On phase semantics and denotational semantics in multiplicative-additive linear logic

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Abstract

We study the notion of logical relation in the coherence space semantics of multiplicative-additive linear logic MALL. We show that, when the ground-type logical relation is "closed under restrictions", the logical relation associated to any type can be seen as a map associating facts of a phase space to families of points of the web of the corresponding coherence space. We introduce a sequent calculus extension of MALL whose formulae denote these families of points. This logic MALL(I) (where I is a set of indexes) admits a truth-value semantics in the previously mentioned phase space, and this truth-value semantics faithfully describes the logical relation model we started from. Then we generalize this notion of phase space, we prove a truth-value completeness result for MALL(I) and we derive from any phase model of MALL(I) a denotational model for MALL. Using the truth-value completeness result, we obtain a weak denotational completeness result based on this new denotational semantics.

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Introduction

Logical relations are a tool introduced by Plotkin (see [Mit90] for a survey) for the purpose of refining the denotational semantics of functional programming languages such as PCF. It corresponds to a standard tool for proving syntactical results in proof theory, the Tait's *reducibility* method.

Among other applications, logical relations have been used for defining *fully complete* denotational models of PCF. A model of PCF is said to be fully complete when any element of the interpretation of any type is the denotation of a PCF term of the corresponding type. Fully complete models, when they are defined in a syntax-free way, provide mathematically appealing tools for reasoning on syntax. Using logical relations, Sieber in [Sie92] obtained a full completeness result for some low types of PCF, and O'Hearn and Riecke in [OR95] obtained a full completeness theorem using some generalization of logical relations (Kripke logical relations).

As this method has proved powerful in the intuitionistic setting of PCF, we investigate here its extension to linear logic, and more precisely, for the time being, to the exponential-free fragment of linear logic, aiming at some kind of full completeness, if possible. Our study is based on the coherence space semantics of linear logic of [Gir87]. As we are dealing with first order propositional logic, we have no atomic formulae but the two multiplicative constants 1 and $\bot$ and the two additive constants, 0 and $\top$. Within this framework, the expressive power of coherence space semantics seems to be too low for reaching full completeness, even if logical relations are at hand. So, following [Gir99a] (in a special case) we consider a generalization of coherence space semantics where cliques are replaced by maps from cliques to a fixed commutative monoid P (such a map is called a P-clique). For practical reasons, we prefer to see a P-clique as a total map from the web to P_0, where P_0 is obtained by adding to P a new element 0 (we extend the monoid structure of P to P_0 by requiring 0 to be the absorbing element), the elements mapped to 0 being those which do not belong to the underlying clique.

If I is a given set of indices, we consider coherence spaces X equipped with a set of I-indexed families of P-cliques, which is intended to be a logical relation on X. As we want the linear negation to be involutive, we require this set of families to satisfy a closure property defined in terms of an orthogonality relation between I-indexed families of P-cliques of X and I-indexed families of P-cliques of $X^\perp$. This notion of orthogonality itself is based on the choice of the relation for the space $\bot$, which is arbitrary, and completely defines the logical relation model[1]: given X and Y coherence spaces endowed with such relations, there is a canonical way of endowing $X \otimes Y$, $X \parr Y$, $X \oplus Y$ and $X \mathbin{\&} Y$ with such a relation. In particular, the relation on $X \multimap Y$, which can be seen as a space of functions, is defined in the usual applicative way, and the relation on $X \mathbin{\&} Y$ (cartesian product) is defined componentwise, like in the semantics of λ-calculus. One can check that the model obtained in that way is just a particular case of the semantics proposed in [Gir99a]. This verification is not included in the present paper as it does not seem to enlighten the following steps of our constructions.

Observe that the logical relation on $\bot$ (which, as a coherence space, has only one element in its web) is nothing but a set of elements of P_0^I. So our logical relation model is defined by the monoid P and a subset $\bot_I$ of P_0^I which itself has a canonical structure of monoid (as the I-product of P_0). These are the plain ingredients of the phase semantics of linear logic [Gir87].

We show that, under the assumption that $\bot_I$ satisfies a natural condition (closure under restriction), all the operations on relations performed in the category of coherence spaces endowed with relations can be described in terms of operations on facts in the phase model $(P_0^I, \bot_I)$. More precisely, we show that a logical relation on a coherence space can be viewed as a map associating to any I-indexed family of the web of this coherence space a fact of $(P_0^I, \bot_I)$ and satisfying a uniformity property. This shift of viewpoint is similar to the introduction of hypercoherences as particular qualitative domains with coherence in [Ehr93] and is discussed more thoroughly in the introductory part of section 3. We observe moreover that, when a connective of MALL is interpreted, the corresponding operation on these maps is similar to the operation performed for this connector in the phase semantics of MALL: this correspondence is exact for the multiplicative constants and connectives, but for the additives, new operations on facts are introduced, which intensively use the indexed structure of the phase model.

This leads to the following idea. Consider a formula F of MALL and a subset J of I, and let

[1] This space $\bot$ plays the same role as the "ground type" (typically, the type of natural numbers, or the type of booleans) in the definition of logical relations for (enriched) simply typed lambda-calculi such as PCF: remember that the relation has only to be defined at ground type, and is then automatically "lifted" to all types.

X be the coherence space interpreting F in the coherence space[2] semantics. Let a be a J-indexed family of $|X|$. Then we can associate to a a formula $F\langle a\rangle$ (depending also on F) in an extension $\mathsf{MALL}(I)$ of MALL where each formula has a "domain" which is a subset of I (of course, the domain of $F\langle a\rangle$ is J). Then we give a sequent calculus for this system $\mathsf{MALL}(I)$, which turns out to be a conservative extension of the system MALL (from this viewpoint, MALL is the fragment of $\mathsf{MALL}(I)$ consisting of all the formulae of empty domain, corresponding to empty families of points of the web). We observe then that this logical system $\mathsf{MALL}(I)$ admits a phase semantics in $(P_0^I, \perp_I)$ such that the fact associated to a in the logical relation semantics described above coincides with the fact associated to $F\langle a\rangle$ in the phase space $(P_0^I, \perp_I)$. We also prove that the formula $F\langle a\rangle$ is provable in $\mathsf{MALL}(I)$ iff the family of points a is contained in the denotation of a proof of F in MALL (we refer below to this property as to the *basic property of* $\mathsf{MALL}(I)$). So we may hope to be able, from a complete model of $\mathsf{MALL}(I)$, to build a denotationally complete model of MALL. But unfortunately, there is no complete model of $\mathsf{MALL}(I)$ of the shape $(P_0^I, \perp_I)$.

