathsf{s}}}$. Soit $u^0 \in U^0$, on a $u^0 \in u^{!_{\mathsf{s}}}$ et $(u^0, U^0) \in \mathsf{dig}^{\mathsf{s}}_S$ ce qui montre que $U^0 \in \mathsf{dig}^{\mathsf{s}}_S\, u^{!_{\mathsf{s}}}$. La naturalité découle de ce qui précède et du lemme 6.4.4. □

**Proposition 6.4.11** *Le foncteur* $!_{\mathsf{s}}\_$*, équipé des transformations naturelles* $\mathsf{der}^{\mathsf{s}}$ *et* $\mathsf{dig}^{\mathsf{s}}$*, est une comonade.*

*Démonstration.* Il s'agit de prouver les trois équations suivantes :

$$\begin{aligned}
\mathsf{der}^{\mathsf{s}}_{!_{\mathsf{s}}S}\, \mathsf{dig}^{\mathsf{s}}_S &= \mathsf{Id}_{!_{\mathsf{s}}S}\\
!_{\mathsf{s}}\mathsf{der}^{\mathsf{s}}_S\, \mathsf{dig}^{\mathsf{s}}_S &= \mathsf{Id}_{!_{\mathsf{s}}S}\\
\mathsf{dig}^{\mathsf{s}}_{!_{\mathsf{s}}S}\, \mathsf{dig}^{\mathsf{s}}_S &= !\mathsf{dig}^{\mathsf{s}}_S\, \mathsf{dig}^{\mathsf{s}}_S
\end{aligned}$$

La vérification de ces équations est immédiate, en utilisant les lemmes 6.4.9, 6.4.10 et 6.4.4. □

6.4.4 Isomorphisme fondamental (isomorphisme de Seely). Soient $S$ et $T$ des préordres. On observe qu'un élément $u^0$ de $|!_{\mathsf{s}}(S \,\&\, T)|$ est de la forme

$$u^0 = \{(1, a_1), \ldots, (1, a_l), (2, b_1), \ldots, (2, b_r)\}$$

avec $a_1, \ldots, a_l \in |S|$ et $b_1, \ldots, b_r \in |T|$. On peut donc associer à $u^0$ le couple $(u^1, u^2) \in |!_{\mathsf{s}}S \otimes !_{\mathsf{s}}T|$ où $u^1 = \{a_1, \ldots, a_l\}$ et $u^2 = \{b_1, \ldots, b_r\}$. Cela définit une bijection $\theta : |!_{\mathsf{s}}(S \,\&\, T)| \to |!_{\mathsf{s}}S \otimes !_{\mathsf{s}}T|$ par $\theta(u^0) = (u^1, u^2)$.

**Exercise 6.4.3** Montrer que $\theta$ est un isomorphisme fort de $!_{\mathsf{s}}(S \,\&\, T)$ vers $!_{\mathsf{s}}S \otimes !_{\mathsf{s}}T$.

Bien sûr cet isomorphisme fort se généralise en arité quelconque : on a un isomorphisme fort $\theta$ de $!_{\mathsf{s}}(S_1 \,\&\, \cdots \,\&\, S_k)$ vers $!_{\mathsf{s}}S_1 \otimes \cdots \otimes !_{\mathsf{s}}S_k$ (voir 4.6.6 pour un traitement catégorique systématique de ces questions, qui s'applique bien sûr ici). Quand $k = 0$, c'est l'isomorphism entre deux préordres à 1 élément ($\emptyset$ et $*$ respectivement).

Si $s \in \mathbf{PoLR}(!_{\mathsf{s}}S, T)$, on rappelle que la promotion de $s$ est $s^{!_{\mathsf{s}}} \in \mathbf{PoLR}(!_{\mathsf{s}}S, !_{\mathsf{s}}T)$ qui est défini par

$$s^{!_{\mathsf{s}}} = !_{\mathsf{s}}s\ \mathsf{dig}^{\mathsf{s}}_S\,.$$

On vérifie facilement que

$$s^{!_{\mathsf{s}}} = \{(u^0, v^0) \in |!_{\mathsf{s}}S \multimap !_{\mathsf{s}}T|) \mid \forall b \in v^0\ (u^0, b) \in s\}\,.$$

Soit $s \in \mathbf{PoLR}(!_{\mathsf{s}}S_1 \otimes \cdots \otimes !_{\mathsf{s}}S_k, T)$. En utilisant l'isomorphisme de Seely et l'opération de promotion définie ci-dessus, on définit une version généralisée de la promotion $s^{!_{\mathsf{s}}} \in \mathbf{PoLR}(!_{\mathsf{s}}S_1 \otimes \cdots \otimes !_{\mathsf{s}}S_k, T)$ qui vérifie

$$s^{!_{\mathsf{s}}} = \{((u^1, \ldots, u^k), v^0) \in |!_{\mathsf{s}}S_1 \otimes \cdots \otimes !_{\mathsf{s}}S_k \multimap !_{\mathsf{s}}T|) \mid \forall b \in v^0 \ ((u^1, \ldots, u^k), b) \in s\} .$$

Enfin, observer qu'un morphisme $s \in \mathbf{PoLR}(!_{\mathsf{s}}S_1 \otimes \cdots \otimes !_{\mathsf{s}}S_k, T)$ est complètement caractérisé par son comportement sur les arguments de la forme $u_1{}^{!_{\mathsf{s}}} \otimes \cdots \otimes u_k{}^{!_{\mathsf{s}}}$. Autrement dit, si $s$ et $s'$ sont deux tels morphismes et si

$$\forall u_1 \in \mathcal{I}(S_1) \ldots \forall u_k \in \mathcal{I}(S_k) \quad s\,(u_1{}^{!_{\mathsf{s}}} \otimes \cdots \otimes u_k{}^{!_{\mathsf{s}}}) = s'\,(u_1{}^{!_{\mathsf{s}}} \otimes \cdots \otimes u_k{}^{!_{\mathsf{s}}})$$

alors $s = s'$. Cela résulte de cet isomorphisme fort et du lemme 6.4.4.

Par exemple, la promotion généralisée ci-dessus est complètement caractérisée par l'équation

$$s^{!_{\mathsf{s}}}\,(u_1{}^{!_{\mathsf{s}}} \otimes \cdots \otimes u_k{}^{!_{\mathsf{s}}}) = (s\,(u_1{}^{!_{\mathsf{s}}} \otimes \cdots \otimes u_k{}^{!_{\mathsf{s}}}))^{!_{\mathsf{s}}} .$$

6.4.5 Catégorie de Kleisli. C'est la "catégorie des coalgèbres libres" de la comonade $(!_{\mathsf{s}}\_, \mathsf{der}^{\mathsf{s}}, \mathsf{dig}^{\mathsf{s}})$. On la notera $\mathbf{PoLR}_{!_{\mathsf{s}}}$ et elle est définie comme suit.

Ses objets sont ceux de $\mathbf{PoLR}$, et $\mathbf{PoLR}_{!_{\mathsf{s}}}(S, T) = \mathbf{PoLR}(!_{\mathsf{s}}S, T)$. L'identité est $\mathrm{Id}^{\mathsf{K}} = \mathsf{der}^{\mathsf{s}}_S \in \mathbf{PoLR}(!_{\mathsf{s}}S, S)$. Etant donnés $s \in \mathbf{PoLR}(!_{\mathsf{s}}S, T)$ et $t \in \mathbf{PoLR}(!_{\mathsf{s}}T, U)$, la composition $t \circ s$ est définie par

$$t \circ s = t\,!_{\mathsf{s}}s\,\mathsf{dig}^{\mathsf{s}}_s .$$

**Exercise 6.4.4** En utilisant les équations de comonade, montrer qu'on a bien défini ainsi une catégorie.

**Exercise 6.4.5** Soient $s \in \mathbf{PoLR}_{!_{\mathsf{s}}}(S, T)$ et $\mathbf{PoLR}_{!_{\mathsf{s}}}(T, U)$. Montrer que

$$t \circ s = \{(u^0, c) \mid \exists b_1, \ldots, b_k \in |T| \ (\{b_1, \ldots, b_k\}, c) \in t \text{ et } \forall i \ (u^0, b_i) \in s\}$$

**Lemme 6.4.12** *On a* $\mathsf{Fun}\,(\mathrm{Id}^{\mathsf{K}}) = \mathsf{Id}$ *et* $\mathsf{Fun}\,(t \circ s) = \mathsf{Fun}(t) \circ \mathsf{Fun}(s)$*, autrement dit, la correspondance qui envoie l'objet* $S$ *de* $\mathbf{PoLR}_{!_{\mathsf{s}}}$ *sur le même objet* $S$ *de* $\mathbf{PoC}$ *et le morphisme* $s \in \mathbf{PoLR}_{!_{\mathsf{s}}}(S, T)$ *sur le morphisme* $\mathsf{Fun}(s) \in \mathbf{PoC}(S, T)$ *est un foncteur de* $\mathbf{PoLR}_{!_{\mathsf{s}}}$ *vers* $\mathbf{PoC}$*, que l'on note encore* $\mathsf{Fun}$*. Réciproquement, si* $f \in \mathbf{PoC}(S, T)$ *et* $g \in \mathbf{PoC}(T, U)$*, on a* $\mathsf{Tr}(g \circ f) = \mathsf{Tr}(g) \circ \mathsf{Tr}(f)$ *et* $\mathsf{Tr}(\mathsf{Id}) = \mathrm{Id}^{\mathsf{K}}$*, c'est-à-dire que* $\mathsf{Tr}$ *définit un foncteur de la catégorie* $\mathbf{PoC}$ *vers la catégorie* $\mathbf{PoLR}_{!_{\mathsf{s}}}$ *(qui est l'identité sur les objets). Ces foncteurs sont inverses l'un de l'autre et définissent donc un isomorphisme entre les catégories* $\mathbf{PoLR}_{!_{\mathsf{s}}}$ *et* $\mathbf{PoC}$*.*

**Exercise 6.4.6** Vérifier la fonctorialité des opérations $\mathsf{Fun}$ et $\mathsf{Tr}$.

**Lemme 6.4.13** *La catégorie* $\mathbf{PoLR}_{!_{\mathsf{s}}}$ *est cartésienne. Le produit cartésien de la famille* $(S_i)_{i \in I}$ *est* $S = \&_{i \in I} S_i$*. La* $i$*-ème projection est* $\pi^{\mathsf{K}}_i = \pi_i\,\mathsf{der}_S \in \mathbf{PoLR}_{!_{\mathsf{s}}}(S, S_i)$*.*

*Démonstration.* Si $t_i \in \mathbf{PoLR}_!(T, S_i)$ pour $i \in I$, alors $t = \langle t_i \rangle_{i \in I} \in \mathbf{PoLR}_!(T, S)$ et vérifie $\pi^{\mathsf{K}}_i \circ t = t_i$, et est caractérisé par cette propriété, comme on le voit facilement. □

**Remarque 6.4.14** On a déjà observé que $\mathcal{I}(\&_{i \in I} S_i) \simeq \prod_{i \in I} \mathcal{I}(S_i)$. Modulo cet isomorphisme d'ordre, on a $\pi^{\mathsf{K}}_i\,(u_j)_{j \in I} = u_i$, et si les $f_i : \mathcal{I}(T) \to \mathcal{I}(S_i)$ sont des fonctions continues, la fonction continue associée $f : \mathcal{I}(T) \to \prod_{i \in I} \mathcal{I}(S_i)$ est donnée par $f(u) = (f_i(u))_{i \in I}$.

**Proposition 6.4.15** *La catégorie* $\mathbf{PoLR}_!$ *est cartésienne fermée. L'objet des morphismes de* $S$ *vers* $T$ *dans* $\mathbf{PoLR}_!$ *est* $S \Rightarrow T = !S \multimap T$*. Le morphisme d'évaluation* $\mathsf{Ev} \in \mathbf{PoLR}_!((S \Rightarrow T) \,\&\, S, T)$ *est*

$$\mathsf{Ev} = \mathsf{ev}\,(\mathsf{der}_{S \Rightarrow T} \otimes \mathsf{Id}_{!S})\,\widehat{\theta} \tag{6.1}$$

*où* $\theta : !((S \Rightarrow T) \,\&\, S) \to !(S \Rightarrow T) \otimes !S$ *est l'isomorphisme fondamental*[1]*. De plus, modulo cet isomorphisme, on a*

$$\begin{aligned} \mathsf{Ev} = \{((v^0, u^0), b) \in (!(S \Rightarrow T) \otimes !S) \multimap T \mid \\ \exists (u^1, b^1) \in v^0 \quad (u^0, b) \leq_{S \Rightarrow T} (v^1, b^1)\} \end{aligned} \tag{6.2}$$

1. On rappelle que $\widehat{\theta} \in \mathbf{PoL}(!((S \Rightarrow T) \,\&\, S), !(S \Rightarrow T) \otimes !S)$ est l'isomorphisme, dans $\mathbf{PoL}$, associé à l'isomorphisme fort $\theta$, voir le paragraphe 6.1.4)

C'est juste une instance de la Proposition 4.6.3. De plus, étant donné $s \in \mathbf{PoLR}_!(U \,\&\, S, T)$, le curryfié $\mathsf{Cur}\, s \in \mathbf{PoLR}_!(U, S \Rightarrow T)$ est donné (toujours modulo l'isomorphisme fondamental) par

$$\mathsf{Cur}\, s = \{(w^0, (u^0, b)) \mid ((w^0, u^0), b) \in s\} \,.$$

6.4.6 FIXPOINT OPERATORS. Let $S$ be a preorder and let $s \in \mathbf{PoLR}_!(S, S)$. We know that $f = \mathsf{Fun}\, s$ is a Scott continuous function $\mathcal{I}(S) \to \mathcal{I}(S)$. Since $f$ is monotonic, we have $\emptyset \subseteq f(\emptyset) \subseteq f^2(\emptyset) \subseteq \cdots$ (by induction). Let $u = \bigcup_{n \in \mathbb{N}} f^n(\emptyset)$, we have $u \in \mathcal{I}(S)$ and moreover $f(u) = u$ (by Scott continuity), that is $u$ is a fixpoint of $f$ (it is actually the least fixpoint of $f$).

**Exercise 6.4.7** Prove that this set is the least set $u$ such that for all $u^0 \in \mathcal{P}_{\mathrm{fin}}(u)$ and $a \in |S|$, if $(u^0, a) \in s$, then $a \in u$.

We want now to prove that the function which maps $s$ to its least fixpoint is itself Scott continuous from $\mathcal{I}(S \Rightarrow S)$ to $\mathcal{I}(S)$. It would be rather easy to prove it directly, but we prefer to present a more elaborate approach which uses the cartesian closeness of $\mathbf{PoLR}_!$ (and the fact that, in that category, any morphism from an object to itself has a least fixpoint). The benefit is that this method can be applied in many other models where a direct proof that the operator mapping a morphism to its least fixpoint is a morphism would be much more painful (we think in particular of game models).

**Exercise 6.4.8** Prove directly that the function $\mathcal{I}(S \Rightarrow S) \to \mathcal{I}(S)$ which maps $s$ to $\bigcup_{n \in \mathbb{N}} (\mathsf{Fun}\, s)^n(\emptyset)$ is Scott continuous.

Given a preorder $S$, we define a morphism

$$\mathcal{Z} \in \mathbf{PoLR}_!((S \Rightarrow S) \Rightarrow S, (S \Rightarrow S) \Rightarrow S)$$

using the cartesian closeness of $\mathbf{PoLR}_!$ (the same definition makes sense in any cartesian closed category). Setting $T = (S \Rightarrow S) \Rightarrow S$, we define $\mathcal{Z} = \mathsf{Cur}\, \mathcal{Z}_0$ where $\mathcal{Z}_0 \in \mathbf{PoLR}_!(T \,\&\, (S \Rightarrow S), S)$ is given by the following composition of morphisms in $\mathbf{PoLR}_!$ :

$$\begin{array}{ccccc} T \,\&\, (S \Rightarrow S) & \xrightarrow{T \,\&\, \langle \mathsf{Id}, \mathsf{Id} \rangle} & T \,\&\, (S \Rightarrow S) \,\&\, (S \Rightarrow S) & \xrightarrow{\mathsf{Ev} \,\&\, \mathsf{Id}} & S \,\&\, (S \Rightarrow S) \\ & & & & \big\downarrow \langle \pi_2, \pi_1 \rangle \\ & & S & \xleftarrow{\mathsf{Ev}} & (S \Rightarrow S) \,\&\, S \end{array}$$

The same morphism $\mathcal{Z}$ can also be defined as $\mathcal{Z} = \mathsf{cur}\, \mathcal{Z}_1$ (curryfication in the symmetric monoidal closed category $\mathbf{PoLR}$ now) where $\mathcal{Z}_1$ the following composition in $\mathbf{PoLR}$

$$\begin{array}{ccc} !_{\mathsf{s}} T \otimes !_{\mathsf{s}}(!_{\mathsf{s}} S \multimap S) & \xrightarrow{!_{\mathsf{s}} T \otimes \mathsf{c}} & !T \otimes !_{\mathsf{s}}(!_{\mathsf{s}} S \multimap S) \otimes !_{\mathsf{s}}(!_{\mathsf{s}} S \multimap S) \\ & & \big\downarrow e_{!S \multimap S, S} \otimes \mathsf{der} \\ S \xleftarrow{\mathsf{ev}} (!_{\mathsf{s}} S \multimap S) \otimes !_{\mathsf{s}} S & \xleftarrow{\sigma} & !_{\mathsf{s}} S \otimes (!_{\mathsf{s}} S \multimap S) \end{array}$$

where, for any preorders $U, V$, we set $e_{U,V} = (\mathsf{ev}\, (\mathsf{der}_{!U \multimap V} \otimes !U))^! : !(!U \multimap V) \otimes !U \to !V$. This morphism is characterized by

$$e_{U,V}\, (w^{!_{\mathsf{s}}} \otimes u^{!_{\mathsf{s}}}) = (w\, u^{!_{\mathsf{s}}})^{!_{\mathsf{s}}}$$

for all $w \in \mathcal{I}(!U \multimap V)$ and $u \in \mathcal{I}(U)$. Therefore $\mathcal{Z}_1$ is characterized by

$$\mathcal{Z}_1\, (Y^{!_{\mathsf{s}}} \otimes t^{!_{\mathsf{s}}}) = t\, (Y\, t^{!_{\mathsf{s}}})^{!_{\mathsf{s}}}$$

for all $Y \in \mathcal{I}(T)$ (intuitively, $Y$ is a candidate for being a fixpoint operator) and $t \in \mathcal{I}(!_{\mathsf{s}} S \multimap S)$. Setting $F = \mathsf{Fun}\, \mathcal{Z}$, the element $F(Y)$ of $\mathcal{I}(T)$ is characterized by

$$\forall t \in \mathcal{I}(!S \multimap S) \quad \mathsf{Fun}(F(Y))(t) = \mathsf{Fun}(t)(\mathsf{Fun}(Y)(t))$$

We define our fixpoint operator $\mathcal{Y}_S \in \mathcal{I}(T)$ as the least fixpoint of $F$ : $\mathcal{Y}_S = \bigcup_{n=0}^{\infty} F^n(\emptyset)$. Therefore, by the property above, it maps any $t \in \mathcal{I}(!S \multimap S)$ to a fixpoint of the Scott continuous function $\mathsf{Fun}(t) : \mathcal{I}(S) \to \mathcal{I}(S)$.

**Exercise 6.4.9** With the notations above, prove that $\mathsf{Fun}(\mathcal{Y}_S)(t)$ is the least fixpoint of $f = \mathsf{Fun}(t)$, that is $\mathsf{Fun}(\mathcal{Y}_S)(t) = \bigcup_{n=0}^{\infty} f^n(\emptyset)$.

**Exercise 6.4.10** Prove that $\mathcal{Y}_S$ is the least element of $\mathcal{I}(T) = \mathcal{I}(!_{\mathsf{s}}(!_{\mathsf{s}}S \multimap S) \multimap S)$ such that, if $(U^0, a) \in \mathcal{Y}_S$, then there exists $(v^0, b) \in U^0$ such that $a \leq_S b$ and $\forall c \in v^0\ (U^0, c) \in \mathcal{Y}_S$. For instance $(\{(\emptyset, a)\}, a) \in \mathcal{Y}_S$ and $(\{(\{b\}, a), (\emptyset, b)\}, a) \in \mathcal{Y}_S$.

6.4.7 Variable types and type fixpoints Given two preorders $S$ and $T$, we say that $S$ is a *sub-preorder* of $T$ and write $S \subseteq T$ if $|S| \subseteq |T|$ and

$$\forall a, a' \in |S| \quad a \leq_S a' \Leftrightarrow a \leq_T a' \,.$$

We denote as $\mathbf{PoLR}_{\subseteq}$ the partially ordered class [2] of all preorders equiped with the $\subseteq$ partial order relation (it is clear that this relation is an order relation).

**Exercise 6.4.11** Prove that any directed subset $\mathcal{D}$ of $\mathbf{PoLR}_{\subseteq}$ has a least upper bound in $\mathbf{PoLR}_{\subseteq}$, for the $\subseteq$ order relation.

This least upper bound will be denoted as $\bigcup \mathcal{D}$, and it is characterized by

$$|\bigcup \mathcal{D}| = \bigcup_{S \in \mathcal{D}} |S|$$
$$\forall a, a' \in |\bigcup \mathcal{D}| \quad a \leq_{\bigcup \mathcal{D}} a' \Leftrightarrow \exists S \in \mathcal{D}\, a \leq_S a' \,.$$

A $k$-ary variable type is a function $\Phi : \mathbf{PoLR}_{\subseteq}^k \to \mathbf{PoLR}_{\subseteq}$ which is monotonic and Scott continuous (for the product order in $\mathbf{PoLR}_{\subseteq}^k$).

**Lemme 6.4.16** *The functions $S \mapsto S^\perp$ and $S \mapsto !_{\mathsf{s}}S$ are variable types or arity 1. The functions $(S_1, S_2) \mapsto S_1 \otimes S_2$ and $(S_1, S_2) \mapsto S_1 \,\&\, S_2$ are variable types of arity 2.*

**Exercise 6.4.12** Prove this lemma.

Remember that all connectives of linear are definable in terms of the connectives mentioned in this lemma, for instance $S \multimap T = (S \otimes T^\perp)^\perp$, and hence are variable types.

Then, given a $k+1$-ary variable type $\Phi$, we define $\mathsf{Rec}\,\Phi : \mathbf{PoLR}_{\subseteq}^k \to \mathbf{PoLR}_{\subseteq}$ by

$$\mathsf{Rec}\,\Phi(S_1, \ldots, S_k) = \bigcup_{n=0}^{\infty} \Psi^n(0)$$

where 0 is the preorder such that $|0| = \emptyset$ (it is the least element of $\mathbf{PoLR}_{\subseteq}$) and $\Psi$ is the variable type of arity 1 defined by $\Psi(S) = \Phi(S_1, \ldots, S_k, S)$. Observe indeed that we have

$$0 \subseteq \Psi(0) \subseteq \Psi^2(0) \subseteq \cdots$$

so that this preorder $\mathsf{Rec}\,\Phi(S_1, \ldots, S_k)$ is well defined.

One proves easily that $\mathsf{Rec}\,\Phi$ is a variable type (of arity $k$). By Scott continuity of $\Phi$, we have

$$\Phi(S_1, \ldots, S_k, \mathsf{Rec}\,\Phi(S_1, \ldots, S_k)) = \mathsf{Rec}\,\Phi(S_1, \ldots, S_k) \,.$$

As an example, consider $\Phi$, the variable type of arity 1 defined by $\Phi(S) = (!_{\mathsf{s}}(\mathsf{N} \multimap S))^\perp$ where $\mathsf{N}$ is the preorder such that $|\mathsf{N}| = \mathbb{N}$ and $\leq_{\mathsf{N}}$ is equality (that is $s \leq_{\mathsf{N}} m$ if $n = m$). Then, since $\mathsf{N} \simeq 1 \oplus \mathsf{N}$ (by the strong isomorphism which maps 0 to $(1, *)$ and $n+1$ to $(2, n)$) we have, setting $D_\infty = \mathsf{Rec}\,\Phi$ :

$$\begin{aligned} D_\infty &= \Phi(D_\infty) \\ &= (!_{\mathsf{s}}(\mathsf{N} \multimap D_\infty))^\perp \\ &\simeq (!_{\mathsf{s}}((1 \oplus \mathsf{N}) \multimap D_\infty))^\perp \\ &\simeq (!_{\mathsf{s}}(D_\infty \,\&\, (\mathsf{N} \multimap D_\infty)))^\perp \\ &\simeq D_\infty \Rightarrow D_\infty \text{ by Seely iso and the definition of } D_\infty \end{aligned}$$

2. It is not a set in the sense of ZF set theory.

which means that we have built a model of the pure $\lambda$-calculus (which is actually isomorphic to Dana Scott's $D_\infty$, historically the first model of the pure $\lambda$-calculus).

A simpler example is obtained by taking $\Phi(S) = 1 \oplus !S$. Then $\mathsf{L} = \mathsf{Rec}\,\Phi$ will be the interpretation of the type of lazy integers of the language LPCF of Chapter 2. We have $|L| = \bigcup_{n\in\mathbb{N}} |\Phi^n(0)|$ and hence for any $a \in |L|$ we can define $\mathsf{ht}(a)$, the height of $a$, as the least $n \in \mathbb{N}$ such that $a \in |\Phi^n(0)|$. Of course we have $\mathsf{ht}(a) \geq 1$ for each $a$.

Any $a \in |\mathsf{L}|$ is of shape either $(1, *)$ (where $*$ is the unique element of $|1|$) and then $\mathsf{ht}(a) = 1$, or of shape $(2, \{a_1, \dots, a_k\})$ where $a_i \in |L|$ and then $\mathsf{ht}(a) = 1 + \max_{i=1}^k(\mathsf{ht}(a_i))$.

We use the following more suggestive notations :

$$\zeta = (1, *) \text{ which represents } 0$$
$$\overline{\mathsf{suc}}\, u^0 = (2, u^0)$$

where $u^0$ is a finite subset of $|\mathsf{L}|$ whose all element's heights are $<$ than $\mathsf{ht}(a)$. This $\overline{\mathsf{suc}}\, u^0$ is a kind of "successor" of $u^0$, but observe that $u^0$ is not really an integer, but should rather be understood as a finite set of potential values for an integer.

Then the preorder relation of $\mathsf{L}$ is characterized by the following clauses :

$$\zeta \leq_{\mathsf{L}} a \text{ iff } a = \zeta$$
$$\overline{\mathsf{suc}}\, u^0 \leq_{\mathsf{L}} a \text{ iff } a = \overline{\mathsf{suc}}\, u^1 \text{ with } u^0 \leq_{!_{\mathsf{s}}\mathsf{L}} u^1 \,.$$

## 6.5 Scott semantics of LPCF as an intersection typing system.

We are now interested in investigating the properties of the Scott semantics of our language LPCF (see Chapter 2). Given a type $A$ of this language, we define $[A]$ as an object of **PoLR** as follows :

$$[\iota] = \mathsf{L}$$
$$[A \to B] = [A] \Rightarrow [B] = !_{\mathsf{s}}[A] \multimap [B]$$

For any type $A$ and any element $a$ of $|A|$, we define $\mathsf{ht}_A(a) \in \mathbb{N}$ by induction on $A$ as follows :

— if $A = \iota$ then $\mathsf{ht}_\iota(a) = \mathsf{ht}(a)$

— and if $A = (B \to C)$ and $a = (\{b_1, \dots, b_k\}, c)$ then $\mathsf{ht}_A(a) = 1 + \max(\mathsf{ht}(c), \mathsf{ht}(b_1), \dots, \mathsf{ht}(b_k))$.

Also, if $u^0 \in |!_{\mathsf{s}}[A]|$, we define $\mathsf{ht}_A(u^0)$ as the max of all $\mathsf{ht}_A(a)$ for $a \in u^0$ (so that $\mathsf{ht}_A(\emptyset) = 0$).

Given a typing context $\Gamma = (x_1 : A_1, \dots, x_k : A_k)$, a type $A$ and a term $M$ such that $\Gamma \vdash M : A$, we define

$$[M]_\Gamma \in \mathbf{PoLR}(!_{\mathsf{s}}[A_1] \otimes \cdots \otimes !_{\mathsf{s}}[A_k], [A])$$

that is, equivalently, $[M]_\Gamma$ is an element of $\mathbf{PoLR}_!([A_1] \,\&\, \cdots \,\&\, [A_k], [A])$ or a Scott continuous function

$$\prod_{i=1}^k \mathcal{I}([A_i]) \to \mathcal{I}([A]) \,.$$

We do not give the details of this categorical interpretation which follows a standard pattern which can be found elsewhere in this document. We prefer to give directly a "type theoretic" presentation of this interpretation and take for granted that it coincides with this denotational semantics.

A *semantic context* is an expression $\Phi = (x_1 : u^1 : A_1, \dots, x_k : u^k, A_k)$ where the $x_i$'s are pairwise distinct variables, the $A_i$'s are types and $u^i \in |!_{\mathsf{s}}[A_i]|$ for $i = 1, \dots, k$. We use $\underline{\Phi}$ for the "underlying typing context" $(x_1 : A_1, \dots, x_k : A_k)$.

Given a typing context $\Gamma = (x_1 : A_1, \dots, x_k : A_k)$ and $u^i \in |!_{\mathsf{s}}[A_i]|$ for $i = 1, \dots, k$ and $a \in |A|$, we write

$$x_1 : u^1 : A_1, \dots, x_k : u^k : A_k \vdash M : a : A$$

instead of $(u^1, \dots, u^k, a) \in [M]_\Gamma$.

More precisely, we admit the following result (whose proof is a simple but boring verification).

**Proposition 6.5.1** *One has $(u^1,\dots,u^k,a)\in[M]_\Gamma$ iff the judgment $x_1:u^1:A_1,\dots,x_k:u^k:A_k\vdash M:a:A$ is derivable in the following typing system.*

$$\frac{\exists a'\in u^0\ a\leq_{[A]}a'}{\Phi,x:u^0:A\vdash x:a:A}\qquad\frac{\Phi\vdash M:(\{a_1,\dots,a_n\},b):A\to B\qquad \Phi\vdash N:a_i:A\text{ for }i=1,\dots,n}{\Phi\vdash(M)\,N:b:B}$$

$$\frac{\Gamma,x:u^0:A\vdash M:b:B}{\Phi\vdash\lambda x^A\,M:(u^0,b):A\to B}$$

$$\frac{\Phi,x:\{a_1,\dots,a_n\}:A\vdash M:a:A\qquad \Phi\vdash \mathsf{fix}\,x^A\cdot M:a_i:A\text{ for }i=1,\dots,n}{\Phi\vdash\mathsf{fix}\,x^A\cdot M:a:A}$$

$$\frac{}{\Phi\vdash\underline{0}:\zeta:\iota}\qquad\frac{\Phi\vdash M:a_i:\iota\text{ for }i=1,\dots,n}{\Phi\vdash\underline{\mathsf{succ}}(M):\overline{\mathsf{suc}}\,\{a_1,\dots,a_n\}:\iota}$$

$$\frac{\Phi\vdash M:\zeta:\iota\qquad\Phi\vdash P:a:A\qquad\underline{\Phi},z:\iota\vdash Q:A}{\Phi\vdash\mathsf{if}(M,P,z\cdot Q):a:A}$$

$$\frac{\Phi\vdash M:\overline{\mathsf{suc}}\,u^0:\iota\qquad\underline{\Phi}\vdash P:A\qquad\Phi,z:u^0:\iota\vdash Q:a:A}{\Phi\vdash\mathsf{if}(M,P,z\cdot Q):a:A}$$

As a consequence of Proposition 6.5.1, this typing system enjoys the following properties.

**Proposition 6.5.2** *Assume that $\Phi\vdash M:a:A$ where $\Phi=(x_1:u^1:A_1,\dots,x_k:u^k:A_k)$.*
- *If $\underline{\Phi}\vdash M':A$ and $M'\sim_\beta M$ then $\Phi\vdash M':a:A$.*
- *If $v^i\in|!_{\mathsf{s}}[A_i]|$ satisfy $u^i\leq_{!_{\mathsf{s}}[A_i]}v^i$ for $i=1,\dots,k$, and if $b\in|[A]|$ satisfy $b\leq_{[A]}a$, then we have $x_1:v^1:A_1,\dots,x_k:v^k:A_k\vdash M:b:A$.*

Therefore, if $\vdash M:\iota$ (so $M$ is a closed term) satisfy $M\sim_\beta\underline{0}$, we have $\vdash M:\zeta:\iota$. We want to prove something which is stronger than the converse property, namely : if $\vdash M:\zeta:\iota$ then $M\ {\beta_{\mathsf{wh}}}^*\ \underline{0}$.

For this purpose we introduce an auxiliary notion. Given any type $A$ and any $a\in|[A]|$, we define a set $|a|^A$ of closed terms $M$ such that $\vdash M:A$. And if $u^0\in|![A]|$, we define a set $|u^0|_!^A$ of closed terms $M$ such that $\vdash M:A$. The definition is by induction on $\mathsf{ht}_A(a)$ :
- $|\zeta|^\iota=\{M\mid\ \vdash M:\iota\text{ and }M\ {\beta_{\mathsf{wh}}}^*\ \underline{0}\}$
- $|\overline{\mathsf{suc}}\,u^0|^\iota=\{M\mid\ \vdash M:\iota\text{ and }\exists N\in|u^0|_!^\iota\ M\ {\beta_{\mathsf{wh}}}^*\ \underline{\mathsf{succ}}(N)\}$.
- $|(u^0,b))|^{A\to B}=\{M\mid\ \vdash M:A\to B\text{ and }\forall N\in|u^0|_!^A\ (M)\,N\in|b|^B\}$.
- If $u^0\in|!_{\mathsf{s}}[A]|$ then $|u^0|_!^A=\{M\mid\ \vdash M:A\text{ and }\forall a\in u^0\ M\in|a|^A\}$.

Observe that $|\emptyset|_!^A$ is the set of all closed terms $M$ such that $\vdash M:A$. So for instance $|\overline{\mathsf{suc}}\,\emptyset|^\iota$ is the set of all closed terms $M$ such that $\vdash M:\iota$ and $M\ {\beta_{\mathsf{wh}}}^*\ \underline{\mathsf{succ}}(N)$ for some term $N$ (with $\vdash N:\iota$). Typically $\underline{\mathsf{succ}}(\Omega^\iota)\in|\overline{\mathsf{suc}}\,\emptyset|^\iota$ where $\Omega^\iota=\mathsf{fix}\,x^\iota\cdot x$ is an everlooping term of type $\iota$.

**Lemme 6.5.3** *Let $M$ and $M'$ be closed terms such that $\vdash M:A$ for some type $A$ and $M\ {\beta_{\mathsf{wh}}}^*\ M'$. Let $a\in|[A]|$ and $u^0\in|!_{\mathsf{s}}[A]|$. If $M'\in|a|^A$ then $M\in|a|^A$ and if $M'\in|u^0|_!^A$ then $M\in|u^0|_!^A$.*

*Démonstration.* The proof is by induction on the height of $a$ and $u^0$. We use the notations of the statement of the lemma.

Assume first that $A=\iota$ and $a=\zeta$. If $M'\in|\zeta|^\iota$, then we know that $M'\ {\beta_{\mathsf{wh}}}^*\ \underline{0}$, and hence $M\ {\beta_{\mathsf{wh}}}^*\ \underline{0}$ since $M\ {\beta_{\mathsf{wh}}}^*\ M'$.

Assume now that $A=\iota$ and $a=\overline{\mathsf{suc}}\,u^0$. If $M'\in|a|^\iota$, then we know that $M'\ {\beta_{\mathsf{wh}}}^*\ \underline{\mathsf{succ}}(N)$ for some $N\in|u^0|_!^\iota$, and hence $M\ {\beta_{\mathsf{wh}}}^*\ \underline{\mathsf{succ}}(N)$ since $M\ {\beta_{\mathsf{wh}}}^*\ M'$ so that $M\in|a|^\iota$.

Assume next that $A=(B\to C)$ and $a=(v^0,c)\in|A|$ and that $M'\in|a|^A$. We must prove that $M\in|a|^A$ so let $N\in|v^0|_!^B$, we have to prove that $(M)\,N\in|c|^C$. But we know that $(M')\,N\in|c|^C$ and since $M\ {\beta_{\mathsf{wh}}}^*\ M'$, we have $(M)\,N\ {\beta_{\mathsf{wh}}}^*\ (M')\,N$ by definition of $\beta_{\mathsf{wh}}$. Therefore we have $(M)\,N\in|c|^C$ by the inductive hypothesis applied to $c$. ☐

**Lemme 6.5.4** *Let $a,a'\in|[A]|$ with $a\leq_{[A]}a'$. Then $|a'|^A\subseteq|a|^A$. Let $u^0,u^1\in|!_{\mathsf{s}}[A]|$ with $u^0\leq_{!_{\mathsf{s}}[A]}u^1$. Then $|u^1|_!^A\subseteq|u^0|_!^A$.*

*Démonstration.* We use the notation of the statement of the lemma. The proof is by induction on $\mathsf{ht}_A(a) + \mathsf{ht}_A(a')$ (and $\mathsf{ht}_A(u^0) + \mathsf{ht}_A(u^1)$).

Assume first that $A = \iota$. The following cases can arise :

— $a = a' = \zeta$ : there is nothing to prove.

— $a = \overline{\mathsf{suc}}\, u^0$ and $a' = \overline{\mathsf{suc}}\, u^1$ with $u^0 \leq_{!\mathsf{L}} u^1$. Let $M \in |a'|^\iota$, we must prove that $M \in |a|^\iota$. There exists $N$ such that $\vdash N : \iota$, $N \in |u^1|^\iota_!$ and $M\ {\beta_{\mathsf{wh}}}^*\ \underline{\mathsf{succ}}(N)$. For each $b \in u^0$ there is an element $m(b) \in u^1$ such that $b \leq_{\mathsf{L}} m(b)$ so that, by inductive hpothesis, $|b|^\iota \supseteq |m(b)|^\iota$. It follows that $|u^0|^\iota_! \supseteq \bigcap_{b\in u^0} |m(b)|^\iota \supseteq |u^1|^\iota_!$. Therefore $N \in |u^0|^\iota_!$ and hence $M \in |a|^\iota$ as expected.

Assume now that $A = (B \to C)$. Then we have $a = (v^0, c)$ and $a' = (v^1, c')$ with $c \leq_{[C]} c'$ and $v^1 \leq_{!_{\mathsf{s}}[B]} v^0$. Let $M \in |a'|^A$ and let us prove that $M \in |a|^A$. So let $N \in |v^0|^B_!$, we must prove that $(M)\, N \in |c|^C$. By inductive hypothesis we have $N \in |v^1|^B_!$ and hence $(M)\, N \in |c'|^C$ by the assumption that $M \in |a'|^A$ and it follows that $(M)\, N \in |c|^C$ by inductive hypothesis again, applied to $c$ and $c'$. □

Now we can prove the main property of this interpretation of the points of the model as sets of terms.

**Proposition 6.5.5 (Interpretation Lemma)** *Assume that $\Phi \vdash M : b : B$ where $\Phi = (x_1 : u^1 : A_1, \dots, x_k : u^k : A_k)$ is a semantic typing context. Then, for any terms $N_1 \in |u^1|^{A_1}_!$,…,$N_k \in |u^k|^{A_k}_!$, one has*

$$M\,[N_1/x_1, \dots N_k/x_k] \in |b|^B\,.$$

*Démonstration.* The proof is by induction on the height of the derivation of the judgment $\Phi \vdash M : b : B$ in our semantic typing sytem.

It is essential to notice that the statement that we prove by induction is the following universally quantified statement : for any terms $N_1 \in |u^1|^{A_1}_!$,…,$N_k \in |u^k|^{A_k}_!$, one has $M\,[N_1/x_1, \dots N_k/x_k] \in |b|^B$. This universal quantification will be used in a crucial way in the inductive hypothesis.

In the sequel, we set $P' = P\,[N_1/x_1, \dots N_k/x_k]$ for any term $P$ such that $\underline{\Phi} \vdash P : C$ for a type $C$, so that we have $\vdash P' : C$.

The first two cases are the axioms of our deduction system, corresponding to the cases where $M$ is a variable or the constant $\underline{0}$.

Assume that $M = x_i$ for some $i \in \{1, \dots, k\}$, so that $B = A_i$ and that there is $a \in u^i$ such that $b \leq_{[A_i]} a$. Then $M' = N_i$ (using the notational convention above) so that $M' \in |u^i|^{A_i}_! = \bigcap_{a'\in u^0} |[a']|^{A_i} \subseteq |[a]|^{A_i}$. By Lemma 6.5.5 (it is the only place of the proof where this lemma is used, but it is crucial) we have $|[a]|^{A_i} \subseteq |[b]|^{A_i}$ and hence $M' \in |[b]|^B$ as expected.

Assume that $M = \underline{0}$, so that $B = \iota$ and $b = \zeta$. Then $M' = \underline{0}$ so that $M'\ {\beta_{\mathsf{wh}}}^*\ \underline{0}$ (in 0 reduction steps actually) and hence $M' \in |\zeta|^\iota$ as required.