This particular class of models is sound for a strict extension $\mathsf{MALL}_\Omega(I)$ of $\mathsf{MALL}(I)$, where any sequent of empty domain is taken as an axiom. This corresponds to introducing partiality in the syntax itself, a common practice in λ-calculus. This extension is strictly stronger than $\mathsf{MALL}(I)$, as a sequent of empty domain is provable in $\mathsf{MALL}(I)$ if, and only if, the corresponding sequent of MALL (obtained by simply forgetting all domain informations) is provable (in MALL). But we also show that this particular class of model is not complete for $\mathsf{MALL}_\Omega(I)$. So we are led to considering a wider class of phase models of $\mathsf{MALL}(I)$, and for this, we are guided by the syntax of $\mathsf{MALL}(I)$. Such a model is just a standard phase model, together with a $\mathcal{P}(I)$-indexed family of idempotent elements. We prove a truth-value completeness result for $\mathsf{MALL}(I)$ in that phase semantics.

Last, coming back to our original denotational motivations, we show how such a phase model can be used for defining a denotational model of MALL. In particular, when the phase model is complete for $\mathsf{MALL}(I)$, it gives rise to a denotational model which is "weakly" denotationally complete: we can characterize in the coherence space model the sets of points which are contained in the interpretation of some proof of MALL. Thus, on one hand the notion of logical relation boils down to phase semantics for an appropriate logic, while the subdefinability property which we want to characterize boils down to provability for the same logic. Hence the weak denotational completeness problem "boils down" to the completeness of the phase semantics for this logic.

The logical system $\mathsf{MALL}(I)$ can also be considered as a new kind of *logic of domains*, very much in the spirit of Abramsky's theory of domains in logical form (see [Abr91], and also [AC98]). This theory, which is based on Coppo-Dezzani's *intersection types discipline* and its connection to denotational semantics (see [CDHL84, Kri90]), develops the idea of considering the (compact) elements of denotational models of the λ-calculus as *formulae*, or as types of a typing system for the λ-calculus (and in that case, a term has a given type if and only if its denotation is greater than the compact element of the model corresponding to this type: compare with the above *basic property of* $\mathsf{MALL}(I)$). These formulae express operational properties of the terms they belong to in the model. In $\mathsf{MALL}(I)$, we have a similar situation since a formula denotes a family of points of the web of the underlying MALL formula. Following strictly the logic of domains philosophy, formulae should correspond to *sets* of points of webs rather than to families thereof. This apparently minor change seems to explain the good logical properties of $\mathsf{MALL}(I)$ (sequent calculus, phase semantics). The precise operational meaning of the $\mathsf{MALL}(I)$ formulae has still to be understood.

Various aspects of the material we develop in this paper have some analogues in previous works

[2] Actually, the coherence relation does not play any role here, the only relevant information is the structure of the web.

by several authors. The deepest connections are to be found with the work of Girard on denotational completeness [Gir99a]. The idea of dealing with *families* of points of the webs rather than with *sets* was present in a generalization of hypercoherences by Winskel [Win94]. The idea of generalizing coherence models by associating "truth values" in a quantale to sets (instead of families) of points of the webs was present in the work of Lamarche [Lam95], but his interpretation of the additives is different from ours.

We assume from the reader some acquaintance with phase semantics and denotational semantics of linear logic. Many references are available on these topics, we mention in particular [Gir87, GLT89, Gir95, AC98].

1 Preliminaries: coherence semantics

Let us start by reminding some basic definitions on coherence spaces. Our purpose here is just to make precise our terminology and notations.

We remind that a coherence space X is a finite or denumerable set $|X|$ (the web) equipped with a binary reflexive and symmetric relation (the coherence relation): we write $a \frown_X b$ when the elements a and b of $|X|$ are coherent. A *clique* of X is a subset x of $|X|$ whose elements are all pairwise coherent:

$$\forall a, b \in x \quad a \frown_X b$$

One denotes by $\mathrm{Cl}(X)$ the set of all cliques of x.

In this paper, we are concerned with the fragment MALL of linear logic consisting of the following formulae:

- The constants 1 and $\bot$ (multiplicative constants).
- The constants 0 and $\top$ (additive constants).
- If A and B are formulae of MALL so are $A \otimes B$ (tensor or times) and $A \mathbin{⅋} B$ (par): these are the two multiplicative connectives of linear logic.
- If A and B are formulae of MALL so are $A \oplus B$ (plus) and $A \mathbin{\&} B$ (with): these are the two additive connectives of linear logic.

If A is a formula of MALL, the formula $A^\perp$ (linear negation) is defined inductively, using the De Morgan law which exchanges the two connectives of each of the four kinds of connectives enumerated above.

Each of these formulae can be interpreted as a coherence space. More precisely, we give for each of the eight connectives above a corresponding operation on coherence spaces (with the same notation).

- The coherence spaces 1 and $\bot$ are identical: they are the unique (up to isomorphism) coherence space whose web is a singleton $\{*\}$.
- The coherence spaces 0 and $\top$ are identical: they are the unique coherence space whose web is empty.
- If X is a coherence space, $X^\perp$ has the same web as X, and two elements of $|X|$ are coherent in $X^\perp$ if they are equal or not coherent in X.

- If X and Y are coherence spaces, the coherence spaces $X \otimes Y$ and $X \parr Y$ have $|X| \times |Y|$ has web. Two elements (a, b) and (a', b') of this web are coherent in $X \otimes Y$ if $a \coh_X a'$ and $b \coh_Y b'$. Using the De Morgan law and the definition above for $X^\perp$, one derives easily the coherence relation for $X \parr Y$. One defines the linear implication of X and Y by $X \multimap Y = X^\perp \parr Y$.
- If X and Y are coherence spaces, the coherence spaces $X \oplus Y$ and $X \& Y$ have $|X| + |Y| = \{1\} \times |X| \cup \{2\} \times |Y|$ as web. Two elements (i, c) and (j, d) of this web are coherent in $X \oplus Y$ if $i = j$ and c and d are coherent in the corresponding component of the sum. Using the De Morgan law and the definition above for $X^\perp$, one derives easily the coherence relation for $X \& Y$.

If X and Y are coherence spaces, there is a canonical bijection between $\mathrm{Cl}(X \& Y)$ and $\mathrm{Cl}(X) \times \mathrm{Cl}(Y)$. And there is a canonical bijection between $\mathrm{Cl}(X \multimap Y)$ and the functions f from $\mathrm{Cl}(X)$ to $\mathrm{Cl}(Y)$, which are *linear* in the sense that for any $V \subset \mathrm{Cl}(X)$ such that $\bigcup V \in \mathrm{Cl}(X)$, $f(\bigcup V) = \bigcup f(V)$ and *stable* in the sense that, for $x, y \in \mathrm{Cl}(X)$ such that $x \cup y \in \mathrm{Cl}(X)$, one has $f(x \cap y) = f(x) \cap f(y)$.

If A is a formula of MALL, let us denote by A^* the coherence space defined by using inductively the constructions above on coherence spaces. If $\vdash \Gamma$ is a sequent of MALL (that is, Γ is a sequence $\Gamma_1, \ldots, \Gamma_n$ of formulae of MALL), its associated coherence space is $\Gamma^* = {\Gamma_1}^* \parr \ldots \parr {\Gamma_n}^*$.

To any proof π in MALL of a sequent $\vdash \Gamma$, one associates a clique π^* of the coherence space Γ^*. This clique is defined by induction on the proof π as follows.

- If π is the axiom

$$\frac{}{\vdash 1}$$

  then $\pi^* = \{*\}$.
- If π is the axiom

$$\frac{}{\vdash \Gamma, \top}$$

  then $\pi^* = \emptyset$.
- If π is the proof

$$\frac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, \bot}$$

  then $\pi^* = \{(\gamma, *) \mid \gamma \in {\pi_1}^*\}$.
- If π is the proof

$$\frac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \parr B}$$

  then $\pi^* = \{(\gamma, (a, b)) \mid (\gamma, a, b) \in {\pi_1}^*\}$.
- If π is the proof

$$\frac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Delta, B\end{array}}{\vdash \Gamma, \Delta, A \otimes B}$$

  then $\pi^* = \{(\gamma, \delta, (a, b)) \mid (\gamma, a) \in {\pi_1}^* \text{ and } (\delta, b) \in {\pi_2}^*\}$.

- If π is the proof

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A\end{array}}{\vdash \Gamma, A \oplus B}$$

then $\pi^* = \{(\gamma, (1, a)) \mid (\gamma, a) \in \pi_1{}^*\}$.

- If π is the proof

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, B\end{array}}{\vdash \Gamma, A \oplus B}$$

then $\pi^* = \{(\gamma, (2, b)) \mid (\gamma, b) \in \pi_1{}^*\}$.

- If π is the proof

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Gamma, B\end{array}}{\vdash \Gamma, A \mathbin{\&} B}$$

then $\pi^* = \{(\gamma, (1, a)) \mid (\gamma, a) \in \pi_1{}^*\} \cup \{(\gamma, (2, b)) \mid (\gamma, b) \in \pi_2{}^*\}$.

- If π is the proof

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Delta, A^\perp\end{array}}{\vdash \Gamma, \Delta}$$

then $\pi^* = \{(\gamma, \delta) \mid \exists a\ (\gamma, a) \in \pi_1{}^* \text{ and } (\delta, a) \in \pi_2{}^*\}$.

We now show how this semantics can be parametrized by an arbitrary commutative monoid. This will give us more possibilities for defining logical relations.

Let P be a commutative monoid (its unit will always be denoted by 1, and the product of two elements p and q of P will be denoted by pq).

Let P_0 be the commutative monoid obtained by adjoining to P a zero element 0, that is an element satisfying $0p = 0$ for all $p \in P$.

If we endow P with a *discrete* structure of coherence space (that is, $p \coh_P q$ iff $p = q$), then it becomes a $⅋$-monoid (see [Gir99a]) that we shall still denote by P, with unit and multiplication given in the most obvious way by the corresponding operations of the monoid P. So, following the constructions given in [Gir99a], we get a $\star$-autonomous category $\mathbf{Coh}(P)$ which has all the structure required for being a model of linear logic. We describe now this category explicitly.

The objects of $\mathbf{Coh}(P)$ are the coherence spaces. If X is a coherence space, a *P-clique* of X is a clique of $X ⅋ P$ (where P is considered as a discrete coherence space). In other terms[3], it is a map $\mu : |X| \to P_0$ such that the set

$$|\mu| = \{a \in |X| \mid \mu(a) \neq 0\}$$

that we shall call the *support* of μ is a clique of X.

[3]Observe that the additional element 0 of P_0 corresponds exactly to the empty clique of the $⅋$-monoid associated to P.

When X is a coherence space, we denote by $\mathrm{Cl}_P(X)$ the set of its P-cliques. If $\mu \in \mathrm{Cl}_P(X)$ and $\mu' \in \mathrm{Cl}_P(X^\perp)$, we can define

$$\langle \mu \mid \mu' \rangle = \sum_{a \in |X|} \mu(a)\mu'(a) \in P_0 \ .$$

This sum makes sense because all of its elements, but possibly only one, are zeros, as $|\mu| \cap |\mu'|$ has at most one element (this is the main technical reason why we are working with coherence spaces instead of arbitrary sets).