Assume that $M = \underline{\mathsf{succ}}(P)$, so that $B = \iota$ and there are $b_1, \dots, b_n \in |[\iota]|$ such that $b = \overline{\mathsf{suc}}\, \{b_1, \dots, b_n\}$ with $\Phi \vdash P : b_j : \iota$ for $j = 1, \dots, n$. By inductive hypothesis we have $P' \in |b_j|^\iota$ for $j = 1, \dots, n$, that is $P' \in |\{b_1, \dots, b_n\}|^\iota_!$. We have $M'\ {\beta_{\mathsf{wh}}}^*\ \underline{\mathsf{succ}}(P')$ (in 0 steps actually) and hence $M' \in |b|^\iota$ as expected, by definition of $|b|^\iota$ in that case.

Assume that $M = (P)\, Q$ and that, for some type $C$ and some $w^0 = \{c_1, \dots, c_n\} \in\ !_{\mathsf{s}}|[C]|$ we have $\Phi \vdash P : (w^0, b) : C \to B$ and $\Phi \vdash Q : c^j : C$ for $j = 1, \dots, n$. By inductive hypothesis we have $P' \in |(w^0, b)|^{C\to B}$ and $Q' \in |w^0|^C_! = \bigcap_{j=1}^n |c_j|^C$. By definition of $|(w^0, b)|^{C\to B}$ we have $M' = (P')\, Q' \in |b|^B$.

Assume that $B = (A \to C)$, $M = \lambda x^A\, P$ and $b = (u^0, c)$ with $\Phi, x : u^0 : A \vdash P : c : C$. We must prove that $\lambda x^A\, P' \in |(u^0, c)|^{A\to C}$. So let $N \in |u^0|^A_!$, we must prove that $\left(\lambda x^A\, P'\right) N \in |c|^C$. By Lemma 6.5.4 and by the definition of $\beta_{\mathsf{wh}}$, it suffices to prove that $P'\,[N/x] \in |c|^C$ which in turn results from the inductive hypothesis applied to the derivation of $\Phi, x : u^0, A \vdash P : c : C$, with substituted terms $N_1, \dots, N_k, N$ (since $P'\,[N/x] = P\,[N_1/x_1, \dots, N_k/x_k, N/x]$). Here we have used in a crucial way the fact that the inductive hypothesis is universally quantified.

Assume that $M = \mathsf{fix}\, x^B \cdot P$ so that, for some $v^0 = \{b_1, \dots, b_n\}$, we have $\Phi, x : v^0 : B \vdash P : b : B$ and $\Phi \vdash M : b_j : B$ for $j = 1, \dots, n$. We must prove that $M' = \mathsf{fix}\, x^B \cdot P' \in |b|^B$. Since $M'\ {\beta_{\mathsf{wh}}}^*\ P'\,[M'/x]$, it suffices to prove that $P'\,[M'/x] \in |b|^B$ (by Lemma 6.5.4). By inductive hypothesis we know that $M' \in |b_j|^B$ for $j = 1, \dots, n$, that is $M' \in |v^0|^B_!$. Therefore, by inductive hypothesis again (applied now to the semantic typing derivation of $\Phi, x : v^0 : B \vdash P : b : B$), we have $P\,[N_1/x_1, \dots, N_k/x_k, M'/x] \in |b|^B$, that is $P'\,[M'/x] \in |b|^B$ as expected. Again we have used in a crucial way the fact that the inductive hypothesis is universally quantified.

Last assume that $M = \mathsf{if}(P, Q, z \cdot R)$ and that we have a derivation of the semantic typing judgment $\Phi \vdash M : b : B$ in our semantic typing system. There are two possibilities as to the last deduction rule of this typing derivation.

In the first case, we know that we have a derivation of the judgments $\Phi \vdash P : \zeta : \iota$, $\Phi \vdash Q : b : B$ and $\underline{\Phi}, z : \iota \vdash R : B$. By inductive hypothesis we have $P' \in |\zeta|^{\iota}$, which means that $P' \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$ and hence $M' = \mathsf{if}(P', Q', z \cdot R') \; {\beta_{\mathsf{wh}}}^* \; \mathsf{if}(\underline{0}, Q', z \cdot R') \; \beta_{\mathsf{wh}} \; Q'$. By inductive hypothesis we have $Q' \in |b|^B$, and hence by Lemma 6.5.4 we have $M' \in |b|^B$.

In the second case we have derivations of the judgments $\Phi \vdash P : \overline{\mathsf{suc}}\, u^0 : \iota$, $\underline{\Phi} \vdash Q : B$ and $\Phi, z : u^0 : \iota \vdash R : b : B$. Then we know by inductive hypothesis that there is a term $N \in |u^0|^{\iota}_{!}$ such that $P' \; {\beta_{\mathsf{wh}}}^* \; \underline{\mathsf{succ}}(N)$. Therefore $M' \; {\beta_{\mathsf{wh}}}^* \; \mathsf{if}(P', Q', z \cdot R') \; {\beta_{\mathsf{wh}}}^* \; \mathsf{if}(\underline{\mathsf{succ}}(N), Q', z \cdot R') \; \beta_{\mathsf{wh}} \; R' [N/z]$. By inductive hypothesis applied to the typing derivation of $R$ (using the fact that $N \in |u^0|^{\iota}_{!}$), we get $R' [N/z] \in |b|^B$. Therefore, by Lemma 6.5.4 we have $M' \in |b|^B$. Here agin we have used crucially the universal quantification in our inductive hypothesis.

This ends the proof of the proposition. □

### 6.5.1 Adequacy and observational equivalence for LPCF, definition of full abstraction.

**Théorème 6.5.6 (Adequacy)** *Let $M$ be a term such that $\vdash M : \iota$. If $\zeta \in [M]$, that is $\vdash M : \zeta : \iota$ is derivable, then $M \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$.*

*Démonstration.* By Proposition 6.5.5, if $\vdash M : \zeta : \iota$ is derivable we have $M \in |\zeta|^{\iota}$, which means exactly that $M \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$. □

From this we can deduce a purely syntact result, sometimes called "standardisation".

**Théorème 6.5.7 (Completeness of $\beta_{\mathsf{wh}}$)** *Assume that $\vdash M : \iota$ and $M \sim_{\beta} \underline{0}$. Then $M \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$.*

*Démonstration.* Since $M \sim_{\beta} \underline{0}$ we have $[M] = [\underline{0}] = \{\zeta\}$ by Proposition 6.5.2, and hence $\vdash M : \zeta : \iota$. Therefore $M \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$ by Theorem 6.5.6. □

This means that the reduction strategy $\beta_{\mathsf{wh}}$ is sufficiently powerful for checking that a closed term of type $\iota$ is equal to $\underline{0}$.

However, notice that if $M \sim_{\beta} \underline{\mathsf{succ}}(\underline{0})$, it is not true that $M \; {\beta_{\mathsf{wh}}}^* \; \underline{\mathsf{succ}}(\underline{0})$. What is true in that case is that $M \; {\beta_{\mathsf{wh}}}^* \; \underline{\mathsf{succ}}(N)$ for some term $N$ such that $N \sim_{\beta} \underline{0}$, and hence $N \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$.

Let now $M_1$ and $M_2$ be two terms such that $\vdash M_i : A$ for $i = 1, 2$. We say that $M_1$ and $M_2$ are *observationally equivalent* and write $M_1 \simeq_{\mathrm{obs}} M_2$ if, for any term $P$ such that $\vdash P : A \to \iota$, we have $(P)\, M_1 \; {\beta_{\mathsf{wh}}}^* \; \underline{0} \Leftrightarrow (P)\, M_2 \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$.

**Exercise 6.5.1** Given $M_1$ and $M_2$ two terms such that $\vdash M_i : A$ for $i = 1, 2$, prove that $M_1 \sim_{\beta} M_2 \Rightarrow M_1 \simeq_{\mathrm{obs}} M_2$.

**Exercise 6.5.2** Let us say that a term $M$ such that $\vdash M : \iota$ converges, and write $M{\downarrow_{\mathsf{wh}}}$, if $M \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$ or $M \; {\beta_{\mathsf{wh}}}^* \; \underline{\mathsf{succ}}(N)$ for some term $N$. Given $M_1$ and $M_2$ two terms such that $\vdash M_i : A$ for $i = 1, 2$, prove that $M_1 \simeq_{\mathrm{obs}} M_2$ iff for any term $P$ such that $\vdash P : A \to \iota$, we have $(P)\, M_1{\downarrow_{\mathsf{wh}}} \Leftrightarrow (P)\, M_2{\downarrow_{\mathsf{wh}}}$.

**Théorème 6.5.8** *Let $M_1$ and $M_2$ be two terms such that $\vdash M_i : A$ for $i = 1, 2$. If $[M_1] = [M_2]$ then $M_1 \simeq_{\mathrm{obs}} M_2$.*

*Démonstration.* Let $P$ be a term such that $\vdash P : A \to \iota$ and assume that $(P)\, M_1 \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$. Then we have $\zeta \in [(P)\, M_1]$ by Lemma 6.5.2 and hence $\zeta \in [(P)\, M_2]$ by our assumption that $[M_1] = [M_2]$ which implies that $[(P)\, M_1] = [(P)\, M_2]$ [3]. Therefore by Theorem 6.5.6 we have $(P)\, M_2 \; {\beta_{\mathsf{wh}}}^* \; \underline{0}$. □

So the model provides a way for proving that two terms are observationally equivalent : it suffices to prove that they have the same interpretation in the model. This is usually easier than proving directly that they are observationally equivalent, this condition involving a painful universal quantification on

3. Actually, one crucial property of denotational semantics is that it is modular in the sense that the semantics of a term can be computed when one knows the semantics of its immediate subterms.

the term $P$. This method is however not complete (there are terms which are observationally equivalent but have not the same semantics).

One says that a model is *fully abstract* if observational equivalence implies equality in the model, that is, if this semantical method for proving observational equivalence is complete.

## 6.6 Une autre présentation de la même exponentielle.

On préfère utiliser une autre façon de décrire le même foncteur exponentiel. On pose $|!_{\mathsf{sm}}S| = \mathcal{M}_{\mathrm{fin}}(|S|)$ muni du préordre suivant : $m \leq m'$ si, pour tout $a \in \mathsf{supp}(m)$ il existe $a' \in \mathsf{supp}(m')$ tel que $a \leq_S a'$. On rappelle que $\mathsf{supp}(m) = \{a \in |S| \mid m(a) \neq 0\}$. Si $s \in \mathbf{PoLR}(S,T)$, on pose

$$!_{\mathsf{sm}}s = \{(m,p) \in |!S| \times |!T| \mid \forall b \in \mathsf{supp}(p)\ \exists a \in \mathsf{supp}(m) \quad (a,b) \in s\}\,.$$

On vérifie facilement qu'il s'agit d'un foncteur $\mathbf{PoLR} \to \mathbf{PoLR}$. De plus, si $x \in \mathcal{I}(S)$ on pose $x^{!_{\mathsf{sm}}} = \mathcal{M}_{\mathrm{fin}}(x)$ et on voit facilement que $!_{\mathsf{sm}}s\, x^{!_{\mathsf{sm}}} = (s\, x)^{!_{\mathsf{sm}}}$.

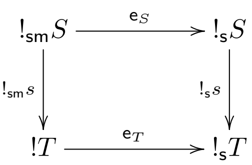
On définit $\mathsf{e}_S \subseteq |!_{\mathsf{sm}}S| \times |!_{\mathsf{s}}S|$ par

$$\mathsf{e}_S = \{(m,u) \in |!_{\mathsf{sm}}S| \times |!_{\mathsf{s}}S| \mid u \leq_{!_{\mathsf{s}}S} \mathsf{supp}(m)\}.$$

Par définition, il est clair que $\mathsf{e} \in \mathbf{PoLR}(!_{\mathsf{sm}}S, !_{\mathsf{s}}S)$.

**Proposition 6.6.1** $\mathsf{e}$ *est un isomorphisme naturel du foncteur* $!_{\mathsf{sm}}\_$ *vers le foncteur* $!_{\mathsf{s}}\_$.

*Démonstration.* On prouve d'abord la naturalité. Soit $s \in \mathbf{PoLR}(S,T)$, on doit montrer que le diagramme suivant commute :




Soient $m \in |!_{\mathsf{sm}}S|$ et $v \in |!_{\mathsf{s}}T|$. Supposons d'abord que $(m,v) \in !_{\mathsf{s}}s\, \mathsf{e}_S$ et montrons que $(m,v) \in \mathsf{e}_T\, !_{\mathsf{sm}}s$. Il existe $u \in |!_{\mathsf{s}}S|$ tel que $u \leq_{!_{\mathsf{s}}S} \mathsf{supp}(m)$ et $(u,v) \in !_{\mathsf{s}}s$. Soit $p \in |!_{\mathsf{sm}}T|$ quelconque tel que $\mathsf{supp}(p) = v$. On a $(m,p) \in !_{\mathsf{sm}}s$ et $(p,v) \in \mathsf{e}_T$, donc $(m,v) \in \mathsf{e}_T\, !_{\mathsf{sm}}s$. Supposons réciproquement que $(m,v) \in \mathsf{e}_T\, !_{\mathsf{sm}}s$. Soit $p$ tel que $(m,p) \in !_{\mathsf{sm}}s$ et $(p,v) \in \mathsf{e}_T$, c'est-à-dire $v \leq_{!_{\mathsf{s}}T} \mathsf{supp}(p)$. Soit $u = \mathsf{supp}(m)$, on a $(m,u) \in \mathsf{e}_S$ et $(u,v) \in !_{\mathsf{s}}s$ et donc $(m,v) \in !_{\mathsf{s}}s\, \mathsf{e}_S$.

Reste à voir que $\mathsf{e}_S$ est un isomorphisme. Soit

$$\mathsf{e}'_S = \{(u,m) \in |!_{\mathsf{s}}S| \times |!_{\mathsf{sm}}S| \mid \mathsf{supp}(m) \leq_{!_{\mathsf{s}}S} u\} \in \mathbf{PoLR}(!_{\mathsf{s}}S, !_{\mathsf{sm}}S)\,.$$

On vérifie facilement que $\mathsf{e}'_S$ est l'inverse de $\mathsf{e}_S$ dans $\mathbf{PoLR}$. □

C'est un très bon exemple d'isomorphisme entre préordres qui ne vient pas d'un isomorphisme fort. A cause de l'existence de cet isomorphisme, on considère à partir de maintenant la version multiensembliste du foncteur $(!_{\mathsf{sm}}\_)$ plutôt que sa version ensembliste. On peut toujours revenir à la version ensembliste en remplaçant partout les opérations multiensemblistes par les opérations ensemblistes correspondantes. Comme ces deux foncteurs sont essentiellement les mêmes, on utilisera désormais la seule notations $!_{\mathsf{s}}\_$ pour la version multiensembliste (ainsi que $x^{!_{\mathsf{s}}}$ pour $\mathcal{M}_{\mathrm{fin}}(x)$ quand $x \in \mathcal{I}(S)$).

## 6.7 Relational semantics

A preorder $S = (|S|, \leq_S)$ is *discrete* if $a \leq_S a' \Leftrightarrow a = a'$. So a discrete preorder is just a set. The full subcategory of $\mathbf{PoLR}$ whose objects are the discrete preorders coincides with the category of sets and relations $\mathbf{Rel}$. The multiplicative and additive operations preserve discreteness of preorders, in other words, they are (very simple) operations on sets. We describe all these constructions explicitly in this section.

The category $\mathbf{Rel}$ has all sets as objects. Identities and composition are defined in the relational way :

$$\mathsf{Id}_E = \{(a,a) \mid a \in E\}$$

and if $s \in \mathbf{Rel}(E,F)$ and $t \in \mathbf{Rel}(F,G)$ then

$$t\,s = \{(a,c) \in E \times G \mid \exists b \in F\, (a,b) \in s \text{ et } (b,c) \in t\}\,.$$

If we consider $s$ and $t$ as (incidence) matrices with coefficients in $\{0,1\}$ (with $1+1 = 1 \times 1 = 1$), this composition can be seen as a simple product of (generally infinite) matrices. See also Remark 8.1.15 for the choice of notations for this product of matrices.

**Exercise 6.7.1** Prove that the isomorphisms of **Rel** are the (graphs of) bijections.

6.7.1 MONOIDAL STRUCTURE. The tensor product of two sets $E$ and $F$ in **Rel** is their cartesian product in the usual set-theoretic sense : $E \otimes F = E \times F$. The tensor product of morphisms is defined as follows. If $s_i \in \mathbf{Rel}(E_i, F_i)$ for $i = 1,2$, then $s_1 \otimes s_2 \in \mathbf{Rel}(E_1 \otimes E_2, F_1 \otimes F_2)$ is given by

$$s_1 \otimes s_2 = \{((a_1,a_2),(b_1,b_2)) \mid (a_i,b_i) \in s_i \text{ pour } i = 1,2\}\,.$$

One checks easily that this operation is functorial and that the associativity and symmetry isomorphisms of the cartesian products in **Set** are isomorphisms in **Rel** which satisfy the axioms of a monoidal category, see 4.6.1. The neutral object of the tensor product is the singleton $1 = \{*\}$.

One defines also $E \multimap F = E \times F$ and then it is very easy to check that $(E \multimap F, \mathsf{ev})$ is the linear internal hom from $E$ to $F$, linear evaluation $\mathsf{ev} \in \mathbf{Rel}((E \multimap F) \otimes E, F)$ is given by

$$\mathsf{ev} = \{(((a,b),a),b) \mid (a,b) \in E \times F\}\,.$$

One checks easily that **Rel**, endowed with this tensor product and the obvious isomorphisms is symmetric monoidal closed. It is a *-autonomous category : one takes $\bot = 1$ as dualizing object. Then one checks that the canonical morphism $\eta_E \in \mathbf{Rel}(E, (E \multimap \bot) \multimap \bot)$ is $\eta_E = \{(a, ((a,*),*)) \mid a \in E\}$ and is thus an isomorphism in **Rel**. Observe that $E$ and $E^\bot = E \multimap \bot$ are isomorphic which is very specific of the absence of structure of the objects of the model (they are bare sets). We identify $E^\bot$ to $E$.

If $s \in \mathbf{Rel}(E,F)$ the *transposed* morphism $s^\mathsf{T} \in \mathbf{Rel}(F^\bot, E^\bot) = \mathcal{P}(F \times E)$ is

$$s^\mathsf{T} = \{(b,a) \in F \times E \mid (a,b) \in s\}\,.$$

6.7.2 PRODUCTS AND COPRODUCTS. Let $(E_i)_{i \in I}$ be a family of sets. Let $\&_{i \in I} E_i = \bigcup_{i \in I}(\{i\} \times E_i)$. This set is the product of the $E_i$'s in the category **Rel** with projections

$$\pi_i = \{((i,a),a) \mid a \in E_i\} \in \mathbf{Rel}\left(\mathop{\&}_{j \in I} E_j, E_i\right)$$

If $s_i \in \mathbf{Rel}(F, E_i)$ is a family of morphisms in **Rel**, the morphism $\langle s_i \rangle_{i \in I} \in \mathbf{Rel}\left(F, \&_{i \in I} E_i\right)$ is given by

$$\langle s_i \rangle_{i \in I} = \{(b,(i,a)) \mid i \in I \text{ et } (b,a) \in s_i\}\,.$$

Since linear negation $E \mapsto E^\bot$ is an involutive contravariant functor which acts as the identity on objects, $\&_{i \in I} E_i$ is also the coproduct $\bigoplus_{i \in I} E_i$ of the $E_i$'s in **Rel**, with $\overline{\pi}_i = \{(a,(i,a)) \mid a \in E_i\}$ as injections.

**Exercise 6.7.2** Prove that the universal property of the cartesian product holds, in other words : $\langle s_i \rangle_{i \in I}$ is the unique element $t$ of $\mathbf{Rel}\left(F, \&_{i \in I} E_i\right)$ such that $\pi_i\, t = s_i$ for each $i \in I$. Explain why the (cartesian) product of $E$ and $F$ is not $E \times F$ (with the usual projections).

6.7.3 EXPONENTIALS. Let $E$ be a set. We set $!E = \mathcal{M}_{\mathrm{fin}}(E)$, the set of all finite *multisets* of elements of $E$. So an element of $\mathcal{M}_{\mathrm{fin}}(E)$ is a function $m : E \to \mathbb{N}$ such that $m(a) = 0$ for all the elements of $E$ but a finite number of them. We use $[]$ for the empty multiset (such that $[](a) = 0$ for all $a \in E$). We use $m + m'$ for the sum of the multisets $m$ and $m'$ defined by $(m+m')(a) = m(a) + m'(a)$. Last $\mathsf{supp}(m)$ denotes the set of all $a \in E$ such that $m(a) \neq 0$, we call the set the *support* of $m$, it is a finite set. We call *cardinality* of $m$ the number of elements $\#m$ of $m$, taking multiplicities into account, in other words $\#m = \sum_{a \in E} m(a) \in \mathbb{N}$.

Given a finite list $a_1, \dots, a_n$ of elements of $E$, we use $[a_1, \dots, a_n]$ for the muliset $m$ such that $m(a) = \#\{i \in \{1, \dots, n\} \mid a_i = a\}$.

Let $x \subseteq E$, we use $x^!$ for the set $\mathcal{M}_{\mathrm{fin}}(x) \subseteq\, !E$ ; it is the *promotion* of $x$. If $s \in \mathbf{Rel}(E, F)$, we set

$$!s = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid n \in \mathbb{N} \text{ et } (a_i, b_i) \in s \text{ pour tout } i\}\,.$$

In other words a pair $(m, p) \in\, !E \times !F$ belongs to $!s$ if the two multisets $m$ and $p$ have the same cardinality $n$ and can be written $m = [a_1, \dots, a_n]$ and $p = [b_1, \dots, b_n]$ with $(a_i, b_i) \in s$ for each $i \in \{1, \dots, n\}$. *Warning* : it is not required that $(a_i, b_j) \in s$ for each pair of indices $(i, j)$.

**Lemme 6.7.1** *The operation $!\_$ is a functor. Moreover if $s \in \mathbf{Rel}(E, F)$ and $x \subseteq E$ then $!s\, x^! = (s\, x)^!$.*

**Exercise 6.7.3** Prove this lemma.

Dereliction and digging are given by

$$\begin{aligned}
\mathsf{der}_E &= \{([a], a) \mid a \in E\} \in \mathbf{Rel}(!E, E) \\
\mathsf{dig}_E &= \{(m_1 + \cdots + m_n, [m_1, \dots, m_n]) \mid m_1, \dots, m_n \in\, !E\} \in \mathbf{Rel}(!E, !!E)\,.
\end{aligned}$$

**Lemme 6.7.2** *These morphisms are natural transformations and turn $!\_$ into a comonad (see Section 4.6.5). Remember that this means that the three following equations hold*

$$\begin{aligned}
\mathsf{der}_{!E}\, \mathsf{dig}_E &= \mathsf{Id}_{!E} \\
!\mathsf{der}_E\, \mathsf{dig}_E &= \mathsf{Id}_{!E} \\
\mathsf{dig}_{!E}\, \mathsf{dig}_E &= !\mathsf{dig}_E\, \mathsf{dig}_E\,.
\end{aligned}$$

**Exercise 6.7.4** Prove these two lemmas.

Let $E$ and $F$ be sets. The function

$$\begin{aligned}
!E \otimes !F &\to\ !(E \,\&\, F) \\
([a_1, \dots, a_p], [b_1, \dots, b_q]) &\mapsto [(1, a_1), \dots, (1, a_p), (2, b_1), \dots, (2, b_q)]
\end{aligned}$$

is a bijection, and hence an isomorphism $\mathsf{m}^2_{E,F} \in \mathbf{Rel}(!E \otimes !F, !(E \,\&\, F))$. One has also a (trivial) iso $\mathsf{m}^0 \in \mathbf{Rel}(1, !\top)$ given by $\mathsf{m}^0 = \{(*, [])\}$. These two isomorphisms satisfy the Seely conditions given in Section 4.6.5.

**Exercise 6.7.5** Check that the diagrams of Seely isomorphisms commute (see Section 4.6.5).

**Exercise 6.7.6** Give an explicit description of the morphisms $\mu^0$, $\mu^2_{E,F}$ and of their generalizations $\mu^n$. For $s \in \mathbf{Rel}(!E_1 \otimes \cdots \otimes !E_n, F)$, give an explicit description of the morphism $s^! \in \mathbf{Rel}(!E_1 \otimes \cdots \otimes !E_n, !F)$. The general definition of these constructions is given in Section 4.6.6.

6.7.4 Kleisli category. The general definition can be found in Section 4.6.7, this category is denoted as $\mathbf{Rel}_!$. An object of this category $\mathbf{Rel}_!$ is a set and $\mathbf{Rel}_!(E, F) = \mathbf{Rel}(!E, F)$. It is not possible to see the elements of $\mathbf{Rel}_!(E, F)$ as functions (at least not directly). We recall that the identity at $E$ is $\mathsf{der}_E \in \mathbf{Rel}_!(E, E)$ and that composition of $s \in \mathbf{Rel}_!(E, F)$ and $t \in \mathbf{Rel}_!(F, G)$ is

$$t \circ s = t\, !s\, \mathsf{dig}_E\,.$$

**Exercise 6.7.7** Let $m \in\, !E$ and $c \in G$. Prove that $(m, c) \in t \circ s$ iff one can find $(m_1, b_1), \dots, (m_n, b_n) \in s$ such that $m_1 + \cdots + m_n = m$ and $([b_1, \dots, b_n], c) \in t$.

**Exercise 6.7.8** Prove that the function $\mathbf{Rel}_!(E, F) \times \mathbf{Rel}_!(F, G) \to \mathbf{Rel}_!(E, G)$ which maps $(s, t)$ to $t \circ s$ is monotone (with respect to inclusion) and Scott-continuous. We recall what it means :

— *Monotonicity* : if $s, s' \in \mathbf{Rel}_!(E, F)$ satisfy $s \subseteq s'$ and $t, t' \in \mathbf{Rel}_!(F, G)$ satisfy $t \subseteq t'$ then $t \circ s \subseteq t' \circ s'$.

— *Continuité :* if $D \subseteq \mathbf{Rel}_!(E,F)$ and $E \subseteq \mathbf{Rel}_!(F,G)$ are directed for inclusion ($D$ directed means : $D \neq \emptyset$ and $\forall s_1, s_2 \in D\, \exists s \in D\ s_1 \subseteq s$ and $s_2 \subseteq s$) then $(\cup E) \circ (\cup D) \subseteq \cup(D \circ E)$ where $D \circ E = \{t \circ s \mid s \in D \text{ et } t \in E\}$. The other inclusion $(\cup E) \circ (\cup D) \supseteq \cup(D \circ E)$ results from monotonicity.

We have seen in Section 4.6.7 that $\mathbf{Rel}_!$ is cartesian closed. The certesian product of a family $(E_i)_{i\in I}$ of sets is the set $E = \&_{i\in I} E_i$ with projections

$$\pi_i\, \mathsf{der}_E \in \mathbf{Rel}_!(\mathop{\&}_{i\in I} E_i, E_i)\,.$$

The internal hom object from $E$ to $F$ is $E \Rightarrow F = {!E} \multimap F$ equipped with the evaluation morphism $\mathsf{Ev} \in \mathbf{Rel}(!(E \Rightarrow F) \otimes {!E}, F)$ (identifying $!((E \Rightarrow F) \,\&\, E)$ and $!(E \Rightarrow F) \otimes {!E}$) given by

$$\mathsf{Ev} = \{(([(m,b)], m), b) \mid m \in {!E} \text{ et } b \in F\}\,.$$

If $s \in \mathbf{Rel}_!(G \,\&\, E, F)$, that is $s \in \mathbf{Rel}(!G \otimes {!E}, F)$, then $\mathsf{Cur}(s) = \{(p,(m,b)) \mid ((p,m),b) \in s\} \in \mathbf{Rel}_!(G, E \Rightarrow F)$ satisfy the universal property of an hom object.

**Exercise 6.7.9** Check that the three corresponding equations hold.

6.7.5 Fixpoint operators. We define a morphism

$$\mathcal{Z} \in \mathcal{L}_!((E \Rightarrow E) \Rightarrow E, (E \Rightarrow E) \Rightarrow E)$$

using the cartesian closeness of $\mathbf{Rel}_!$ (the same definition makes sense in any cartesian closed category), setting $F = (E \Rightarrow E) \Rightarrow E$. We set $\mathcal{Z} = \mathsf{Cur}\,\mathcal{Z}_0$ where $\mathcal{Z}_0 \in \mathbf{Rel}_!(F \,\&\, (E \Rightarrow E), E)$ is defined as the following composition of morphisms in $\mathbf{Rel}_!$, where $F = (E \Rightarrow E) \Rightarrow E$ :

$$\begin{array}{ccccc}
F \,\&\, (E \Rightarrow E) & \xrightarrow{F \,\&\, \langle \mathsf{Id}, \mathsf{Id}\rangle} & F \,\&\, (E \Rightarrow E) \,\&\, (E \Rightarrow E) & \xrightarrow{\mathsf{Ev} \,\&\, \mathsf{Id}} & E \,\&\, (E \Rightarrow E) \\
 & & & & \big\downarrow {\scriptstyle \langle \pi_2, \pi_1\rangle} \\
 & & & E \xleftarrow{\mathsf{Ev}} & (E \Rightarrow E) \,\&\, E
\end{array}$$

The morphism $\mathcal{Z}$ can also be defined as $\mathcal{Z} = \mathsf{cur}\,\mathcal{Z}_1$ where $\mathcal{Z}_1$ the following composition in $\mathcal{L}$.

$$\begin{array}{ccccc}
{!F} \otimes {!(!E \multimap E)} & \xrightarrow{!F \otimes \mathsf{c}} & {!F} \otimes {!(!E \multimap E)} \otimes {!(!E \multimap E)} & & \\
 & & \big\downarrow {\scriptstyle e_{!E \multimap E, E} \otimes \mathsf{der}} & & \\
E \xleftarrow{\mathsf{ev}} (!E \multimap E) \otimes {!E} & \xleftarrow{\sigma} & {!E} \otimes (!E \multimap E) & &
\end{array}$$

where, for any object $G, T$, we set $e_{G,T} = (\mathsf{ev}\,(\mathsf{der}_{!G \multimap T} \otimes {!T}))^! : {!(!G \multimap T)} \otimes {!G} \to {!T}$

**Exercise 6.7.10** Prove first that

$$\begin{aligned} e_{G,T} = \{((&[(m_1,d_1),\dots,(m_k,d_k)], m_1 + \cdots + m_k), [d_1,\dots,d_k]) \mid k \in \mathbb{N}, \\ &m_1,\dots,m_k \in {!G} \text{ and } d_1,\dots,d_k \in T\}\,. \end{aligned}$$

Using this result, prove that

$$\begin{aligned} \mathcal{Z}_1 = \{((&[(m_1,a_1),\dots,(m_k,a_k)], m_1 + \cdots + m_k + [([a_1,\dots,a_k],a)]), a) \mid \\ &k \in \mathbb{N},\ a, a_1,\dots,a_k \in E \text{ and } m_1,\dots,m_k \in {!(!E \multimap E)}\} \end{aligned}$$

We know that any $f \in \mathbf{Rel}(!G, H)$ induces a function $\tilde{f} : \mathcal{P}(G) \to \mathcal{P}(H)$ defined by $\tilde{f}(w) = \{d \in H \mid \exists m \in \mathcal{M}_{\mathrm{fin}}(G)\ (m,d) \in f\}$ and that this function is Scott continuous.

**Exercise 6.7.11** Defining $\mathcal{Y} \in \mathcal{P}(!(!E \multimap E) \multimap E)$ as the least fixpoint of $\tilde{\mathcal{Z}}$, prove that $\mathcal{Y}$ is the least set such that

$$\mathcal{Y} = \{(m_1 + \cdots + m_k + [([a_1,\dots,a_k],a))], a) \mid k \in \mathbb{N} \text{ and } \forall i\, (m_i, a_i) \in \mathcal{Y}\}$$

6.7.6 THE EILENBERG-MOORE CATEGORY. We recall that the general category of coalgebras is defined in Section 4.6.8. An object is a pair $P = (\underline{P}, \mathsf{h}_P)$ where $\mathsf{h}_P \in \mathbf{Rel}(\underline{P}, !\underline{P})$ satisfies two commutations.

**Exercise 6.7.12** Prove that $(E, h)$ is a coalgebra iff $h \in \mathbf{Rel}(E, !E)$ satisfies :
- $\forall a, b \in E\ (a, [b]) \in h \Leftrightarrow a = b$
- For all $a \in E$ and $m_1, \ldots, m_k \in !E$, one has $(a, m_1 + \cdots + m_k) \in h$ iff there are $a_1, \ldots, a_k \in E$ such that $(a, [a_1, \ldots, a_k]) \in h$ and $(a_i, m_i) \in h$ for all $i = 1, \ldots, k$.

**Exercise 6.7.13** Let $P$ and $Q$ be coalgebras. Prove that $f \in \mathbf{Rel}(\underline{P}, \underline{Q})$ belongs to $\mathbf{Rel}^!(P, Q)$ iff forall $a \in E$ and $b_1, \ldots, b_k \in F$, there is $b \in F$ such that $(a, b) \in f$ and $(b, [b_1, \ldots, b_k]) \in \mathsf{h}_Q$ iff there are $a_1, \ldots, a_k \in E$ such that $(a, [a_1, \ldots, a_k]) \in \mathsf{h}_Q$ and $(a_i, b_i) \in f$ for each $i = 1, \ldots, k$.

**Exercise 6.7.14** Let $E$ be a set and $P$ be a coalgebra. Let $f \in \mathbf{Rel}(\underline{P}, E)$. Prove that $f^! = \{(b, [a_1, \ldots, a_k]) \mid \exists b_1, \ldots, b_k\ (b, [b_1, \ldots, b_k]) \in \mathsf{h}_P \text{ and } \forall i\, (a_i, b_i) \in f\}$.

**Exercise 6.7.15** Prove that $\mathsf{h}_1 = \{(*, k[*]) \mid k \in \mathbb{N}\}$. Given two coalgebras $P$ and $Q$, prove that

$$\begin{aligned}
\mathsf{h}_{P \otimes Q} = \{((a, b), [(a_1, b_1), \ldots, (a_k, b_k)]) \mid\ & (a, [a_1, \ldots, a_k]) \in \mathsf{h}_P \\
& \text{and } (b, [b_1, \ldots, b_k]) \in \mathsf{h}_Q\} \\
\mathsf{h}_{P \oplus Q} = \{((1, a), [(1, a_1), \ldots, (1, a_k)]) \mid\ & (a, [a_1, \ldots, a_k]) \in \mathsf{h}_P\} \\
\cup\, \{((2, b), [(2, b_1), \ldots, (2, b_k)]) \mid\ & (b, [b_1, \ldots, b_k]) \in \mathsf{h}_Q\}
\end{aligned}$$

6.7.7 EMEBDDING AND RETRACTIONS. The cpoc of sets $\mathbf{Rel}_{\sqsubseteq}$ is simply the class of all sets, equiped with the order relation $\subseteq$. It has $\emptyset$ as least element and has lubs of all countable directed families : if $(E_\gamma)_{\gamma \in \Gamma}$ is such a family, its lub is its union $\bigcup_{\gamma \in \Gamma} E_\gamma$ (actually, all unions exist, but only countable directed ones are used here).

Let $E, F$ be such that $E \subseteq F$, then we set

$$\begin{aligned}
\mathsf{i}^+_{E,F} &= \{(a, a) \in E \times F \mid a \in E\} \in \mathbf{Rel}(E, F) \\
\mathsf{i}^-_{E,F} &= \{(a, a) \in F \times E \mid a \in E\} \in \mathbf{Rel}(F, E)
\end{aligned}$$

**Exercise 6.7.16** Prove that all the conditions of Section 4.6.11 are satisfied by these definition.

As explained in Section 4.6.11, it is important to extend this notion of subobject to the category $\mathbf{Rel}^!$. Remember that we define $P \sqsubseteq Q$ as meaning that $\underline{P} \subseteq \underline{Q}$ and $\mathsf{i}^+_{\underline{P},\underline{Q}} \in \mathbf{Rel}^!(P, Q)$, that is, iff $\mathsf{h}_Q\, \mathsf{i}^+_{\underline{P},\underline{Q}} = !(\mathsf{i}^+_{\underline{P},\underline{Q}})\, \mathsf{h}_P$.

**Exercise 6.7.17** Prove that $P \sqsubseteq Q$ holds iff, for all $a \in \underline{P}$ and $b_1, \ldots, b_k \in \underline{Q}$ one has $(a, [b_1, \ldots, b_k]) \in \mathsf{h}_Q$ iff $b_1, \ldots, b_k \in \underline{P}$ and $(a, [b_1, \ldots, b_k]) \in \mathsf{h}_P$.

# Chapitre 7

# Sémantique relationnelle de CBPV et de PCF

## 7.1 Semantics of CBPV in Rel

We have seen in Section 4.7 how to interpret the language CBPV of Chapter 3. We want here to describe this interpretation in the particular case where the model of LL under consideration is **Rel**.

There is nothing to say about types : each closed type $\sigma$ is interpreted as a set $[\sigma]$, and if $\sigma$ is a positive type $\varphi$, this set $[\varphi]$ is equipped with a coalgebra structure simply denoted as $\mathsf{h}_\varphi \in \mathbf{Rel}([\varphi], ![\varphi])$. We have seen above how the various connectives of LL are interpreted in **Rel**. All the point of the use of continuous functionals for interpreting types is to make this interpretation compatible with the equation on types induced by the recursive types construction.

For instance, if this equation tells us that a closed positive type $\varphi$ satisfies $\varphi = \varphi_1 \otimes \varphi_2$ (where $\varphi_1$ and $\varphi_2$ are positive types) then we know the $[\varphi] = [\varphi_1] \times [\varphi_2]$ and that $((a_1, a_2), [(a_{1,1}, a_{2,1}), \dots, (a_{1,k}, a_{2,k})]) \in \mathsf{h}_\varphi$ iff $(a_i, [a_{i,1}, \dots, a_{i,k}]) \in \mathsf{h}_{\varphi_i}$ for $i = 1, 2$.

Here are some examples of types.

- $[\top] = \emptyset$
- $1 = !\top$ so $[1] = \{[]\}$ with $\mathsf{h}_1 = \{([], k[[]]) \mid k \in \mathbb{N}\}$ (remember that if $m$ is a multiset and $k \in \mathbb{N}$ then $km = m + \cdots + m$ ($k$ times)).
- The type of flat of natural numbers $\iota = 1 \oplus \iota$ (that is $\iota = \mathsf{Fix}\,\zeta \cdot (1 \oplus \zeta)$). Then $[\iota] = \{(1, [])\} \cup \{(2, a) \mid a \in [\iota]\}$, this set being defined as a least fixpoint. In other words, an element of $[\iota]$ is a sequence $(2, \dots, 2, 1, [])$ that we denote as $\overline{n}$ (where $n$ is the number of 2's in this sequence). Then we have $\mathsf{h}_\iota = \{(\overline{n}, k[\overline{n}]) \mid k, n \in \mathbb{N}\}$.
- The type of streams of natural numbers $\rho = \iota \otimes !\rho$. Then $[\rho] = \{(\overline{n}, [a_1, \dots, a_k]) \mid n, k \in \mathbb{N}$ and $a_1, \dots, a_k \in [\rho]\}$, again it is a definition as lest fixpoint. And we have

$$\mathsf{h}_\rho = ((\overline{n}, m_1 + \cdots + m_k), [(\overline{n}, m_1), \dots, (\overline{n}, m_k)]) \mid n, k \in \mathbb{N} \\ \text{and } m_1, \dots, m_k \in \mathcal{M}_{\mathrm{fin}}([\rho])\}\,.$$

To describe simply the interpretation of terms, we introduce a non-idempotent intersection typing system.

A semantic typing judgment is an expression $\Phi = (x_1 : a_1 : \varphi_1, \dots, x_k : a_k : \varphi_k)$ where the variables $x_i$ are pairwise distinct, the $\varphi_i$'s are positive types and $a_i \in [\varphi_i]$. Given such a semantic judgment $\Phi$, we define its underlying typing judgment $\underline{\Phi} = (x_1 : \varphi_1, \dots, x_k : \varphi_k)$ and the tuple of points $\widehat{\Phi} = (a_1, \dots, a_k) \in [\underline{\Phi}]$.

A semantic judgment is then an expression of shape $\Phi \vdash M : a : \sigma$ where $\Phi$ is a semantic judgment, $M$ is a term, $\sigma$ is a type and $a \in [\sigma]$. We give now the rules of the semantic typing system.

*Warning :* in each of the following rules, it is assumed that all the semantic contexts which appear have the same underlying typing context. Given a typing context $\mathcal{P}$, we use $\mathsf{h}_\mathcal{P}$ for $\mathsf{h}_{[\mathcal{P}]^!}$.

$$\frac{(\widehat{\Phi}, []) \in \mathsf{h}_{\underline{\Phi}}}{\Phi, x : a : \varphi \vdash x : a : \varphi}$$

$$\frac{\Phi_i \vdash M : a_i : \sigma \text{ for } i = 1, \dots, k \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \dots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash M^! : [a_1, \dots, a_k] : !\sigma}$$

Remember that we assume that $\underline{\Phi} = \underline{\Phi_i}$ for each $i$.

$$\frac{\Phi_1 \vdash M_1 : a_1 : \varphi_1 \qquad \Phi_2 \vdash M_2 : a_2 : \varphi_2 \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \langle M_1, M_2 \rangle : (a_1, a_2) : \varphi_1 \otimes \varphi_2}$$

$$\frac{\Phi \vdash M : a : \varphi_i}{\Phi \vdash \mathsf{in}_i M : (i, a) : \varphi_1 \oplus \varphi_2} \qquad \frac{\Phi, x : a : \varphi \vdash M : b : \sigma}{\Phi \vdash \lambda x^\varphi\, M : (a, b) : \varphi \multimap \sigma}$$

$$\frac{\Phi_1 \vdash M : (a, b) : \varphi \multimap \sigma \qquad \Phi_2 \vdash N : a : \varphi \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \langle M \rangle N : b : \sigma}$$

$$\frac{\Phi \vdash M : [a] : !\sigma}{\Phi \vdash \mathsf{der}(M) : a : \sigma} \qquad \frac{\Phi \vdash M : (a_1, a_2) : \varphi_1 \otimes \varphi_2 \qquad (a_2, []) \in \mathsf{h}_{\varphi_2}}{\Phi \vdash \mathsf{pr}_1 M : a_1 : \varphi_1}$$

$$\frac{\Phi \vdash M : (a_1, a_2) : \varphi_1 \otimes \varphi_2 \qquad (a_1, []) \in \mathsf{h}_{\varphi_1}}{\Phi \vdash \mathsf{pr}_2 M : a_2 : \varphi_2}$$

$$\frac{\Phi_0 \vdash M : (1, a_1) : \varphi_1 \oplus \varphi_2 \quad \Phi_1, x : a_1 ; \varphi_1 \vdash N_1 : b : \sigma \qquad \underline{\Phi}, x_2 : \varphi_2 \vdash N_2 : \varphi_2 \qquad (\widehat{\Phi}, [\widehat{\Phi_0}, \widehat{\Phi_1}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) : b : \sigma}$$

$$\frac{\Phi_0 \vdash M : (2, a_2) : \varphi_1 \oplus \varphi_2 \quad \Phi_2, x : a_2 ; \varphi_2 \vdash N_2 : b : \sigma \qquad \underline{\Phi}, x_1 : \varphi_1 \vdash N_1 : \varphi_1 \qquad (\widehat{\Phi}, [\widehat{\Phi_0}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) : b : \sigma}$$

$$\frac{\Phi_0, x : [a_1, \dots, a_k] : !\sigma \vdash M : a : \sigma \qquad \forall i\ \Phi_i \vdash \mathsf{fix}\, x^{!\sigma}\, M : a_i : \sigma \qquad (\widehat{\Phi}, [\widehat{\Phi_0}, \dots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \mathsf{fix}\, x^{!\sigma}\, M : a : \sigma}$$

This typing system is motivated by the following result.

**Théorème 7.1.1** *Assume that $\mathcal{P} \vdash M : \sigma$. Let $\vec{a} \in [\mathcal{P}]$ and $b \in [\sigma]$. Let $\Phi$ be the semantic context characterized by $\underline{\Phi} = \mathcal{P}$ and $\widehat{\Phi} = \vec{a}$. Then $(\vec{a}, b) \in [M]_{\mathcal{P}}$ holds iff the typing judgment $\Phi \vdash M : b : \sigma$ is derivable.*

**Exercice 7.1.1** Prove this theorem. This not particularly interesting, it is a good way though to review the definition of the semantics.

7.1.1 Adequacy. Our goal is to prove that, if $\vdash M : \varphi$ and $[M] \neq \emptyset$, then the reduction $\to_{\mathsf{w}}$ terminates on $M$, that is : there is no infinite sequence $(M_i)_{i \in \mathbb{N}}$ such that $M_0 = M$ and $M_i \to_{\mathsf{w}} M_{i+1}$ for each $i \in \mathbb{N}$.

For this pupose, we use a realizability method. Given a type $\sigma$ and a point $a \in [\sigma]$, we define a set $|a|^\sigma$ of closed terms $M$ such that $\vdash M : \sigma$ and, if $\sigma$ is a positive type $\varphi$, we define moreover a set $|a|^\varphi_{\mathsf{v}}$ of closed values $V$ such that $\vdash V : \varphi$. The definition is by induction on the structure of $a$.

$$\begin{aligned}
|[a_1, \dots, a_k]|^{!\sigma}_{\mathsf{v}} &= \{!M \mid \vdash M : \sigma \text{ and } M \in \bigcap_{i=1}^{k} |a_i|^\sigma\} \\
|(a_1, a_2)|^{\varphi_1 \otimes \varphi_2}_{\mathsf{v}} &= \{\langle V_1, V_2 \rangle \mid V_i \in |a_i|^{\varphi_i}_{\mathsf{v}} \text{ for } i = 1, 2\} \\
|(i, a)|^{\varphi_1 \oplus \varphi_2}_{\mathsf{v}} &= \{\mathsf{in}_i V \mid V \in |a|^{\varphi_i}_{\mathsf{v}}\} \\
|a|^\varphi &= \{M \mid \vdash M : \varphi \text{ and } \exists V \in |a|^\varphi_{\mathsf{v}}\ M \to^*_{\mathsf{w}} V\} \\
|(a, b)|^{\varphi \multimap \sigma} &= \{M \mid \vdash M : \varphi \multimap \sigma \text{ and } \forall V \in |a|^\varphi_{\mathsf{v}}\ \langle M \rangle V \in |b|^\sigma\}\,.
\end{aligned}$$

**Lemme 7.1.2** *If $\vdash M : \sigma$, $a \in [\sigma]$, $M \to_{\mathsf{w}} M'$ and $M' \in |a|^\sigma$, then $M \in |a|^\sigma$.*

*Démonstration.* By induction on the structure of $a$. If $\sigma$ is a positive type $\varphi$, the property results immediately from the definition. If $\sigma = \varphi \multimap \tau$ and $a = (b, c)$, then let $V \in |b|^{\varphi}$, we have to prove that $\langle M \rangle V \in |c|^{\sigma}$. This results from the inductive hypothesis and from the fact that, by definition of $\to_{\mathsf{w}}$, we have $\langle M \rangle V \to_{\mathsf{w}}^{*} \langle M' \rangle V$. □

**Lemme 7.1.3** *Let $\varphi$ be a positive type and let $(a, [a_1, \dots, a_k]) \in \mathsf{h}_{\varphi}$. Then for all $i$ one has $|a|_{\mathsf{v}}^{\varphi} \subseteq |a_i|_{\mathsf{v}}^{\varphi}$.*

*Démonstration.* By induction on the structure of $a$. If $\varphi = !\sigma$ and $a = [b_1, \dots, b_n]$ then $a_1, \dots, a_k \in [!\sigma]$ are finite multisets such that $a = a_1 + \cdots + a_k$. So we have $|a|_{\mathsf{v}}^{!\sigma} = \bigcap_{i=1}^{k} |a_i|_{\mathsf{v}}^{!\sigma}$ and hence $|a|_{\mathsf{v}}^{!\sigma} \subseteq |a_i|_{\mathsf{v}}^{!\sigma}$ for each $i$. □

**Exercice 7.1.2** Complete the proof.

Then we can state and prove the main result.

**Théorème 7.1.4** *Assume that $x_1 : a_1, \varphi_1, \dots, x_n : a_n, \varphi_n \vdash M : b : \sigma$ and that $V_i \in |a_i|_{\mathsf{v}}^{\varphi_i}$ for $i = 1, \dots, n$. Then $M\,[V_1/x_1, \dots, V_k/x_k] \in |b|^{\sigma}$.*

*Démonstration.* By induction on the semantic typing derivation of $x_1 : a_1, \varphi_1, \dots, x_n : a_n, \varphi_n \vdash M : b : \sigma$. We denote as $M'$ the term $M\,[V_1/x_1, \dots, V_k/x_k]$.

Assume that $M = x_i$ for somme $i \in \{1, \dots, n\}$ so that the derivation consists of the axiom $\Phi \vdash x : a_i : \varphi_i$ with $(\widehat{\Phi}, []) \in \mathsf{h}_{\underline{\Phi}}$ so that $b = a_i$ and $\sigma = \varphi_i$. Then $M' = V_i \in |a_i|_{\mathsf{v}}^{\varphi_i} \subseteq |a_i|^{\varphi_i}$ by definition of this latter set.

Assume that $M = N^{!}$, $\sigma = !\tau$, $a = [b_1, \dots, b_k]$ and that we have $\Phi_i \vdash N : b_i : \tau$ for $i = 1, \dots, k$ with $(\widehat{\Phi}, [\widehat{\Phi_1}, \dots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}$. For each $i = 1, \dots, n$, we have assumed that $V_i \in |a_i|_{\mathsf{v}}^{\varphi_i}$ but if we write explicitely $\Phi_j = (x_1 : a_{j,1} : \varphi_1, \dots, x_1 : a_{j,n} : \varphi_n)$, we know for each $i = 1, \dots, n$ one has $(a_i, [a_{1,i}, \dots, a_{k,i}]) \in \mathsf{h}_{\varphi_i}$ from which we deduce by Lemma 7.1.3 that $V_i \in |a_{j,i}|_{\mathsf{v}}^{\varphi_i}$ for each $j = 1, \dots, k$. Therefore by inductive hypothesis we have that $N' \in |b_j|^{\sigma}$ for $j = 1, \dots, k$. Hence $M' = (N')^{!} \in |a|_{\mathsf{v}}^{!\sigma}$ by definition of this latter set. Therefore $M' \in |a|^{!\sigma}$ as contended.

Assume that $M = \langle M_1, M_2 \rangle$, $\sigma = \varphi_1 \otimes \varphi_2$, $a = (a_1, a_2)$ and that we have $\Phi_j \vdash M_j : a_j : \varphi_j$ for $j = 1, 2$ with $(\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}$. As above, applying Lemma 7.1.3 and using the fact that $(\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}$, we get by inductive hypothesis that $M_j' \in |a_j|_{\mathsf{v}}^{\varphi_j}$ for $j = 1, 2$ and hence $M' = \langle M_1', M_2' \rangle \in |a|_{\mathsf{v}}^{\sigma} \subseteq |a|^{\sigma}$.

The case $M = \mathsf{in}_j N$ is completely similar.

Assume that $M = \langle N \rangle R$ and that we have $\Phi_1 \vdash N : (b, a) : \varphi \multimap \sigma$ and $\Phi_2 \vdash R : b : \varphi$ with $(\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}$. Therefore, by inductive hypothesis (using Lemma 7.1.3 as usual), we get $N' \in |(b, a)|^{\varphi \multimap \sigma}$ and $R' \in |b|^{\varphi}$. This latter property means that $R' \to_{\mathsf{w}}^{*} V$ for some $V \in |b|_{\mathsf{v}}^{\varphi}$. By definition of $|(b, a)|^{\varphi \multimap \sigma}$ we have $\langle N' \rangle V \in |a|^{\sigma}$. By definition of $\to_{\mathsf{w}}$ for CBPV, we get $M' \to_{\mathsf{w}}^{*} \langle N' \rangle V$ and hence $M' \in |a|^{\sigma}$ by Lemma 7.1.2.

Assume that $M = \mathsf{der}(N)$ and that we have $\Phi \vdash N : [a] : !\sigma$. By inductive hypothesis we have $N' \in |[a]|^{!\sigma}$, which means that there is $V \in |[a]|_{\mathsf{v}}^{!\sigma}$ such that $N' \to_{\mathsf{w}}^{*} V$. By definition of $|[a]|_{\mathsf{v}}^{!\sigma}$ there exists $R \in |a|^{\sigma}$ such that $V = R^{!}$. By definition of $\to_{\mathsf{w}}$ we have $M' = \mathsf{der}(N') \to_{\mathsf{w}}^{*} \mathsf{der}(R^{!}) \to_{\mathsf{w}} R$ and hence $M' \in |a|^{\sigma}$ by Lemma 7.1.2.

Assume that $M = \lambda x^{\varphi} N$, $\sigma = \varphi \multimap \tau$, $a = (b, c)$ and that we have $\Phi, x : b : \varphi \vdash N : c : \tau$. Let $V \in |b|_{\mathsf{v}}^{\varphi}$, we have $\langle M' \rangle V \to_{\mathsf{w}} M'\,[V/x] = M\,[V_1/x_1, \dots, V_n/x_n, V/x]$ and we know that this latter term belongs to $|c|^{\tau}$ by inductive hypothesis. Therefore we have $\langle M' \rangle V \in |c|^{\tau}$ by Lemma 7.1.2. Hence $M' \in |(b, c)|^{\varphi \multimap \tau}$ as contended.

Assume that $M = \mathsf{pr}_j N$ for some $j \in \{1, 2\}$, $\sigma = \varphi_j$, $a = a_j$ and $\Phi \vdash N : (a_1, a_2) : \varphi_1 \otimes \varphi_2$ (and also $(a_{2-j}, []) \in \mathsf{h}_{\underline{\Phi}}$ though we do not use this fact here; this property is crucial in the proof of soundness of the semantics wrt. reduction). By inductive hypothesis we have $N' \in |(a_1, a_2)|^{\varphi_1 \otimes \varphi_2}$. So there are values $V_j \in |a_j|_{\mathsf{v}}^{\varphi_j}$ for $j = 1, 2$ such that $N' \to_{\mathsf{w}}^{*} \langle V_1, V_2 \rangle$. By definition of $\to_{\mathsf{w}}$ we have $\mathsf{pr}_j M' \to_{\mathsf{w}}^{*} \mathsf{pr}_j \langle V_1, V_2 \rangle \to_{\mathsf{w}} V_j$ from which we deduce that $M' = \mathsf{pr}_j N' \in |a_j|^{\varphi_j}$ by definition of $|a_j|^{\varphi_j}$.

Assume that $M = \mathsf{case}(N, x_1 \cdot R_1, x_2 \cdot R_2)$ with $\Phi_0 \vdash N : (j, b_j) : \varphi_1 \oplus \varphi_2$, $\Phi_1, x_j : b_j : \varphi_j \vdash R_j : a : \sigma$ and $\underline{\Phi}, x_{2-j} : \varphi_{2-j} \vdash R_{2-j} : \sigma$ for some $j \in \{1, 2\}$, with $(\widehat{\Phi}, [\widehat{\Phi_0}, \widehat{\Phi_j}]) \in \mathsf{h}_{\underline{\Phi}}$. Applying Lemma 7.1.3 we get, by inductive hypothesis, $N' \in |(j, b_j)|^{\varphi_1 \oplus \varphi_2}$, which means that there exists $V \in |a_j|_{\mathsf{v}}^{\varphi_j}$ such that $N' \to_{\mathsf{w}}^{*} \mathsf{in}_j V$, and hence, by definition of $\to_{\mathsf{w}}$, we have $M' \to_{\mathsf{w}}^{*} \mathsf{case}(\mathsf{in}_j V, x_1 \cdot R_1', x_2 \cdot R_2') \to_{\mathsf{w}} R_j'\,[V/x_j]$, and this latter term belongs to $|a|^{\sigma}$ by inductive hypothesis. So by Lemma 7.1.2 we have $M' \in |a|^{\sigma}$.

Assume last that $M = \mathsf{fix}\, x^{!\sigma}\, N$ with $\Phi_0, x : [a_1, \dots, a_k] : !\sigma \vdash N : a : \sigma$ and $\Phi_j \vdash M : a_j : \sigma$ for $j = 1, \dots, k$, and $(\widehat{\Phi}, [\widehat{\Phi_0}, \dots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}$. As usual, we get $M' \in |a_j|^\sigma$ for $j = 1, \dots, k$ and hence $(M')^! \in |[a_1, \dots, a_k]|_{\mathsf{v}}^{!\sigma}$. By inductive hypothesis again (applied now to $N$) we get therefore $N'\left[(M')^!/x\right] \in |a|^\sigma$. By definition of $\rightarrow_{\mathsf{w}}$ (and of substitution) we have $M' \rightarrow_{\mathsf{w}} N'\left[(M')^!/x\right]$ and hence $M' \in |a|^\sigma$ by Lemma 7.1.2. □

7.1.2 Observational equivalence for CBPV. We can say that two closed terms $M_1$ and $M_2$ are observationally equivalent if, whenever plugged in any context, the resulting processes behave in the same way. This context can use the plugged term various times, and therefore a promotion will be applied before plugging the term in the context.

More precisely, assume that $\vdash M_i : \sigma$ for $i = 1, 2$. Then we say that $M_1$ and $M_2$ are observationally equivalent (written $M_1 \simeq_{\mathrm{obs}} M_2$) if, for any term $C$ such that $\vdash C : !\sigma \multimap 1$, the term $\langle C \rangle M_1^!$ is $\rightarrow_{\mathsf{w}}$-normalizable iff $\langle C \rangle M_2^!$ is $\rightarrow_{\mathsf{w}}$-normalizable. Remember that $1 = !\top$.

The main drawback of this notion is that *a priori* it is not compositional. For instance, it is not clear from the definition, whether $M_1 \simeq_{\mathrm{obs}} M_2,\ N_1 \simeq_{\mathrm{obs}} N_2 \Rightarrow \langle M_1 \rangle N_1 \simeq_{\mathrm{obs}} \langle M_2 \rangle N_2$. This makes it difficult to prove that two terms are equivalent.

Fortunately, one can prove that two terms are observationally equivalent by proving that they are denotationally equivalent in some model of CBPV (that is, they have the same interpretation in this model) as soon as (an analogue of) Theorem 7.1.4 holds.

**Lemme 7.1.5** *Assume that* $\vdash M : !\sigma$*. Then* $[] \in [M]$ *iff there is* $N$ *such that* $\vdash N : \sigma$ *and* $M \rightarrow_{\mathsf{w}} N^!$*.*

*Démonstration.* If $M \rightarrow_{\mathsf{w}} N^!$ then $[M] = [N^!]$ by Theorem 4.7.4. We have $[] \in [N^!]$ by definition of the interpretation (see the semantic typing rule above for $N^!$ with $k = 0$). Conversely, if $[] \in [M]$ then $M \in |[]|^{!\sigma}$ by Theorem 7.1.4 which means that $M \rightarrow_{\mathsf{w}}^* N^!$ for some term $N$ such that $\vdash N : \sigma$. □

**Théorème 7.1.6** *Assume that* $\vdash M_i : \sigma$ *for* $i = 1, 2$*. If* $[M_1] = [M_2]$ *then* $M_1 \simeq_{\mathrm{obs}} M_2$*.*

*Démonstration.* Assume that $\vdash M_i : \sigma$ for $i = 1, 2$ and that $[M_1] = [M_2]$. Let $C$ be a term such that $\vdash C : !\sigma \multimap 1$. We have $[\langle C \rangle M_1^!] = [\langle C \rangle M_2^!]$ and therefore $\langle C \rangle M_1^!$ $\rightarrow_{\mathsf{w}}$-normalizes iff $\langle C \rangle M_2^!$ $\rightarrow_{\mathsf{w}}$-normalizes by Lemma 7.1.5. □

The converse implication does not hold simply because there are other models of CBPV which equate more terms than **Rel** and satisfy an analogue of 7.1.4. Such a model can be defined using the Scott semantics of LL.

When the converse implication holds, one says that the model is *fully abstract*.

## 7.2 Sémantics of PCF in $\mathbf{Rel}_!$

We define a denotational interpretation of PCF in the CCC $\mathbf{Rel}_!$. We first exhibit the basic ingredients of this interpretation.

7.2.1 Natural numbers. We know that the category **Rel** has arbitrary countable coproducts, and therefore, it has an object of natural numbers $\mathsf{N}$ which is a coproduct of $\omega$ copies of 1. One has $\mathsf{N} = \mathbb{N}$ and for each $n \in \mathbb{N}$, one a morphism $\overline{n} = \{(*, n)\} \in \mathbf{Rel}(1, \mathsf{N})$ which represents the integer $n$. There is a successor morphism $\overline{\mathsf{suc}} \in \mathbf{Rel}(\mathsf{N}, \mathsf{N})$ given by

$$\overline{\mathsf{suc}} = \{(n, n+1) \mid n \in \mathbb{N}\}\,.$$

The canonical coalgebra structure of 1, see Section 4.6.8, is the morphism $h_1 = \mu^0$ defined as the following composition of morphisms in **Rel** :

$$1 \xrightarrow{\mathsf{m}^0} !\top \xrightarrow{\mathsf{dig}_\top} !!\top \xrightarrow{!(\mathsf{m}^0)^{-1}} !1$$

It results from this definition that

$$h_1 = \{(*, k[*]) \mid k \in \mathbb{N}\}$$

(which is an infinite set). It follows by the fact that $\mathbf{Rel}^!$ has all countable coproducts, see Section 4.6.9, that the canonical coalgebra structure of $\mathsf{N}$ is given by the following morphism $h_{\mathsf{N}} \in \mathbf{Rel}(\mathsf{N}, !\mathsf{N})$ :

$$h_{\mathsf{N}} = \{(n, k[n]) \mid n, k \in \mathbb{N}\}$$

since $h_{\mathsf{N}} = [!\overline{n}\, h_1]_{n \in \mathbb{N}}$.

Another crucial property of $\mathsf{N}$ is that we have an iso between $1 \oplus \mathsf{N}$ and $\mathsf{N}$ given by the following bijection

$$\begin{aligned} \varphi : 1 \oplus \mathsf{N} &\to \mathsf{N} \\ (1, *) &\mapsto 0 \\ (2, n) &\mapsto n + 1 \end{aligned}$$

Let $E$ be an object of $\mathbf{Rel}$, that is, a set. We define $\overline{\mathsf{if}}_0 \in \mathbf{Rel}(1 \otimes !E \otimes !(!\mathsf{N} \multimap E), E)$ as the following composition of morphisms in $\mathbf{Rel}$

$$1 \otimes !E \otimes !(!\mathsf{N} \multimap E) \xrightarrow{\lambda} !E \otimes !(!\mathsf{N} \multimap E) \xrightarrow{!E \otimes \mathsf{w}_{!\mathsf{N} \multimap E}} !E \xrightarrow{\mathsf{der}_E} E$$

and therefore

$$\overline{\mathsf{if}}_0 = \{(*, [a], [], a) \mid a \in E\}\,.$$

We also define $\overline{\mathsf{if}}_+ \in \mathcal{L}(\mathsf{N} \otimes !E \otimes !(!\mathsf{N} \multimap E), E)$ as the following composition of morphisms

$$\mathsf{N} \otimes !E \otimes !(!\mathsf{N} \multimap E) \xrightarrow{\mathsf{N} \otimes \mathsf{w}_E \otimes \mathsf{der}_{!\mathsf{N} \multimap E}} \mathsf{N} \otimes (!\mathsf{N} \multimap E) \xrightarrow{h_{\mathsf{N}} \otimes \mathsf{Id}} !\mathsf{N} \otimes (!\mathsf{N} \multimap E) \xrightarrow{\mathsf{ev}\ \sigma} \mathsf{N}$$

so that

$$\overline{\mathsf{if}}_+ = \{(n, [], [(k[n], a)], a) \mid k, n \in \mathbb{N} \text{ and } a \in E\}\,.$$

Using the iso $\varphi$ and the distributivity of $\otimes$ of $\oplus$, we can define

$$\overline{\mathsf{if}} \in \mathcal{L}(\mathsf{N} \otimes !E \otimes !(!\mathsf{N} \multimap E), E)$$

using $\overline{\mathsf{if}}^0$ and $\overline{\mathsf{if}}^+$; we have therefore

$$\overline{\mathsf{if}} = \{(0, [a], [], a) \mid a \in E\} \cup \{(n+1, [], [(k[n], a)], a) \mid k, n \in \mathbb{N} \text{ and } a \in E\}\,.$$

7.2.2 Fixpoints. This is a reminder of Section 6.7.5. Consider the operator $\mathcal{Y} : \mathbf{Rel}_!(E \Rightarrow E, E) \to \mathbf{Rel}_!(E \Rightarrow E, E)$ as defined in section 4.6.12. Considered as a function $\mathcal{P}(\mathcal{M}_{\mathrm{fin}}(E \Rightarrow E) \times E) \to \mathcal{P}(\mathcal{M}_{\mathrm{fin}}(E \Rightarrow E) \times E)$, this function is Scott continuous. Therefore, it has a least fixpoint

$$\overline{\mathsf{fix}} = \bigcup_{n=0}^{\infty} \mathcal{Y}^n(\emptyset)$$

that is $\overline{\mathsf{fix}} = \bigcup_{n=0}^{\infty} \overline{\mathsf{fix}}_n$ where $\overline{\mathsf{fix}}_0 = \emptyset$ and

$$\begin{aligned} \overline{\mathsf{fix}}_{n+1} &= \mathcal{Y}(\overline{\mathsf{fix}}_n) \\ &= \mathsf{Ev} \circ \langle \mathsf{Id}_{E \to E}, \overline{\mathsf{fix}}_n \rangle \\ &= \mathsf{ev}\,(\mathsf{der}_{!E \multimap E} \otimes \overline{\mathsf{fix}}_n^{\,!})\ \mathsf{c}_{!E \multimap E} \end{aligned}$$

and this sequence of morphisms is monotone with respect to inclusion.

Applying the last equation, we get that an element $(m, a)$ of $(E \Rightarrow E) \Rightarrow E$ belongs to $\overline{\mathsf{fix}}_{n+1}$ iff $m$ can be written

$$m = [([a_1, \ldots, a_k], a)] + m_1 + \cdots + m_k$$

with $(m_i, a_i) \in \overline{\mathsf{fix}}_n$ for $i = 1, \ldots, k$. So $\overline{\mathsf{fix}}_0 = \emptyset$, $\overline{\mathsf{fix}}_1 = \{([([], a)], a) \mid a \in E\}$ etc.

In other words, $\overline{\mathsf{fix}}$ is inductively defined as

$$\overline{\mathsf{fix}} = \{([([a_1, \ldots, a_k], a)] + m_1 + \cdots + m_k, a) \mid k \in \mathbb{N},\ a \in E \text{ and } \forall i\ (m_i, a_i) \in \overline{\mathsf{fix}}\}\,.$$

7.2.3 Definition and invariance of the interpretation. With any type $A$, we associate an object $[A]$ of $\mathbf{Rel}_!$, by the obvious inductive definition :

$$[\iota] = \mathsf{N} \quad \text{and} \quad [A \to B] = [A] \Rightarrow [B] = ![A] \multimap [B]$$

Given a typing context $\Gamma = (x_1 : A_1, \ldots, x_l : A_l)$, we set

$$[\Gamma] = [A_1] \mathbin{\&} \cdots \mathbin{\&} [A_l]$$

Given a typing context $\Gamma = (x_1 : A_1, \ldots, x_l : A_l)$, a type $A$ and a term $M$ such that $\Gamma \vdash M : A$, we define a morphism $[M]_\Gamma \in \mathbf{Rel}_!([\Gamma], [A])$ by induction on $M$. This definition makes sense actually in any Kleisli category $\mathcal{L}_!$ with the required structure and uses the fact that this Kleisli category is cartesian closed, see Section 4.6.7 ; we present it first in the most abstract and general way, using purely categorical notations.

If $M = x_i$ for some $i \in \{1, \ldots, l\}$, then $[M]_\Gamma = \pi_i\, \mathsf{der}$.

If $M = \underline{n}$ for $n \in \mathbb{N}$ then $[M]_\Gamma = \overline{n} \circ \mathsf{t}_{[\Gamma]}$, considering $\overline{n}$ as an element of $\mathcal{L}_!(\top, \mathsf{N})$ by identifying $!\top$ with 1.

If $M = \underline{\mathsf{succ}}(P)$ with $\Gamma \vdash P : \iota$, then we have $[P]_\Gamma \in \mathcal{L}_!([\Gamma], \mathsf{N})$ and we set

$$[M]_\Gamma = \overline{\mathsf{suc}}\, [P]_\Gamma \in \mathcal{L}_!([\Gamma], \mathsf{N})$$

If $M = \mathsf{if}(P, Q, z \cdot R)$ with $\Gamma \vdash P : \iota$, $\Gamma \vdash Q : A$ and $\Gamma, z : \iota \vdash R : A$ then we have $[P]_\Gamma \in \mathcal{L}_!([\Gamma], \mathsf{N})$, $[Q]_\Gamma \in \mathcal{L}_!([\Gamma], [A])$ and $[R]_{\Gamma, z:\iota} \in \mathcal{L}_!([\Gamma] \mathbin{\&} \mathsf{N}, [A])$ so that $\mathsf{Cur}\,([R]_{\Gamma, z:\iota}) \in \mathcal{L}_!([\Gamma], !\mathsf{N} \multimap [A])$. Therefore we have $[Q]_\Gamma{}^! \in \mathcal{L}_!(![\Gamma], ![A])$ and $\mathsf{Cur}\,([R]_{\Gamma, z:\iota})^! \in \mathcal{L}(![\Gamma], !(!\mathsf{N} \multimap [A]))$. So we set

$$[M]_\Gamma = \overline{\mathsf{if}}\, ([P]_\Gamma \otimes [Q]_\Gamma{}^! \otimes \mathsf{Cur}\,([R]_{\Gamma, z:\iota})^!)\; \mathsf{c} \in \mathcal{L}(![\Gamma], [A])$$

where $\mathsf{c} \in \mathcal{L}(![\Gamma], ![\Gamma] \otimes ![\Gamma] \otimes ![\Gamma])$ is a ternary version of contraction (uniquely defined up to $\alpha$ isos, for instance $\mathsf{c} = (![\Gamma] \otimes \mathsf{c}_{[\Gamma]})\, \mathsf{c}_{[\Gamma]}$).

If $M = \lambda x^B\, P$ with $\Gamma, x : B \vdash P : C$ (and $A = B \to C$) then we have $[P]_{\Gamma, x:B} \in \mathcal{L}_!([\Gamma] \mathbin{\&} [B], [C])$ and we set $[M]_\Gamma = \mathsf{Cur}\,([P]_{\Gamma, x:B}) \in \mathcal{L}_!([\Gamma], [B] \Rightarrow [C])$.

If $M = (P)\, Q$ with $\Gamma \vdash P : B \to A$ and $\Gamma \vdash Q : B$ then we have $[P]_\Gamma \in \mathcal{L}_!([\Gamma], [B] \Rightarrow [A])$ and $[Q]_\Gamma \in \mathcal{L}_!([\Gamma], [B])$ and we set

$$\begin{aligned} [M]_\Gamma &= \mathsf{Ev} \circ \langle [P]_\Gamma, [Q]_\Gamma \rangle \\ &= \mathsf{ev}\, ([P]_\Gamma \otimes [Q]_\Gamma{}^!)\; \mathsf{c}_{[\Gamma]} \end{aligned}$$

If $M = \mathsf{fix}(P)$ with $\Gamma \vdash P : A \to A$, then we have $[P]_\Gamma \in \mathcal{L}([\Gamma], [A] \Rightarrow [A])$ and we set

$$\begin{aligned} [M]_\Gamma &= \overline{\mathsf{fix}} \circ [P]_\Gamma \\ &= \overline{\mathsf{fix}}\, [P]_\Gamma{}^! \end{aligned}$$

The following result is a tool for proving that the semantics is invariant under term reduction. It is also interesting *per se* as it means that the semantics is modular : the interepretation of a term is known as soon as one knows the interpretation of its subterms. This is an essential feature of denotational semantics.

**Théorème 7.2.1 (Substitution Lemma)** *Assume that* $\Gamma, x : A \vdash M : B$ *and that* $\Gamma \vdash P : A$*. Then*

$$\begin{aligned} [M\,[P/x]]_\Gamma &= [M]_{\Gamma, x:A} \circ \langle \mathsf{Id}_{[\Gamma]}, [Q]_\Gamma \rangle \\ &= [M]_{\Gamma, x:A}\, (![\Gamma] \otimes [P]_\Gamma{}^!)\; \mathsf{c}_{[\Gamma]} \end{aligned}$$

The proof is a simple induction on $M$ (or, equivalently, on the derivation that $\Gamma, x : A \vdash M : B$).

**Théorème 7.2.2** *Assume that* $\Gamma \vdash M : A$ *and that* $M\ \beta\ M'$*. Then* $[M']_\Gamma = [M]_\Gamma$*.*

The proof is a straightforward induction on the derivation of the fact that $M\ \beta\ M'$ in the deduction system presented in Section 1.3. We deal here with three cases.

Assume first that $M = \left(\lambda x^B\, P\right) Q$ with $\Gamma, x : B \vdash P : A$ and $\Gamma \vdash Q : B$, and that $M' = P\,[Q/x]$. We have

$$\begin{aligned}[M]_\Gamma &= \mathsf{Ev} \circ \langle \mathsf{Cur}\,[P]_{\Gamma,x:B}, [Q]_\Gamma\rangle \\ &= [P]_{\Gamma,x:B} \circ \langle \mathsf{Id}_{[\Gamma]}, [Q]_\Gamma\rangle \quad \text{by cartesian closedness of } \mathcal{L}_! \\ &= [M']_\Gamma \quad \text{by Theorem 7.2.1.}\end{aligned}$$

Assume now that $M = \mathsf{if}(\underline{n+1}, P, z \cdot Q)$ with $\Gamma \vdash P : A$ and $\Gamma, z : \iota \vdash Q : A$, and that $M' = Q\,[\underline{n}/z]$. Then we have $[P]_\Gamma{}^! \in \mathcal{L}(![\Gamma], ![A])$ and $\mathsf{Cur}\,[Q]_{\Gamma,z:\iota}{}^! \in \mathcal{L}(![\Gamma], !(!\mathsf{N} \multimap [A]))$. Remember also that $[\underline{n+1}]_\Gamma = \overline{n+1}\ \mathsf{w}_{[\Gamma]}$. We have

$$\begin{aligned}[M]_\Gamma &= \overline{\mathsf{if}}\,([\underline{n+1}]_\Gamma \otimes [P]_\Gamma{}^! \otimes \mathsf{Cur}\,[Q]_{\Gamma,z:\iota}{}^!)\ \mathsf{c} \\ &= \overline{\mathsf{if}}\,(\overline{n+1} \otimes ![\Gamma] \otimes !(!\mathsf{N} \multimap [A]))\,(\mathsf{w}_{[\Gamma]} \otimes [P]_\Gamma{}^! \otimes \mathsf{Cur}\,[Q]_{\Gamma,z:\iota}{}^!)\ \mathsf{c} \\ &= \mathsf{ev}\,(\mathsf{Cur}\,[Q]_{\Gamma,z:\iota} \otimes [\underline{n}]_\Gamma{}^!)\ \mathsf{c}_{[\Gamma]} \quad \text{by definition of } \overline{\mathsf{if}} \\ &= [Q]_{\Gamma,z:\iota}\,(![\Gamma] \otimes [\underline{n}]_\Gamma{}^!)\ \mathsf{c}_{[\Gamma]} \quad \text{by cartesian closedness of } \mathcal{L}_! \\ &= [Q\,[\underline{n}/z]]_\Gamma = [M']_\Gamma \quad \text{by Theorem 7.2.1}\end{aligned}$$

Assume last that $M = \mathsf{fix}(P)$ with $\Gamma \vdash P : A \to A$ and that $M' = (P)\,M$. Then we have $[P]_\Gamma \in \mathcal{L}_!([\Gamma], [A] \Rightarrow [A])$ and

$$\begin{aligned}[M]_\Gamma &= \overline{\mathsf{fix}} \circ [P]_\Gamma \\ &= \mathsf{Ev} \circ \langle [P]_\Gamma, \overline{\mathsf{fix}} \circ [P]_\Gamma\rangle \quad \text{as seen in Section 4.6.12} \\ &= [(P)\,M]_\Gamma\end{aligned}$$

7.2.4 Relational interpretation as a typing system. We provide now a more concrete presentation of that interpretation, in the relational model.

A *semantic context* is a sequence $\Phi = (x_1 : m_1 : A_1, \dots, x_l : m_l : A_l)$ where the $x_i$s are pairwise distinct variable and $m_i \in ![A_i]$ for $i = 1, \dots, l$. We use $\underline{\Gamma}$ for the underlying typing context $\underline{\Gamma} = (x_1 : A_1, \dots, x_l : A_l)$. Given a typing context $\Gamma = (x_1 : A_1, \dots, x_l : A_l)$, we use $0_\Gamma$ for the semantic context $0_\Gamma = (x_1 : [] : A_1, \dots, x_l : [] : A_l)$. Observe that $\underline{0_\Gamma} = \Gamma$. More generally, given a finite family $(\Phi_i)_{i \in I}$ of semantic contexts such that $\forall i\ \underline{\Phi_i} = \Gamma$ for some given typing context $\Gamma = (x_1 : A_1, \dots, x_l : A_l)$ (so that $\Phi_i = (x_1 : m_1^i :, A_1, \dots, x_l : m_l^i : A_l))$, we define $\Phi = \sum_{i\in I} \Phi_i = (x_1 : \sum_{i\in I} m_1^i : A_1, \dots, x_l : \sum_{i\in I} m_l^i : A_l)$.

A *semantic judgment* is a statement of shape $\Phi \vdash M : a : A$ where $\Phi$ is a semantic context, $M$ is a term, $A$ is a type and $a \in [A]$.

We give now a deduction system for these judgments.

$$\frac{}{0_\Gamma, x : [a] : A \vdash x : a : A} \qquad \frac{}{0_\Gamma \vdash \underline{n} : n : \iota} \qquad \frac{\Phi \vdash M : n : \iota}{\Phi \vdash \mathsf{succ}(M) : n+1 : \iota}$$

$$\frac{\Phi_0 \vdash M : 0 : \iota \qquad \Phi_1 \vdash P : a : A \qquad \Gamma, z : \iota \vdash Q : A \text{ where } \Gamma = \underline{\Phi_0} = \underline{\Phi_1}}{\Phi_0 + \Phi_1 \vdash \mathsf{if}(M, P, z \cdot Q) : a : A}$$

$$\frac{\Phi_0 \vdash M : n+1 : \iota \qquad \Phi_2, z : k[n] : \iota \vdash Q : a : A \qquad \Gamma \vdash P : A \text{ where } \Gamma = \underline{\Phi_0} = \underline{\Phi_2}}{\Phi_0 + \Phi_2 \vdash \mathsf{if}(M, P, z \cdot Q) : a : A}$$

In that rule, $k$ and $n$ are arbitrary elements of $\mathbb{N}$.

$$\frac{\Phi, x : m : A \vdash M : b : B}{\Phi \vdash \lambda x^A\, M : (m, b) : A \to B}$$

$$\frac{\Phi_0 \vdash M : ([a_1, \dots, a_n], b) : A \to B \qquad \Phi_i \vdash P : a_i : A \text{ and } \underline{\Phi_i} = \underline{\Phi_0} \text{ for } i = 1, \dots, n}{\sum_{i=0}^n \Phi_i \vdash (M)\,P : b : B}$$

$$\frac{\Phi_0 \vdash M : ([a_1,\dots,a_n],a) : A \to A \qquad \Phi_i \vdash \mathsf{fix}(M) : a_i : A \text{ and } \underline{\Phi_i} = \underline{\Phi_0} \text{ for } i = 1,\dots,n}{\sum_{i=0}^{n} \Phi_i \vdash \mathsf{fix}(M) : a : A}$$

The proofs of the two next statements are simple inductions of semantic derivations.

**Proposition 7.2.3** *If $\Phi \vdash M : a : A$ then $\underline{\Phi} \vdash M : A$.*

**Théorème 7.2.4** *Assume that $\Gamma \vdash M : A$ with $\Gamma = (x_1 : A_1, \dots, x_l : A_l)$. Let $a \in [A]$ and let $m_i \in ![A_i]$ for $i = 1,\dots,l$. The two following statements are equivalent :*

— $(m_1,\dots,m_l,a) \in [M]_\Gamma$

— *the judgment $(x_1 : m_1 : A_1, \dots, x_l : m_l : A_l) \vdash M : a : A$ is derivable.*

7.2.5 The adequacy Theorem. Let $M$ be a term such that $\vdash M : \iota$, that is, $M$ is closed and of base type. Then if we apply the $\beta_{\mathsf{wh}}$ reduction strategy (see Section 1.3.2) to $M$, there are two possibilities :

— either $M\ \beta^*_{\mathsf{wh}}\ \underline{n}$ for a uniquely determined $n \in \mathbb{N}$

— or the computation does not terminate.

In the first case, we know by Theorem 7.2.2 that $[M] = \overline{n}$, but what can we say in the second case ? The answer is that, in the second case, $[M] = \emptyset$. This result can be proved under rather general hypotheses about the model (basically, one needs a CCC with an "object of natural numbers", and where each hom-set is a cpo with a least element).

We give here a proof in this particular model, which is more direct than the general one and can be adapted to many similar contexts.