If X and Y are coherence spaces, a morphism from X to Y is a P-clique of $X \multimap Y$. The identity morphism from X to X is the P-clique $\mathrm{Id} : |X| \times |X| \to P_0$ given by

$$\mathrm{Id}(a, a') = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{otherwise} \end{cases}$$

Composition of morphisms is defined as a product of matrices: if φ is a morphism from X to Y and ψ is a morphism from Y to Z, the function

$$\begin{array}{rcl} \psi \circ \varphi : |X| \times |Z| & \to & P_0 \\ (a, c) & \mapsto & \displaystyle\sum_{b \in |Y|} \varphi(a, b)\psi(b, c) \end{array}$$

is a morphism from X to Z (the sum above makes sense, due to the fact that both φ and ψ are cliques).

In particular, as there is an obvious bijective correspondence between the morphisms from 1 to X and the P-cliques of X, when φ is a P-clique of $X \multimap Y$ and μ is a P-clique of X, we can apply φ to μ, getting a map $\nu : |Y| \to P_0$ given by

$$\nu(b) = \sum_{(a,b) \in |X \multimap Y|} \varphi(a, b)\mu(a)$$

which is a P-clique of Y. This P-clique will be denoted in the sequel by $\varphi[\mu]$.

If φ is a morphism from X to Y, its transpose $\varphi^\perp$ is a morphism from $Y^\perp$ to $X^\perp$ defined (as a P-clique) by $\varphi^\perp(b, a) = \varphi(a, b)$ for $(a, b) = |X| \times |Y|$. If $\mu \in \mathrm{Cl}_P(X)$ and $\nu' \in \mathrm{Cl}_P(Y^\perp)$, then the following standard adjunction equation holds

$$\langle \varphi[\mu] \mid \nu' \rangle = \langle \mu \mid \varphi^\perp[\nu'] \rangle \ .$$

Conversely, one can check that if $f : \mathrm{Cl}_P(X) \to \mathrm{Cl}_P(Y)$ and $f' : \mathrm{Cl}_P(Y^\perp) \to \mathrm{Cl}_P(X^\perp)$ are maps satisfying

$$\langle f(\mu) \mid \nu' \rangle = \langle \mu \mid f'(\nu') \rangle$$

for all $\mu \in \mathrm{Cl}_P(X)$ and $\nu' \in \mathrm{Cl}_P(Y^\perp)$, there exists exactly one $\varphi \in \mathrm{Cl}_P(X \multimap Y)$ such that $f(\mu) = \varphi[\mu]$ for all $\mu \in \mathrm{Cl}_P(X)$.

Let $\varphi_1 : X_1 \multimap Y_1$ and $\varphi_2 : X_2 \multimap Y_2$ be two morphisms. Their tensor product is defined by

$$(\varphi_1 \otimes \varphi_2)((a_1, a_2), (b_1, b_2)) = \varphi_1(a_1, b_1)\varphi_2(a_2, b_2)$$

for $a_i \in |X_i|$ and $b_i \in |Y_i|$ $(i = 1, 2)$. It is straightforwardly a P-clique of $(X_1 \otimes X_2) \multimap (Y_1 \otimes Y_2)$, and so the tensor product becomes a bifunctor on the category $\mathbf{Coh}(P)$.

In particular, if $\mu_i \in \mathrm{Cl}_P(X_i)$ (for $i = 1, 2$), one has $\mu_1 \otimes \mu_2 \in \mathrm{Cl}_P(X_1 \otimes X_2)$, defined by $(\mu_1 \otimes \mu_2)(a_1, a_2) = \mu_1(a_1)\mu_2(a_2)$. Then $\varphi_1 \otimes \varphi_2$ is completely characterized by the following functional behavior:

$$(\varphi_1 \otimes \varphi_2)[\mu_1 \otimes \mu_2] = \varphi_1[\mu_1] \otimes \varphi_2[\mu_2]$$

for all μ_1, μ_2.

Any morphism $\varphi : (X \otimes Y) \multimap Z$ can be transposed into a morphism $\varphi' : X \multimap (Y \multimap Z)$ by setting simply $\varphi'(a, (b, c)) = \varphi((a, b), c)$. This operation turns $\mathbf{Coh}(P)$ into a symmetric monoidal closed category, which is actually a $\star$-autonomous category, the dualizing object being the one point coherence space $\perp$ whose web is the singleton $\{*\}$.

We now describe the additive structure of $\mathbf{Coh}(P)$. This category has a terminal object, namely the coherence space $\top$, whose web is the empty set. The cartesian product of two coherence spaces X_1 and X_2 is $X_1 \;\&\; X_2$, the projections are $\pi_i : (X_1 \;\&\; X_2) \multimap X_i$, given by

$$\pi_i((j, a), b) = \begin{cases} 1 & \text{if } j = i \text{ and } a = b \\ 0 & \text{otherwise} \end{cases}$$

In particular, if $\rho \in \mathrm{Cl}_P(X_1 \;\&\; X_2)$, one has $\pi_i[\rho](a) = \rho(i, a)$ (for $a \in |X_i|$).

If $\varphi_i : Y \multimap X_i$ (for $i = 1, 2$) are two morphisms, their pairing $\langle \varphi_1, \varphi_2 \rangle : Y \multimap (X_1 \;\&\; X_2)$ is given by

$$\langle \varphi_1, \varphi_2 \rangle (b, (j, a)) = \varphi_j(b, a)$$

Observe that we have a bijection between $\mathrm{Cl}_P(X_1 \;\&\; X_2)$ and $\mathrm{Cl}_P(X_1) \times \mathrm{Cl}_P(X_2)$.