For any type $A$ we define a relation $\Vdash_A$ between closed terms of type $A$ and elements of $[A]$. The definition is by induction on $A$.

If $A = \iota$, $M$ such that $\vdash M : \iota$ and $n \in \mathbb{N}$, we say that $M \Vdash_\iota n$ if $M\ \beta^*_{\mathsf{wh}}\ \underline{n}$.

If $A = B \to C$, $M$ such that $\vdash M : B \to C$, $b_1,\dots,b_n \in [B]$ and $c \in [C]$, we say that $M \Vdash_{B\to C} ([b_1,\dots,b_n],c)$ if, for any $P$ such that $\vdash P : B$ and $P \Vdash_B b_i$ for $i = 1,\dots,n$, one has $(M)\,P \Vdash_C c$.

The next lemma states the main property of the relation $\Vdash_A$ for the proof of Theorem 7.2.6.

**Lemme 7.2.5** *Assume that $\vdash M : A$, $M\ \beta_{\mathsf{wh}}\ M'$ and $a \in [A]$. If $M' \Vdash_A a$ then $M \Vdash_A a$.*

*Démonstration.* By induction on $A$.

Assume that $A = \iota$. The assumption $M' \Vdash_\iota n$ means that $M'\ \beta_{\mathsf{wh}}\ \underline{n}$. If $M\ \beta_{\mathsf{wh}}\ M'$, we have $M\ \beta_{\mathsf{wh}}\ \underline{n}$, that is $M \Vdash_\iota n$.

Assume that $A = B \to C$, $a = ([b_1,\dots,b_n],c)$ and $M'$ is such that $M' \Vdash_{B\to C} a$. We assume that $M\ \beta_{\mathsf{wh}}\ M'$ and we want to prove $M \Vdash_{B\to C} a$. So let $P$ be such that $\vdash P : B$ and $P \Vdash_B b_i$ for $i = 1,\dots,n$. Since $M\ \beta_{\mathsf{wh}}\ M'$, we have $(M)\,P\ \beta_{\mathsf{wh}}\ (M')\,P$ (see the definition of $\beta_{\mathsf{wh}}$ in Section 1.3.2). But $(M')\,P \Vdash_C c$ because $M' \Vdash_{B\to C} ([b_1,\dots,b_n],c)$ and hence $(M)\,P \Vdash_C c$ by inductive hypothesis. □

We write $M \Vdash_A [a_1,\dots,a_n]$ if $M \Vdash_A a_i$ for $i = 1,\dots,n$.

**Théorème 7.2.6 (Adequacy)** *Assume that $(x_1 : m_1 : A_1, \dots, x_l : m_l : A_l) \vdash M : a : A$. For any closed terms $P_1,\dots,P_n$ such that $\vdash P_i : A_i$ for $i = 1,\dots,l$, if $P_i \Vdash_{A_i} m_i$ for $i = 1,\dots,l$, then we have $M\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$.*

*Démonstration.* By induction on the derivation that $(x_1 : m_1 : A_1, \dots, x_l : m_l : A_l) \vdash M : a : A$, we prove the following universally quantified statement :

For any closed terms $P_1,\dots,P_n$ such that $\vdash P_i : A_i$ for $i = 1,\dots,l$, if $P_i \Vdash_{A_i} m_i$ for $i = 1,\dots,l$, then we have $M\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$.

We use $\Phi$ for the semantic context $(x_1 : m_1 : A_1, \dots, x_l : m_l : A_l)$.

Assume that $M = x_i$ for some $i \in \{1,\dots,l\}$. Then we have $m_j = []$ for $j \neq i$ and $m_i = [a]$. We have $M\,[P_1/x_1,\dots,P_l/x_l] = P_i$ and hence $M\,[P_1/x_1,\dots,P_l/x_l] \Vdash_{A_i} a$ by our assumption about the $P_j$s.

Assume that $M\,[P_1/x_1,\dots,P_l/x_l] = \underline{n}$ so that $M\,[P_1/x_1,\dots,P_l/x_l] \Vdash_\iota n$ because $\underline{n}\ \beta^*_{\mathsf{wh}}\ \underline{n}$ and by the definition of $\Vdash_\iota$.

Assume that $M = \underline{\mathsf{succ}}(P)$, $A = \iota$ and $a = \underline{n+1}$ for some $n \in \mathbb{N}$, so that we have $\Phi \vdash P : n : \iota$ and we know that $P\,[P_1/x_1,\dots,P_l/x_l] \Vdash_\iota n$ by inductive hypothesis. This means that $P\,[P_1/x_1,\dots,P_l/x_l]\ \beta^*_{\mathsf{wh}}\ \underline{n}$.

By definition of $\beta_{\mathsf{wh}}$, it follows that $\underline{\mathsf{succ}}(P\,[P_1/x_1,\dots,P_l/x_l])\ \beta^*_{\mathsf{wh}}\ \underline{n+1}$, that is $M\,[P_1/x_1,\dots,P_l/x_l] \Vdash_\iota n+1$ as required.

Assume that $M = \mathsf{if}(P,Q,z\cdot R)$, that $\Phi = \Phi_0 + \Phi_1$, with $\Phi_0 \vdash P : 0 : \iota$, $\Phi_1 \vdash Q : a : A$ and $\underline{\Phi}, z:\iota \vdash R : A$. The contexts $\Phi_p$ (for $p = 0,1$) can be written $\Phi_p = (x_1 : m_1^p : A_1,\dots,x_l : m_l^p : A_l)$ with $m_i = m_i^0 + m_i^1$ for $i = 1,\dots,l$. For each $i$, we have assumed that $P_i \Vdash_{A_i} m_i$ so that we have $P_i \Vdash_{A_i} m_i^p$ for $p = 0,1$, and for each $i = 1,\dots,l$. By inductive hypothesis, it follows that $P\,[P_1/x_1,\dots,P_l/x_l] \Vdash_\iota 0$ and $Q\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$. Therefore $P\,[P_1/x_1,\dots,P_l/x_l]\ \beta^*_{\mathsf{wh}}\ \underline{0}$. Hence, by definition of $\beta_{\mathsf{wh}}$, we have $M\,[P_1/x_1,\dots,P_l/x_l]\ \beta^*_{\mathsf{wh}}\ P\,[P_1/x_1,\dots,P_l/x_l]$. Since $Q\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$ we have $M\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$ by Lemma 7.2.5.

Assume that $M = \mathsf{if}(P,Q,z\cdot R)$, that $\Phi = \Phi_0 + \Phi_2$, with $\Phi_0 \vdash P : n+1 : \iota$, $\Phi_2, z : k[n] : \iota \vdash R : a : A$ and $\underline{\Phi} \vdash Q : A$, for some $k,n \in \mathbb{N}$. The contexts $\Phi_p$ (for $p = 0,2$) can be written $\Phi_p = (x_1 : m_1^p : A_1,\dots,x_l : m_l^p : A_l)$ with $m_i = m_i^0 + m_i^2$ for $i = 1,\dots,l$. For each $i$, we have assumed that $P_i \Vdash_{A_i} m_i$ so that we have $P_i \Vdash_{A_i} m_i^p$ for $p = 0,2$, and for each $i = 1,\dots,l$. By inductive hypothesis, it follows that $P\,[P_1/x_1,\dots,P_l/x_l] \Vdash_\iota n+1$ and $R\,[P_1/x_1,\dots,P_l/x_l,\underline{n}/z] \Vdash_A a$ (because $\underline{n} \Vdash_\iota n$ by definition of $\Vdash_\iota$, and hence $\underline{n} \Vdash_\iota k[n]$). Therefore we have $P\,[P_1/x_1,\dots,P_l/x_l]\ \beta^*_{\mathsf{wh}}\ \underline{n+1}$. Hence, coming back to the definition of $\beta_{\mathsf{wh}}$, we see that $M\,[P_1/x_1,\dots,P_l/x_l]\ \beta^*_{\mathsf{wh}}\ R\,[P_1/x_1,\dots,P_l/x_l,\underline{n}/z]$. But we have seen that $R\,[P_1/x_1,\dots,P_l/x_l,\underline{n}/z] \Vdash_A a$ and hence we have $M\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$ by Lemma 7.2.5.

Assume that $A = B \to C$, $M = \lambda x^B\,P$, that $a = (m,c)$, and that we have $\Phi, x : m : B \vdash P : c : C$. Given $P_1,\dots,P_l$ as in the statement of the theorem, we must prove that $\lambda x^B\,P\,[P_1/x_1,\dots,P_l/x_l] \Vdash_{B\to C} (m,c)$. So let $Q$ be a term such that $Q \Vdash_B m$, we must prove that

$$\left(\lambda x^B\,P\,[P_1/x_1,\dots,P_l/x_l]\right) Q \Vdash_C c\,.$$

Since $\left(\lambda x^B\,P\,[P_1/x_1,\dots,P_l/x_l]\right) Q\ \beta_{\mathsf{wh}}\ P\,[P_1/x_1,\dots,P_l/x_l,Q/x]$, by Lemma 7.2.5, it suffices to prove that $P\,[P_1/x_1,\dots,P_l/x_l,Q/x] \Vdash_C c$ which is a direct consequence of our inductive hypothesis (observe here that it is very important to write this inductive hypothesis as a statement universally quantified over the $P_i$'s).

Assume that $M = (P)\,Q$, that we have $\Phi_0 \vdash P : (m,a) : B \to A$ with $m = [b_1,\dots,b_n]$, that $\Phi_p \vdash Q : b_p : B$ for $p = 1,\dots,n$ and $\Phi = \sum_{p=0}^n \Phi_p$. For $p = 0,\dots,n$ we can write $\Phi_p = (x_1 : m_1^p : A_1,\dots,x_l : m_l^p : A_l)$ and we have $m_i = \sum_{p=0}^n m_i^p$ for $i = 1,\dots,l$. Since we know that $P_i \Vdash_{A_i} m_i$, for $i = 1,\dots,l$, we have $P_i \Vdash_{A_i} m_i^p$ for each $i = 1,\dots,l$ and each $p = 0,\dots,n$. By inductive hypothesis, we have therefore $P\,[P_1/x_1,\dots,P_l/x_l] \Vdash_{B\to A} (m,a)$ and $Q\,[P_1/x_1,\dots,P_l/x_l] \Vdash_B b_p$ for $p = 1,\dots,n$. By definition of $\Vdash_{B\to A}$, it follows that

$$M\,[P_1/x_1,\dots,P_l/x_l] = (P\,[P_1/x_1,\dots,P_l/x_l])\,Q\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$$

as required.

Assume last that $M = \mathsf{fix}(P)$, that we have $\Phi_0 \vdash P : (m,a) : A \to A$ with $m = [a_1,\dots,a_n]$, $\Phi_p \vdash \mathsf{fix}(P) : a_p : A$ for $p = 1,\dots,n$ and $\Phi = \sum_{i=0}^n \Phi_i$. For $p = 0,\dots,n$ we can write $\Phi_p = (x_1 : m_1^p : A_1,\dots,x_l : m_l^p : A_l)$ and we have $m_i = \sum_{p=0}^n m_i^p$ for $i = 1,\dots,l$. Since we know that $P_i \Vdash_{A_i} m_i$, for $i = 1,\dots,l$, we have $P_i \Vdash_{A_i} m_i^p$ for each $i = 1,\dots,l$ and each $p = 0,\dots,n$. By inductive hypothesis, we have therefore $P\,[P_1/x_1,\dots,P_l/x_l] \Vdash_{A\to A} (m,a)$ and $\mathsf{fix}(P)\,[P_1/x_1,\dots,P_l/x_l] \Vdash_B a_p$ for $p = 1,\dots,n$. By definition of $\Vdash_{A\to A}$, it follows that

$$(P\,[P_1/x_1,\dots,P_l/x_l])\,\mathsf{fix}(P)\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$$

We have

$$\begin{aligned}M\,[P_1/x_1,\dots,P_l/x_l] &= \mathsf{fix}(P\,[P_1/x_1,\dots,P_l/x_l])\\ &\beta_{\mathsf{wh}}\ (P\,[P_1/x_1,\dots,P_l/x_l])\,\mathsf{fix}(P\,[P_1/x_1,\dots,P_l/x_l])\\ &=\ \ (P\,[P_1/x_1,\dots,P_l/x_l])\,\mathsf{fix}(P)\,[P_1/x_1,\dots,P_l/x_l]\end{aligned}$$

and hence $M\,[P_1/x_1,\dots,P_l/x_l] \Vdash_A a$ by Lemma 7.2.5, as required. □

In particular, when $M$ is closed, we have

$$\vdash M : a : A \Rightarrow M \Vdash_A a$$

and in the particular case where $A = \iota$, we get the announced answer to our initial question :

$$\vdash M : n : \iota \Rightarrow M \ \beta^*_{\mathsf{wh}} \ \underline{n}$$

so that conversely, if the reduction of $M$ in the $\beta_{\mathsf{wh}}$ strategy does not terminate, we must have $[M] = \emptyset$.

In section 1.3.4, for any type $A$, we have defined an observational preorder on closed terms of type $A$. Let $M$ and $N$ be closed terms of type $A$. We write $M \sqsubseteq_{\mathrm{obs}} N$ if, for any closed term $C$ of type $A \to \iota$, one has

$$\forall n \in \mathbb{N} \quad (C)\, M \ \beta^*_{\mathsf{wh}} \ \underline{n} \Rightarrow (C)\, N \ \beta^*_{\mathsf{wh}} \ \underline{n}\,.$$

**Théorème 7.2.7** *Let $M$ and $N$ be closed terms such that $\vdash M : A$ and $\vdash N : A$. If $[M] \subseteq [N]$ (as subsets of $[A]$) then $M \sqsubseteq_{\mathrm{obs}} N$.*

*Démonstration.* Assume that $[M] \subseteq [N]$ and let $C$ be a closed term such that $\vdash C : A \to \iota$. Assume that $(C)\, M \ \beta^*_{\mathsf{wh}} \ \underline{n}$ for some $n \in \mathbb{N}$. Then we have $[(C)\, M] = \{n\}$ by Theorem 7.2.2, that is

$$\mathsf{Ev} \circ \langle [C], [M] \rangle = \{n\}$$

Since $[M] \subseteq [N]$, we have $\mathsf{Ev} \circ \langle [C], [M] \rangle \subseteq \mathsf{Ev} \circ \langle [C], [N] \rangle = [(C)\, N]$ and hence $\vdash (C)\, N : n : \iota$ (by monotonicity of the interpretation in the relational model, see for instance Exercice 6.7.8). Therefore $(C)\, N \ \beta^*_{\mathsf{wh}} \ \underline{n}$ by Theorem 7.2.6. This shows that $M \sqsubseteq_{\mathrm{obs}} N$. □

In particular, if $[M] = [N]$, then $M \simeq_{\mathrm{obs}} N$, that is, two terms which have the same interpretation in the model are observationally equivalent. When the converse implication is also true (which is not the case in $\mathbf{Rel}_!$ but could be the case in some other model), one says that the model is (equationally) *fully abstract*. If the model is equiped with an order relation $\leq$ (as here the inclusion relation) on morphisms "compatible with the CCC structure" and if $[M] \leq [N] \Rightarrow M \sqsubseteq_{\mathrm{obs}} N$, one says that the model is *inequationally fully abstract*, which is of course a stronger condition than being fully abstract.

As an exemple of application, it is easy to see (exercise) that the two terms $G$ and $D$ of Section 1.3.4 satisfy

$$[G] = [D] = \{([0], [0], 0)\} \cup \{([n], [n'], 1) \mid n + n' \neq 0)\}$$

and hence $G \simeq_{\mathrm{obs}} D$.

It is also easy to see that $\mathbf{Rel}_!$ is not fully abstract. First, for each type $A$, we set $\Omega_A = \mathsf{fix}(\lambda x^A\, x)$ which satisfies $\vdash \Omega_A : A$.

**Exercice 7.2.1** Prove that $[\Omega_A] = \emptyset$.

Let $M_1 = \lambda x^\iota \, \mathsf{if}(x, \underline{0}, z \cdot \Omega_\iota)$ and $M_2 = \lambda x^\iota \, \mathsf{if}(x, \mathsf{if}(x, \underline{0}, z \cdot \Omega_\iota), z \cdot \Omega_\iota)$ so that $\vdash M_i : \iota \to \iota$ for $i = 1, 2$.

**Exercice 7.2.2** Prove that $[M_i] = \{(i[0], 0)\}$ for $i = 1, 2$, so that $[M_1] \neq [M_2]$. Prove that $M_1 \simeq_{\mathrm{obs}} M_2$ (by a syntactic analysis of a context $C$, or using the Adequacy Theorem for the Scott model of $\mathsf{PCF}$).

# Chapitre 8

# Probabilistic coherence spaces

## 8.1 The category of probabilistic coherence spaces and linear morphisms

Let $I$ be an at most countable set. Remember that, equipped with the product order (according to which $u \leq v$ means $\forall i \in I\ u_i \leq v_i$), the set $\overline{\mathbb{R}_{\geq 0}}^I$ is a complete lattice. That is, any subset $A$ of $\overline{\mathbb{R}_{\geq 0}}^I$ has a least upper bound $\sup A \in \overline{\mathbb{R}_{\geq 0}}^I$ given by

$$\sup A = \left( \sup_{u \in A} u_i \right)_{i \in I} .$$

We refer to Section 1.4.1 for basic results about computations on elements of $\overline{\mathbb{R}_{\geq 0}}^I$.

Given $u, u' \in \overline{\mathbb{R}_{\geq 0}}^I$ we set

$$\langle u, u' \rangle = \sum_{i \in I} u_i u'_i \in \overline{\mathbb{R}_{\geq 0}} .$$

**Lemme 8.1.1** *The operation $\langle \_ , \_ \rangle$ is bilinear, that is*

— $\forall u, u'(1), u'(2) \in \mathbb{R}_{\geq 0}^I\ \forall \lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}\ \ \langle u, \lambda_1 u'(1) + \lambda_2 u'(2) \rangle = \lambda_1 \langle u, u'(1) \rangle + \lambda_2 \langle u, u'(2) \rangle$.

— $\forall u', u(1), u(2) \in \mathbb{R}_{\geq 0}^I\ \forall \lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}\ \ \langle \lambda_1 u(1) + \lambda_2 u(2), u' \rangle = \lambda_1 \langle u(1), u' \rangle + \lambda_2 \langle u(2), u' \rangle$.

*Démonstration.* We prove the second statement, the first one being completely similar. Let $u = \lambda_1 u(1) + \lambda_2 u(2)$, we have

$$\begin{aligned}
\langle u, u' \rangle &= \sum_{i \in I} u_i u'_i \\
&= \sum_{i \in I} (\lambda_1 u(1)_i u'_i + \lambda_2 u(2)_i u'_i) \\
&= \lambda_1 \sum_{i \in I} u(1)_i u'_i + \lambda_2 \sum_{i \in I} u(2)_i u'_i \\
&= \lambda_1 \langle u(1), u' \rangle + \lambda_2 \langle u(2), u' \rangle .
\end{aligned}$$

□

**Lemme 8.1.2** *The operation $\langle \_ , \_ \rangle$ is separately Scott-continuous, that is*

— *for all $u$ and all non-decreasing (see Section 0.1) sequence $(u'(n))_{n \in \mathbb{N}}$ in $\overline{\mathbb{R}_{\geq 0}}^I$ one has $\langle u, \sup_{n \in \mathbb{N}} u'(n) \rangle = \sup_{n \in \mathbb{N}} \langle u, u'(n) \rangle$*

— *and for all $u'$ and all non-decreasing sequence $(u(n))_{n \in \mathbb{N}}$ in $\overline{\mathbb{R}_{\geq 0}}^I$ one has $\langle \sup_{n \in \mathbb{N}} u(n), u' \rangle = \sup_{n \in \mathbb{N}} \langle u(n), u' \rangle$.*

*Démonstration.* We prove the second statement, the first one being completely similar. We have

$$\begin{aligned}\left\langle \sup_{n\in\mathbb{N}} u(n), u' \right\rangle &= \sum_{i\in I} (\sup_{n\in\mathbb{N}} u(n)_i) u'_i \\ &= \sum_{i\in I} \sup_{n\in\mathbb{N}} (u(n)_i u'_i) \quad \text{scalar multiplication commutes with lubs} \\ &= \sup_{n\in\mathbb{N}} \sum_{i\in I} u(n)_i u'_i \quad \text{by Theorem 1.4.4.}\end{aligned}$$

□

Given $\mathcal{P} \in \overline{\mathbb{R}_{\geq 0}}^I$ we define its *orthogonal*[1] by

$$\mathcal{P}^\perp = \left\{ u' \in \overline{\mathbb{R}_{\geq 0}}^I \mid \forall \langle u, u' \rangle \leq 1 \right\} .$$

**Lemme 8.1.3** *If $\mathcal{P}$ and $\mathcal{Q}$ are subsets of $\overline{\mathbb{R}_{\geq 0}}^I$ then $\mathcal{P} \subseteq \mathcal{Q} \Rightarrow \mathcal{Q}^\perp \subseteq \mathcal{P}^\perp$. For any $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ one has $\mathcal{P} \subseteq \mathcal{P}^{\perp\perp}$ and $\mathcal{P}^\perp = \mathcal{P}^{\perp\perp\perp}$.*

*Démonstration.* Assume that $\mathcal{P} \subseteq \mathcal{Q}$ and let $v' \in \mathcal{Q}^\perp$, we prove that $v' \in \mathcal{P}^\perp$. So let $u \in \mathcal{P}$. We have $u \in \mathcal{Q}$ and hence $\langle u, v' \rangle \leq 1$. Since this holds for all $u \in \mathcal{P}$ we have $v' \in \mathcal{P}^\perp$.

Let $u \in \mathcal{P}$. Let $u' \in \mathcal{P}^\perp$, we have $\langle u, u' \rangle \leq 1$. Since this holds for all $u' \in \mathcal{P}^\perp$, we have $u \in \mathcal{P}^{\perp\perp}$. We have shown that $\mathcal{P} \subseteq \mathcal{P}^{\perp\perp}$.

Since $\mathcal{P} \subseteq \mathcal{P}^{\perp\perp}$, we have $\mathcal{P}^{\perp\perp\perp} \subseteq \mathcal{P}^\perp$ by the first property we have proven. The inclusion $\mathcal{P}^\perp \subseteq \mathcal{P}^{\perp\perp\perp}$ is an instance of the second property (applied to $\mathcal{P}^\perp$). This proves the third property. □

So to check that $\mathcal{P} = \mathcal{P}^{\perp\perp}$, it suffices to prove that $\mathcal{P} = \mathcal{Q}^\perp$ for some $\mathcal{Q} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ (*not necessarily* $Q = \mathcal{P}^\perp$; such a $\mathcal{Q}$ is called sometimes a *predual* of $\mathcal{P}$). This observation will be used extremely often, tacitely.

We recall that the order relation $\leq$ we consider on $\overline{\mathbb{R}_{\geq 0}}^I$ is always the product order relation, so that $u \leq v$ means $\forall i \in I\ u_i \leq v_i$.

**Lemme 8.1.4** *Let $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ be such that $\mathcal{P} = \mathcal{P}^{\perp\perp}$. Then the following properties hold.*

- *If $u \in \mathcal{P}$ and $v \in \overline{\mathbb{R}_{\geq 0}}^I$ satisfy $v \leq u$ then $v \in \mathcal{P}$.*
- *If $u(1), u(2) \in \mathcal{P}$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ are such that $\lambda_1 + \lambda_2 = 1$ then $\lambda_1 u(1) + \lambda_2 u(2) \in \mathcal{P}$.*
- *If $u(1), \dots, u(k) \in \mathcal{P}$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$ are such that $\lambda_1 + \cdots + \lambda_k \leq 1$ then $\sum_{j=1}^k \lambda_j u(j) \in \mathcal{P}$.*
- *If $(u(n))_{n\in\mathbb{N}}$ is a non-decreasing sequence of elements of $\mathcal{P}$, then $u = \sup_{n\in\mathbb{N}} u(n) \in \mathcal{P}$.*

*For the last statement observe that $u = (\sup_{n\in\mathbb{N}} u(n)_i)_{i\in I}$ because the order relation we consider on $\overline{\mathbb{R}_{\geq 0}}^I$ is the product order.*

*Démonstration.* Assume that $v \leq u \in \mathcal{P}$. We want to prove that $v \in \mathcal{P}$. Since $\mathcal{P} = \mathcal{P}^{\perp\perp}$, it suffice to prove that $\forall u' \in \mathcal{P}^\perp\ \langle v, u' \rangle \leq 1$, so let $u' \in \mathcal{P}^\perp$. We have

$$\begin{aligned}\langle v, u' \rangle &= \sum_{i\in I} v_i u'_i \\ &\leq \sum_{i\in I} u_i u'_i \quad \text{since } v \leq u \\ &\leq 1 \quad \text{since } u \in \mathcal{P} \text{ and } u' \in \mathcal{P}^\perp .\end{aligned}$$

Let $u(1), \dots, u(k) \in \mathcal{P}$ and let $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\geq 0}$ be such that $\lambda_1 + \cdots + \lambda_k \leq 1$. We want to prove that $u = \lambda_1 u(1) + \cdots + \lambda_k u(k) \in \mathcal{P} = \mathcal{P}^{\perp\perp}$. So let $u' \in \mathcal{P}^\perp$, it suffices to prove that $\langle u, u' \rangle \leq 1$. We have $\langle u, u' \rangle = \sum_{j=1}^k \lambda_j \langle u(j), u' \rangle$ by Lemma 8.1.1 and $\langle u(j), u' \rangle \leq 1$ because $u(j) \in \mathcal{P} = \mathcal{P}^{\perp\perp}$ for $j = 1, \dots, k$ and hence $\langle u, u' \rangle \leq \lambda_1 + \cdots + \lambda_k \leq 1$. Since this holds for any $u' \in \mathcal{P}^\perp$, we have proven that $u = \sum_{j=1}^k \lambda_j u(j)$.

1. The word should not be taken in its geometrical sense, but in its linear logical sense. From a geometrical point of view, the world *polar* would much better.

Let $(u(n))_{n\in\mathbb{N}}$ be a non-decreasing sequence of elements of $\mathcal{P}$, we prove that $u = \sup_{n\in\mathbb{N}} u(n) \in \mathcal{P} = \mathcal{P}^{\perp\perp}$ so let $u' \in \mathcal{P}^{\perp}$. For all $n \in \mathbb{N}$ we have $u(n) \in \mathcal{P} = \mathcal{P}^{\perp\perp}$ and hence $\langle u(n), u'\rangle \leq 1$ and hence $\sup_{n\in\mathbb{N}} \langle u(n), u'\rangle \leq 1$. But $\sup_{n\in\mathbb{N}} \langle u(n), u'\rangle = \langle \sup_{n\in\mathbb{N}} u(n), u'\rangle$ by Lemma 8.1.2, hence $\langle \sup_{n\in\mathbb{N}} u(n), u'\rangle \leq 1$. Since this holds for all $u' \in \mathcal{P}^{\perp}$, we have proven that $\sup_{n\in\mathbb{N}} u(n) \in \mathcal{P}$. □

**Remarque 8.1.5** Using the Hahn-Banach theorem it is possible to prove that if $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ satisfies the three properties expressed in Lemma 8.1.4, then $\mathcal{P}^{\perp\perp} = \mathcal{P}$, but we will not use this fact in the sequel.

**Lemme 8.1.6** *If $\mathcal{P} = \mathcal{P}^{\perp\perp}$, $(u(n))_{n\in\mathbb{N}}$ is a family of elements of $\mathcal{P}$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a family of elements of $\mathbb{R}_{\geq 0}$ such that $\sum_{n\in\mathbb{N}} \lambda_n \leq 1$, one has $\sum_{n\in\mathbb{N}} \lambda_n u(n) \in \mathcal{P}$.*

*Démonstration.* It suffices to observe that $\sum_{n\in\mathbb{N}} \lambda_n u(n) = \sup_{k\in\mathbb{N}} \sum_{n=0}^{k} \lambda_n u(n)$ and use Lemma 8.1.4. □

Let $i \in I$, we use $\mathsf{e}_i$ for the element of $\mathbb{R}_{\geq 0}^I$ defined by $(\mathsf{e}_i)_j = \delta_{i,j}$.

Let $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ and $i \in I$.

- Assume that $\sup_{u\in\mathcal{P}} u_i < \infty$. So let $\lambda \in \mathbb{R}_{\geq 0}$ be such that $\forall u \in \mathcal{P}\ \ u_i \leq \lambda$, we take $\lambda \neq 0$. Then $\frac{1}{\lambda}\mathsf{e}_i \in \mathcal{P}^{\perp}$ since $\forall u \in \mathcal{P}\ \ \langle u, \frac{1}{\lambda}\mathsf{e}_i\rangle = \frac{1}{\lambda} u_i \leq 1$. Hence $\sup_{u'\in\mathcal{P}^{\perp}} u'_i \geq \frac{1}{\lambda} > 0$.
- Assume that $\sup_{u\in\mathcal{P}} u_i > 0$, so let $u \in \mathcal{P}$ be such that $\varepsilon = u_i > 0$. Then for all $u' \in \mathcal{P}^{\perp}$ we have $\langle u, u'\rangle \leq 1$, hence $\varepsilon u'_i \leq 1$, that is $u'_i \leq \frac{1}{\varepsilon}$. Therefore $\sup_{u'\in\mathcal{P}^{\perp}} u'_i \leq \frac{1}{\varepsilon} < \infty$.

So we have proven the following.

**Lemme 8.1.7** *Let $\mathcal{P} \subseteq \mathbb{R}_{\geq 0}^I$ and $i \in I$ be such that $0 < \sup_{u\in\mathcal{P}} u_i < \infty$. Then $0 < \sup_{u'\in\mathcal{P}^{\perp}} u'_i < \infty$.*

So if $\forall i \in I\ 0 < \sup_{u\in\mathcal{P}} u_i < \infty$ then $\forall i \in I\ 0 < \sup_{u'\in\mathcal{P}^{\perp}} u'_i < \infty$ and in particular $\mathcal{P}^{\perp} \subseteq \mathbb{R}_{\geq 0}^I$ (that is one never has $u'_i = \infty$ for $u' \in \mathcal{P}^{\perp}$ and $i \in I$). In order to be sure to extract meaningful coefficients from our semantics, we want to avoid the appearance of $\infty$ coefficients. This justifies the following definition.

**Definition 8.1.8** A *probabilistic coherence space* (PCS for short) is a pair $X = (|X|, \mathsf{P}(X))$ where $|X|$ is an at most countable set and $\mathsf{P}(X) \subseteq \mathbb{R}_{\geq 0}^{|X|}$ satisfies $\mathsf{P}(X)^{\perp\perp} = \mathsf{P}(X)$ and $\forall a \in |X|\ 0 < \sup_{u\in\mathsf{P}(X)} u_a < \infty$.

**Remarque 8.1.9** The purpose of the condition $\forall a \in |X|\ 0 < \sup_{u\in\mathsf{P}(X)} u_a < \infty$ is to avoid the appearance of $\infty$ coefficients in our objects and morphisms.

**Lemme 8.1.10** *Let $X^{\perp} = (|X|, \mathsf{P}(X)^{\perp})$. Then $X^{\perp}$ is a PCS.*

*Démonstration.* We just have to prove that $\forall a \in |X|\ 0 < \sup_{u'\in\mathsf{P}(X)^{\perp}} u'_a < \infty$ and this results from Lemma 8.1.7. □

Here are some examples.

- $0$ is the unique PCS such that $|0| = \emptyset$ and $\mathsf{P}(0)$ consists of the empty function $\emptyset \to \mathbb{R}_{\geq 0}$.
- $1$ is the PCS such that $|1| = \{*\}$ and $\mathsf{P}(1) = [0, 1]$ (identifying $\mathbb{R}_{\geq 0}^{\{*\}}$ and $[0, 1]$). Notice that $1^{\perp} = 1$.
- **Bool** is defined by $|\mathbf{Bool}| = \{0, 1\}$ and $u \in \mathsf{P}(\mathbf{Bool})$ if $u_0 + u_1 \leq 1$. In other words $\mathsf{P}(\mathbf{Bool}) = \{\mathsf{e}_0 + \mathsf{e}_1\}^{\perp}$ and hence **Bool** is a PCS. The elements of $\mathsf{P}(\mathbf{Bool})$ are sub-probability distributions on the booleans 0 and 1. Then $\mathbf{Bool}^{\perp}$ is characterized by $|\mathbf{Bool}^{\perp}| = \{0, 1\}$ and $u' \in \mathsf{P}(\mathbf{Bool}^{\perp})$ iff $u'_0, u'_1 \leq 1$.
- Similarly $\mathsf{N} = (\mathbb{N}, \{u \in \mathbb{R}_{\geq 0}^{\mathbb{N}} \mid \sum_{n\in\mathbb{N}} u_n \leq 1\})$ is a PCS since this $\mathsf{P}(\mathsf{N})$ is equal to $\{\sum_{n=0}^{\infty} \mathsf{e}_n\}^{\perp}$. Then we have $|\mathsf{N}^{\perp}| = \mathbb{N}$ and $\mathsf{P}(\mathsf{N}^{\perp}) = \{u' \in \mathbb{R}_{\geq 0}^{\mathbb{N}} \mid \forall n \in \mathbb{N}\ u'_n \leq 1\}$.

**Remarque 8.1.11** The example of $\mathsf{N}$ is most typical of the intuitions supported by this semantics : the elements of types are probability sub-distributions (and not probability distributions because as it is usual in denotational semantics, we take partiality into account) on basic values, here integers. But the example of $\mathsf{N}^{\perp}$ shows that there are also probability coherence space which do not support this intuition : there are elements $u'$ of $\mathsf{P}(\mathsf{N}^{\perp})$ whose total weight $\sum_{n\in\mathbb{N}} u'_n$ is infinite. Such "non-probabilistic" objects will show up naturally when interpreting cartesian products and function types.

Let $u \in \overline{\mathbb{R}_{\geq 0}}^I$ and $v \in \overline{\mathbb{R}_{\geq 0}}^J$, we define $u \otimes v \in \overline{\mathbb{R}_{\geq 0}}^{I \times J}$ by $(u \otimes v)_{i,j} = u_i v_j$.

**Lemme 8.1.12** *Let $X_1$ and $X_2$ be PCSs, one defines a PCS $X_1 \otimes X_2$ by setting $|X_1 \otimes X_2| = |X_1| \times |X_2|$ and $\mathsf{P}(X_1 \otimes X_2) = \{u_1 \otimes u_2 \mid u_i \in \mathsf{P}(X_1) \text{ for } i = 1, 2\}^{\perp\perp}$.*

*Démonstration.* Let $a_i \in |X_i|$ for $i = 1, 2$ and let $\mathcal{P} = \{u_1 \otimes u_2 \mid u_i \in \mathsf{P}(X_1) \text{ for } i = 1, 2\}$, we have $\sup_{v \in \mathcal{P}} v_{a_1,a_2} = (\sup_{u(1) \in \mathsf{P}(X_1)} u(1)_{a_1})(\sup_{u(2) \in \mathsf{P}(X_2)} u(2)_{a_2})$ and hence $0 < \sup_{v \in \mathcal{P}} v_{a_1,a_2} < \infty$. Therefore $0 < \sup_{v \in \mathcal{P}^{\perp\perp}} v_{a_1,a_2} < \infty$ by Lemma 8.1.7. □

If $X$ and $Y$ are PCSs, we define $X \multimap Y = (X \otimes Y^{\perp})^{\perp}$, which is a PCS by Lemma 8.1.10. Given $s \in \mathbb{R}_{\geq 0}^{|X \multimap Y|}$ and $u \in \mathbb{R}_{\geq 0}^{|X|}$ we define $s \cdot u \in \mathbb{R}_{\geq 0}^{|Y|}$ by

$$(s \cdot u)_{b \in |Y|} = \sum_{a \in |X|} s_{a,b} u_a \,.$$

**Lemme 8.1.13** *Let $s \in \mathbb{R}_{\geq 0}^{|X \multimap Y|}$, $u \in \mathbb{R}_{\geq 0}^{|X|}$ and $v' \in \mathbb{R}_{\geq 0}^{|Y|}$. We have $\langle s \cdot u, v' \rangle = \langle s, u \otimes v' \rangle \in \overline{\mathbb{R}_{\geq 0}}$.*

*Démonstration.*

$$\begin{aligned}
\langle s \cdot u, v' \rangle &= \sum_{b \in |Y|} \left( \sum_{a \in |X|} s_{a,b} u_a \right) v'_b \\
&= \sum_{b \in |Y|} \sum_{a \in |X|} s_{a,b} u_a v'_b \\
&= \sum_{(a,b) \in |X \multimap Y|} s_{a,b} (u \otimes v')_{a,b} \,.
\end{aligned}$$

□

**Lemme 8.1.14** *Let $s \in \mathbb{R}_{\geq 0}^{|X \multimap Y|}$. One has $s \in \mathsf{P}(X \multimap Y)$ if and only if for any $u \in \mathsf{P}(X)$, $s \cdot u \in \mathsf{P}(Y)$.*

*Démonstration.* Assume first that $s \in \mathsf{P}(X \multimap Y)$. Let $u \in \mathsf{P}(X)$, we must prove that $s \cdot u \in \mathsf{P}(Y) = \mathsf{P}(Y)^{\perp\perp}$, so let $v' \in \mathsf{P}(X)^{\perp}$, it suffices to prove that $\langle s \cdot u, v' \rangle \leq 1$. This results from our assumption on $s$, from the fact that $u \otimes v' \in \mathsf{P}(X \otimes Y^{\perp}) = \mathsf{P}(X \otimes Y^{\perp})^{\perp\perp} = \mathsf{P}(X \multimap Y)^{\perp}$ and from $\langle s \cdot u, v' \rangle = \langle s, u \otimes v' \rangle$ by Lemma 8.1.13.

Conversely assume that $s \cdot u \in \mathsf{P}(Y)$ for all $u \in \mathsf{P}(X)$, we must prove that $s \in \mathsf{P}(X \multimap Y) = \mathsf{P}(X \otimes Y^{\perp})^{\perp} = \left\{ u \otimes v' \mid u \in \mathsf{P}(X) \text{ and } v' \in \mathsf{P}(Y)^{\perp} \right\}^{\perp}$, so let $u \in \mathsf{P}(X)$ and $v' \in \mathsf{P}(Y)^{\perp}$, we must prove that $\langle s, u \otimes v' \rangle \leq 1$. This results from our assumption on $s$ and from Lemma 8.1.13. □

In particular the element $\mathsf{Id}$ of $\mathbb{R}_{\geq 0}^{|X \multimap X|}$ given by $\mathsf{Id}_{a,b} = \delta_{a,b}$ satisfies $\mathsf{Id} \in \mathsf{P}(X \multimap X)$ since $\forall u \in \mathsf{P}(X)$ $\mathsf{Id} \cdot u = u$. Let $s \in \mathsf{P}(X \multimap Y)$ and $t \in \mathsf{P}(Y \multimap Z)$, we define $t\, s \in \overline{\mathbb{R}_{\geq 0}}^{|X \multimap Z|}$ by

$$(t\, s)_{a,c} = \sum_{b \in |Y|} s_{a,b} t_{b,c} \,.$$

Then $t\, s \in \mathsf{P}(X \multimap Z)$. Indeed let $u \in \mathsf{P}(X)$, we have

$$\begin{aligned}
(t\, s) \cdot u &= \left( \sum_{a \in |X|} \sum_{b \in |Y|} s_{a,b} t_{b,c} u_a \right)_{c \in |Z|} \\
&= \left( \sum_{b \in |Y|} t_{b,c} \sum_{a \in |X|} s_{a,b} u_a \right)_{c \in |Z|} \\
&= \left( \sum_{b \in |Y|} t_{b,c} (s \cdot u)_b \right)_{c \in |Z|} \\
&= t \cdot (s \cdot u)
\end{aligned}$$

so that $(t\,s)\cdot u \in \mathsf{P}(Z)$ since $s \cdot u \in \mathsf{P}(Y)$. So we have defined an operation of composition

$$\begin{aligned} \mathsf{P}(Y \multimap Z) \times \mathsf{P}(X \multimap Y) &\to \mathsf{P}(X \multimap Z) \\ (t,s) &\mapsto t\,s \end{aligned}$$

and this operation is associative and has $\mathsf{Id}$ as a left and right unit, so we have defined a category **Pcoh** whose objects are the probabilistic coherence spaces and $\mathbf{Pcoh}(X,Y) = \mathsf{P}(X \multimap Y)$.