The sum $X_1 \oplus X_2$ is given by $X_1 \oplus X_2 = ({X_1}^\perp \;\&\; {X_2}^\perp)^\perp$. Let $\mu_i \in \mathrm{Cl}_P(X_i)$. We denote by $\lambda_i(\mu_i)$ the corresponding P-clique in $X_1 \oplus X_2$ which, as easily checked, is given by

$$\lambda_i(\mu_i)(j, a) = \begin{cases} \mu_i(a) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Observe that the plain model of coherence spaces presented at the beginning of this section is just the model $\mathbf{Coh}(P)$ when P is the one element monoid.

If X is a coherence space, then to any clique x of X, we associate a P-clique x^P given by

$$x^P(a) = \begin{cases} 1 & \text{if } a \in x \\ 0 & \text{otherwise} \end{cases}$$

2 Logical relations

Let I be a set which is fixed, once and for all.

We define a notion of I-indexed logical relation in the model $\mathbf{Coh}(P)$ previously described. The first thing to do is to give the relation on the one point space $\perp$: this will be the only parameter for the logical relation model we are aiming at. So we endow $\perp$ with a subset $\perp_I$ of $\mathrm{Cl}_P(\perp)^I$. Observe that $\perp_I$ can also be considered as a subset of the product monoid ${P_0}^I$.

Let X be a coherence space. Let $\mu \in \mathrm{Cl}_P(X)^I$ and $\mu' \in \mathrm{Cl}_P(X^\perp)^I$. We shall say that μ and μ' are *orthogonal* if the family $(\langle \mu_i \mid \mu'_i \rangle)_{i \in I}$ (also denoted $\langle \mu \mid \mu' \rangle$), which is an element of P_0^I, belongs to $\perp_I$. In that case, we shall write $\mu \perp \mu'$.

If $\mathcal{R} \subseteq \mathrm{Cl}_P(X)^I$, we denote by $\mathcal{R}^\perp$ the set

$$\mathcal{R}^\perp = \{\mu' \in \mathrm{Cl}_P(X^\perp)^I \mid \forall \mu \in \mathcal{R}\ \mu \perp \mu'\} .$$

It is clear that one always has $\mathcal{R} \subseteq \mathcal{R}^{\perp\perp}$. We shall say that the relation $\mathcal{R}$ is *prelogical* if $\mathcal{R} = \mathcal{R}^{\perp\perp}$. Observe that for any $\mathcal{R}' \subset \mathrm{Cl}_P(X^\perp)^I$, the relation $\mathcal{R}'^\perp$ is prelogical (on X). Observe also that $\perp_I$ is always prelogical on $\perp$. This is due to the fact that the relation ${\perp_I}^\perp$ contains the family $(v_i)_{i\in I}$ of P-cliques of $1 = \perp^\perp$ defined by $v_i(*) = 1$ for all $i \in I$ (the unit of the monoid ${P_0}^I$).

We now explain how, given $\perp_I$, to endow the interpretation of any formula of linear logic in $\mathbf{Coh}(P)$ with a prelogical relation. Rather than giving an inductive definition on formulae, we prefer to define a new category $\mathbf{Coh}(P, \perp_I)$. An object of this category is a coherence space X endowed with a subset $\mathcal{R}_X$ of $\mathrm{Cl}_P(X)^I$ which is prelogical. A morphism from $(X, \mathcal{R}_X)$ to $(Y, \mathcal{R}_Y)$ is just a morphism in $\mathbf{Coh}(P)$ from X to Y.

We shall still denote by A^* the interpretation in this category of a formula A of MALL, as the underlying coherence space of this relational coherence space will be the interpretation of A in the category of coherence spaces.

Again, what we present here can be considered as a special case of the general construction of [Gir99a].

The space $\perp$ is endowed with the relation $\perp_I$, and the space 1 is endowed with the relation ${\perp_I}^\perp$. More generally, $X^\perp$ will always be endowed with the relation $(\mathcal{R}_X)^\perp$.

Observe that $\mathrm{Cl}_P(\top)^I$ has one element, namely the family ζ given by $\zeta_i = 0$ where 0 is the empty map from $\emptyset$ to P_0. We take $\mathcal{R}_\top = \{\zeta\}$. Then $\mathcal{R}_0 = (\mathcal{R}_\top)^\perp$ is empty if $\perp_I$ does not contain the constantly zero family, and non-empty (that is, equal to $\{\zeta\}$) otherwise.

Let $\mu \in \mathcal{R}_X$ and $\nu \in \mathcal{R}_Y$. We denote by $\mu \otimes \nu$ the family $(\mu_i \otimes \nu_i)_{i\in I}$. We shall set

$$\mathcal{R}_X \bar{\otimes} \mathcal{R}_Y = \{\mu \otimes \nu \mid \mu \in \mathcal{R}_X \text{ and } \nu \in \mathcal{R}_Y\} \ .$$

Then we set

$$\mathcal{R}_{X\otimes Y} = (\mathcal{R}_X \bar{\otimes} \mathcal{R}_Y)^{\perp\perp} \ .$$

So, for respecting the duality constraints of linear logic, we must set

$$\mathcal{R}_{X \invamp Y} = (\mathcal{R}_{X^\perp} \bar{\otimes} \mathcal{R}_{Y^\perp})^\perp \ .$$

As we have $X \multimap Y = X^\perp \invamp Y$, the previous formula gives us the relation $\mathcal{R}_{X\multimap Y}$. If $\varphi \in \mathrm{Cl}_P(X \multimap Y)^I$ and $\mu \in \mathrm{Cl}_P(X)^I$, we shall denote by $\varphi[\mu]$ the family $(\varphi_i[\mu_i])_{i\in I}$.

Proposition 1 *Let $\varphi \in Cl_P(X \multimap Y)^I$. We have $\varphi \in \mathcal{R}_{X\multimap Y}$ iff*

$$\forall \mu \in \mathcal{R}_X \quad \varphi[\mu] \in \mathcal{R}_Y \ .$$

Proof: Let $\varphi \in \mathcal{R}_{X\multimap Y}$ and let $\mu \in \mathcal{R}_X$. We must show that $\varphi[\mu] \in \mathcal{R}_Y$. For this it is enough to show that

$$\forall \nu' \in (\mathcal{R}_Y)^\perp \quad \varphi[\mu] \perp \nu'$$

since we know that $\mathcal{R}_Y = (\mathcal{R}_Y)^{\perp\perp}$. But one can check that

$$\langle \varphi[\mu] \mid \nu' \rangle = \langle \varphi \mid \mu \otimes \nu' \rangle$$

and we have $\mu \otimes \nu' \in \mathcal{R}_X \bar{\otimes} \mathcal{R}_{Y^\perp}$.

One proves similarly that, if $\varphi \in \mathrm{Cl}_P(X \multimap Y)^I$ satisfies $\varphi[\mu] \in \mathcal{R}_Y$ for all $\mu \in \mathcal{R}_X$, then $\varphi \in \mathcal{R}_{X\multimap Y}$. ∎

If $\mu \in \mathcal{R}_X$, we set $\lambda_1(\mu) = (\lambda_1(\mu_i))_{i \in I}$, which is an element of $\mathrm{Cl}_P(X \oplus Y)^I$. Then, let

$$\mathcal{R}_X \bar{\oplus} \mathcal{R}_Y = \{\lambda_1(\mu) \mid \mu \in \mathcal{R}_X\} \cup \{\lambda_2(\nu) \mid \nu \in \mathcal{R}_Y\}$$

which is a subset of $\mathrm{Cl}_P((X \oplus Y))^I$. We set

$$\mathcal{R}_{X \oplus Y} = (\mathcal{R}_X \bar{\oplus} \mathcal{R}_Y)^{\perp\perp} ,$$

and so,

$$\mathcal{R}_{X \& Y} = (\mathcal{R}_{X^\perp} \bar{\oplus} \mathcal{R}_{Y^\perp})^{\perp} .$$

As usual, if $\rho \in \mathrm{Cl}_P(X \& Y)^I$, we denote by $\pi_j(\rho)$ the family $(\pi_j(\rho_i))_{i \in I}$.

Proposition 2 *Let $\rho \in Cl_P(X \& Y)^I$. We have $\rho \in \mathcal{R}_{X \& Y}$ iff $\pi_1(\rho) \in \mathcal{R}_X$ and $\pi_2(\rho) \in \mathcal{R}_Y$.*

The proof is a straightforward verification.

An object of $\mathbf{Coh}(P, \perp_I)$ will be called a *relational coherence space.*

Let X be a relational coherence space. A clique x of X will be said to be *invariant* if the I-indexed family μ such that $\mu_i = x^P$ for all $i \in I$ belongs to $\mathcal{R}_X$, that is, if x^P belongs to the diagonal of $\mathcal{R}_X$. It is easy to check that the interpretation of any proof in MALL of a formula A is always an invariant clique of the corresponding relational coherence space A^* (this corresponds to the standard "lemma of logical relations").

3 The phase space viewpoint

We shall develop now another viewpoint on the logical relations introduced in the previous section. The idea behind this change of perspective is not new, and might be summarized by the following slogan, that we consider as a powerful methodological guideline in denotational semantics:

As much as possible, express everything at the level of atoms.

Let us present two applications of this paradigm (see [AC98, Ehr93] for more details).

- *From dI-domains to coherence spaces.* Guided by deep computational intuitions, Berry introduced the notion of stable function. The need of cartesian closedness and ω-algebraicity led him to consider a more restricted class of domains than Scott domains: the *dI-domains.* A crucial step has been taken by Winskel, who observed that all the structure of a dI-domain can be expressed at the level of its prime elements, which constitute an *event structure.* In particular, he observed that coherence can be described by a binary predicate on prime elements. The next[4] shift of viewpoint has been achieved by Girard. He introduced qualitative domains, which can be considered as event structures where the order relation is trivial, and then made the same observation as Winskel, namely that coherence can be limited to be pairwise, and this led him to the notion of *coherence space.* The elements of the web of a coherence space are the prime elements of the corresponding dI-domain, and all these primes are atomic (this is of course not the case in a general dI-domain). Coherence spaces constitute probably the smallest "natural" cartesian closed category of stable functions. Focusing his attention on this particular atomic case, Girard discovered linear logic.

[4]We do not give here an historical account. A large part of these results have been obtained independently, with different motivations.

- *From qualitative domains with coherence to hypercoherences.* The authors of the present paper introduced some years ago the idea of strong stability, another approach to sequentiality. The objects of their first model were qualitative domains equipped with an additional structure called "coherence", a predicate on the finite sets of *finite elements* of the domain. For various technical reasons, the second author has been led to restrict his attention to particular qualitative domains with coherence, where the coherence is determined by its restriction to the sets of *singletons*, and where this "atomic coherence" determines the elements of the qualitative domain itself. He called *hypercoherences* these particular qualitative domains with coherence, and observed that they were not only a model of the λ-calculus, but also of linear logic. This was already a surprising outcome, but moreover, this shift of viewpoint allowed him to prove results relating tightly strongly stable functions to sequential algorithms, leading to an operational interpretation of strong stability.