**Remarque 8.1.15** Our notation $t\,s$ for composition of morphisms departs slightly from the ordinary notation for product of matrices in linear algebra. Given $n, m \in \mathbb{N}$, an $(n,m)$-matrix (with coefficients in $\mathbb{R}$, say) is an array $A = (A_{i,j})_{\substack{1\le i\le n\\ 1\le j\le m}}$ with $n$ lines and $m$ columns. If $A$ is an $(n,m)$-matrix and $B$ is an $(m,p)$ matrix then $A\,B = \left(\sum_{j=1}^{m} A_{i,j}B_{j,k}\right)_{\substack{1\le i\le n\\ 1\le k\le p}}$ is an $(n,p)$-matrix. If we want to see $A$ as a linear operator $f_A : \mathbb{R}^n \to \mathbb{R}^m$ then the application of $A$ to a vector $x \in \mathbb{R}^n$ (which is usually considered as a column matrix $(x_{i,1})_{i=1}^{n}$) is $x^{\mathsf{T}} A = (\sum_{i=1}^{n} x_{i,1}A_{i,j})_{1\le j\le m}$, turning $x$ into a line matrix $x^{\mathsf{T}}_{1,i} = x_{i,1}$ by transposition. With these conventions we have $f_{A\,B} = f_B \circ f_A$ and so we have an inversion of order between the two notations for composition. This definition of $f_A$ seems quite standard in applied mathematics, but in algebra it seems more usual to see $f_A$ as a function acting in the reverse order

$$\begin{aligned} f_A : \mathbb{R}^m &\to \mathbb{R}^n \\ y &\mapsto \left(\sum_{j=1}^{m} A_{i,j}y_{j,1}\right)_{i=1}^{n} \end{aligned}$$

since then we have $f_{A\,B} = f_A \circ f_B$ (again we see $y \in \mathbb{R}^m$ as a column vector).

In our case an $s \in \mathsf{P}(X \multimap Y)$ (which is a matrix) induces naturally a morphism $f_s : \mathsf{P}(X) \to \mathsf{P}(Y)$ (by $f_s(u) = s \cdot u$ defined above) and we have chosen our reversed notation for matrix composition $(t\,s)_{a,c} = \sum_{b\in|Y|} s_{a,b}t_{b,c}$ in order to have $f_{t\,s} = f_t \circ f_s$, with the same notational coherence as in linear algebra.

Let $s \in \mathbb{R}_{\ge 0}^{I\times J}$, we define $s^\perp$, the transpose of $s$, as $s^\perp \in \mathbb{R}_{\ge 0}^{J\times I}$ by $s^\perp_{j,i} = s_{i,j}$.

**Lemme 8.1.16** *Let $s \in \mathbf{Pcoh}(X,Y)$. Then $s^\perp \in \mathbf{Pcoh}(Y^\perp, X^\perp)$. Moreover, for any $u \in \mathsf{P}(X)$ and $v' \in \mathsf{P}(Y^\perp)$ we have*

$$\langle s\cdot u, v'\rangle = \left\langle u, s^\perp \cdot v'\right\rangle = s\,(u \otimes v')\,.$$

*Démonstration.* The stated equations are obvious : all three expressions are equal to $\sum_{a\in|X|,b\in|Y|} s_{a,b}u_a v'_b$. For the first part of the lemma, we must prove that $s^\perp \in \mathsf{P}(Y^\perp \multimap X^\perp) = \mathsf{P}(Y^\perp \otimes X)^\perp$. So let $v' \in \mathsf{P}(Y^\perp)$ and $u \in \mathsf{P}(X)$, we have $\left\langle s^\perp, v'\otimes u\right\rangle = \langle s, u\otimes v'\rangle \le 1$ since $s \in \mathsf{P}(X \multimap Y) = \mathsf{P}(X \otimes Y^\perp)^\perp$. □

**Lemme 8.1.17** $\mathsf{Id}^\perp = \mathsf{Id}$ *and* $(t\,s)^\perp = s^\perp t^\perp$ *so that* $\_^\perp$ *is an involutive functor* $\mathbf{Pcoh} \to \mathbf{Pcoh}^{\mathsf{op}}$.

**Exercice 8.1.1** Prove this lemma.

**Lemme 8.1.18** *Let $s, t \in \mathbf{Pcoh}(X,Y)$. If $\forall u \in \mathsf{P}(X)\ s\cdot u = t\cdot u$ then $s = t$.*

*Démonstration.* Let $a \in |X|$. Let $\varepsilon \in \mathbb{R}_{\ge 0}$ be such that $0 < \varepsilon < \sup_{u\in\mathsf{P}(X)} u_a$ (such an $\varepsilon$ exists by definition of a PCS). So let $u \in \mathsf{P}(X)$ be such that $\varepsilon < u_a$. We have $\varepsilon\mathsf{e}_a \le u$ and hence $\varepsilon\mathsf{e}_a \in \mathsf{P}(X)$ by Lemma 8.1.4. We have $s\cdot\varepsilon\mathsf{e}_a = (\varepsilon s_{a,b})_{b\in|Y|}$ and $t\cdot\varepsilon\mathsf{e}_a = (\varepsilon t_{a,b})_{b\in|Y|}$ so if $\forall u \in \mathsf{P}(X)\ s\cdot u = t\cdot u$ we have in particular $(\varepsilon s_{a,b})_{b\in|Y|} = (\varepsilon t_{a,b})_{b\in|Y|}$ and hence $\forall b \in |Y|\ s_{a,b} = t_{a,b}$ since $\varepsilon \ne 0$. Since this holds for all $a \in |X|$ (although $\varepsilon$ depends on $a$), we have $s = t$. □

**Théorème 8.1.19** *Let $f : \mathsf{P}(X) \to \mathsf{P}(Y)$. The two following properties are equivalent.*

— *There exists $s \in \mathsf{P}(X \multimap Y)$ such that $\forall u \in \mathsf{P}(X)\ f(u) = s\cdot u$.*

— *The function $f$ is non-decreasing ($u \leq v \Rightarrow f(u) \leq f(v)$), Scott-continuous (given any non-decreasing sequence $(u(n))_{n\in\mathbb{N}}$ of elements of $\mathsf{P}(X)$ one has $f(\sup_{n\in\mathbb{N}} u(n)) = \sup_{n\in\mathbb{N}} f(u(n))$ or, equivalently, $f(\sup_{n\in\mathbb{N}} u(n)) \leq \sup_{n\in\mathbb{N}} f(u(n))$) and commutes with convex linear combinations, that is, given $u(i) \in \mathsf{P}(X)$ and $\lambda_i \in \mathbb{R}_{\geq 0}$ for $i = 1, 2$ such that $\lambda_1 + \lambda_2 = 1$, one has $f(\lambda_1 u(1) + \lambda_2 u(2)) = \lambda_1 f(u(1)) + \lambda_2 f(u(2))$.*

**Exercise 8.1.1** Prove this theorem.

## 8.2 The monoidal structure of Pcoh.

**Lemme 8.2.1** *Let $X_1$, $X_2$ and $Y$ be PCSs and $s \in \mathbb{R}_{\geq 0}^{|X_1 \otimes X_2 \multimap Y|}$. If $\forall u^1 \in \mathsf{P}(X_1), u^2 \in \mathsf{P}(X_2)$ $s \cdot (u^1 \otimes u^2) \in \mathsf{P}(Y)$, then $s \in \mathbf{Pcoh}(X_1 \otimes X_2, Y)$.*

*Démonstration.* By Lemma 8.1.16, it suffices to prove that $s^\perp \in \mathbf{Pcoh}(Y^\perp, (X_1 \otimes X_2)^\perp)$, so let $v' \in \mathsf{P}(Y^\perp)$, we prove that $s^\perp \cdot v' \in \mathsf{P}(X_1 \otimes X_2)^\perp$. So let $u^i \in \mathsf{P}(X_i)$ for $i = 1, 2$, we have $\langle u^1 \otimes u^2, s^\perp \cdot v' \rangle = \langle s \cdot (u^1 \otimes u^2), v' \rangle \leq 1$ by our assumption about $s$. □

**Lemme 8.2.2** *Let $s, t \in \mathbf{Pcoh}(X_1 \otimes X_2, Y)$. If $\forall u^1 \in \mathsf{P}(X_1)$, $u^2 \in \mathsf{P}(Y_2)$ $s \cdot (u^1 \otimes u^2) = t \cdot (u^1 \otimes u^2)$ then $s = t$.*

*Démonstration.* The proof is essentially the same as that of Lemma 8.1.18 : let $a_i \in |X_i|$ for $i = 1, 2$ and $\varepsilon_i > 0$ such that there is $u^i \in \mathsf{P}(X_i)$ with $u^i_{a_i} \geq \varepsilon_i$. Then $\varepsilon_i \mathsf{e}_{a_i} \in \mathsf{P}(X_i)$ for $i = 1, 2$. It follows that $\varepsilon_1 \mathsf{e}_{a_1} \otimes \varepsilon_2 \mathsf{e}_{a_2} = \varepsilon_1 \varepsilon_2 \mathsf{e}_{a_1, a_2} \in \mathsf{P}(X_1 \otimes X_2)$ and hence $s \cdot \varepsilon_1 \varepsilon_2 \mathsf{e}_{a_1, a_2} = t \cdot \varepsilon_1 \varepsilon_2 \mathsf{e}_{a_1, a_2}$ by our assumption on $s$ and $t$. That is $\forall b \in |Y|$ $\varepsilon_1 \varepsilon_2 s_{(a_1, a_2), b} = \varepsilon_1 \varepsilon_2 t_{(a_1, a_2), b}$ and hence $s = t$ since $\varepsilon_1 \varepsilon_2 \neq 0$. □

Let $X_i$ and $Y_i$ be PCSs and $s^i \in \mathbf{Pcoh}(X_i, Y_i)$ for $i = 1, 2$. Then we define $s^1 \otimes s^2 \in \mathbb{R}_{\geq 0}^{|X_1 \otimes X_2 \multimap Y_1 \otimes Y_2|}$ by

$$s^1 \otimes s^2{}_{(a_1, a_2), (b_1, b_2)} = s^1_{a_1, b_1} s^2_{a_2, b_2} \,.$$

Let $u^i \in \mathbb{R}_{\geq 0}^{X_i}$ for $i = 1, 2$, we have

$$\begin{aligned}
(s^1 \otimes s^2) \cdot (u^1 \otimes u^2) &= \Big( \sum_{a_1 \in |X_1|, a_2 \in |X_2|} s^1_{a_1, b_1} s^2_{a_2, b_2} u^1_{a_1} u^2_{a_2} \Big)_{b_1 \in |Y_1|, b_2 \in |Y_2|} \\
&= \Big( \sum_{a_1 \in |X_1|} \sum_{a_2 \in |X_2|} s^1_{a_1, b_1} s^2_{a_2, b_2} u^1_{a_1} u^2_{a_2} \Big)_{b_1 \in |Y_1|, b_2 \in |Y_2|} \\
&= \Big( \sum_{a_1 \in |X_1|} s^1_{a_1, b_1} u^1_{a_1} \sum_{a_2 \in |X_2|} s^2_{a_2, b_2} u^2_{a_2} \Big)_{b_1 \in |Y_1|, b_2 \in |Y_2|} \\
&= (s^1 \cdot u^1) \otimes (s^2 \cdot u^2) \,.
\end{aligned}$$

**Lemme 8.2.3** *We have $s^1 \otimes s^2 \in \mathbf{Pcoh}(X_1 \otimes X_2, Y_1 \otimes Y_2)$.*

*Démonstration.* By Lemma 8.2.1 it suffices to prove that $\forall u^1 \in \mathsf{P}(X_1), u^2 \in \mathsf{P}(X_2)$ $(s^1 \otimes s^2) \cdot (u^1 \otimes u^2) \in \mathsf{P}(Y_1 \otimes Y_2)$ which is obvious since $(s^1 \otimes s^2) \cdot (u^1 \otimes u^2) = (s^1 \cdot u^1) \otimes (s^2 \cdot u^2)$ and $s^i \cdot u^i \in \mathsf{P}(Y_i)$ for $i = 1, 2$. □

If $\mathsf{Id}^i$ is the identity of $X_i$ (for $i = 1, 2$), it is easy to see that $\mathsf{Id}^1 \otimes \mathsf{Id}^2$ is the identity of $X_1 \otimes X_2$. Now let $s^i \in \mathbf{Pcoh}(X_i, Y_i)$ and $t_i \in \mathbf{Pcoh}(Y_i, Z_i)$ for $i = 1, 2$. Then a computation similar to the previous one shows that

$$(t^1 \otimes t^2)\,(s^1 \otimes s^2) = (t_1\, s_1) \otimes (t_2\, s_2)$$

and hence $\otimes$ is a bifunctor $\mathbf{Pcoh}^2 \to \mathbf{Pcoh}$.

**Exercice 8.2.1** Prove the equation above, namely $(t^1 \otimes t^2)\,(s^1 \otimes s^2) = (t_1\, s_1) \otimes (t_2\, s_2)$.

Given sets $I$, $J$ and $K$, let $\alpha_{I,J,K} = \mathbb{R}_{\geq 0}^{((I\times J)\times K)\times(I\times(J\times K))}$ given by $\alpha_{((i,j),k),(i',(j',k'))} = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}$ (the characteristic function of the associativity isomorphisms of $\otimes$ in **Rel**). We also define similarly $\beta_{I,J,K} \in \mathbb{R}_{\geq 0}^{((I\times(J\times K)\times((I\times J)\times K)}$ by $\beta_{(i,(j,k)),((i',j'),k')} = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}$.

**Lemme 8.2.4** *Let $X_1$, $X_2$ and $Y$ be PCSs. Then $\alpha_{|X_1|,|X_2|,|Y|} \in \mathbf{Pcoh}(X_1 \otimes X_2 \multimap Y, X_1 \multimap (X_2 \multimap Y))$, and this morphism is an isomorphism whose inverse is $\beta_{|X_1|,|X_2|,|Y|}$.*

*Démonstration.* We apply Theorem 8.1.19. So let $s \in \mathsf{P}(X_1 \otimes X_2 \multimap Y)$, we prove that $t = \alpha \cdot s \in \mathsf{P}(X_1 \multimap (X_2 \multimap Y))$. For this we use again Theorem 8.1.19. So let $u^1 \in \mathsf{P}(X_1)$ and let us prove that $t \cdot u^1 \in \mathsf{P}(X_2 \multimap Y)$. For this we use again Theorem 8.1.19. So let $u^2 \in \mathsf{P}(X_2)$, it suffices to observe that $(t \cdot u^1)\cdot u^2 \in \mathsf{P}(Y)$. This results from the fact that $(t \cdot u^1)\cdot u^2 = s\cdot(u^1 \otimes u^2)$ and that $u^1 \otimes u^2 \in \mathsf{P}(X_1 \otimes X_2)$.

To finish the proof, it suffices to show that $\beta_{|X_1|,|X_2|,|Y|} \in \mathbf{Pcoh}(X_1 \multimap (X_2 \multimap Y), X_1 \otimes X_2 \multimap Y)$. So let $t \in \mathsf{P}(X_1 \multimap (X_2 \multimap Y))$ and let us prove that $s = \beta \cdot t \in \mathsf{P}(X_1 \otimes X_2 \multimap Y)$, this will prove our contemption by Theorem 8.1.19. For this, we use Lemma 8.2.1. So let $u^i \in \mathsf{P}(X_i)$ for $i = 1, 2$, it suffices to prove that $s \cdot (u^1 \otimes u^2)$. This results from the fact that $s \cdot (u^1 \otimes u^2) = (t \cdot u^1) \cdot u^2$ and from our assumption that $t \in \mathsf{P}(X_1 \multimap (X_2 \multimap Y))$.

The fact that $\beta$ and $\alpha$ are inverse of each other is obvious. □

Let $X_i$ be PCSs for $i = 1, 2, 3$. We prove that $\alpha_{|X_1|,|X_2|,|X_3|} \in \mathbf{Pcoh}((X_1 \otimes X_2)\otimes X_3, X_1\otimes(X_2 \otimes X_3))$. For this it suffices to observe that, by Lemma 8.2.4,

$$\beta_{|X_1|,|X_2|,|X_3|} = {\alpha_{|X_1|,|X_2|,|X_3|}}^{\perp} \in \mathbf{Pcoh}((X_1 \otimes (X_2 \otimes X_3))^{\perp}, ((X_1 \otimes X_2) \otimes X_3)^{\perp})$$

since $((X_1 \otimes X_2) \otimes X_3)^{\perp} = (X_1 \otimes X_2) \multimap {X_3}^{\perp}$ and $(X_1 \otimes (X_2 \otimes X_3))^{\perp} = X_1 \multimap (X_2 \multimap {X_3}^{\perp})$. By the same argument we have $\beta_{|X_1|,|X_2|,|X_3|} \in \mathbf{Pcoh}(X_1 \otimes (X_2 \otimes X_3), (X_1 \otimes X_2) \otimes X_3)$.

Let $\lambda_I \in \mathbb{R}_{\geq 0}^{(\{*\}\times I)\times I}$ be dedfined by $\lambda_{(*,i),i'} = \delta_{i,i'}$, then $\lambda_{|X|} \in \mathbf{Pcoh}(1 \otimes X, X)$. Again we apply Lemma 8.2.1 : given $r \in \mathsf{P}(1)$ (that is, $r \in [0,1]$) and $u \in \mathsf{P}(X)$ we have $\lambda_{|X|} \cdot (r \otimes u) = ru \in \mathsf{P}(X)$ since $r \in [0,1]$. This map is an iso with inverse the map $u \mapsto 1 \otimes u$ (because $1 \otimes ru = r \otimes u$). We define similarly an iso $\rho_{|X|} \in \mathbf{Pcoh}(X \otimes 1, X)$.

Last let $\sigma_{I,J} \in \mathbb{R}_{\geq 0}^{(I\times J)\times(J\times I)}$ given by $\sigma_{(i,j),(j',i')} = \delta_{i,i'}\delta_{j,j'}$. Then, given PCSs $X_1$ and $X_2$ we have $\sigma_{|X_1|,|X_2|} \in \mathbf{Pcoh}(X_1 \otimes X_2, X_2 \otimes X_1)$ by the same kind of argument. It is an isomorphism which has $\sigma_{|X_2|,|X_1|}$ as inverse.

**Théorème 8.2.5** $(\mathbf{Pcoh}, 1, \otimes, \alpha, \lambda, \rho)$ *is a symmetric monoidal closed category.*

*Démonstration.* To prove that we have a symmetric monoidal category, one has just to check the McLane's coherence diagrams, they commute because the involved isomorphisms are defined as in **Rel** (with coefficients 0 and 1).

Given PCSs $X$ and $Y$, we have already defined a PCS $X \multimap Y$ by $X \multimap Y = (X \otimes Y^{\perp})^{\perp}$. Then we define $\mathsf{ev}_{X,Y} \in \mathbb{R}_{\geq 0}^{|((X\multimap Y)\otimes X)\multimap Y|}$ by

$$(\mathsf{ev}_{X,Y})_{((a,b),a'),b'} = \delta_{a,a'}\delta_{b,b'}$$

We have $\mathsf{ev}_{X,Y} \in \mathbf{Pcoh}((X \multimap Y) \otimes X, Y)$ by Lemma 8.2.1 and by the observation that, given $s \in \mathsf{P}(X \multimap Y)$ and $u \in \mathsf{P}(X)$, one has

$$\mathsf{ev}_{X,Y}\,(s \otimes u) = s \cdot u \in \mathsf{P}(Y)\,.$$

Let $t \in \mathbf{Pcoh}(Z\otimes X, Y)$, so that actually $t \in \mathsf{P}((Z \otimes X) \multimap Y)$ then by Lemma 8.2.4 we have $\alpha_{|Z|,|X|,|Y|} \cdot t \in \mathsf{P}(Z \multimap (X \multimap Y))$ so we set $\mathsf{cur}(t) = \alpha_{|Z|,|X|,|Y|} \cdot t \in \mathbf{Pcoh}(Z, X \multimap Y)$. This morphism is uniquely characterized by

$$\forall w \in \mathsf{P}(Z)\, \forall u \in \mathsf{P}(X) \quad ((\mathsf{cur}(t)) \cdot w) \cdot u = t \cdot (w \otimes u) \tag{8.1}$$

Therefore it is the unique morphism $t' \in \mathbf{Pcoh}(Z, X \multimap Y)$ such that

$$\mathsf{ev}\,(t' \otimes X) = t\,. \tag{8.2}$$

Indeed, let $w \in \mathsf{P}(Z)$ and $u \in \mathsf{P}(X)$, we have

$$\begin{aligned}(\mathsf{ev}\,(\mathsf{cur}(t) \otimes X)) \cdot (w \otimes u) &= \mathsf{ev}\,((\mathsf{cur}(t) \cdot w) \otimes u)\\ &= ((\mathsf{cur}(t)) \cdot w) \cdot u\\ &= t \cdot (w \otimes u)\end{aligned}$$

and hence $\mathsf{ev}\,(\mathsf{cur}(t) \otimes X) = t$ by Lemma 8.2.2. Given $t' \in \mathbf{Pcoh}(Z, X \multimap Y)$ such that $\mathsf{ev}\,(t' \otimes X) = t$ we have $((t') \cdot w) \cdot u = t \cdot (w \otimes u)$ for all $w \in \mathsf{P}(Z)$ and $u \in \mathsf{P}(X)$ and hence applying twice Lemma 8.2.2 we get $t' = \mathsf{cur}(t)$. So we have shown that $(X \multimap Y, \mathsf{ev})$ is a linear hom object (see Section 4.6.2) our symmetric monoidal category $\mathbf{Pcoh}$ is closed. □

**Lemme 8.2.6** *Let $\perp = 1$. Equipped with the dualizing object $\perp$, the symmetric monoidal closed category* $\mathbf{Pcoh}$ *is $*$-autonomous.*

*Démonstration.* Observe first that there is an isomorphism $\theta_X \in \mathbf{Pcoh}(X^\perp, X \multimap \perp)$ given by $(\theta_X)_{(a,(a',*))} = \delta_{a,a'}$ : given $u' \in \mathsf{P}(X^\perp)$, $\theta \cdot u' \in \mathsf{P}(X \multimap \perp)$ is characterizd by $\forall u \in \mathsf{P}(X)$, $(\theta \cdot u') \cdot u = \langle u, u' \rangle \in \mathsf{P}(\perp)$ (that we identify as usual with $[0,1]$). Since $X^{\perp\perp} = X$, $(\theta_X)^\perp \in \mathbf{Pcoh}((X \multimap \perp)^\perp, X)$ is an iso, let $\theta'_X \in \mathbf{Pcoh}(X, (X \multimap \perp)^\perp)$ be its inverse. Hence $\eta_X = \theta_{X \multimap \perp}\, \theta'_X \in \mathbf{Pcoh}(X, (X \multimap \perp) \multimap \perp)$ is an iso and it is easily checked that $(\eta_X)_{a,((a',*),*)} = \delta_{a,a'}$ and hence $\eta_X = \mathsf{cur}(\mathsf{ev}\,\sigma_{X, X \multimap \perp}) \in \mathbf{Pcoh}(X, (X \multimap \perp) \multimap \perp)$. □

## 8.3 Additive structure.

Given a collection of sets $(J_i)_{i \in I}$ we set

$$J = \sum_{i \in I} J_i = \bigcup_{i \in I} \{i\} \times J_i$$

and for each $i \in I$ we define $\pi_i \in \mathbb{R}_{\geq 0}^{J \times J_i}$ and $\overline{\pi}_i \in \mathbb{R}_{\geq 0}^{J_i \times J}$ by $(\pi_i)_{(i',j'),j} = (\overline{\pi}_i)_{j,(i',j')} = \delta_{i,i'}\delta_{j,j'}$. Observe that, if $v \in \mathbb{R}_{\geq 0}^{\sum_{i \in I} J_i}$, $i \in I$ and $u' \in \mathbb{R}_{\geq 0}^{J_i}$, then

$$\langle v, \overline{\pi}_i\, u' \rangle = \langle \pi_i\, v, u' \rangle$$

Let $(X_i)_{i \in I}$ be an at most countable family of PCSs. We define $X = \&_{i \in I} X_i$ as follows : $|X| = \sum_{i \in I} |X_i|$ and $\mathsf{P}(X) = \left\{ v \in \mathbb{R}_{\geq 0}^{|X|} \mid \forall i \in I\ \pi_i \cdot v \in \mathsf{P}(X_i) \right\}$.

**Lemme 8.3.1** *We have $\mathsf{P}(X) = \left\{ \overline{\pi}_i \cdot u'_i \mid i \in I \text{ and } u'_i \in \mathsf{P}({X_i}^\perp) \right\}^\perp$ and hence $X = \&_{i \in I} X_i$ is a PCS.*

*Démonstration.* Let $\mathcal{P} = \left\{ \overline{\pi}_i \cdot u'_i \mid i \in I \text{ and } u'_i \in \mathsf{P}({X_i}^\perp) \right\}$. For any $i \in I$ and $a \in |X_i|$ we have $\sup \left\{ v'_{i,a} \mid v' \in \mathcal{P} \right\} = \sup \left\{ u'_a \mid u' \in \mathsf{P}({X_i}^\perp) \right\}$ and hence $0 < \sup \left\{ v'_{i,a} \mid v' \in \mathcal{P} \right\} < \infty$ since each ${X_i}^\perp$ is a PCS. Therefore $(|X|, \mathcal{P}^\perp)$ is a PCS, it suffices to prove that $\mathcal{P}^\perp = \left\{ v \in \mathbb{R}_{\geq 0}^{|X|} \mid \forall i \in I\ \pi_i \cdot v \in \mathsf{P}(X_i) \right\}$.

Let first $v \in \mathcal{P}^\perp$ and $i \in I$, we have $\pi_i\, v \in \mathsf{P}(X_i)$ because for any $u' \in \mathsf{P}({X_i}^\perp)$, $\langle \pi_i\, v, u' \rangle = \langle v, \overline{\pi}_i\, u' \rangle \leq 1$ since $\overline{\pi}_i\, u' \in \mathcal{P}$. Conversely let $v \in \mathbb{R}_{\geq 0}^{|X|}$ be such that $\forall i \in I\ \pi_i \cdot v \in \mathsf{P}(X_i)$ and let us prove that $v \in \mathcal{P}^\perp$, so let $i \in I$ and $u' \in \mathsf{P}({X_i}^\perp)$, we have $\langle v, \overline{\pi}_i \cdot u' \rangle = \langle \pi_i \cdot v, u' \rangle \leq 1$. □

It follows from this lemma that $\pi_i \in \mathbf{Pcoh}(\&_{k \in I} X_k, X_i)$ for each $i \in I$.

**Théorème 8.3.2** *The PCS $X = \&_{i \in I} X_i$, together with the projections $(\pi_i)_{i \in I}$, is the cartesian product of the $X_i$s in* $\mathbf{Pcoh}$.

*Démonstration.* It suffices to show that for any family of morphisms $(s_i \in \mathbf{Pcoh}(Y, X_i))_{i\in I}$, the matrix

$$s = \langle s^i \rangle_{i\in I} \in \mathbb{R}_{\geq 0}^{|Y|\times|X|}$$

defined by $s_{b,(i,a)} = s^i_{a,b}$ belongs to $\mathbf{Pcoh}(Y, X)$. This results from the fact that, for any $v \in \mathsf{P}(Y)$ and any $i \in I$, $\pi_i \cdot (s \cdot v) = s^i \cdot v \in \mathsf{P}(X_i)$ and hence $s \cdot v \in \mathsf{P}(X)$. Then it is clear that $s$ is the unique element of $\mathbf{Pcoh}(Y, X)$ such that $\forall i \in I\ \pi_i\, s = s^i$. □

It is important to observe also that $\mathsf{P}(\&_{i\in I} X_i)$ is isomorphic, as a partially ordered set, to $\prod_{i\in I} \mathsf{P}(X_i)$. This isomorphism maps $v \in \mathsf{P}(\&_{i\in I} X_i)$ to $(\pi_i \cdot v)_{i\in I} \in \prod_{i\in I} \mathsf{P}(X_i)$. Conversely an element $(u^i)_{i\in I}$ of $\prod_{i\in I} \mathsf{P}(X_i)$ is mapped to

$$\sum_{i\in I} \overline{\pi}_i \cdot u^i \in \mathsf{P}(\mathop{\&}_{i\in I} X_i)\,.$$

Using the isomorphism $\_^\perp$ between $\mathbf{Pcoh}$ and $\mathbf{Pcoh}^{\mathsf{op}}$ we define the coproduct of the $X_i$s by setting

$$\bigoplus_{i\in I} X_i = \left(\mathop{\&}_{i\in I} {X_i}^\perp\right)^\perp$$

equipped with the injections $\overline{\pi}_k = {\pi_k}^\perp \in \mathbf{Pcoh}(X_k, \bigoplus_{i\in I} X_i)$.

8.3.1 Norm and distance. We can characterize this latter PCS more directly. For this purpose we equip any PCS $X$ with a "norm" $\|\_\|_X : \mathsf{P}(X) \to [0,1]$. Let $u \in \mathsf{P}(X)$, we set

$$\|u\|_X = \sup_{u' \in \mathsf{P}(X^\perp)} \langle u, u' \rangle \in [0,1]\,.$$

**Exercice 8.3.1** Prove that this operation features the usual properties of a norm, namely :
— $\|u\|_X = 0 \Rightarrow u = 0$ (we recall that 0 is the element of $\mathsf{P}(X)$ which maps each element of $|X|$ to 0).
— If $u^1, u^2 \in \mathsf{P}(X)$ satisfy $u^1 + u^2 \in \mathsf{P}(X)$, then $\|u^1 + u^2\|_X \leq \|u^1\|_X + \|u^2\|_X$.
— If $u \in \mathsf{P}(X)$ and $\lambda \in [0,1]$ then $\|\lambda u\|_X = \lambda \|u\|_X$.
Prove that, if $u \leq v \in \mathsf{P}(X)$, then $\|u\|_X \leq \|v\|_X$. Prove also that the norm is Scott-continuous (that is if $(u(n))_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathsf{P}(X)$, then $\|\sup_{n\in\mathbb{N}} u(n)\|_X = \sup_{n\in\mathbb{N}} \|u(n)\|_X$).

**Exercice 8.3.2** Let $s \in \mathbf{Pcoh}(X, Y)$, prove that

$$\|s\|_{X \multimap Y} = \sup_{u \in \mathsf{P}(X)} \|s \cdot u\|_Y$$

**Exercice 8.3.3** Given $u^i \in \mathsf{P}(X_i)$ for $i = 1, 2$, prove that

$$\|u^1 \otimes u^2\|_{X_1 \otimes X_2} = \|u_1\|_{X_1} \|u_2\|_{X_2}\ .$$

**Exercice 8.3.4** Let $v \in \mathsf{P}(\&_{i\in I} X_i)$. Prove that

$$\|v\|_{\&_{i\in I} X_i} = \sup_{i\in I} \|\pi_i \cdot v\|_{X_i}\ .$$

**Théorème 8.3.3** *$\mathsf{P}(\bigoplus_{i\in I} X_i)$ is the set of all the elements $v$ of $\mathsf{P}(\&_{i\in I} X_i)$ which satisfy*

$$\sum_{i\in I} \|\pi_i \cdot v\|_{X_i} \leq 1\,.$$

*Moreover* $\|v\|_{\bigoplus_{i\in I} X_i} = \sum_{i\in I} \|\pi_i \cdot v\|_{X_i}$.

*Démonstration.* Let $u \in \mathsf{P}(X)$. By commutation of finite sums with lubs in $\overline{\mathbb{R}_{\geq 0}}$ we have

$$\begin{aligned}
\sum_{i\in I} \|\pi_i \cdot u\|_{X_i} &= \sup \left\{ \sum_{i\in I_0} \|\pi_i \cdot u\|_{X_i} \mid I_0 \subseteq I \text{ finite} \right\} \\
&= \sup \left\{ \sum_{i\in I_0} \sup_{u'\in \mathsf{P}(X_i{}^\perp)} \langle \pi_i \cdot u, u' \rangle \mid I_0 \subseteq I \text{ finite} \right\} \\
&= \sup \left\{ \sum_{i\in I_0} \langle \pi_i \cdot u, u'(i) \rangle \mid I_0 \subseteq I \text{ finite and } (u'(i))_{i\in I} \in \prod_{i\in I_0} \mathsf{P}(X_i{}^\perp) \right\} \\
&= \sup \left\{ \sum_{i\in I_0} \langle \pi_i \cdot u, \pi_i \cdot u' \rangle \mid I_0 \subseteq I \text{ finite and } u' \in \mathsf{P}(X^\perp) \right\} \\
&= \sup \left\{ \langle u, u' \rangle \mid u' \in \mathsf{P}(X^\perp) \right\} \quad \text{by definition of } \langle u, u' \rangle \\
&= \|u\|_X \leq 1 \,.
\end{aligned}$$

This shows also in particular that $\forall i \in I \ \|\pi_i \cdot u\|_{X_i} \leq 1$ and hence $u \in \mathsf{P}(\&_{i\in I} X_i)$.

Conversely let $u \in \mathsf{P}(\&_{i\in I} X_i)$ be such that $\sum_{i\in I} \|\pi_i \cdot v\|_{X_i} \leq 1$, then for any $u' \in \mathsf{P}(X^\perp)$ we have

$$\begin{aligned}
\langle u, u' \rangle &= \sum_{i\in I} \langle \pi_i \cdot u, \pi_i \cdot u' \rangle \\
&\leq \sum_{i\in I} \|\pi_i \cdot u\|_X \\
&\leq 1
\end{aligned}$$

and hence $u \in \mathsf{P}(X)$. □

Given a PCS $X$, the norm $\|\_\|_X : \mathsf{P}(X) \to [0,1]$ induces a distance $\mathsf{d}_X$ on $\mathsf{P}(X)$. We cannot set $\mathsf{d}_X(u,v) = \|v-u\|_X$ as one would do in a normed vector space because the difference $v - u$ is not an element of $\mathsf{P}(X)$ unless $u \leq v$, but we can set

$$\mathsf{d}_X(u,v) = \inf \left\{ \|u-w\|_X + \|v-w\| \mid w \in \mathsf{P}(X) \ w \leq u \text{ and } w \leq v \right\} .$$

The point of this definition is that it is not specific to PCSs but makes sense in more general structures such as *positive cones* [?].

It turns out that any two elements $u$ and $v$ of $\mathsf{P}(X)$ have a greatest lower bound (glb) $u \wedge v \in \mathsf{P}(X)$ obviously given by $(u \wedge v)_a = \min(u_a, v_a) \in \mathbb{R}_{\geq 0}$ for each $a \in |X|$. Therefore we have

$$\mathsf{d}_X(u,v) = \|u - (u \wedge v)\| + \|v - (u \wedge v)\| \,.$$

**Lemme 8.3.4** *Given $u, v \in \mathsf{P}(X)$, $u + v - (u \wedge v)$ is the least upper bound (lub) of $u$ and $v$ in $\mathbb{R}_{\geq 0}^{|X|}$ (ordered with the product order).*

*Démonstration.* We have $u \wedge v \leq v$, hence $u + (u \wedge v) \leq u + v$ and therefore $u \leq u + v - (u \wedge v)$. Similarly $v \leq u + v - (u \wedge v)$.

Let now $w \in \mathbb{R}_{\geq 0}^{|X|}$ such that $u \leq w$ and $v \leq w$. We have $u + v \leq v + w$ and $u + v \leq u + w$, hence $u + v \leq (u + w) \wedge (v + w) = (u \wedge v) + w$ which shows that $u + v - (u \wedge v) \leq w$.

Let us prove that $(u + w) \wedge (v + w) = (u \wedge v) + w$, a property that we just used without proof. First $(u \wedge v) + w \leq u + w, v + w$ by monotonicity of $+$ and hence $(u + w) \wedge (v + w) \geq (u \wedge v) + w$. Next $u + w - w = u$ and since $w \leq (u + w) \wedge (v + w) \leq u + w$ we have $(u + w) \wedge (v + w) - w \leq u$. Similarly $(u+w) \wedge (v+w) - w \leq v$ and hence $(u+w) \wedge (v+w) - w \leq u \wedge v$ and therefore $(u+w) \wedge (v+w) \leq (u \wedge v) + w$. □

**Exercice 8.3.5** Given $u, v \in \mathsf{P}(X)$, let $|v - u| \in \mathbb{R}_{\geq 0}^{|X|}$ be given by $|v-u|_a = |v_a - u_a|$. Observe that $|v - u| = u - (u \wedge v) + v - (u \wedge v)$ in $\mathbb{R}_{\geq 0}^{|X|}$ and hence

$$w = \frac{1}{2} |v - u| \in \mathsf{P}(X) \,.$$

Since $u-(u\wedge v), v-(u\wedge v)\in \mathsf{P}(X)$. Prove that

$$\|w\|_X \leq \frac{1}{2}\,\mathsf{d}_X(u,v) \leq 2\,\|w\|_X \ .$$

**Solution :** For the second inequation, we have

$$\begin{aligned}
\mathsf{d}_X(u,v) &= \sup_{u'\in\mathsf{P}(X)^\perp} \langle u-(u\wedge v)), u'\rangle + \sup_{u'\in\mathsf{P}(X)^\perp} \langle v-(u\wedge v)), u'\rangle \\
&= \sup_{u'\in\mathsf{P}(X)^\perp} \sum_{v_a<u_a} (u_a-v_a)u'_a + \sup_{u'\in\mathsf{P}(X)^\perp} \sum_{u_a<v_a} (v_a-u_a)u'_a \\
&\leq \sup_{u'\in\mathsf{P}(X)^\perp} \langle |u-v|\,, u'\rangle + \sup_{u'\in\mathsf{P}(X)^\perp} \langle |v-u|\,, u'\rangle \\
&= 4\,\|w\|_X \ .
\end{aligned}$$

For the first inequation he have

$$\begin{aligned}
\|w\|_X &= \left\|\frac{1}{2}(u-(u\wedge v)) + \frac{1}{2}(v-(u\wedge v))\right\|_X \\
&\leq \left\|\frac{1}{2}(u-(u\wedge v))\right\|_X + \left\|\frac{1}{2}(v-(u\wedge v))\right\|_X \\
&= \frac{1}{2}\,\mathsf{d}_X(u,v)\,.
\end{aligned}$$

Notice that for any $u'\in\mathsf{P}(X^\perp)$ we have

$$\forall u,v\in\mathsf{P}(X) \quad |\langle u,u'\rangle - \langle v,u'\rangle| \leq \mathsf{d'}_X(u,v)$$

8.3.2 Banach space associated with a PCS. Let $X$ be a PCS, we define $\widetilde{\mathsf{P}}(X)$ as the set of all $x\in\mathbb{R}^{|X|}$ such that, for some $\varepsilon>0$, one has $\varepsilon\,|x|\in\mathsf{P}(X)$ (where $|x| = (|x_a|)_{a\in|X|}$). Then $\widetilde{\mathsf{P}}(X)$ is a $\mathbb{R}$-vector space, operations being defined pointwise, that is $\lambda x = (\lambda x_a)_{a\in|X|}$ and $x+y=(x_a+y_a)_{a\in|X|}$. Indeed given $\varepsilon>0$ such that $\varepsilon\,|x|\,,\varepsilon\,|y|\in\mathsf{P}(X)$ we have

$$\frac{\varepsilon}{2}(|x+y|) \leq \frac{\varepsilon}{2}(|x|+|y|)\in\mathsf{P}(X)$$

and hence $x+y\in\widetilde{\mathsf{P}}(X)$.

We define $\|\_\|_{\widetilde{\mathsf{P}}(X)} : \widetilde{\mathsf{P}}(X)\to\mathbb{R}_{\geq 0}$ by

$$\|x\|_{\widetilde{\mathsf{P}}(X)} = \sup_{u'\in\mathsf{P}(X)^\perp} \langle |x|\,, u'\rangle \ .$$

We have $\|x\|_{\widetilde{\mathsf{P}}(X)} < \infty$ for all $x\in\widetilde{\mathsf{P}}(X)$ : let $\varepsilon>0$ be such that $\varepsilon\,|x|\in\mathsf{P}(X)$, for any $u'\in\mathsf{P}(X)^\perp$ we have

$$\langle\varepsilon\,|x|\,, u'\rangle \leq 1$$

so that $\|x\|_{\widetilde{\mathsf{P}}(X)} \leq \varepsilon^{-1}$.

**Exercice 8.3.6** Prove that

$$\|x\|_{\widetilde{\mathsf{P}}(X)} = \inf\left\{\varepsilon^{-1} \mid \varepsilon\,|x|\in\mathsf{P}(X)\right\} \ .$$

This map $\|\_\|_{\widetilde{\mathsf{P}}(X)}$ is a norm on $\widetilde{\mathsf{P}}(X)$ by Exercise 8.3.1.

**Exercice 8.3.7** Let $\mathcal{B}' = \left\{x'\in\mathbb{R}^{|X|} \mid |x'|\in\mathsf{P}(X)^\perp\right\}$, prove that

$$\forall x\in\widetilde{\mathsf{P}}(X) \quad \|x\|_{\widetilde{\mathsf{P}}(X)} = \sup_{x'\in\mathcal{B}'} \langle x,x'\rangle \ .$$

**Proposition 8.3.5** *The normed vector space $\widetilde{\mathsf{P}}(X)$ is a Banach space.*

*Démonstration.* It remains only to prove completeness. So let $(x(n))_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widetilde{\mathsf{P}}(X)$. This implies in particular that the family $(x(n))_{n\in\mathbb{N}}$ is bounded, so there is $\alpha > 0$ such that $\forall n \in \mathbb{N}\ \|x(n)\| \leq \alpha$.