We shall now adopt a similar approach. A prelogical relation on a coherence space X is a set of I-indexed families of P-cliques of X. We want to restrict our attention to the "atomic" prelogical relations. We shall say that a prelogical relation is atomic when it is completely determined by those of its elements which are I-indexed families of P-cliques whose support contains at most one element. We shall show that, under the hypothesis that $\perp_I$ is closed under restriction, *any* prelogical relation is atomic in that sense. This shift of viewpoint will have interesting outcomes:

- Logical relations will be naturally described in terms of phase semantics, and this will lead to the idea of a "logic of web-based domains".
- The closure condition imposed to $\perp_I$ will have another consequence, which is the introduction of partiality in the model. This will appear clearly later, when we shall state our "weak" completeness result in section 7. This result do not state that an invariant clique *is equal* to the interpretation of a proof, but that an invariant clique *is contained* in the interpretation of a proof: it is weak in that sense.
- Most importantly, phase semantics will become more primitive than logical relations, and even than coherence semantics itself. The coherence space structure of the webs associated to formulae played a crucial role in the logical relation setting: remember that coherence is essential in the definition of the orthogonality relation between families of P-cliques of X and families of P-cliques of $X^\perp$. After "atomic reduction", the notion of logical relation, expressed now in terms of fact-valued functions on $|X|^I$ (where X is the coherence space associated to some formula) will not depend anymore on the coherence relation of X, and this is easy to understand: this atomic reduction consists in focusing ones attention on cliques having at most one element, that is, on the empty set and on all the singletons of the web. These are precisely the only cliques which do not depend on the coherence relation. The phase-valued notion of coherence becomes more primitive than the notion of clique we started from: the similarity with hypercoherences is striking here. We shall see in particular in section 7 how various natural coherence relations on the web (not only the binary notion of coherence space) can be retrieved as particular phase valued functions.

If U and V are two subsets of a monoid, we denote by UV the set $\{mn \mid m \in U,\, n \in V\}$ and if m is an element of the monoid, we denote by mV the set $\{m\}V$. A phase space is a pair $(M, \perp)$ where M is a commutative monoid and $\perp$ is a subset of M. If U is a subset of M, one defines $U^\perp$ as the set of all the elements m of M such that $mU \subseteq \perp$. One has always $U \subseteq U^{\perp\perp}$, and one says that U is a fact when $U = U^{\perp\perp}$. Since the equality $U^\perp = U^{\perp\perp\perp}$ always holds, one can also say that a fact is a subset of M which is the orthogonal of some subset of M. In the phase space truth-value semantics of linear logic, formulae are always interpreted by facts. See [Gir95] for more details on the phase space semantics of linear logic.

We now consider $({P_0}^I, \perp_I)$ as a phase space, that we shall simply denote by ${P_0}^I$. We suppose, in all the sequel, that $\perp_I$ satisfies the following closure condition:

$$\forall \varepsilon \in \{0,1\}^I \quad \varepsilon \perp_I \subseteq \perp_I \ .$$

It is straightforward to check that any fact F of ${P_0}^I$ has the same closure property: $\varepsilon F \subseteq F$ for all $\varepsilon \in \{0,1\}^I$.

This closure property of $\perp_I$ has a useful consequence.

Let E be a set. If $a, b \in E^I$, we denote by $\delta(a,b)$ the element of $\{0,1\}^I$ defined by

$$\delta(a,b)_i = \begin{cases} 1 & \text{if } a_i = b_i \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3 *Let X be a coherence space, and let $\mu \in Cl_P(X)^I$ and $\mu' \in Cl_P(X^\perp)^I$. Then $\mu \perp \mu'$ iff*

$$\forall a \in |X|^I \quad \mu(a)\mu'(a) \in \perp_I \ .$$

Proof: The condition is obviously sufficient, as there exists $a \in |X|^I$ such that $\langle \mu \mid \mu' \rangle = \mu(a)\mu'(a)$. Assume that $\langle \mu \mid \mu' \rangle \in \perp_I$ and let $b \in |X|^I$ be such that $\langle \mu \mid \mu' \rangle = \mu(b)\mu'(b)$. Let $a \in |X|^I$. We have

$$\mu(a)\mu'(a) = \delta(a,b)\mu(b)\mu'(b) \ ,$$

which we prove as follows: let $i \in I$, and assume that $\mu_i(a_i)\mu'_i(a_i) \neq 0$. Then $a_i \in |\mu_i| \cap |\mu'_i|$. But then we necessarily have $a_i = b_i$ (by definition of b).

Since $\mu(b)\mu'(b) \in \perp_I$, and since $\delta(a,b)\perp_I \subseteq \perp_I$ (closure condition on $\perp_I$), we have $\mu(a)\mu'(a) \in \perp_I$. ∎

Let $a \in |X|^I$ and let $p \in {P_0}^I$. We denote by $[a,p]$ the element of $\mathrm{Cl}_P(X)^I$ given by

$$[a,p]_i(b) = \begin{cases} p_i & \text{if } b = a_i \\ 0 & \text{otherwise} \end{cases}$$

for any $b \in |X|$. In other words, for $b \in |X|^I$, one has $[a,p](b) = \delta(a,b)p$. For any $\mu' \in \mathrm{Cl}_P(X^\perp)^I$, we have

$$\langle [a,p] \mid \mu' \rangle = p\mu'(a) \ .$$

Let X be a coherence space, and let $\mathcal{R} \subseteq \mathrm{Cl}_P(X)^I$. For any $a \in |X|^I$, we set

$$\widetilde{\mathcal{R}}(a) = \{\mu(a) \mid \mu \in \mathcal{R}\} \ .$$

It is a subset of ${P_0}^I$.

Lemma 4 *Let X be a coherence space, and let $\mathcal{R} \subseteq Cl_P(X)^I$. For any $a \in |X|^I$, one has*

$$\widetilde{\mathcal{R}^\perp}(a) = \widetilde{\mathcal{R}}(a)^\perp \ .$$

Proof: Let $p' \in \widetilde{\mathcal{R}^\perp}(a)$, and let $p \in \widetilde{\mathcal{R}}(a)$. We must prove that $pp' \in \perp_I$. So let $\mu' \in \mathcal{R}^\perp$ and $\mu \in \mathcal{R}$ be such that $p = \mu(a)$ and $p' = \mu'(a)$. Since $\mu \perp \mu'$, we have $\mu(a)\mu'(a) \in \perp_I$ by lemma 3.

Conversely, let $p' \in \widetilde{\mathcal{R}}(a)^\perp$. It is enough to show that $[a, p'] \in \mathcal{R}^\perp$. Let $\mu \in \mathcal{R}$. We have

$$\langle \mu \mid [a, p'] \rangle = \mu(a) p'$$

and we are done, since $\mu(a) \in \widetilde{\mathcal{R}}(a)$. ∎

As an immediate corollary, we get the first statement of the next proposition.