For each $a \in |X|$ let $c_a > 0$ such that $c_a \mathsf{e}_a \in \mathsf{P}(X)^\perp$. The map $x \mapsto x_a$ from $\widetilde{\mathsf{P}}(X)$ to $\mathbb{R}$ is ${c_a}^{-1}$-Lipschitz and hence the sequence $(x(n)_a)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ and hence has a limit $x_a \in \mathbb{R}$, in this way we have defined $x \in \mathbb{R}^{|X|}$.

Now we prove that $x \in \widetilde{\mathsf{P}}(X)$. Let $u' \in \mathsf{P}(X)^\perp$. Let $\varepsilon > 0$ and $I$ be a finite subset of $|X|$, we can find $n \in \mathbb{N}$ such that

$$\forall a \in I \quad |x(n)_a - x_a| \leq \frac{\varepsilon}{\#I(1 + \sum_{a\in I} u'_a)}\,.$$

We have

$$\begin{aligned}\sum_{a\in I} |x_a|\, u'_a &\leq \sum_{a\in I}(|x(n)_a| + |x_a - x(n)_a|)u'_a \\ &\leq \alpha + \sum_{a\in I} |x_a - x(n)_a|\, u'_a \\ &\leq \alpha + \varepsilon\end{aligned}$$

and since this inequation holds for all finite $I \subseteq |X|$, we have $\langle |x|\,, u'\rangle \leq \alpha + \varepsilon$. Since this holds for all $\varepsilon > 0$ we have $\langle |x|\,, u'\rangle \leq \alpha$ and hence $x \in \widetilde{\mathsf{P}}(X)$.

Last we prove that $x(n) \to x$ in $\widetilde{\mathsf{P}}(X)$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ be such that

$$\forall n, k \geq N \quad \|x(n) - x(k)\|_{\widetilde{\mathsf{P}}(X)} \leq \varepsilon$$

We use the notations of Exercise 8.3.7. Given $x' \in \mathcal{B}'$, the map $\mathsf{ev}_{x'} : \widetilde{\mathsf{P}}(X) \to \mathbb{R}$ defined by $\mathsf{ev}_{x'}(x) = \langle x, x'\rangle$ is 1-Lipschitz by the characterization of the norm proven in that exercise and hence by Lemma 9.1.1 we have

$$\forall n \geq N \quad |\langle x - x(n), x'\rangle| = |\langle x, x'\rangle - \langle x(k), x'\rangle| \leq \varepsilon$$

and since this holds for all $x' \in \mathcal{B}'$ we have

$$\|x - x(n)\|_{\widetilde{\mathsf{P}}(X)} \leq \varepsilon$$

proving our contention. □

**Proposition 8.3.6** *For any PCS $X$, $(\mathsf{P}(X), \mathsf{d}_X)$ is a complete metric space.*

*Démonstration.* It is obvious that $\mathsf{d}_X(u, v) = \mathsf{d}_X(v, u)$ and that $\mathsf{d}_X(u, v) = 0 \Rightarrow u = v$ so let us prove the triangular inequality. So let $u, v, w \in \mathsf{P}(X)$, we have $u - (u \wedge w) \leq u - (u \wedge v \wedge w) = u - (u \wedge v) + (u \wedge v) - (u \wedge v \wedge w)$ and hence

$$\|u - (u \wedge w)\|_X \leq \|u - (u \wedge v)\|_X + \|(u \wedge v) - (u \wedge v \wedge w)\|_X$$

Now we have $(u \wedge v) \vee (v \wedge w) \leq v$, that is $(u \wedge v) + (v \wedge w) - (u \wedge v \wedge w) \leq v$, that is $(u \wedge v) - (u \wedge v \wedge w) \leq v - (v \wedge w)$. It follows that

$$\|u - (u \wedge w)\|_X \leq \|u - (u \wedge v)\|_X + \|v - (v \wedge w)\|_X$$

and symmetrically

$$\|w - (u \wedge w)\|_X \leq \|w - (w \wedge v)\|_X + \|v - (v \wedge u)\|_X$$

and summing up we get, as expected $\mathsf{d}_X(u, w) \leq \mathsf{d}_X(u, v) + \mathsf{d}_X(v, w)$.

Now we prove completeness of $\mathsf{P}(X)$ for this distance. Instead, but equivalently, we prove completeness for the distance $\mathsf{d'}_X$ given by

$$\mathsf{d'}_X(u,v) = 2\left\|\frac{1}{2}\,|v-u|\right\|_X$$

which is equivalent to $\mathsf{d}_X$ by Exercise 8.3.5. The announced completeness results from the following facts :

— $\widetilde{\mathsf{P}}\,(X)$ is a Banach space.

— $\mathsf{P}(X)$ is a closed subset of the Banach space $\widetilde{\mathsf{P}}\,(X)$ because

$$\mathsf{P}(X) = \left\{x \in \widetilde{\mathsf{P}}\,(X) \mid \|x\|_{\widetilde{\mathsf{P}}(X)} \leq 1 \text{ and } \forall a \in |X|\ x_a \geq 0\right\}$$

and the maps $x \mapsto x_a$ are continuous $\widetilde{\mathsf{P}}\,(X) \to \mathbb{R}$.

— On $\mathsf{P}(X)$, the distance $\mathsf{d'}_X$ coincides with the distance induced by the norm $\|\_\|_{\widetilde{\mathsf{P}}(X)}$, since $\frac{1}{2}\,\mathsf{d'}_X(u,v) = \left\|\frac{1}{2}\,|u-v|\right\|_X = \frac{1}{2}\,\|u-v\|_{\widetilde{\mathsf{P}}(X)}$.

□

## 8.4 Exponentials.

Given $u \in \overline{\mathbb{R}_{\geq 0}}^I$ and $m \in \mathcal{M}_{\mathrm{fin}}(I)$ (finite multiset of elements of $I$), we define

$$u^m = \prod_{i \in I} u_i^{m(i)}\,.$$

In other words, if $m = [i_1, \dots, i_k]$, then $u^m = \prod_{j=1}^k u_{i_j}$. Next we define $u^{(!)} \in \overline{\mathbb{R}_{\geq 0}}^{\mathcal{M}_{\mathrm{fin}}(I)}$ by

$$u^{(!)}_m = u^m\,.$$

Let $X$ be a PCS and let

$$\mathcal{P} = \left\{u^{(!)} \mid u \in \mathsf{P}(X)\right\} \subseteq \mathbb{R}_{\geq 0}^{\mathcal{M}_{\mathrm{fin}}(|X|)}\,.$$

Then, for any $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$ we have

$$0 < \sup_{u \in \mathsf{P}(X)} u^m < \infty\,.$$

Indeed we can write $m = [a_1, \dots, a_k]$ and we know that for each $i = 1, \dots, k$, there are $\varepsilon_i > 0$ and $A_i < \infty$ such that

$$\forall i \in \{1, \dots, k\} \quad \varepsilon_i \leq \sup_{u \in \mathsf{P}(X)} u_{a_i} \leq A_i\,.$$

Therefore

$$\forall i \in \{1, \dots, k\} \quad 0 < \prod_{i=1}^k \varepsilon_i \leq \sup_{u \in \mathsf{P}(X)} u^m \leq \prod_{i=1}^k A_i < \infty\,.$$

So we define a PCS $!X$ by setting $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and $\mathsf{P}(!X) = \left\{u^{(!)} \mid u \in \mathsf{P}(X)\right\}^{\perp\perp}$.

**Lemme 8.4.1** *Let $s \in \overline{\mathbb{R}_{\geq 0}}^{|!X \multimap Y|}$. Then $s \in \mathsf{P}(!X \multimap Y)$ if and only if $\forall u \in \mathsf{P}(X)\ s \cdot u^{(!)} \in \mathsf{P}(Y)$.*

*Démonstration.* The condition is clearly necessary, so let us prove that it is sufficient. Let $s \in \overline{\mathbb{R}_{\geq 0}}^{|!X \multimap Y|}$ be such that $\forall u \in \mathsf{P}(X)\ s \cdot u^{(!)} \in \mathsf{P}(Y)$ and let us prove that $s \in \mathsf{P}(!X \multimap Y)$. For this it suffices to prove that $s^\perp \in \mathsf{P}(Y^\perp \multimap (!X)^\perp)$. So let $v' \in \mathsf{P}(Y^\perp)$, we prove that $s^\perp \cdot v' \in \mathsf{P}((!X)^\perp) = \left\{u^{(!)} \mid u \in \mathsf{P}(X)\right\}^\perp$. This results clearly from the fact that $\left\langle s^\perp \cdot v', u^{(!)}\right\rangle = \left\langle v', s \cdot u^{(!)}\right\rangle \leq 1$ since we have $s \cdot u^{(!)} \in \mathsf{P}(Y)$. □

**Lemme 8.4.2** *Let* $s \in \overline{\mathbb{R}_{\geq 0}}^{|!X_1 \otimes \cdots \otimes !X_n \multimap Y|}$. *Then* $s \in \mathsf{P}(!X_1 \otimes \cdots \otimes !X_n \multimap Y)$ *if and only if* $\forall u(1) \in \mathsf{P}(X_1), \dots, u(n) \in \mathsf{P}(X_n)$ $s \cdot (u(1)^{(!)} \otimes \cdots \otimes u(n)^{(!)}) \in \mathsf{P}(Y)$.

*Démonstration.* By induction on $n \geq 1$. For $n = 1$, this is just Lemma 8.4.1. Assume that $n > 1$. By Lemma 8.2.4 it suffices to prove that $t = \alpha \cdot s \in \mathsf{P}(!X_1 \multimap (!X_2 \otimes \cdots \otimes !X_n \multimap Y))$. Let $u(1) \in \mathsf{P}(X_1)$, we prove that $t \cdot u(1) \in \mathsf{P}(!X_2 \otimes \cdots \otimes !X_n \multimap Y)$. By inductive hypothesis, it suffices to prove that, for any $u(2) \in \mathsf{P}(X_2), \dots, u(n) \in \mathsf{P}(X_n)$, one has $(t \cdot u(1)) \cdot (u(2)^{(!)} \otimes \cdots \otimes u(n)^{(!)}) \in \mathsf{P}(Y)$. This latter property holds because

$$(t \cdot u(1)) \cdot (u(2)^{(!)} \otimes \cdots \otimes u(n)^{(!)}) = s \cdot (u(1)^{(!)} \otimes \cdots \otimes u(n)^{(!)})$$

and by our assumtion about $s$. □

**Lemme 8.4.3** *Let* $s^1, s^2 \in \mathsf{P}(!X \multimap Y)$. *If* $\forall u \in \mathsf{P}(X)$ $s^1 \cdot u^{(!)} = s^2 \cdot u^{(!)}$ *then* $s^1 = s^2$.

*Démonstration.* Let $m \in |!X|$ and $b \in |Y|$, we prove that $s^1_{m,b} = s^2_{m,b}$. Let $\{a_1, \dots, a_k\}$ be the support of $m$ (that is the *set* of all $a$'s such that $m(a) \neq 0$), with the $a_j$ pairwise distinct. Let $n = m(a_1) + \cdots + m(a_k)$ be the size of $m$. Let $\varepsilon > 0$ be such that $\varepsilon \mathsf{e}_{a_i} \in \mathsf{P}(X)$ for $i = 1, \dots, k$. Then we consider the function

$$\begin{aligned}\varphi^i &: [-\varepsilon, \varepsilon]^k \to \mathbb{R} \\ (x_1, \dots, x_k) &\mapsto (s^i \cdot (x_1 \mathsf{e}_{a_1} + \cdots + x_k \mathsf{e}_{a_k})^{(!)})_b = \sum_{p \in \mathcal{M}_{\mathrm{fin}}(|X|)} s^i_{p,b} \prod_{i=1}^{k} x_i^{p(a_i)}\end{aligned}$$

which is well defined since the involved (usually infinite) summation converges absolutely by our assumption about $s$ and $\varepsilon$. Then we have

$$s^i_{m,b} = \frac{1}{m(a_1)! \cdots m(a_k)!} \frac{\partial^n \varphi^i(x_1, \dots, x_k)}{\partial x_1^{m(a_1)} \cdots \partial x_k^{m(a_k)}}(0, \dots, 0)$$

for $i = 1, 2$ and hence $s^1_{m,b} = s^2_{m,b}$ since $\varphi^1 = \varphi^2$. Since this holds for all $m$ and $b$, we have proven that $s^1 = s^2$. □

**Lemme 8.4.4** *Let* $s^1, s^2 \in \mathsf{P}(!X_1 \otimes \cdots \otimes !X_n \multimap Y)$. *If, for all* $u(1) \in \mathsf{P}(X_1), \dots u(n) \in \mathsf{P}(X_n)$, *one has* $s^1 \cdot (u(1)^{(!)} \otimes \cdots \otimes u(n)^{(!)}) = s^2 \cdot (u(1)^{(!)} \otimes \cdots \otimes u(n)^{(!)})$ *then* $s^1 = s^2$.

The proof is by an induction similar to that of the proof of Lemma 8.4.2.

**Exercice 8.4.1** Write the proof of Lemma 8.4.4.

Now we show that !_ is actually a functor. For this we need some notations. First if $n \in \mathbb{N}$, the factorial of $n \in \mathbb{N}$ is $n! = n \times (n-1) \times \cdots \times 1 \in \mathbb{N}$. If $n_1, \dots, n_k \in \mathbb{N}$ and $n = n_1 + \cdots + n_k$, then the number

$$\binom{n}{n_1, \dots, n_k} = \frac{n!}{n_1! \cdots n_k!}$$

belongs to $\mathbb{N}$. It is the number of tuples $(L_1, \dots, L_k)$ of pairwise disjoint subsets of $\{1, \dots, n\}$ such that $\# L_1 = n_1, \dots, \# L_k = n_k$ (and hence $L_1 \cup \cdots \cup L_k = \{1, \dots, n\}$). It is the multinomial coefficient of $n, n_1, \dots, n_k$.

Next if $m \in \mathcal{M}_{\mathrm{fin}}(I)$ we set $m! = \prod_{i \in I} m(i)! \in \mathbb{N}$.

**Lemme 8.4.5** *Let* $s \in \mathbb{R}_{\geq 0}^{I \times J}$. *There is a* $!s \in \mathbb{R}_{\geq 0}^{\mathcal{M}_{\mathrm{fin}}(I) \times \mathcal{M}_{\mathrm{fin}}(J)}$ *such that* $\forall u \in \mathbb{R}^I_{\geq 0}$ $!s \cdot u^{(!)} = (s \cdot u)^{(!)}$.

*Démonstration.* We explain first the definition of $!s$. Using the multinomial equation :

$$(r_1 + \cdots + r_k)^n = \sum_{\substack{(n_1, \dots, n_k) \in \mathbb{N}^k \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \dots, n_k} r_1^{n_1} \cdots r_k^{n_k}.$$

Let $m \in \mathcal{M}_{\mathrm{fin}}(I)$ and $p \in \mathcal{M}_{\mathrm{fin}}(J)$. We use $\mathsf{L}(m,p)$ for the set of all $h \in \mathcal{M}_{\mathrm{fin}}(I \times J)$ such that

$$\forall i \in I \; m(i) = \sum_{j \in J} h(i,j) \quad \text{and} \quad \forall j \in J \; p(j) = \sum_{i \in I} h(i,j)\,.$$

Notice that if such an $h$ exists, we must have $\#m = \#p = \#h$ since $\#m = \sum_{i \in I} m(i)$, $\#p = \sum_{j \in I} p(j)$ and $\#h = \sum_{i \in I, j \in J} h(i,j)$. For $h \in \mathsf{L}(m,p)$, we set

$$\binom{p}{h} = \prod_{j \in J} \frac{p(j)!}{\prod_{i \in I} h(i,j)!} = \frac{p!}{h!} \in \mathbb{N}$$

which can be seen as a product of multinomial coefficients since $p(j) = \sum_{i \in I} h(i,j)$ for each $j \in J$, by our assumption that $h \in \mathsf{L}(m,p)$. Then we define $!s \in \mathbb{R}_{\geq 0}^{\mathcal{M}_{\mathrm{fin}}(I) \times \mathcal{M}_{\mathrm{fin}}(J)}$ by

$$(!s)_{m,p} = \sum_{h \in \mathsf{L}(m,p)} \binom{p}{h} s^h = p! \sum_{h \in \mathsf{L}(m,p)} \frac{s^h}{h!}\,.$$

Let us prove that, for any $u \in \mathsf{P}(X)$, one has $!s \cdot u^{(!)} = (s \cdot u)^{(!)}$. Let $p \in |!Y|$, we have

$$\begin{aligned}
(s \cdot u)^{(!)}_p &= \prod_{b \in |Y|} \left( \sum_{a \in |X|} s_{a,b} u_a \right)^{p(b)} \\
&= \prod_{b \in |Y|} \left( \sum_{\substack{m \in \mathcal{M}_{\mathrm{fin}}(|X|) \\ \#m = p(b)}} \binom{p(b)}{m} \prod_{a \in |X|} (s_{a,b} u_a)^{m(a)} \right) \\
&= \prod_{b \in |Y|} \left( \sum_{\substack{m \in \mathcal{M}_{\mathrm{fin}}(|X|) \\ \#m = p(b)}} \binom{p(b)}{m} u^m \prod_{a \in |X|} s_{a,b}^{m(a)} \right) \\
&= \sum_{\substack{\varphi \in \mathcal{M}_{\mathrm{fin}}(|X|)^{|Y|} \\ \forall b \; \#\varphi(b) = p(b)}} \left( \prod_{b \in |Y|} \binom{p(b)}{\varphi(b)} u^{\varphi(b)} \prod_{a \in |X|} s_{a,b}^{\varphi(b)(a)} \right) \\
&= \sum_{m \in \mathcal{M}_{\mathrm{fin}}(|X|)} u^m \sum_{h \in \mathsf{L}(m,p)} \binom{p}{h} s^h = !s \cdot u^{(!)}
\end{aligned}$$

observing that, given $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$, there is an obvious bijection between $\mathsf{L}(m,p)$ and the set of all $\varphi \in \mathcal{M}_{\mathrm{fin}}(|X|)^{|Y|}$ such that $\forall b \; \sum_{a \in |X|} \varphi(b)(a) = \#\varphi(b) = p(b)$ and $\forall a \in |X| \; \sum_{b \in |X|} \varphi(b)(a) = m(a)$, that is $\sum_{b \in |Y|} \varphi(b) = m$. □

**Remarque 8.4.6** If we write $p = [b_1, \dots, b_k]$ where $b_1, \dots, b_k$ is an (arbitrary) enumeration of the elements of $p$ (taking of course multiplicities into account) then we have

$$\begin{aligned}
(s \cdot u)^{(!)}_p &= \prod_{i=1}^{k} \left( \sum_{a \in |X|} s_{a,b_i} u_a \right) \\
&= \sum_{f : \{1,\dots,k\} \to |X|} \prod_{i=1}^{k} s_{f(i),b_i} u_{f(i)} \\
&= \sum_{f : \{1,\dots,k\} \to |X|} u^{[f(1),\dots,f(k)]} \prod_{i=1}^{k} s_{f(i),b_i}
\end{aligned}$$

so that we can write

$$\forall m \in |!X|, p \in |!Y| \quad (!s)_{m,p} = \sum_{\substack{f:\{1,\dots,n\}\to|X| \\ [f(1),\dots,f(k)]=m}} \prod_{i=1}^{k} s_{f(i),b_i}\,.$$

This shows that, for any $m \in |!X|$ and $p = [b_1, \dots, b_k] \in |!Y|$, then for any $h \in \mathsf{L}(m,p)$, the number of functions $f : \{1, \dots, k\} \to |X|$ such that $[(f(1), b_1), \dots, (f(k), b_k)] = h$ is equal to $\binom{p}{h} = \frac{p!}{h!}$. Indeed we have proven that

$$\sum_{\substack{f:\{1,\dots,n\}\to|X| \\ [f(1),\dots,f(k)]=m}} s^{[(f(1),b_1),\dots,(f(k),b_k)]} = \sum_{h\in\mathsf{L}(m,p)} \binom{p}{h} s^h$$

and this holds for any $s \in \mathbf{Pcoh}(X, Y)$.

**Lemme 8.4.7** *Let $s \in \mathbf{Pcoh}(X, Y)$. Then $!s \in \mathbf{Pcoh}(!X, !Y)$ and it is the unique element $t$ of $\mathbf{Pcoh}(!X, !Y)$ such that $t \cdot u^{(!)} = (s \cdot u)^{(!)}$ for all $u \in \mathsf{P}(X)$.*

*Démonstration.* By Lemma 8.4.5 we know that for any $u \in \mathsf{P}(X)$ one has $!s \cdot u^{(!)} = (s \cdot u)^{(!)} \in \mathsf{P}(!Y)$ since $s \cdot u \in \mathsf{P}(Y)$. By Lemma 8.4.1 it follows that $!s \in \mathbf{Pcoh}(!X, !Y)$. The uniqueness follows from Lemma 8.4.3. □

**Théorème 8.4.8** *The operation defined on objects by $X \mapsto !X$ and on morphisms by $s \in \mathbf{Pcoh}(X, Y) \mapsto !s \in \mathbf{Pcoh}(!X, !Y)$ is a functor.*

*Démonstration.* First we have $!\mathsf{Id}_X = \mathsf{Id}_{!X}$ by Lemma 8.4.3 since both morphisms map $u^{(!)}$ to itself by any $u \in \mathsf{P}(X)$.

Next let $s \in \mathbf{Pcoh}(X, Y)$ and $t \in \mathbf{Pcoh}(Y, Z)$ then $!(t\,s) = !t\,!s$ results again easily from Lemma 8.4.3 : for any $u \in \mathsf{P}(X)$ we have

$$\begin{aligned}(!t\,!s) \cdot u^{(!)} &= !t \cdot (!s \cdot u^{(!)}) \\ &= !t \cdot (s \cdot u)^{(!)} \\ &= (t \cdot (s \cdot u))^{(!)} \\ &= ((t\,s) \cdot u)^{(!)} \\ &= !(t\,s) \cdot u^{(!)}\,.\end{aligned}$$

□

We define $\mathsf{der}_X \in \mathbb{R}_{\geq 0}^{|!X\multimap X|}$ by $(\mathsf{der}_X)_{m,a} = \delta_{m,[a]}$ and $\mathsf{dig}_X \in \mathbb{R}_{\geq 0}^{|!X\multimap !!X|}$ by $(\mathsf{dig}_X)_{m,M} = \delta_{m,\Sigma(M)}$ (where, given $M \in |!!X|$, the element $\Sigma(M) \in |!X|$ is defined by $\Sigma(M)(a) = \sum_{p\in|!X|} M(p)p(a) \in \mathbb{N}$).

**Lemme 8.4.9** *For each PCS $X$ we have $\mathsf{der}_X \in \mathbf{Pcoh}(!X, X)$ and $\mathsf{dig}_X \in \mathbf{Pcoh}(!X, !!X)$ which satisfy $\mathsf{der}_X \cdot u^{(!)} = u$ and $\mathsf{dig}_X\, u^{(!)} = u^{(!)(!)}$*

*Démonstration.* It suffices to prove the two equations, the two first statements result from Lemma 8.4.1. Given $a \in |X|$ we have

$$\begin{aligned}(\mathsf{der}_X \cdot u^{(!)})_a &= \sum_{m\in|!X|} (\mathsf{der}_X)_{m,a} u^m \\ &= \sum_{m\in|!X|} \delta_{m,[a]} u^m \\ &= u^{[a]} = u_a\end{aligned}$$

which proves the first equation. Next, given $M \in |!!X|$, we have

$$\begin{aligned}
(\mathsf{dig}_X \cdot u^{(!)})_M &= \sum_{m \in |!X|} (\mathsf{dig}_X)_{m,M} u^m \\
&= \sum_{m \in |!X|} \delta_{m,\Sigma(M)} u^m \\
&= u^{\Sigma(M)} \\
&= \prod_{p \in |!X|} u^{M(p)p} \\
&= \prod_{p \in |!X|} (u_p^{(!)})^{M(p)} \\
&= u_M^{(!)(!)}
\end{aligned}$$

which proves the second equation. □

**Théorème 8.4.10** *The morphisms* $\mathsf{der}_X$ *and* $\mathsf{dig}_X$ *define natural transformations and the functor* $!\_$*, equipped with these natural transformations, is a comonad on* **Pcoh**.

*Démonstration.* Let us prove for instance that $\mathsf{dig}_X$ is natural, so let $s \in \mathbf{Pcoh}(X, Y)$, we must show that $\mathsf{dig}_Y\, !s = !!s\, \mathsf{dig}_X$. Given $u \in \mathsf{P}(X)$, we have

$$\begin{aligned}
(\mathsf{dig}_Y\, !s) \cdot u^{(!)} &= \mathsf{dig}_Y \cdot (!s \cdot u^{(!)}) \\
&= \mathsf{dig}_Y \cdot (s \cdot u)^{(!)} \\
&= (s \cdot u)^{(!)(!)} \\
&= (!s \cdot u^{(!)})^{(!)} \\
&= !!s \cdot u^{(!)(!)} \\
&= (!!s\, \mathsf{dig}_X)\, u^{(!)}
\end{aligned}$$

and the equataion follows by Lemma 8.4.3. □

**Exercice 8.4.2** Prove in the same way the other commutations (naturality of dereliction and the three comonad diagrams).

Let $s \in \mathbf{Pcoh}(!X, Y)$, remember that the lifting (or *promotion*) $s^! \in \mathbf{Pcoh}(!X, !Y)$ is given by

$$s^! = !s\ \mathsf{dig}_X$$

Let $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$ and $p \in \mathcal{M}_{\mathrm{fin}}(|Y|)$, we have

$$\begin{aligned}
s^!_{m,p} &= \sum_{\substack{M \in |!!X| \\ \Sigma(M) = m}} (!s)_{M,p} \\
&= \sum_{\substack{M \in |!!X| \\ \Sigma(M) = m}} \sum_{h \in \mathsf{L}(M,p)} \binom{p}{h} s^h \\
&= \sum_{h \in \mathcal{L}(m,p)} \binom{p}{h} s^h \\
&= p! \sum_{h \in \mathcal{L}(m,p)} \frac{s^h}{h!}
\end{aligned}$$

where $\mathcal{L}(m,p)$ is the set of all $h \in \mathcal{M}_{\mathrm{fin}}(|!X| \times |Y|)$ such that

$$\sum_{l \in |!X|, b \in |Y|} h(l,b)l = m \text{ and } \forall b \in |Y| \sum_{l \in |!X|} h(l,b) = p(b)\,.$$

Notice that $\mathcal{L}(m,p)$ is finite since, for any $h \in \mathcal{L}(m,p)$ we must have $\mathsf{supp}(h) \subseteq \mathcal{P}(\mathsf{supp}(m)) \times \mathsf{supp}(p)$ and, for any $(l,b) \in \mathsf{supp}(h)$, we must have $h(l,b) \leq p(b)$.

**Remarque 8.4.11** Following Remark 8.4.6, let us write $p = [b_1, \dots, b_k]$, then we have

$$\begin{aligned} s^!_{m,p} &= \sum_{\substack{M \in |!!X| \\ \Sigma(M)=m}} \sum_{\substack{f:\{1,\dots,k\} \to |!X| \\ [f(1),\dots,f(k)]=M}} \prod_{i=1}^{k} s_{f(i),b_i} \\ &= \sum_{f \in \mathcal{S}(m,k)} \prod_{i=1}^{k} s_{f(i),b_i} \end{aligned}$$

where, given $k \in \mathbb{N}$ and a multiset $m \in \mathcal{M}_{\mathrm{fin}}(I)$, we use $\mathcal{S}(m,k)$ for the (finite) set of all functions $f : \{1,\dots,k\} \to \mathcal{M}_{\mathrm{fin}}(I)$ such that $\sum_{i=1}^{k} f(i) = m$.

8.4.1 Monoidality of the exponential (Seely isomorphisms). Given $p \in \mathcal{M}_{\mathrm{fin}}(\sum_{i \in I} J_i)$ and $i \in I$, we use $\mathsf{pm}_i(p) \in \mathcal{M}_{\mathrm{fin}}(J_i)$ for the projection of $p$ on the component $i$ of the sum of sets, that is $\mathsf{pm}_i(p) : J_i \to \mathbb{N}$ is given by $\mathsf{pm}_i(p)(j) = p(i,j)$.

Just as for the comonad structure of $!\_$, there is absolutely no surprise : the Seely isomorphisms of **Rel** become Seely isomorphisms in **Pcoh**. More precisely we define $\mathsf{m}^0 \in \mathbb{R}_{\geq 0}^{|1 \multimap !\top|}$ by $\mathsf{m}^0_{*,[]} = 1$ and, given two PCSs $X_1$ and $X_2$, we define $\mathsf{m}^2_{X_1,X_2} \in \mathbb{R}_{\geq 0}^{|!X_1 \otimes !X_2 \multimap !(X_1 \& X_2)|}$ by $(\mathsf{m}^2_{X_1,X_2})_{m_1,m_2,p} = \delta_{m_1,\mathsf{pm}_1(p)}\delta_{m_2,\mathsf{pm}_2(p)}$.

**Lemme 8.4.12** *One has $\mathsf{m}^0 \in \mathbf{Pcoh}(1, !\top)$ and $\mathsf{m}^2_{X_1,X_2} \in \mathbf{Pcoh}(!X_1 \otimes !X_2, !(X_1 \& X_2))$ and these morphisms are isos.*

*Démonstration.* Concerning $\mathsf{m}^0$ it suffices to observe that $0^{(!)}_{[\emptyset]} = 1$, where $0$ is the unique element of $\mathsf{P}(\top)$ so that $\mathsf{P}(!\top) = [0,1]$ up to the usual identification of $\mathbb{R}^{[]}_{\geq 0}$ with $\mathbb{R}_{\geq 0}$.

Let $u(i) \in \mathsf{P}(X_i)$, we have

$$\mathsf{m}^2_{X_1,X_2} \cdot (u(1)^{(!)} \otimes u(2)^{(!)}) = (\overline{\pi}_1 \cdot u(1) + \overline{\pi}_2 \cdot u(2))^{(!)} \in \mathsf{P}(X_1 \& X_2)$$

since $\overline{\pi}_1 \cdot u(1) + \overline{\pi}_2 \cdot u(2) \in \mathsf{P}(X_1 \& X_2)$ by definition of the PCS $X_1 \& X_2$. This shows that $\mathsf{m}^2_{X_1,X_2} \in \mathbf{Pcoh}(!X_1 \otimes !X_2, !(X_1 \& X_2))$ by Lemma 8.4.1.

Let now $t \in \mathbb{R}_{\geq 0}^{|!(X_1 \& X_2) \multimap !X_1 \otimes !X_2|}$ be defined by $t_{p,m_1,m_2} = \delta_{\mathsf{pm}_1(p),m_1}\delta_{\mathsf{pm}_2(p),m_2}$. Let $v \in \mathsf{P}(X_1 \& X_2)$, we have

$$t \cdot v^{(!)} = (\pi_1 \cdot v) \otimes (\pi_2 \cdot v) \in \mathsf{P}(!X_1 \otimes !X_2)$$

since $\pi_j\, v \in \mathsf{P}(X_i)$ for $i = 1,2$ and hence by Lemma 8.4.1 again we have $t \in \mathbf{Pcoh}(!(X_1 \& X_2), !X_1 \otimes !X_2)$. Since $t$ is the inverse of $\mathsf{m}^2_{X_1,X_2}$ (by Lemma 8.4.3), this shows that $\mathsf{m}^2_{X_1,X_2}$ is an iso. □

**Théorème 8.4.13** *The isomorphism $\mathsf{m}^2_{X_1,X_2}$ is natural in $X_1$ and $X_2$ and $!\_$ is a strong symmetric monoidal comonad from the symmetric monoidal category $(\mathbf{Pcoh}, \top, \&)$ to the symmetric monoidal category $(\mathbf{Pcoh}, 1, \otimes)$.*

*Démonstration.* Remember that this means that the following diagrams must commute.

$$\begin{array}{ccc} 1 \otimes !X & \xrightarrow{\lambda} & !X \\ \downarrow{\scriptstyle \mathsf{m}^0 \otimes !X} & & \downarrow{\scriptstyle !\langle \mathsf{t}_X, X\rangle} \\ !\top \otimes !X & \xrightarrow{\mathsf{m}^2_{\top,X}} & !(\top \& X) \end{array} \qquad \begin{array}{ccc} !X \otimes 1 & \xrightarrow{\rho} & !X \\ \downarrow{\scriptstyle !X \otimes \mathsf{m}^0} & & \downarrow{\scriptstyle !\langle X, \mathsf{t}_X\rangle} \\ !X \otimes !\top & \xrightarrow{\mathsf{m}^2_{X,\top}} & !(X \& \top) \end{array}$$

where $\mathtt{t}_X$ is the unique element of $\mathbf{Pcoh}(X, \top)$, so that $\langle \mathtt{t}_X, X\rangle$ is an iso which expresses that $\top$ is left neutral for $\&$.

$$\begin{array}{ccc}
(!X_1 \otimes !X_2) \otimes !X_3 & \xrightarrow{\alpha_{!X_1,!X_2,!X_3}} & !X_1 \otimes (!X_2 \otimes !X_3) \\
\downarrow{\scriptstyle \mathsf{m}^2_{X_1,X_2} \otimes !X_3} & & \downarrow{\scriptstyle !X_1 \otimes \mathsf{m}^2_{X_2,X_3}} \\
!(X_1 \mathbin{\&} X_2) \otimes !X_3 & & !X_1 \otimes !(X_2 \mathbin{\&} X_3) \\
\downarrow{\scriptstyle \mathsf{m}^2_{X_1 \& X_2, X_3}} & & \downarrow{\scriptstyle \mathsf{m}^2_{X_1, X_2 \& X_3}} \\
!((X_1 \mathbin{\&} X_2) \mathbin{\&} X_3) & \xrightarrow{\langle \pi_1\,\pi_1, \langle \pi_2\,\pi_1, \pi_2\rangle\rangle} & !(X_1 \mathbin{\&} (X_2 \mathbin{\&} X_3))
\end{array}$$

$$\begin{array}{ccc}
!X_1 \otimes !X_2 & \xrightarrow{\sigma_{!X_1,!X_2}} & !X_2 \otimes !X_1 \\
\downarrow{\scriptstyle \mathsf{m}^2_{X_1,X_2}} & & \downarrow{\scriptstyle \mathsf{m}^2_{X_2,X_1}} \\
!(X_1 \mathbin{\&} X_2) & \xrightarrow{!\langle \pi_2,\pi_1\rangle} & !(X_2 \mathbin{\&} X_1)
\end{array}$$

$$\begin{array}{ccc}
!X \otimes !Y & \xrightarrow{\mathsf{m}^2_{X,Y}} & !(X \mathbin{\&} Y) \\
 & & \downarrow{\scriptstyle \mathsf{dig}_{X \& Y}} \\
\downarrow{\scriptstyle \mathsf{dig}_X \otimes \mathsf{dig}_Y} & & !!(X \mathbin{\&} Y) \\
 & & \downarrow{\scriptstyle !\langle !\pi_1, !\pi_2\rangle} \\
!!X \otimes !!Y & \xrightarrow{\mathsf{m}^2_{!X,!Y}} & !(!X \mathbin{\&} !Y)
\end{array}$$

Again these commutations can be proved easily using now Lemma 8.4.4. For instance, to prove the last commutation, one shows that given $u \in \mathsf{P}(X)$ and $v \in \mathsf{P}(Y)$, both sides of the diagram map $u^{(!)} \otimes v^{(!)}$ to $(\overline{\pi}_1 \cdot u^{(!)} + \overline{\pi}_2 \cdot v^{(!)})^{(!)}$. Indeed we have

$$\begin{aligned}
&(!\langle !\pi_1, !\pi_2\rangle \ \mathsf{dig}_{X\&Y} \ \mathsf{m}^2_{X,Y}) \cdot (u^{(!)} \otimes v^{(!)}) \\
&\qquad = (!\langle !\pi_1, !\pi_2\rangle \ \mathsf{dig}_{X\&Y}) \cdot (\overline{\pi}_1 \cdot u + \overline{\pi}_2 \cdot v)^{(!)} \\
&\qquad = (!\langle !\pi_1, !\pi_2\rangle) \cdot (\overline{\pi}_1 \cdot u + \overline{\pi}_2 \cdot v)^{(!)(!)} \\
&\qquad = (\langle !\pi_1, !\pi_2\rangle \cdot (\overline{\pi}_1 \cdot u + \overline{\pi}_2 \cdot v)^{(!)})^{(!)} \\
&\qquad = (\overline{\pi}_1 \cdot (!\pi_1 \cdot (\overline{\pi}_1 \cdot u + \overline{\pi}_2 \cdot v)^{(!)}) + \overline{\pi}_2 \cdot (!\pi_1 \cdot (\overline{\pi}_1 \cdot u + \overline{\pi}_2 \cdot v)^{(!)}))^{(!)} \\
&\qquad = (\overline{\pi}_1 \cdot u^{(!)} + \overline{\pi}_2 \cdot v^{(!)})^{(!)}
\end{aligned}$$

where we use the fact that, given $s^i \in \mathbf{Pcoh}(Z, X_i)$ for $i = 1, 2$ and $w \in \mathsf{P}(Z)$, we have $\langle s^1, s^2\rangle \cdot w = \overline{\pi}_1 \cdot (s^1 \cdot w) + \overline{\pi}_2 \cdot (s^2 \cdot w)$. □

**Exercice 8.4.3** In the same way, prove the other commutations.

8.4.2 THE DERIVED STRUCTURAL STRUCTURES. In the specific model **Pcoh** under consideration, we make explicit the additional structures presented in Section 4.6.6.

Remember that more generally, for any $k \in \mathbb{N}$, we can define $\mathsf{m}_{X_1,\dots,X_k} \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, !(X_1 \mathbin{\&} \cdots \mathbin{\&} X_k))$ for all $k \in \mathbb{N}$ by induction on $k$ :

— $\mathsf{m} \in \mathbf{Pcoh}(1, !\top)$ is $\mathsf{m}^0$

— $\mathsf{m}_X \in \mathbf{Pcoh}(!X, !X)$ is the identity

— $\mathsf{m}_{X_1,\dots,X_k,X}$ is the following composition of morphisms

$$!X_1 \otimes \cdots \otimes !X_k \otimes !X \xrightarrow{\mathsf{m}_{X_1,\dots,X_k} \otimes !X} !(X_1 \mathbin{\&} \cdots \mathbin{\&} X_k) \otimes !X \xrightarrow{\mathsf{m}^2_{X_1 \& \cdots \& X_k, X}} !(X_1 \mathbin{\&} \cdots \mathbin{\&} X_k \mathbin{\&} X)$$

and we have

$$(\mathsf{m}_{X_1,\dots,X_k})_{(m_1,\dots,m_k),p} = \prod_{i=1}^{k} \delta_{m_i, \mathsf{pm}_i(p)}$$

as easily checked.

Each object $!X$ is endowed with a symmetric comonoid structure given by the counit (weakening) $\mathsf{w}_X \in \mathbf{Pcoh}(!X, 1)$ and comultiplication (contraction) $\mathsf{c}_X \in \mathbf{Pcoh}(!X, !X \otimes !X)$ given by

$$(\mathsf{w}_X)_{m,*} = \delta_{m,[]} \quad \text{and} \quad (\mathsf{c}_X)_{m,(m_1,m_2)} = \delta_{m,m_1+m_2}$$

Also, as explained in Section 4.6.6, the functor !_ is equipped with a lax symmetric monoidal structure from the SMC $(\mathbf{Pcoh}, \otimes, 1)$ to itself, given by the morphism $\mu^0 \in \mathbf{Pcoh}(1, !1)$ and $\mu^2_{X_1,X_2} \in \mathbf{Pcoh}(!X_1 \otimes !X_2, !(X_1 \otimes X_2))$ which are given by

$$\mu^0_{*,n} = 1 \quad \text{for all} \quad n \in |!1| \simeq \mathbb{N}$$

and

$$(\mu^2_{X_1,X_2})_{(m_1,m_2),p} = \begin{cases} 1 & \text{if } \forall a_1 \in |X_1| \; m_1(a_1) = \sum_{a_2 \in |X_2|} m(a_1, a_2) \\ & \text{and } \forall a_2 \in |X_2| \; m_2(a_2) = \sum_{a_1 \in |X_1|} m(a_1, a_2) \\ 0 & \text{otherwise.} \end{cases}$$

**Exercice 8.4.4** Coming back to the general definition of $\mu^0$ and $\mu^2_{X_1,X_2}$ in Section 4.6.6 prove that the decsription above of these morphisms in **Pcoh** are correct.

On defines inductively $\mu^k_{X_1,\dots,X_k} \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, !(X_1 \otimes \cdots \otimes X_k))$ as follows
— $\mu^0 \in \mathbf{Pcoh}(1, !1)$ is $\mu^0$,
— $\mu^1 \in \mathbf{Pcoh}(!X, !X)$ is $\mathsf{Id}_{!X}$ and
— $\mu^{k+1}_{X_1,\dots,X_k,X}$ is the following composition of morphisms

$$!X_1 \otimes \cdots \otimes !X_k \otimes !X \xrightarrow{\mu^k_{X_1,\dots,X_k} \otimes !X} !(X_1 \otimes \cdots \otimes X_k) \otimes !X \xrightarrow{\mu^2_{X_1 \otimes \cdots \otimes X_k, X}} !(X_1 \otimes \cdots \otimes X_k \otimes X)$$

An easy induction on $k$ shows that

$$\mu^k_{m_1,\dots,m_k,m} = \begin{cases} 1 & \text{if } \forall i \in \{1,\dots,k\} \, \forall a_i \in |X_i| \; m_i(a_i) = \sum_{\substack{j=1 \\ j \neq i}}^k \sum_{a_j \in |X_j|} m(a_1, \dots, a_k) \\ 0 & \text{otherwise} \end{cases}$$

First there are generalized dereliction $\mathsf{der}_{X_1,\dots,X_k} = \mathsf{der}_{X_1} \otimes \cdots \otimes \mathsf{der}_{X_k} \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, X_1 \otimes \cdots \otimes X_k)$ and digging $\mathsf{dig}_{X_1,\dots,X_k} \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, !(!X_1 \otimes \cdots \otimes !X_k))$. This latter is defined as the following composition of morphisms

$$!X_1 \otimes \cdots \otimes !X_k \xrightarrow{\mathsf{dig}_{X_1} \otimes \cdots \otimes \mathsf{dig}_{X_k}} !!X_1 \otimes \cdots \otimes !!X_k \xrightarrow{\mu^k_{X_1,\dots,X_k}} !(!X_1 \otimes \cdots \otimes !X_k)$$

So that we have

$$(\mathsf{der}_{X_1,\dots,X_k})_{(\vec{m}),(\vec{a})} = \prod_{i=1}^k \delta_{m_i,[a_i]}$$
$$(\mathsf{dig}_{X_1,\dots,X_k})_{m_1,\dots,m_k,M} = \delta_{\vec{m},\Sigma(M)}$$

where

$$\Sigma(M) = \sum_{\vec{l} \in \prod_{i=1}^k \mathcal{M}_{\mathrm{fin}}(|X_i|)} M(\vec{l})\vec{l}.$$

Then we can define the *generalized structural weakening* and *contraction* maps

$$\mathsf{w}_{X_1,\dots,X_k} \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, 1)$$
$$\mathsf{c}_{X_1,\dots,X_k} \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, (!X_1 \otimes \cdots \otimes !X_k) \otimes (!X_1 \otimes \cdots \otimes !X_k))$$

as follows. The map $\mathsf{w}_{X_1,\dots,X_k}$ is the following composition of morphisms

$$!X_1 \otimes \cdots \otimes !X_k \xrightarrow{\mu^k_{X_1,\dots,X_k}} !(X_1 \otimes \cdots \otimes X_k) \xrightarrow{\mathsf{w}_{X_1 \otimes \cdots \otimes X_k}} 1$$

It follows that

$$(\mathsf{w}_{X_1,\dots,X_k})_{(m_1,\dots,m_k),*} = \prod_{i=1}^{k} \delta_{m_i,[]}\,.$$

The map $\mathsf{c}_{X_1,\dots,X_k}$ is the following composition of morphisms

$$\begin{array}{c} !X_1 \otimes \cdots \otimes !X_k \\ \Big\downarrow \mathsf{dig}_{X_1,\dots,X_k} \\ !(!X_1 \otimes \cdots \otimes !X_k) \\ \Big\downarrow \mathsf{c}_{X_1 \otimes \cdots \otimes X_k} \\ !(!X_1 \otimes \cdots \otimes !X_k) \otimes !(!X_1 \otimes \cdots \otimes !X_k) \\ \Big\downarrow \mathsf{der}_{!X_1 \otimes \cdots \otimes !X_k} \otimes \mathsf{der}_{!X_1 \otimes \cdots \otimes !X_k} \\ (!X_1 \otimes \cdots \otimes !X_k) \otimes (!X_1 \otimes \cdots \otimes !X_k) \end{array}$$

It follows that

$$(\mathsf{c}_{X_1,\dots,X_k})_{\vec{m},(\vec{m^1},\vec{m^2})} = \prod_{i=1}^{k} \delta_{m_i, m^1_i + m^2_i}$$

where $\vec{m} = (m_1, \dots, m_k)$ and similarly for $\vec{m^1}$ and $\vec{m^2}$.

The *generalized promotion* of a morphism $t \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, X)$ is the morphism $t^! \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, !X)$ defined as the following composition of morphisms

$$!X_1 \otimes \cdots \otimes !X_k \xrightarrow{\mathsf{dig}_{X_1,\dots,X_k}} !(!X_1 \otimes \cdots \otimes !X_k) \xrightarrow{!t} !Y$$

Therefore we have, for any $\vec{m} \in |!X_1 \otimes \cdots \otimes !X_k|$ and $m \in |!X|$ :

$$t^!_{\vec{m},m} = m! \sum_{h \in \mathcal{L}(\vec{m},m)} \frac{t^h}{h!}$$

where $\mathcal{L}(\vec{m}, m)$ is the set of all $h \in \mathcal{M}_{\mathrm{fin}}(\prod_{i=1}^k |!X_i| \times |X|)$ such that

$$\sum_{\vec{l} \in \prod_{i=1}^k |!X_i|, a \in |X|} h(\vec{l}, a)\vec{l} = \vec{m} \quad \text{and} \quad \forall a \in |X| \sum_{\vec{l} \in \prod_{i=1}^k |!X_i|} h(\vec{l}, a) = m(a)\,.$$

**Remarque 8.4.14** Let $m = [a_1, \dots, a_l]$, as in Remark 8.4.11, we can write

$$s^!_{\vec{m},m} = \sum_{f \in \mathcal{S}(\vec{m},l)} \prod_{j=1}^{l} s_{f(j),a_j}$$

where, given $l \in \mathbb{N}$ and a tuple of multisets $\vec{m} \in \prod_{j=1}^l \mathcal{M}_{\mathrm{fin}}(I_j)$, we use $\mathcal{S}(\vec{m}, l)$ for the (finite) set of all functions $f : \{1, \dots, k\} \to \prod_{j=1}^l \mathcal{M}_{\mathrm{fin}}(I_j)$ such that $\sum_{j=1}^l f(j) = \vec{m}$, generalizing the notation introduced in Remark 8.4.11.

Let now $s \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, !X \multimap Y)$, the application of $s$ to $t$ is the morphism $(s)\,t \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, Y)$ obtained as the following composition (where we set $P = !X_1 \otimes \cdots \otimes !X_k$)

$$P \xrightarrow{\mathsf{c}_{X_1,\dots,X_k}} P \otimes P \xrightarrow{s \otimes t^!} (!X \multimap Y) \otimes !X \xrightarrow{\mathsf{ev}} Y$$

and therefore we have

$$((s)\,t)_{\vec{m},b} = \sum_{m\in|!X|} m! \sum_{\substack{\vec{p}\leq\vec{m}\\ h\in\mathcal{L}(\vec{p},m)}} s_{(\vec{m}-\vec{p},(m,b))}\frac{t^h}{h!}$$

where $\vec{p}\leq\vec{m}$ means that $p_i \leq m_i$ (as multisets) for all $i\in\{1,\dots,k\}$ and $\vec{m}-\vec{p}=(m_i-p_i)_{i=1}^k$.

We can write this expression in a slightly different way. Let

$$\mathcal{L}^{\leq}(\vec{p},m) = \{h \in \mathcal{M}_{\mathrm{fin}}(\prod_{i=1}^k |!X_i|\times|X|) \mid \sum_{\vec{l}\in\prod_{i=1}^k |!X_i|, a\in|X|} h(\vec{l},a)\vec{l}\leq\vec{m} \text{ and } \forall a\in|X| \sum_{\vec{l}\in\prod_{i=1}^k |!X_i|} h(\vec{l},a) = m(a)\}\,.$$

Then we have

$$((s)\,t)_{\vec{m},b} = \sum_{m\in|!X|} m! \sum_{h\in\mathcal{L}^{\leq}(\vec{m},m)} s_{(\vec{m}-\mathsf{pm}_1^{\Sigma}(h),(m,b))}\frac{t^h}{h!} \tag{8.3}$$

where $\mathsf{pm}_1^{\Sigma}(h) = \sum_{\vec{l}\in\prod_{i=1}^k |!X_i|, a\in|X|} h(\vec{l},a)\vec{l}$.

**Remarque 8.4.15** This can be expressed in another equivalent way, following Remark 8.4.14. Notice that, for a given $m\in|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ with $\#m = l$, there are exactly $\frac{l!}{m!}$ tuples $(a_1,\dots,a_l)\in|X|^l$ such that $m=[a_1,\dots,a_l]$ (for instance if all the $a_j$'s are equal, there is exactly one such tuple, and if they are pairwise distinct, there are $l!$ such tuples). So we have

$$((s)\,t)_{\vec{m},b} = \sum_{l=0}^{\infty}\frac{1}{l!}\sum_{(a_1,\dots,a_l)\in|X|^l}[a_1,\dots,a_l]!\sum_{f\in\mathcal{S}^{\leq}(\vec{m},l)} s_{\vec{m}-\Sigma(f),b}\prod_{j=1}^{l} t_{f(j),a_j} \tag{8.4}$$

where, given $f:\{1,\dots,l\}\to\prod_{i=1}^k\mathcal{M}_{\mathrm{fin}}(I_i)$, we define $\Sigma(f)=\sum_{j=1}^l f(j)\in\prod_{i=1}^k\mathcal{M}_{\mathrm{fin}}(I_i)$ and, given $\vec{m}\in\prod_{i=1}^k\mathcal{M}_{\mathrm{fin}}(I_i)$ we set $\mathcal{S}^{\leq}(\vec{m},l)=\left\{f:\{1,\dots,l\}\to\prod_{i=1}^k\mathcal{M}_{\mathrm{fin}}(I_i)\mid\Sigma(f)\leq\vec{m}\right\}$.

8.4.3 Cartesian closed category and fixed points. Thanks to what we have proven so far, we know that the Kleisli category $\mathbf{Pcoh}_!$ is cartesian closed, just as $\mathbf{Rel}_!$. However the situation is better in $\mathbf{Pcoh}_!$ because the morphisms of this category can be considered as functions, which is not the case of $\mathbf{Rel}_!$ (a morphism in $\mathbf{Rel}_!(E,F)$ induces a function $\mathcal{P}(E)\to\mathcal{P}(F)$, but different morphisms can induce the same function so that morphisms in $\mathbf{Rel}_!$ cannot be considered as simple functions).

Given $s\in\mathbf{Pcoh}_!(X,Y)=\mathbf{Pcoh}(!X,Y)$, we define $\widehat{s}:\mathsf{P}(X)\to\mathsf{P}(Y)$ by $\widehat{s}(u)=s\cdot u^{(!)}$. More explicitely, this function is characterized by

$$\forall u\in\mathsf{P}(X)\,\forall b\in|Y|\quad \widehat{s}(u)_b=\sum_{m\in|!X|}s_{m,b}u^m$$

Lemma 8.4.3 expresses exactly that if $s,s'\in\mathbf{Pcoh}_!(X,Y)$ satisfy $\widehat{s}=\widehat{s'}$ then $s=s'$.

Let us denote with $\circ^!$ composition of morphisms in $\mathbf{Pcoh}_!$ and with $\mathsf{Id}^!_X$ the identity morphism at $X$, so that

$$\mathsf{Id}^!_X=\mathsf{der}_X\quad\text{and}\quad t\circ^! s = t\,!s\,\mathsf{dig}_X$$

(where $s\in\mathbf{Pcoh}_!(X,Y)$ and $t\in\mathbf{Pcoh}_!(Y,Z)$). Then, given $u\in\mathsf{P}(X)$, we have $\widehat{\mathsf{Id}^!_X}(u)=\mathsf{der}_X\cdot u^{(!)}=u$ so that $\widehat{\mathsf{Id}^!_X}$ is the identity function $\mathsf{P}(X)\to\mathsf{P}(X)$ and

$$\begin{aligned}(t\circ^! s)\cdot u^{(!)} &= (t\,!s\,\mathsf{dig}_X)\cdot u^{(!)}\\ &= (t\,!s)\cdot u^{(!)(!)}\\ &= t\cdot(s\cdot u^{(!)})^{(!)}\\ &= \widehat{t}(\widehat{s}(u))\end{aligned}$$

so that $\widehat{t \circ^! s} = \widehat{t} \circ \widehat{s}$. This shows that not only the morphisms of $\mathbf{Pcoh}_!$ can be considered as functions, but that the basic operations of that category (identities and composition) are compatible with this identification.

We shall say that a function $f : \mathsf{P}(X) \to \mathsf{P}(Y)$ is *a stable function from X to Y* if there exists $s \in \mathbf{Pcoh}(X, Y)$ such that $f = \widehat{s}$.

**Remarque 8.4.16** The explicit mention of $X$ and $Y$ in the expression *stable function from X to Y* is necessary because this notion depends on $X$ and $Y$ and not only on the sets $\mathsf{P}(X)$ and $\mathsf{P}(Y)$. Recent results show that the algebraic structure of $\mathsf{P}(X)$ and $\mathsf{P}(Y)$ is sufficient to characterize stable functions [**?**].

**Théorème 8.4.17** *Let $f : \mathsf{P}(X) \to \mathsf{P}(Y)$ be stable from $X$ to $Y$. Then $f$ is monotone (that is $\forall u, v \in \mathsf{P}(X)\ u \leq v \Rightarrow f(u) \leq f(v)$) and Scott-continuous, that is, given any non-decreasing sequence $(u(n))_{n\in\mathbb{N}}$ in $\mathsf{P}(X)$, one has $f(\sup_{n\in\mathbb{N}} u(n)) = \sup_{n\in\mathbb{N}} f(u(n))$.*

*Démonstration.* Let $s \in \mathbf{Pcoh}(!X, Y)$ be such that $f = \widehat{s}$. Let $g : \mathsf{P}(!X) \to \mathsf{P}(Y)$ be the function defined by $g(w) = s \cdot w$, we know by Theorem 8.1.19 that this function is monotone and Scott continuous. Let $\delta_X : \mathsf{P}(X) \to \mathsf{P}(!X)$ be given the stable function given by $\delta = \widehat{\mathsf{Id}}_{!X}$, that is $\forall u \in \mathsf{P}(X)\ \delta_X(u) = u^{(!)}$. Then we have $f = \delta_X \circ g$ so it suffices to prove that $\delta_X$ is monotone and Scott-continuous, what we do now.

Assume that $u, v \in \mathsf{P}(X)$ with $u \leq v$ and let $m \in |!X|$, we have $\delta_X(u) = u^m = \prod_{a\in|X|} u_a^{m(a)} \leq \prod_{a\in|X|} v_a^{m(a)} = \delta_X(v)$ (since $\forall a \in |X|\ u_a \leq v_a$) and hence $\delta_X(u) \leq \delta_X(v)$ in $\mathsf{P}(!X)$. Let $(u(n))_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathsf{P}(X)$ and let $v = \sup_{n\in\mathbb{N}} u(n)$. We know that $\forall a \in |X|\ v_a = \sup_{n\in\mathbb{N}} u(n)_a$. For any $m \in |!X|$ and let $A = \{a \in |X| \mid m(a) = 0\} \subseteq |X|$, which is a finite set (the *domain* of $m$), we have

$$\begin{aligned}\delta_X(v)_m &= \prod_{a\in A} v_a^{m(a)} \\ &= \prod_{a\in A} (\sup_{n\in\mathbb{N}} u(n)_a)^{m(a)} \\ &= \sup_{n\in\mathbb{N}} u(n)^m\end{aligned}$$

by Scott-continuity of multiplication $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. □

**Remarque 8.4.18** If $s \in \mathbf{Pcoh}(X, Y)$, we have a stable function $g = \widehat{s\,\mathsf{der}_X} : \mathsf{P}(X) \to \mathsf{P}(Y)$. Such a stable function will be called a *linear function from X to Y*. Using this terminology, any $f : \mathsf{P}(X) \to \mathsf{P}(Y)$ which is stable from $X$ to $Y$ can be written uniquely $f = g \circ \delta_X$ where $g : \mathsf{P}(!X) \to \mathsf{P}(Y)$ is linear from $!X$ to $Y$. So the $\delta_X$'s can be thought of as the "less linear stable functions", or the "universal stable functions".

**Exercice 8.4.5** Prove that a stable function from $X$ to $Y$ is linear iff for all $u(1), u(2) \in \mathsf{P}(X)$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ such that $\lambda_1 + \lambda_2 \leq 1$, one has $f(\lambda_1 u(1) + \lambda_2 u(2)) = \lambda_1 f(u(1)) + \lambda_2(u(2))$.

We know that $\mathbf{Pcoh}_!$ is cartesian closed, with product of the family $(X_i)_{i\in I}$ the object $X = \&_{i\in I} X_i$ and projections $\pi_i\,\mathsf{der}_X \in \mathbf{Pcoh}_!(X, X_i)$. Upon identifying $\mathsf{P}(X)$ with $\prod_{i\in I} \mathsf{P}(X_i)$ (these partially ordered sets are indeed isomorphic), the linear map $\pi_i = \widehat{\pi_i\,\mathsf{der}_X} : \mathsf{P}(X) \to \mathsf{P}(X_i)$ is given by $\pi_i(\vec{u}) = u(i)$ (where $\vec{u} = (u(k))_{k\in I} \in \prod_{k\in I} \mathsf{P}(X_k)$). Given $(f_i)_{i\in I}$ where $f_i$ is stable from $Y$ to $X_i$ (and $s_i \in \mathbf{Pcoh}(!Y, X_i)$ such that $f_i = \widehat{s_i}$ for each $i$) then the unique stable function $f : \mathsf{P}(Y) \to \mathsf{P}(X)$ such that $\forall i \in I\ \pi_i \circ f = f_i$ (given by $f = \widehat{\langle s_i \rangle_{i\in I}}$) is characterized by $f(u) = (f_i(u))_{i\in I}$.

We can summarize these properties by saying that the cartesian product in $\mathbf{Pcoh}_!$ is defined as in **Set** (upon considering morphisms in $\mathbf{Pcoh}_!$ as stable functions). Notice that a similar statement also holds for **Pcoh**, upon considering morphisms as linear functions.

The object of morphisms from $X$ to $Y$ is the pair $(X \Rightarrow Y, \mathsf{Ev}_{X,Y})$ where $(X \Rightarrow Y) = (!X \multimap Y)$ and $\mathsf{Ev}_{X,Y} \in \mathbf{Pcoh}_!((X \Rightarrow Y) \,\&\, X, Y)$ is the following composition of morphisms in **Pcoh** :

$$!((X \Rightarrow Y) \,\&\, X) \xrightarrow{(\mathsf{m}^2_,)^{-1}} !(!X \multimap Y) \otimes !X \xrightarrow{\mathsf{der} \otimes !X} (!X \multimap Y) \otimes !X \xrightarrow{\mathsf{ev}} Y$$

from which it follows that, given $s \in \mathsf{P}(X \Rightarrow Y)$ and $u \in \mathsf{P}(X)$, we have

$$\begin{aligned}
\widehat{\mathsf{Ev}}(s,u) &= (\mathsf{ev}\,(\mathsf{der} \otimes {!X})) \cdot (s^{(!)} \otimes u^{(!)}) \\
&= \mathsf{ev}\,((\mathsf{der}\ \cdot s^{(!)}) \otimes u^{(!)}) \\
&= \mathsf{ev} \cdot (s \otimes u^{(!)}) \\
&= s \cdot u^{(!)} \\
&= \widehat{s}(u)\,.
\end{aligned}$$

Hence $\widehat{\mathsf{Ev}_{X,Y}}$ takes a pair $(f,u)$ of a stable function from $X$ to $Y$ and an element of $\mathsf{P}(X)$ and maps it to $f(u)$, just as the evaluation map of the cartesian closed category of sets and functions.

Given now $s \in \mathbf{Pcoh}_!(Z \,\&\, X, Y)$, then $\mathsf{Cur}(s) \in \mathbf{Pcoh}_!(Z, X \Rightarrow Y)$ is $\mathsf{cur}(s\,\mathsf{m}^2_{Z,X})$ which makes sense since $\mathsf{m}^2_{Z,X} \in \mathbf{Pcoh}({!Z} \otimes {!X}, {!(Z \,\&\, X)})$. Let $w \in \mathsf{P}(Z)$ and $u \in \mathsf{P}(X)$, we have

$$\begin{aligned}
\widehat{\widehat{s}(w)}(u) &= (\mathsf{cur}(s\,\mathsf{m}^2_{Z,X}) \cdot w^{(!)}) \cdot u^{(!)} \\
&= (s\,\mathsf{m}^2_{Z,X}) \cdot (w^{(!)} \otimes u^{(!)}) \\
&= s \cdot (\overline{\pi}_1 \cdot w + \overline{\pi}_2 \cdot u)^{(!)} \\
&= \widehat{s}(w,u)\,.
\end{aligned}$$

In other words, given a stable map $f : \mathsf{P}(Z) \times \mathsf{P}(X) \simeq \mathsf{P}(Z \,\&\, X) \to \mathsf{P}(Y)$, $\mathsf{Cur}(f)$ is the stable function $\mathsf{P}(Z) \to \mathsf{P}(X \Rightarrow Y)$ which maps $w \in \mathsf{P}(Z)$ to the function $\mathsf{P}(X) \to \mathsf{P}(Y)$ which maps $u$ to $f(w,u)$, which turns out to be stable. Again curryfication is defined exactly as in the cartesian closed category of sets and functions.

Given any $s \in \mathbf{Pcoh}({!X}, X)$, the function $\widehat{s} : \mathsf{P}(X) \to \mathsf{P}(X)$ is monotonic and Scott-continuous, therefore it has a least fixed points which is $\sup_{n \in \mathbb{N}} \widehat{s}^n(0) \in \mathsf{P}(X)$.

Taking $X = ({!({!Y} \multimap Y)} \multimap Y)$ for a PCS $Y$, we define $\mathcal{Z} \in \mathbf{Pcoh}_!(Y,Y)$ by $\mathcal{Z} = \mathsf{cur}\,\mathcal{Z}'$ where $\mathcal{Z}' \in \mathbf{Rel}({!X} \otimes {!({!Y} \multimap Y)}, Y)$ is the following composition of morphisms in $\mathbf{Pcoh}$ :

$$\begin{array}{c}
{!E} \otimes {!({!Y} \multimap Y)} \\
\Big\downarrow {\scriptstyle {!X} \otimes \mathsf{c}_{!Y \multimap Y}} \\
{!X} \otimes {!({!Y} \multimap Y)} \otimes {!({!Y} \multimap Y)} \\
\Big\downarrow {\scriptstyle e \otimes \mathsf{der}_{!Y \multimap Y}} \\
{!Y} \otimes ({!Y} \multimap Y) \\
\Big\downarrow {\scriptstyle \mathsf{ev}\ \sigma} \\
Y
\end{array}$$

where $e = h^!$ (generalized promotion) with $h = \mathsf{ev}\,(\mathsf{der}_X \otimes {!({!Y} \multimap Y)}) \in \mathbf{Rel}({!X} \otimes {!({!Y} \multimap Y)}, Y)$.

Then one can check that the least fixed point $\mathcal{Y} = \sup_{n \in \mathbb{N}} \widehat{\mathcal{Z}}(0)$ which is an element of $\mathsf{P}({!({!Y} \multimap Y)}, Y)$, that is, an element of $\mathbf{Pcoh}_!(Y \Rightarrow Y, Y)$, satisfies

$$\forall s \in \mathsf{P}(Y \Rightarrow Y) \quad \widehat{\mathcal{Y}}(s) = \sup_{n \in \mathbb{N}} \widehat{s}^n(0)$$

that is, $\mathcal{Y}$ is a stable function which maps any stable function from $Y$ to $Y$ to its least fixed point. This is a very important outcome of cartesian closeness because proving directly the stability of the map which sends any $s \in \mathsf{P}(Y \Rightarrow Y)$ to its least fixed point $\sup_{n \in \mathbb{N}} \widehat{s}^n(0)$ is not easy at all.

## 8.5 Interpreting probabilistic PCF

We refer to Section 1.4 for a description of the language we are interpreting in the model.

The interpretation of types is obvious : $[\iota] = \mathsf{N}$ and $[A \to B] = ([A] \Rightarrow [B])$. Remember that $\mathsf{N} = (\mathbb{N}, \{u \in \mathbb{R}^{\mathbb{N}}_{\geq 0} \mid \sum_{n=0}^{\infty} u_n \leq 1\})$.

Then given a term $M$ such that $\Gamma \vdash M : A$ where $\Gamma$ is a typing context $\Gamma = (x_1 : A_1, \dots, x_n : A_l)$, we define $[M]_\Gamma \in \mathbf{Pcoh}_!([A_1] \& \cdots \& [A_l], [A])$ or, equivalently, $[M]_\Gamma \in \mathbf{Pcoh}(![A_1] \otimes \cdots \otimes ![A_l], [A])$ up the Seely isomorphism. In this second format, the interpretation can be defined categorically exactly as we did for PCF in **Rel**.

We adopt the first presentation thus describing $[M]_\Gamma$ as a stable function from $[A_1] \& \cdots \& [A_l]$ to $[A]$, that is, as a function

$$\begin{aligned} \prod_{i=1}^{l} \mathsf{P}(A_i) &\to \mathsf{P}(A) \\ \vec{u} = (u(1), \dots, u(l)) &\mapsto [M]_\Gamma(\vec{u})\,. \end{aligned}$$

We describe now this functional interpretation. Of course to check that the morphisms we are describing are indeed stable, one must refer to the categorical description of the semantics, which is always possible and relies essentially on the fact that $\mathbf{Pcoh}_!$ is cartesian closed with fixed points operators, as explained above.

If $M = x_i$ and $A = A_i$ then $[M]_\Gamma(\vec{u}) = u(i)$.

If $M = \underline{n}$ and $A = \iota$ then $[M]_\Gamma(\vec{u}) = \mathsf{e}_n \in \mathsf{P}(\mathsf{N})$.

If $M = \mathsf{rand}(r)$ then $[M]_\Gamma(\vec{u}) = r\mathsf{e}_0 + (1-r)\mathsf{e}_1$.

If $M = \underline{\mathsf{succ}}(P)$ with $\Gamma \vdash P : \iota$ then $[M]_\Gamma(\vec{u}) = \overline{\mathsf{suc}} \cdot ([P]_\Gamma(\vec{u}))$ where $\overline{\mathsf{suc}} \in \mathsf{P}(\mathsf{N} \multimap \mathsf{N})$ is given by $\overline{\mathsf{suc}}_{n,p} = \delta_{n+1,p}$. In other words $([M]_\Gamma(\vec{u}))_0 = 0$ and $([M]_\Gamma(\vec{u}))_{n+1} = ([P]_\Gamma(\vec{u}))_n$.

If $M = \mathsf{if}(P, Q, x \cdot R)$ with $\Gamma \vdash P : \iota$, $\Gamma \vdash Q : A$ and $\Gamma, x : \iota \vdash R : A$ then

$$[M]_\Gamma(\vec{u}) = [P]_\Gamma(\vec{u})_0 [Q]_\Gamma(\vec{u}) + \sum_{n=0}^{\infty} [P]_\Gamma(\vec{u})_{n+1} [R]_{\Gamma, x:\iota}(\vec{u}, \mathsf{e}_n)\,.$$

Categorically, just as in **Rel**, the interpretation of this construct uses a morphism $h_{\mathsf{N}} \in \mathbf{Pcoh}(\mathsf{N}, !\mathsf{N})$ which is such that $(h_{\mathsf{N}})_{n,m} = 1$ if $m = [n, \dots, n]$ (an arbitrary number of $n$'s) and $(h_{\mathsf{N}})_{n,m} = 0$ otherwise.

Assume that $M = \lambda x^B\, P$ with $\Gamma, x : B \vdash P : C$ and $A = (B \Rightarrow C)$ so that $[P]_{\Gamma, x:B} \in \mathbf{Pcoh}_!([A_1] \& \cdots \& [A_l] \& [B], [C])$ and we set

$$[M]_\Gamma = \mathsf{Cur}([P]_{\Gamma, x:B}) \in \mathbf{Pcoh}_!([A_1] \& \cdots \& [A_l], [B] \Rightarrow [C])\,,$$

in other words $[M]_\Gamma(\vec{u})$ is the stable function which maps $v \in \mathsf{P}([B])$ to $[P]_{\Gamma, x:B}(\vec{u}, v) \in \mathsf{P}([C])$.

Assume that $M = (P)\,Q$ with $\Gamma \vdash P : B \to A$ and $\Gamma \vdash Q : B$ then we have $[P]_\Gamma \in \mathbf{Pcoh}_!([A_1] \& \cdots \& [A_l], [B] \Rightarrow [A])$ and $[Q]_\Gamma \in \mathbf{Pcoh}_!([A_1] \& \cdots \& [A_l], [B])$ then $[M]_\Gamma \in \mathbf{Pcoh}_!([A_1] \& \cdots \& [A_l], [A])$ is given by $[M]_\Gamma(\vec{u}) = \widehat{[P]_\Gamma(\vec{u})}([Q]_\Gamma(\vec{u}))$.

Assume last that $M = \mathsf{fix}(P)$ where $\Gamma \vdash P : A \to A$. Then

$$[M]_\Gamma(\vec{u}) = \sup_{n \in \mathbb{N}} \widehat{[P]_\Gamma(\vec{u})}^{\,n}(0)\,.$$

The fact that this interpretation is invariant by reduction is phrased as follows, and the proof uses quite straightforwardly the fact that $\mathbf{Pcoh}_!$ is a cartesian closed category with fixed point operators.

**Théorème 8.5.1** *Let $M$ be a term of* pPCF *such that $\Gamma \vdash M : A$. Then*

$$[M]_\Gamma = \sum_{M' \text{ s.t. } \Gamma \vdash M' : A} \mathsf{Red}(\Gamma \vdash A)_{M,M'} [M']_\Gamma\,.$$

Iterating this formula one can prove the following result.

**Lemme 8.5.2** *Assume that $\vdash M : \iota$ so that $[M] \in \mathsf{P}(\mathsf{N})$ is a subprobability distribution on the natural numbers. Then for any $n \in \mathbb{N}$*

$$\forall n \in \mathbb{N} \quad \mathsf{Red}(\vdash \iota)^{\infty}_{M,\underline{n}} \leq [M]_n\,.$$

But actually we can even prove an adequacy theorem :

**Théorème 8.5.3** *Assume that* $\vdash M : \iota$ *so that* $[M] \in \mathsf{P}(\mathsf{N})$ *is a subprobability distribution on the natural numbers. Then for any* $n \in \mathbb{N}$

$$\forall n \in \mathbb{N} \quad \mathsf{Red}(\vdash \iota)^{\infty}_{M,\underline{n}} = [M]_n\,.$$

One proves the $\geq$ inequation using a *logical relation* between closed terms of type $A$ and elements of $\mathsf{P}([A])$.

8.5.1 The adequacy theorem. For each type $A$ we define a relation

$$\triangleright^A = \{(M,u) \mid \vdash M : A \text{ and } u \in \mathsf{P}([A])\}$$

by induction on $A$.

Assume first that $A = \iota$, let $M \in \Lambda(\vdash \iota)$ and $u \in \mathsf{P}(\mathsf{N}) \subseteq [0,1]$ (remember that $[\iota] = \mathsf{N}$). Then $M \triangleright^A u$ if

$$\forall n \in \mathbb{N} \quad \mathsf{Red}(\vdash \iota)^{\infty}_{M,\underline{n}} \geq u_n\,.$$

Next

$$\triangleright^{B \to C} = \big\{(M,t) \in \Lambda(\vdash B \to C) \times \mathsf{P}([B \to A]) \mid \forall (N,u) \in \triangleright^B \ (M)\,N \triangleright^C \widehat{t}(u)\big\}\,.$$

**Lemme 8.5.4** *For all type* $A$ *and all* $M \in \Lambda(\vdash A)$ *we have*

— $M \triangleright^A 0$

— *and if* $(u(i))_{i\in\mathbb{N}}$ *is a non-decreasing sequence of elements of* $\mathsf{P}([A])$, *if* $\forall i \in \mathbb{N}\ M \triangleright^A u(i)$ *then* $M \triangleright^A \sup_{i\in\mathbb{N}} u(i)$.

*Démonstration.* By induction on $A$. The properties are obvious for $A = \iota$. Assume that it holds for $C$, we prove it for $A = (B \to C)$. Let $M \in \Lambda(\vdash B \to C)$. We prove first that $M \triangleright^{B\to C} 0$ so let $(N,u) \in \triangleright^B$, we have $\widehat{0}(u) = 0$ and hence $(M)\,N \triangleright^C \widehat{0}(u)$ by inductive hypothesis.

The second property is dealt with similarly, using the fact that if $(t(i))_{i\in\mathbb{N}}$ is a non-decreasing sequence in $\mathsf{P}([B \to C])$, then for any $u \in \mathsf{P}([B])$ we have $\widehat{\sup_{i\in\mathbb{N}} t(i)}(u) = \sup_{i\in\mathbb{N}} \widehat{t(i)}(u)$ (the sequence $(\widehat{t(i)}(u))_{i\in\mathbb{N}}$ being non-decreasing in $\mathsf{P}([C])$). □

**To be completed.**

8.5.2 Examples of term interpretation. Consider the term

$$M = \lambda x^\iota\, \mathsf{if}(x, \mathsf{if}(x, \underline{1}, z \cdot z), z \cdot \mathsf{if}(z, \underline{0}, d \cdot x))$$

so that $\vdash M : \iota \to \iota$. Then given $u \in \mathsf{P}(\mathsf{N})$ (identifying elements of $\mathsf{P}(X \Rightarrow Y)$ with stable functions) we have

$$\begin{aligned}
[M](u) &= u_0[\mathsf{if}(x,\underline{1},z\cdot z)]_{x:\iota}(u) + \sum_{n=0}^{\infty} u_{n+1}[\mathsf{if}(z,\underline{0},d\cdot x)]_{x:\iota,z:\iota}(u,\mathsf{e}_n)\\
&= u_0(u_0\mathsf{e}_1 + \sum_{n=0}^{\infty} u_{n+1}\mathsf{e}_n) + \sum_{n=0}^{\infty} u_{n+1}((\mathsf{e}_n)_0\mathsf{e}_0 + \sum_{k=0}^{\infty}(\mathsf{e}_n)_{k+1}u)\\
&= u_0^2\mathsf{e}_1 + u_0\sum_{n=0}^{\infty} u_{n+1}\mathsf{e}_n + u_1\mathsf{e}_0 + (\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} u_{n+1}\delta_{n,k+1})u\\
&= u_0^2\mathsf{e}_1 + u_0\sum_{n=0}^{\infty} u_{n+1}\mathsf{e}_n + u_1\mathsf{e}_0 + (\sum_{n=0}^{\infty} u_{n+2})u\\
&= (u_1 + u_0u_1 + u_0\sum_{n=0}^{\infty} u_{n+2})\mathsf{e}_0 + (u_0^2 + u_0u_2 + u_1\sum_{n=0}^{\infty} u_{n+2})\mathsf{e}_1 + \sum_{k=2}^{\infty}(u_0u_{k+1} + u_k\sum_{n=0}^{\infty} u_{n+2})\mathsf{e}_k\,.
\end{aligned}$$

So for instance

$$\begin{aligned}
[M](\frac{1}{2}\mathsf{e}_0 + \frac{1}{6}\mathsf{e}_1 + \frac{1}{3}\mathsf{e}_2) &= (\frac{1}{6} + \frac{1}{12} + \frac{1}{6})\mathsf{e}_0 + (\frac{1}{4} + \frac{1}{6} + \frac{1}{18})\mathsf{e}_1 + \frac{1}{9}\mathsf{e}_2\\
&= \frac{5}{12}\mathsf{e}_0 + \frac{17}{36}\mathsf{e}_1 + \frac{1}{9}\mathsf{e}_2
\end{aligned}$$

8.5.3 PROBABILISTIC INTERSECTION TYPES. For the sake of completeness, we provide a presentation of the semantics of pPCF in the intersection typing style that we have used for the other models **Rel** and **Scott**. However the presentation is more complicated here due to infinite sums involved in interpreting the application operation, see the explicit computation (8.3).

We use relational typing contexts as defined in Section 7.2.4. Given such a typing context $\Phi = (x_1 : m_1 : A_1, \dots, x_k : m_l : A_k)$, a type $A$ and $a \in |[A]|$ and $r \in \mathbb{R}_{\geq 0}$, we write

$$\Phi \vdash_r M : a : A$$

to express that $[M]_{\underline{\Phi}} \geq r$. We give deduction rules for deriving this kind of quantitative semantic judgment.

Given a typing context $\Gamma = (x_1 : A_1, \dots, x_k : A_k)$, we use $\mathsf{SC}(\Gamma)$ for the set of all semantic context $\Phi$ such that $\underline{\Phi} = \Gamma$. We consider $\mathsf{SC}(\Gamma)$ as a commutative monoid, with neutral element $0_\Gamma$ and addition defined in Section 7.2.4 :

$$\begin{aligned} &0_\Gamma = (x_i : [] : A_i)_{i=1}^k \\ &(x_i : m_i : A_i)_{i=1}^k + (x_i : m'_i : A_i)_{i=1}^k = (x_i : m_i + m'_i : A_i)_{i=1}^k \,. \end{aligned}$$

We write $\Phi \leq \Phi'$ if there is a $\Psi$ such that $\Phi + \Psi = \Phi'$, and when this holds, this uniquely determined $\Psi$ is denoted $\Phi' - \Phi$. Given $l \in \mathbb{N}$ we use $\mathcal{S}^{\leq}(\Phi, l)$ for the set of all functions $f : \{1, \dots, l\} \to \mathsf{SC}(\underline{\Phi})$ such that $\Sigma(f) \leq \Phi$ where $\Sigma(f) = \sum_{j=1}^{l} f(j)$.

The first rule is probably not strictly necessary but it his harmless and convenient.

$$\frac{}{\Phi \vdash_0 M : a : A}$$

Now we give the rules corresponding to the various constructs of the language.

$$\frac{\Phi = (x_1 : [] : A_1, \dots, x_i : [a_i] : A_i, \dots, x_k : [] : A_k)}{\Phi \vdash_1 x_i : a_i : A_i} \qquad \frac{}{0_\Gamma \vdash_1 \underline{n} : n : \iota}$$

$$\frac{}{0_\Gamma \vdash_r \mathsf{rand}(r) : 0 : \iota} \qquad \frac{}{0_\Gamma \vdash_{1-r} \mathsf{rand}(r) : 1 : \iota}$$

$$\frac{\Phi \vdash_r M : n : \iota}{\Phi \vdash_r \underline{\mathsf{succ}}(M) : n+1 : \iota} \qquad \frac{\Phi, x : m : A \vdash_r M : b : B}{\Phi \vdash_r \lambda x^A\, M : (m, b) : A \to B}$$

The next rules are more difficult to describe. One reason for this complication is that we have now to take into account the fact that when we have several deductions leading to conclusion of a given shape (for instance $\Phi \vdash_r \mathsf{if}(M, N, x \cdot P) : a : A$), we must sum up the corresponding probabilities and not consider these possibilities independently as we did in the deduction system associated with **Rel** in Section 7.2.4.

Here is the simplest example. We can derive

$$\Phi \vdash_r \mathsf{if}(M, N, x \cdot P) : a : A$$

if $r \leq \sum_{n=0}^{\infty} r_n s_n$ with

$$\forall n \in \mathbb{N}\ \Phi^n \vdash_{r_n} M : \underline{n} : \iota, \quad \Psi^0 \vdash_{s_0} N : a : A \quad \text{and} \quad \forall n \in \mathbb{N}\ \Psi^{n+1} \vdash_{s_{n+1}} N\,[\underline{n}/x] : a : A$$

and, moreover

$$\forall n \in \mathbb{N} \quad \underline{\Phi^n} = \underline{\Psi^n} = \underline{\Phi} \text{ and } \Phi = \Phi^n + \Psi^n \,.$$

Now we deal with application, that is we want to implement Formula (8.3), or rather Formula (8.4) as a "deduction rule". Assume that $\Gamma \vdash M : A \to B$ and $\Gamma \vdash N : A$. Let $\Phi$ be such that $\underline{\Phi} = \Gamma$.

We can derive

$$\Phi \vdash_r (M)\, N : b : B$$

if we can find a set $H$ of pairs $h = (s^h, f^h)$ where

— $s^h = (a(h)_1, \dots, a(h)_{l(h)})$ is a sequence of elements of $|[A]|$

— and $f^h \in \mathcal{S}^{\leq}(\Phi, l(h))$
and a family $(r(h,j))_{(h,j)\in S}$ of elements of $\mathbb{R}_{\geq 0}$ indexed by $S = \sum_{h\in H}\{1,\dots,l(h)\}$ such that

$$\forall h \in H\, \forall j \in \{1,\dots,l(h)\} \quad f^h(j) \vdash_{r(h,j)} N : a_j : A$$

as well as a family $(r(h))_{h\in H}$ of elements of $\mathbb{R}_{\geq 0}$ such that

$$\forall h \in H \quad \Phi - \Sigma(f^h) \vdash_{r(h)} M : ([a_1,\dots,a_{l(h)}], b) : A \to B$$

and

$$r \leq \sum_{h\in H} \frac{[a(h)_1,\dots,a(h)_{l(h)}]!}{l(h)!} r(h) \prod_{j=1}^{l(h)} r(h,j)$$

Fixed points are dealt with similarly. We have

$$\Phi \vdash_r \mathsf{fix}(M) : a : A$$

if we can find a set $H$ of pairs $h = (s^h, f^h)$ where
— $s^h = (a(h)_1,\dots,a(h)_{l(h)})$ is a sequence of elements of $|[A]|$
— and $f^h \in \mathcal{S}^{\leq}(\Phi, l(h))$
and a family $(r(h,j))_{(h,j)\in S}$ of elements of $\mathbb{R}_{\geq 0}$ indexed by $S = \sum_{h\in H}\{1,\dots,l(h)\}$ such that

$$\forall h \in H\, \forall j \in \{1,\dots,l(h)\} \quad f^h(j) \vdash_{r(h,j)} \mathsf{fix}(M) : a_j : A$$

as well as a family $(r(h))_{h\in H}$ of elements of $\mathbb{R}_{\geq 0}$ such that

$$\forall h \in H \quad \Phi - \Sigma(f^h) \vdash_{r(h)} M : ([a_1,\dots,a_{l(h)}], a) : A \to A$$

and

$$r \leq \sum_{h\in H} \frac{[a(h)_1,\dots,a(h)_{l(h)}]!}{l(h)!} r(h) \prod_{j=1}^{l(h)} r(h,j)$$

## 8.6 Probabilistic coherence spaces with totality

Probabilistic coherence spaces admit partial elements, for instance, in $\mathsf{P}(\mathsf{N})$, we have all the $u \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} u_n \leq 1$, that is, the sub-probability distributions on $\mathbb{N}$. Only the element $u$ of $\mathsf{P}(\mathsf{N})$ such that $\sum_{n\in\mathbb{N}} u_n = 1$ correspond to true probability distributions. The very idea of totality is to single out generalizations to all types of such probability distributions.

Let $X$ be a PCS and $\mathcal{U} \subseteq \mathsf{P}(X)$, let

$$\mathcal{U}^\perp = \left\{u' \in \mathsf{P}(X^\perp) \mid \langle u, u'\rangle = 1\right\} .$$

Notice that if $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathsf{P}(X)$ then $\mathcal{V}^\perp \subseteq \mathcal{U}^\perp$ and that $\mathcal{U} \subseteq \mathcal{U}^{\perp\perp}$. It follows that $\mathcal{U}^{\perp\perp\perp} = \mathcal{U}^\perp$ for any $\mathcal{U} \subseteq \mathsf{P}(X)$.

A probabilistic coherence space with totality (PCST) is a pair $X = (\underline{X}, \mathcal{T}(X))$ where $\underline{X}$ is a PCS and $\mathcal{T}(X) \subseteq \mathsf{P}(\underline{X})$ satisfies $\mathcal{T}(X)^{\perp\perp} = \mathcal{T}(X)$.

**Proposition 8.6.1** *Let $X$ be a PCST. Then $\mathcal{T}(X)$ is convex*[2] *in the following sense : for any at most countable set $I$, and for any families $(u_i)_{i\in I} \in \mathcal{T}(X)^I$ and $(\lambda_i)_{i\in I} \in \mathbb{R}_{\geq 0}^I$ such that $\sum_{i\in I}\lambda_i = 1$, one has*

$$\sum_{i\in I} \lambda_i u_i \in \mathcal{T}(X) .$$

2. Actually there is also some closeness in this condition.

*Démonstration.* With these notations, we have, for any $u' \in \mathcal{T}(X)^{\perp}$

$$\left\langle \sum_{i\in I} \lambda_i u_i, u' \right\rangle = \sum_{i\in I} \lambda_i \langle u_i, u' \rangle = \sum_{i\in I} \lambda_i = 1$$

hence we have $\sum_{i\in I} \lambda_i u_i \in \mathcal{T}(X)^{\perp\perp} = \mathcal{T}(X)$. □

As suggested above an example is $\mathsf{N}$ where

$$\underline{\mathsf{N}} = \left(\mathbb{N}, \left\{u \in \mathbb{R}_{\geq 0}^{\mathbb{N}} \mid \sum_{n\in\mathbb{N}} u_n \leq 1\right\}\right)$$

$$\mathcal{T}(\mathsf{N}) = \left\{u \in \mathsf{P}(\underline{\mathsf{N}}) \mid \sum_{n\in\mathbb{N}} u_n = 1\right\}$$

so that $\mathcal{T}(\mathsf{N})^{\perp}$ has only one element, namely $\sum_{n=0}^{\infty} \mathsf{e}_n$ (the element $u' \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}\ u'_n = 1$) and indeed $\mathcal{T}(\mathsf{N})^{\perp\perp} = \mathcal{T}(\mathsf{N})$.

**Lemme 8.6.2** *Let $X$ be a PCST, $u \in \mathcal{T}(X)$ and $v \in \mathsf{P}(\underline{X})$ be such that $u \leq v$. Then $v \in \mathcal{T}(X)$.*

*Démonstration.* Let $u' \in \mathcal{T}(X)^{\perp}$, we have

$$1 = \langle u, u' \rangle \leq \langle v, u' \rangle \leq 1$$

and hence $\langle v, u' \rangle = 1$. □

**Remarque 8.6.3** The following properties are equivalent :
- $0 \in \mathcal{T}(X)$
- $\mathsf{P}(\underline{X}) = \mathcal{T}(X)$
- $\mathcal{T}(X)^{\perp} = \emptyset$.

We say that $X$ is non-degenerate if $0 \notin \mathcal{T}(X) \neq \emptyset$. Notice that the objects $\top = (\emptyset, \{0\}, \{0\})$ and $0 = \top^{\perp} = (\emptyset, \{0\}, \emptyset)$ are degenerate objects, they are the only objects of **Pcoht** with an empty web. Using them it is easy to build other degenerate objects such as $0 \,\&\, X$ where $X$ is a an arbitrary PCST.

8.6.1 Metric properties of totality. We refer to Section 8.3.2. Let $X$ be a PCST. Given $u' \in \mathcal{T}(X^{\perp})$, the map

$$\begin{aligned} \mathsf{ev}_{u'} : \widetilde{\mathsf{P}}(\underline{X}) &\to \mathbb{R} \\ u &\mapsto \langle u, u' \rangle \end{aligned}$$

is continuous (by definition of the norm of $\widetilde{\mathsf{P}}(X)$) and hence

$$\mathcal{T}(X) = \mathsf{P}(X) \cap \bigcap_{u' \in \mathcal{T}(X^{\perp})} \mathsf{ev}_{u'}{}^{-1}(\{1\})$$

is a closed subset of $\widetilde{\mathsf{P}}(X)$ and hence of $\mathsf{P}(X)$ which is closed in $\widetilde{\mathsf{P}}(X)$.

This means that $\mathcal{T}(X)$ is a closed subset of the complete metric space $(\mathsf{P}(X), \mathsf{d}_X)$, see Proposition 8.3.6.

8.6.2 Density and maximality of total elements. One could expect the elements of $\mathcal{T}(X)$ to be "large", typically to be maximal in $\mathsf{P}(\underline{X})$. The existence of degenerate objects $X$ such that $0 \in \mathcal{T}(X)$, and hence $\mathcal{T}(X) = \mathsf{P}(X)$, shows that this property does not hold in general. A natural question is whether the objects $X$ such that all elements of $\mathcal{T}(X)$ are maximal in $\mathsf{P}(\underline{X})$ are stable under the constructions of linear logic. Let us call the property that all total elements are maximal the *maximality property* for an object $X$ of **Pcoht**. It is related to the following *density property* : an object $X$ of **Pcoht** has the density property if for all $a \in |\underline{X}|$ there exists $u \in \mathcal{T}(X)$ such that $u_a \neq 0$.

**Lemme 8.6.4** *Let $X$ be a PCST. Then $X$ has the maximality property iff* $\forall u, v \in \mathcal{T}(X)\ \ u \leq v \Rightarrow u = v$.

*Démonstration.* This is an obvious consequence of Lemma 8.6.2. □

**Proposition 8.6.5** *If $X$ has the maximality and the density properties then so does $X^{\perp}$.*

*Démonstration.* Assume that $X$ has the two properties. Let $u', v' \in \mathcal{T}(X)^{\perp}$ be such that $u' \leq v'$, by Lemma 8.6.4 it suffices to prove that $u' = v'$ for concluding that $X^{\perp}$ has the maximality property. Let $a \in |X|$. We have $u'_a \leq v'_a$. Let $u \in \mathcal{T}(X)$ be such that $u_a > 0$. We have $\langle u, v' - u' \rangle = \langle u, v' \rangle - \langle u, u' \rangle = 1 - 1 = 0$ and $\langle u, v' - u' \rangle = (v'_a - u'_a)u_a + \sum_{b \in |X| \setminus \{a\}} u_b(v'_b - u'_b)$ and since all the summands are $\geq 0$ (as $u' \leq v'$), we must have in particular $(v'_a - u'_a)u_a = 0$ and hence $u'_a = v'_a$ since $u_a \neq 0$. This shows that $v' = u'$ and hence the elements of $\mathcal{T}(X)^{\perp}$ are maximal in $\mathsf{P}(\underline{X})$.

Assume now that $X^{\perp}$ has not the density property, so let $a \in |\underline{X}|$ be such that $\forall u' \in \mathcal{T}(X)^{\perp}\ u'_a = 0$. By the density property of $X$ there exists $v \in \mathcal{T}(X)$ such that $v_a > 0$. Let $u \in \mathbb{R}_{\geq 0}^{|\underline{X}|}$ be defined by

$$u_b = \begin{cases} v_b & \text{if } b \neq a \\ 0 & \text{if } b = a\,. \end{cases}$$

Since $u \leq v \in \mathsf{P}(\underline{X})$ we have $u \in \mathsf{P}(\underline{X})$. Let $u' \in \mathsf{P}(X^{\perp})$, we have

$$\begin{aligned} \langle u, u' \rangle &= u_a u'_a + \sum_{b \in |\underline{X}| \setminus \{a\}} v_b u'_b \\ &= \sum_{b \in |\underline{X}| \setminus \{a\}} v_b u'_b \quad \text{since } u'_a = 0 \\ &= \langle v, u' \rangle \quad \text{for the same reason} \\ &= 1 \quad \text{since } v \in \mathcal{T}(X)\,. \end{aligned}$$

This shows that $u \in \mathcal{T}(X)$. Since $u \leq v$ and $u \neq v$ this contradicts our maximality assumption about $X$. □

We will say that an object $X$ of **Pcoht** is regular if $|\underline{X}| \neq \emptyset$ and $X$ has the density and the maximality properties.

**Proposition 8.6.6** *If $X$ is regular and $u \in \mathcal{T}(X)$ then $\|u\|_{\underline{X}} = 1$.*

*Démonstration.* We know that $u$ is maximal in $\mathsf{P}(\underline{X})$ and $|\underline{X}| \neq \emptyset$ hence $u \neq 0$. Let $\lambda = \|u\|_{\underline{X}} > 0$. We have $\lambda^{-1}u \in \mathsf{P}(\underline{X})$ and $u \leq \lambda^{-1}u$ because $\lambda \leq 1$. If follows that $\lambda = 1$ by maximality of $u$. □

Notice that the converse is very far from being true. Take for instance $X = 1 \,\&\, 1$ (with web $\{1, 2\}$). Then $\|\mathsf{e}_1\| = 1$ but $\mathsf{e}_1 < \mathsf{e}_1 + \mathsf{e}_2$, this latter element being the only element of $\mathcal{T}(X)$ in that case.

**Remarque 8.6.7** Let $X$ be regular and let $u \in \mathsf{P}(\underline{X})$. Is it true that there exists $v \in \mathcal{T}(X)$ such that $u \leq v$? Although this property seems quite natural, we don't know at all if it holds in general. In some sense, density is a very weak form of this property of totality. A related question is the following : is it true that, for all $u \in \mathsf{P}(\underline{X})$, one has

$$\|u\|_{\underline{X}} = \sup\left\{\langle u, u' \rangle \mid u' \in \mathcal{T}(X)^{\perp}\right\}?$$

8.6.3 The *-autonomous category of probabilistic coherence spaces with totality. Of course we define $X^{\perp} = (\underline{X}^{\perp}, \mathcal{T}(X)^{\perp})$ so that $X^{\perp\perp} = X$.

Given PCST $X$ and $Y$, let $\mathcal{T}$ be the set of all $t \in \mathsf{P}(\underline{X} \multimap \underline{Y})$ such that

$$\forall u \in \mathcal{T}(X) \quad t \cdot u \in \mathcal{T}(Y)\,.$$

Then we have

$$\mathcal{T} = \left\{u \otimes v' \mid u \in \mathcal{T}(X) \text{ and } v' \in \mathcal{T}(Y)^{\perp}\right\}^{\perp}.$$

Indeed, given $t \in \mathsf{P}(\underline{X} \multimap \underline{Y})$ one has

$$\begin{aligned} t \in \mathcal{T} &\Leftrightarrow \forall u \in \mathcal{T}(X) \;\; t \cdot u \in \mathcal{T}(Y) \\ &\Leftrightarrow \forall u \in \mathcal{T}(X) \;\; t \cdot u \in \mathcal{T}(Y)^{\perp\perp} \\ &\Leftrightarrow \forall u \in \mathcal{T}(X) \; \forall v' \in \mathcal{T}(Y)^{\perp} \;\; \langle t \cdot u, v' \rangle = 1 \\ &\Leftrightarrow \forall u \in \mathcal{T}(X) \; \forall v' \in \mathcal{T}(Y)^{\perp} \;\; \langle t \cdot, u \otimes v' \rangle = 1 \,. \end{aligned}$$

Therefore we have $\mathcal{T}^{\perp\perp} = \mathcal{T}$ and hence we define a PCST by setting

$$\underline{X \multimap Y} = \underline{X} \multimap \underline{Y} \text{ and } \mathcal{T}(X \multimap Y) = \mathcal{T}$$

Notice that $\mathsf{Id}_{\underline{X}} \in \mathcal{T}(X \multimap X)$ and that, if $s \in \mathcal{T}(X \multimap Y)$ and $t \in \mathcal{T}(Y \multimap Z)$, one has $t\,s \in \mathcal{T}(X \multimap Z)$. As a consequence we can define a category **Pcoht** whose objects are the PCST and such that $\mathbf{Pcoht}(X,Y) = \mathcal{T}(X \multimap Y)$.

For instance, an element $t \in \mathbf{Pcoht}(\mathsf{N},\mathsf{N})$ is a matrix such that $\forall n \in \mathbb{N}\ t \cdot \mathsf{e}_n \in \mathcal{T}(\mathsf{N})$, that is $t$ is a stochastic $\mathbb{N} \times \mathbb{N}$-matrix.

**Lemme 8.6.8** *Let $X$ and $Y$ be PCST and $t \in \mathbf{Pcoht}(X,Y)$. Then $t^{\perp} \in \mathbf{Pcoht}(Y^{\perp}, X^{\perp})$.*

*Démonstration.* Let $v' \in \mathcal{T}(Y)^{\perp}$, we prove that $t^{\perp} \cdot v' \in \mathcal{T}(X)^{\perp}$. So let $u \in \mathcal{T}(X)$, we have $\langle t^{\perp} \cdot v', u \rangle = \langle t \cdot u, v' \rangle = 1$ by our assumptions on $t$, $u$ and $v'$. □

We define $X_1 \otimes X_2 = \left(X_1 \multimap {X_2}^{\perp}\right)^{\perp}$ and $1$ is the usual PCS $1$ together with the totality $\mathcal{T}(1) = \{1\}$.

**Lemme 8.6.9** *Let $X_1$, $X_2$ and $Y$ be PCST and let $t \in \mathbf{Pcoh}(\underline{X_1} \otimes \underline{X_2}, \underline{Y})$. The following conditions are equivalent.*

1. $t \in \mathbf{Pcoht}(X_1 \otimes X_2, Y)$
2. $\forall u_1 \in \mathcal{T}(X_1)\ \forall u_2 \in \mathcal{T}(X_2)\ \ t \cdot (u_1 \otimes u_2) \in \mathcal{T}(Y)$
3. $\mathsf{cur}(t) \in \mathbf{Pcoht}(X_1, X_2 \multimap Y)$.

*Démonstration.* Since $(\mathsf{cur}(t) \cdot u_1) \cdot u_1 = t \cdot (u_1 \otimes u_2)$ we have (1) ⇒ (3). For the same reason we have (3) ⇒ (2) so let us prove that (2) ⇒ (1). By Lemma 8.6.8 it suffices to prove that $t^{\perp} \in \mathbf{Pcoht}(Y^{\perp}, (X_1 \otimes X_2)^{\perp})$ so let $v' \in \mathcal{T}(Y)^{\perp}$. We prove that $t^{\perp} \cdot v' \in \mathcal{T}\left((X_1 \otimes X_2)^{\perp}\right) = \mathcal{T}\left(X_1 \multimap {X_2}^{\perp}\right)$ so let $u_1 \in \mathcal{T}(X_1)$, we prove that $\left(t^{\perp} \cdot v'\right) \cdot u_1 \in \mathcal{T}(X_2)^{\perp}$. So let finally $u_2 \in \mathcal{T}(X_2)$, we have

$$\begin{aligned} \left\langle \left(\left(t^{\perp} \cdot v'\right) \cdot u_1\right), u_2 \right\rangle &= \left\langle t^{\perp} \cdot v', u_1 \otimes u_2 \right\rangle \\ &= \langle t \cdot (u_1 \otimes u_2), v' \rangle \\ &= 1 \end{aligned}$$

since $t \cdot (u_1 \otimes u_2) \in \mathcal{T}(Y)$ by (2). □

As a consequence, given $t_i \in \mathbf{Pcoht}(X_i, Y_i)$ for $i = 1, 2$ we have $t_1 \otimes t_2 \in \mathbf{Pcoht}(X_1 \otimes X_2, Y_1 \otimes Y_2)$ : given $u_i \in \mathcal{T}(X_i)$ for $i = 1, 2$ it suffices to prove that $(t_1 \otimes t_2) \cdot (u_1 \otimes u_2) \in \mathcal{T}(Y_1 \otimes Y_2)$ which results from $(t_1 \otimes t_2) \cdot (u_1 \otimes u_2) = (t_1 \cdot u_1) \otimes (t_2 \cdot u_2)$. Hence $\otimes$ is a functor $\mathbf{Pcoht}^2 \to \mathbf{Pcoht}$. By Lemma 8.6.9 we also have

$$\begin{aligned} \lambda_X &\in \mathbf{Pcoht}(1 \otimes X, X) \\ \rho_X &\in \mathbf{Pcoht}(X \otimes 1, X) \\ \alpha_{X_1,X_2,X_3} &\in \mathbf{Pcoht}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3)) \\ \sigma_{X_1,X_2} &\in \mathbf{Pcoht}(X_1 \otimes X_2, X_2 \otimes X_1) \end{aligned}$$

and this shows that **Pcoht** is an SMC. Let us prove for instance that $\alpha_{X_1,X_2,X_3} \in \mathbf{Pcoht}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))$. It suffices to prove that $\mathsf{cur}\left(\alpha_{X_1,X_2,X_3}\right) \in \mathbf{Pcoht}(X_1 \otimes X_2, X_3 \multimap Y)$ where $Y = X_1 \otimes (X_2 \otimes X_3)$. We use again Lemma 8.6.9 : let $u_i \in \mathcal{T}(X_i)$ for $i = 1, 2$, it suffices to prove that

$$\mathsf{cur}\left(\alpha_{X_1,X_2,X_3}\right) \cdot (u_1 \otimes u_2) \in \mathcal{T}(X_3 \multimap Y)$$

so let $u_3 \in \mathcal{T}(X_3)$. We have

$$(\mathsf{cur}(\alpha_{X_1,X_2,X_3}) \cdot (u_1 \otimes u_2)) \cdot u_3 = u_1 \otimes (u_2 \otimes u_3) \in \mathcal{T}(X_1 \otimes (X_2 \otimes X_3)) \ .$$

Actually we have also shown that this SMC $(\mathbf{Pcoht}, 1, \otimes, \lambda, \rho, \alpha, \sigma)$ is closed with internal hom from $X$ to $Y$ the pair $(X \multimap Y, \mathsf{ev})$. Indeed given $t \in \mathcal{T}(X \multimap Y)$ and $u \in \mathcal{T}(X)$ we have $\mathsf{ev} \cdot (t \otimes u) = t \cdot u \in \mathcal{T}(Y)$ and hence $\mathsf{ev} \in \mathbf{Pcoht}((X \multimap Y) \otimes X, Y)$ by Lemma 8.6.9. And given $t \in \mathbf{Pcoht}(Z \otimes X, Y)$ we have $\mathsf{cur}(t) \in \mathbf{Pcoht}(Z, X \multimap Y)$ by the "easy part" of Lemma 8.6.9.

Let $\perp = 1$, the structure $(\mathbf{Pcoht}, 1, \otimes, \lambda, \rho, \alpha, \sigma, \perp)$ is a *-autonomous category. Coming back to the notations of the proof of Lemma 8.2.6, it suffices to observe that $\theta_X$ is an isomorphism from $X^\perp$ to $X \multimap \perp$ in $\mathbf{Pcoht}$.

**Proposition 8.6.10** *If $X$ and $Y$ are regular, so is $X \multimap Y$.*

*Démonstration.* First we obviously have $|\underline{X \multimap Y}| = |\underline{X}| \times |\underline{Y}| \neq \emptyset$. Let $s, t \in \mathcal{T}(X \multimap Y)$ be such that $s \leq t$ (we use Lemma 8.6.2). Let $(a, b) \in |\underline{X \multimap Y}|$. Let $u \in \mathcal{T}(X)$ be such that $u_a \neq 0$ and let $v' \in \mathcal{T}(Y)^\perp$ be such that $v'_b \neq 0$. We have $s \cdot u, t \cdot u \in \mathcal{T}(Y)$ and hence

$$\langle s, u \otimes v' \rangle = \langle t, u \otimes v' \rangle = 1$$

and hence by the same reasoning as in the proof of 8.6.5 we get $s_{a,b} = t_{a,b}$ : we have

$$0 = \langle t, u \otimes v' \rangle - \langle s, u \otimes v' \rangle = (t_{a,b} - s_{a,b}) u_a v'_b + \sum_{(a',b') \in |\underline{X}| \times |\underline{Y}| \setminus \{(a,b)\}} (t_{a',b'} - s_{a',b'}) u_{a'} v'_{b'}$$

and since all the summands are $\geq 0$ by our assumption that $s \leq t$, they must all vanish, and since $u_a v'_b \neq 0$, we are done. This proves that $X \multimap Y$ has the maximality property.

Now we prove that the density property holds in $X \multimap Y$ so let $(a, b) \in |\underline{X \multimap Y}|$. Let $u' \in \mathcal{T}(X)^\perp$ and $v \in \mathcal{T}(Y)$. We have $u' \otimes v \in \mathsf{P}(\underline{X \multimap Y})$ since

$$(u' \otimes v) \cdot u = \langle u, u' \rangle \, v \in \mathsf{P}(Y)$$

since $\langle u, u' \rangle \leq 1$. By the same property we have $u' \otimes v \in \mathcal{T}(X \multimap Y)$. And since $(u' \otimes v)_{a,b} = u'_a v_b \neq 0$ this proves that the density property holds in $X \multimap Y$. ☐

It follows by 8.6.5 that $X_1 \otimes X_2$ is regular as soon as $X_1$ and $X_2$ are regular. Observe also that 1 is regular.

8.6.4 Additive structure. Let $(X_i)_{i \in I}$ be an at most countable family of PCST. We define $X = \&_{i \in I} X_i$ by

$$\underline{X} = \&_{i \in I} \underline{X_i}$$
$$\mathcal{T}(X) = \{v \in \mathsf{P}(\underline{X}) \mid \forall i \in I \ \pi_j \cdot v \in \mathcal{T}(X_i)\} \ .$$

Indeed, we have

$$\mathcal{T}(X) = \left\{v \in \mathsf{P}(\underline{X}) \mid \forall i \in I \, \forall u' \in \mathcal{T}(X_i)^\perp \ \langle v, \overline{\pi}_j \cdot u' \rangle = 1\right\}^\perp .$$

This shows that $\mathcal{T}(X) = \mathcal{T}(X)^{\perp\perp}$. By definition we have $\forall i \in I \ \pi_i \in \mathbf{Pcoht}(X, X_i)$ and if $t_i \in \mathbf{Pcoht}(Y, X_i)$ for each $i \in I$ we have $\langle t_i \rangle_{i \in I}$.

**Lemme 8.6.11** *The pair $(\&_{i \in I} X_i, (\pi_i)_{i \in I})$ is the cartesian product of the $X_i$'s in* **Pcoht**.

**Lemme 8.6.12** *If $(X_i)_{i \in I}$ is an at most countable family of regular objects of* **Pcoht** *with $I \neq \emptyset$ then the object $\&_{i \in I} X_i$ is regular.*

*Démonstration.* Maximality is obvious since, for $u \in \mathcal{T}\left(\&_{i\in I} X_i\right)$, each $\pi_i \cdot u$ is a maximal element of $\mathsf{P}(\underline{X_i})$ by our assumption that the $X_i$'s are regular. We prove density, so let $i \in I$ and let $a \in |\underline{X_i}|$. By regularity of $X_i$ we can find an element $u_i \in \mathcal{T}(X_i)$ such that $(u_i)_a \neq 0$. For $j \neq i$, we know that $X_j$ is regular and hence $\mathcal{T}(X_j) \neq \emptyset$ so we pick some $u_j \in \mathcal{T}(X_j)$. Then $u = \cup_{j\in I}\{j\} \times u_j \in \mathcal{T}\left(\&_{j\in I} X_j\right)$ satisfies $u_{(i,a)} \neq 0$ proving our contention. □

So **Pcoht** is cartesian. The terminal object is $(\top, \{0\})$. Since **Pcoht** is *-autonomous, it is also cocartesian with coproduct

$$\bigoplus_{i\in I} X_i = \left(\left(\&_{i\in I} \underline{X_i}^{\perp}\right)^{\perp}, (\overline{\pi}_i)_{i\in I}\right)$$

The following is actually a conjecture which seems very likely.

**Proposition 8.6.13** *Let $I \neq \emptyset$ be at most countable and le the $X_i$'s be regular. The elements of $\mathcal{T}\left(\bigoplus_{i\in I} X_i\right)$ are the linear combinations*

$$\sum_{i\in I} \lambda_i\,(\overline{\pi}_i \cdot v_i)$$

*where $\forall i \in I\ v_i \in \mathcal{T}(X_i)$ and the $\lambda_i \in \mathbb{R}_{\geq 0}$ satisfy $\sum_{i\in I} \lambda_i = 1$.*

*Démonstration.* Given a family $(v_i)_{i\in I}$ with $v_i \in \mathcal{T}(X_i)$ and $(\lambda_i)_{i\in I} \in \mathbb{R}^I_{\geq 0}$ with $\sum_{i\in I}\lambda_i = 1$ we have $\forall i \in I\ \overline{\pi}_i \cdot v_i \in \mathcal{T}\left(\bigoplus_{i\in I} X_i\right)$ and hence $\sum_{i\in I}\lambda_i\,(\overline{\pi}_i \cdot v_i) \in \mathcal{T}\left(\bigoplus_{i\in I} X_i\right)$ by Lemma 8.6.1.

Conversely let $v \in \mathcal{T}\left(\bigoplus_{i\in I} X_i\right)$. We have $u_i = \pi_i \cdot v \in \mathsf{P}(X_i)$ for each $i \in I$ and $\sum_{i\in I}\|u_i\|_{\underline{X_i}} \leq 1$.

Let $j \in I$. Let $u' \in \mathcal{T}(X_j)^{\perp}$. Then since $v \in \mathcal{T}\left(\bigoplus_{i\in I} X_i\right)$ we know that for any family $\vec{u'} = (u'_i)_{i\in I\setminus\{j\}}$ with $\forall i \in I \setminus \{j\}\ \ u'_i \in \mathcal{T}(X_i)^{\perp}$

$$\langle u_j, u'\rangle + \sum_{i\in I\setminus\{j\}} \langle u_i, u'_i\rangle = 1\,.$$

This equation and these universal quantifications show that

- — the value of $\langle u_j, u'\rangle$ does not depend on the choice of $u'$
- — and the value of $\sum_{i\in I\setminus\{j\}}\langle u_i, u'_i\rangle$ does not depend on the choice of $\vec{u'}$.

Let $\lambda_j \in \mathbb{R}_{\geq 0}$ be the unique element of $\left\{\langle u_j, u'\rangle \mid u' \in \mathcal{T}(X)^{\perp}\right\}$. We have $\lambda_j \leq \|u_j\|_{\underline{X_j}}$ and if $\lambda_j = 0$ then $u_j = 0$ (take $a \in |\underline{X}|$ such that $(u_j)_a \neq 0$; there is $u' \in \mathcal{T}(X)^{\perp}$ such that $u'_a$ by density in $X_j{}^{\perp}$, then we have $\langle u_j, u'\rangle \geq (u_j)_a u'_a > 0$).

We have $\sum_{j\in I}\lambda_j = 1$ and to end the proof it would be sufficient to prove that $\lambda_j = \|u_j\|_{\underline{X_j}}$.

**To be completed.** □

8.6.5 The exponential. We define $!X = (!\underline{X}, \{u^{(!)} \mid u \in \mathcal{T}(X)\}^{\perp\perp})$.

**Lemme 8.6.14** *We have $t \in \mathbf{Pcoht}(!X, Y)$ iff $\forall u \in \mathcal{T}(X)\ \ t \cdot u^{(!)} \in \mathcal{T}(Y)$.*

*Démonstration.* The condition is obviously necessary, let us assume that it holds and prove that $t \in \mathbf{Pcoht}(!X, Y)$. For this it is sufficient to prove that $t^{\perp} \in \mathcal{T}\left(Y^{\perp} \multimap (!X)^{\perp}\right)$. Let $v' \in \mathcal{T}(Y)^{\perp}$. For all $u \in \mathcal{T}(X)$ we have

$$\left\langle t^{\perp} \cdot v', u^{(!)}\right\rangle = \left\langle t \cdot u^{(!)}, v'\right\rangle = 1$$

by our assumption. □

**Proposition 8.6.15** *If $X$ is regular, so is $!X$*

*Démonstration.* It suffices to prove that $!X \multimap 1$ is regular. Let first $s, t \in \mathcal{T}(!X \multimap 1)$ be such that $s \leq t$ (we use Lemma 8.6.2). Let $m = [a_1, \dots, a_k] \in \mathcal{M}_{\text{fin}}(|\underline{X}|)$, we know that $s_{m,*} \leq t_{m,*}$. By density we can find $u_1, \dots, u_k \in \mathcal{T}(X)$ such that $(u_i)_{a_i} \neq 0$ for $i = 1, \dots, k$. If $k \neq 0$ let

$$u = \frac{1}{k} \sum_{i=1}^{k} u_i$$

we have $u \in \mathcal{T}(X)$ and $u_{a_i} \neq 0$ for each $i$ by Lemma 8.6.1. If $m = []$, we take for $u$ any element of $\mathcal{T}(X)$ (which is non-empty by regularity). We have $u_m^{(!)} \neq 0$ and since $u$ is total we have

$$s \cdot u^{(!)} = t \cdot u^{(!)} = 1 .$$

It follows as in the proof of Proposition 8.6.5 that $s_{m,*} = t_{m,*}$ which proves the maximality property.

Now we prove density. Let $m = [a_1, \dots, a_k] \in \mathcal{M}_{\text{fin}}(|\underline{X}|)$. For each $i = 1, \dots, k$ we choose $u'(i) \in \mathcal{T}\left(X^{\perp}\right)$ such that $u'(i)_{a_i} \neq 0$ for $i = 1, \dots, k$. Let $f : \mathsf{P}(\underline{X}) \to \mathbb{R}_{\geq 0}$ be defined by

$$f(u) = \prod_{i=1}^{k} \langle u, u'(i) \rangle .$$

Obviously $\forall u \in \mathsf{P}(\underline{X})\ f(u) \in [0,1]$ and $\forall u \in \mathcal{T}(X)\ \ f(u) = 1$. We have

$$\begin{aligned} f(u) &= \prod_{i=1}^{k} \sum_{a \in |\underline{X}|} u_a u'(i)_a \\ &= \sum_{\varphi : \{1,\dots,k\} \to |\underline{X}|} \prod_{i=1}^{k} u_{\varphi(i)} u'(i)_{\varphi(i)} \\ &= \sum_{q \in \mathcal{M}_{\text{fin}}(|\underline{X}|)} \alpha_q u^q \end{aligned}$$

where, for any $q \in \mathcal{M}_{\text{fin}}(|\underline{X}|)$, $\alpha_q \in \mathbb{R}_{\geq 0}$ is equal to 0 if $\# q \neq k$ and, otherwise

$$\alpha_q = \sum_{\substack{\varphi : \{1,\dots,k\} \to |\underline{X}| \\ [\varphi(1),\dots,\varphi(k)] = q}} \prod_{i=1}^{k} u'(i)_{\varphi(i)} .$$

Let $t \in \mathbb{R}_{\geq 0}^{|\underline{!X \multimap 1}|}$ be given by $t_{q,*} = \alpha_q$. We have $\forall u \in \mathsf{P}(\underline{X})\ t \cdot u^{(!)} = f(u) \in [0,1]$ and hence $t \in \mathsf{P}(\underline{!X \multimap 1})$. Moreover $\forall u \in \mathcal{T}(X)\ \ t \cdot u^{(!)} = 1$ and hence $t \in \mathcal{T}(!X \multimap 1)$.

By our assumption about the $u'(i)$ for $i = 1, \dots, k$, we have $\alpha_m \neq 0$, that is $t_{m,*} \neq 0$. This proves that $!X \multimap 1$ has the density property, and hence is regular. This also shows that $!X$ is regular since it is isomorphic to $(!X \multimap 1)^{\perp}$. □

**Lemme 8.6.16** *Let $X_1$, $X_2$ and $Y$ be objects of* **Pcoht**. *Let $t \in \mathbb{R}_{\geq 0}^{|(!X_1 \otimes !X_2) \multimap Y|}$. One has $t \in$* **Pcoht**$((!X_1 \otimes !X_2), Y)$ *iff*

$$\forall u_1 \in \mathcal{T}(X_1)\ \forall u_2 \in \mathcal{T}(X_2) \quad t \cdot (u_1^{(!)} \otimes u_2^{(!)}) \in \mathcal{T}(Y) .$$

*Démonstration.* The condition being clearly necessary, one assumes that it holds. It suffices to prove that $\mathsf{cur}(t) \in \mathbf{Pcoht}(!X_1, !X_2 \multimap Y)$, which is done by using twice Lemma 8.6.14. □

**Proposition 8.6.17** *For any object $X$ de* **Pcoht** *we have*

$$\mathsf{der}_{\underline{X}} \in \mathbf{Pcoht}(!X, X) \quad \textit{and} \quad \mathsf{dig}_{\underline{X}} \in \mathbf{Pcoht}(!X, !!X) .$$

*Moreover $\mathsf{m}^0 \in \mathbf{Pcoht}(1, !\top)$ and for any two objects $X_1$ and $X_2$ of* **Pcoht**, *one has*

$$\mathsf{m}^2_{\underline{X_1}, \underline{X_2}} \in \mathbf{Pcoht}(!X_1 \otimes !X_2, !(X_1 \,\&\, X_2)) .$$

# Chapitre 9

# Complement

## 9.1 Topology

9.1.1 Metric spaces.

**Lemme 9.1.1** *Let $(X, d)$ be a metric space and $f : X \to \mathbb{R}$ be 1-Lipschitz. Let $(x_n)_{n\in\mathbb{N}}$ be Cauchy in $X$, so that $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, and let $\lambda \in \mathbb{R}$ be the limit of $(f(x_n))_{n\in\mathbb{N}}$. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $\forall n, k \geq N\ d(x_n, x_k) \leq \varepsilon$. Then $\forall n \geq N\ \ |f(x_n) - \lambda| \leq \varepsilon$.*

*Démonstration.* Let $\varepsilon' > 0$. Let $n \geq N$. Let $k \geq N$ be such that $|f(x_k) - \lambda| \leq \varepsilon'$. We have

$$\begin{aligned} |f(x_n) - \lambda| &\leq |f(x_n) - f(x_k)| + |f(x_k) - \lambda| \\ &\leq d(x_n, x_k) + \varepsilon' \quad \text{by the Lipschitz property of } f \\ &\leq \varepsilon + \varepsilon' \end{aligned}$$

and since this can be done for any $\varepsilon'$ the contention is proven. □

9.1.2 Dual of a Banach space. If $E$ is a normed vector space, we use $\mathsf{d}_E$ for the associated distance $\mathsf{d}_E(x, y) = \|x - y\|_E$.

Let $E$ be a Banach space. The (topological) dual $E'$ is the normed vector space of all linear and continuous functions $f : E \to \mathbb{R}$, equipped with the norm

$$\|x'\| = \sup_{x\in\mathcal{B}_1(E)} |\langle x, x'\rangle|$$

We prove that $E'$ is a Banach space. It is obvious that $\|\_\|$ is a norm, let us check the triangular inequality, so let $x', y' \in E'$, we have

$$\begin{aligned} \|x' + y'\| &= \sup_{x\in\mathcal{B}_1(E)} |\langle x, x' + y'\rangle| \\ &\leq \sup_{x\in\mathcal{B}_1(E)} (|\langle x, x'\rangle| + |\langle x, y'\rangle|) \\ &\leq \|x'\| + \|y'\| \ . \end{aligned}$$

Now we prove completeness. So let $(x'(n))_{n\in\mathbb{N}}$ be a Cauchy sequence in $E'$. We define a function $x' : E \to \mathbb{R}$.

Let $x \in E$, let $\varepsilon > 0$ be such that $\varepsilon x \in \mathcal{B}_1(E)$. Given any $y' \in E'$ we have $|\langle \varepsilon x, y'\rangle| \leq \|y'\|$, hence the sequence $(\langle \varepsilon x, x'(n)\rangle)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ with a limit $\lambda$, we set

$$x'(x) = \frac{\lambda}{\varepsilon}\,.$$

First, this value does not depend on $\varepsilon$ : if $\varepsilon' > 0$ also satisfies $\varepsilon' x \in \mathcal{B}_1(E)$ then $\langle \varepsilon' x, x'(n)\rangle = \frac{\varepsilon'}{\varepsilon} \langle \varepsilon x, x'(n)\rangle$ goes to $\frac{\varepsilon'}{\varepsilon}\lambda$ when $n \to \infty$, and we have $\frac{1}{\varepsilon'}\frac{\varepsilon'}{\varepsilon}\lambda = \frac{\lambda}{\varepsilon} = x'(x)$. By the same reasoning we see that $x'(\alpha x) =$

$\alpha x'(x)$. Now let $x_1, x_2 \in E$ and let us prove that $x'(x_1 + x_2) = x'(x_1) + x'(x_2)$. Wlog. we can assume that $x_1, x_2, x_1 + x_2 \in \mathcal{B}_1()$. Let $\lambda_i = x'(x_i)$ be the limit of $x'(n)(x_i)$ for $i = 1, 2$, by linearity we have that $x'(n)(x_1 + x_2) = x'(n)(x_1) + x'(n)(x_2) \to \lambda_1 + \lambda_2$.

Being Cauchy, the sequence $(x'(n))_{n\in\mathbb{N}}$ is clearly bounded, let $\alpha \in \mathbb{R}_{\geq 0}$ be such that $\forall n \in \mathbb{N}\ \|x'(n)\| \leq \alpha$. Then $\forall x \in \mathcal{B}_1(E)\ \|x'(n)(x)\| \leq \alpha$ and it follows that $\|x'(x)\| \leq \alpha$ for all $x \in \mathcal{B}_1(E)$ which proves that $x'$ is bounded and hence continuous since it is linear.

We prove that $(x'(n))_{n\in\mathbb{N}}$ converges to $x'$ in $E'$. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that

$$\forall n, k \geq N \quad \mathsf{d}_{E'}(x'(n), x'(k)) = \|x'(n) - x'(k)\|_{E'} \leq \varepsilon\,.$$

Let $x \in \mathcal{B}_1(E)$, the map $\mathsf{ev}_{:} E' \to \mathbb{R}$ given by $\mathsf{ev}_x(x') = \langle x, x' \rangle$ is 1-Lipschitz by definition of $\|\_\|_{E'}$ and hence by Lemma 9.1.1 we have

$$\forall n \geq N \quad |\langle x, x'(n)\rangle - \langle x, x'\rangle| \leq \varepsilon$$

and since this holds for all $x \in \mathcal{B}_1(E)$ we have

$$\forall n \geq N \quad \|x'(n) - x'\|\, E' \leq \varepsilon\,.$$

□

**To be completed.**

# Index

# Bibliographie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