Proposition 5 *Let X be a coherence space, and let $\mathcal{R}$ be a prelogical I-relation on X. Then for all $a \in |X|^I$, the set $\widetilde{\mathcal{R}}(a)$ is a fact of ${P_0}^I$.*

Moreover, we have

$$\widetilde{\mathcal{R}}(a) = \{p \in {P_0}^I \mid [a, p] \in \mathcal{R}\} \ .$$

The proof of the latter statement is straightforward (one uses the fact that $\mathcal{R}$ is prelogical). This statement expresses that, for defining $\widetilde{\mathcal{R}}$, the "atomic part" of $\mathcal{R}$ suffices.

Let $\mathcal{F}({P_0}^I)$ denote the set of all facts of the phase space ${P_0}^I$. Given a prelogical I-relation $\mathcal{R}$ on X, we have defined a map

$$\widetilde{\mathcal{R}} : |X|^I \to \mathcal{F}({P_0}^I)$$

The following result expresses a uniformity property of this map.

Lemma 6 *If $a, b \in |X|^I$, then*

$$\delta(a, b)\widetilde{\mathcal{R}}(a) = \delta(a, b)\widetilde{\mathcal{R}}(b)$$

The proof is straightforward.

This motivates the following definition.

Definition 7 Let X be a coherence space, and let R be a map from $|X|^I$ to $\mathcal{P}({P_0}^I)$ (we say that R is relative to X). One says that R is *uniform* if for all $a, b \in |X|^I$,

$$\delta(a, b)R(a) = \delta(a, b)R(b) \ .$$

Given a map $R : |X|^I \to \mathcal{P}({P_0}^I)$ relative to the coherence space X, one defines $\overline{R} \subseteq \mathrm{Cl}_P(X)^I$ as follows:

$$\overline{R} = \{\mu \in \mathrm{Cl}_P(X)^I \mid \forall a \in |X|^I \quad \mu(a) \in R(a)\} \ .$$

One also defines $R^\perp : |X|^I \to \mathcal{P}({P_0}^I)$, relative to $X^\perp$, by

$$R^\perp(a) = R(a)^\perp$$

where the orthogonal on the right side is taken in the phase model ${P_0}^I$.

Lemma 8 *If $R : |X|^I \to \mathcal{P}({P_0}^I)$ is uniform and satisfies $\varepsilon R(a) \subseteq R(a)$ for all $a \in |X|^I$ and $\varepsilon \in \{0, 1\}^I$, then*

$$\overline{R^\perp} = \overline{R}^\perp \ .$$

So this equation holds in particular when R takes only facts as values.

Proof: Let $\mu' \in \overline{R^\perp}$. We prove that $\mu' \in \overline{R}^\perp$. So let $\mu \in \overline{R}$. Let $a \in |X|^I$ be such that

$$\langle \mu \mid \mu' \rangle = \mu(a)\mu'(a) \ .$$

By definition of $\overline{R}$ and $\overline{R^\perp}$, we have $\mu(a) \in R(a)$ and $\mu'(a) \in R(a)^\perp$, and we conclude. Observe that we have used none of the hypotheses on R here.

Conversely, let $\mu' \in \overline{R}^\perp$. We have to prove that $\mu' \in \overline{R^\perp}$, so let $a \in |X|^I$, and let $p \in R(a)$. We have to prove that $p\mu'(a) \in \perp_I$. We show that $[a,p] \in \overline{R}$. Let $b \in |X|^I$. We have

$$[a,p](b) = \delta(b,a)p$$

but $\delta(b,a)p \in \delta(b,a)R(a) = \delta(b,a)R(b)$ by uniformity of R. Since $\delta(b,a)R(b) \subseteq R(b)$, we conclude that $[a,p](b) \in R(b)$, and so $[a,p] \in \overline{R}$. Now, since

$$\langle [a,p] \mid \mu' \rangle = p\mu'(a)$$

we conclude that $\mu' \in \overline{R^\perp}$, as required. ∎

Definition 9 Let X be a coherence space. A *factual I-valuation* on X is a uniform map from $|X|^I$ to $\mathcal{F}({P_0}^I)$.

By the previous lemma, if R is a factual I-valuation on X, then $\overline{R}$ is a prelogical I-relation on X. Actually, we have established a bijective correspondence between the factual I-valuations on X and the prelogical I-relations on X.

Proposition 10 *If $\mathcal{R}$ is a prelogical I-relation on X, then $\widetilde{\overline{\mathcal{R}}} = \mathcal{R}$, and if R is a factual I-valuation on X, then $\widetilde{\overline{R}} = R$.*

Proof: Let $\mathcal{R}$ be a prelogical I-relation. It is clear that $\mathcal{R} \subseteq \widetilde{\overline{\mathcal{R}}}$. Let $\mu \in \widetilde{\overline{\mathcal{R}}}$. We use the fact that $\mathcal{R} = \mathcal{R}^{\perp\perp}$. So let $\mu' \in \mathcal{R}^\perp$. Let $a \in |X|^I$ be such that $\langle \mu \mid \mu' \rangle = \mu(a)\mu'(a)$. We have $\mu(a) \in \widetilde{\mathcal{R}}(a)$, so let $\nu \in \mathcal{R}$ be such that $\mu(a) = \nu(a)$. We know that $\nu \perp \mu'$. Let $b \in |X|^I$ be such that $\langle \nu \mid \mu' \rangle = \nu(b)\mu'(b)$. We conclude by proving that

$$\mu(a)\mu'(a) = \delta(a,b)\nu(b)\mu'(b) \ .$$

So let $i \in I$ and suppose that $a_i \neq b_i$ (otherwise, there is nothing to prove). Assume, towards a contradiction, that $\mu_i(a_i)\mu'_i(a_i) \neq 0$. Then $a_i \in |\mu_i| \cap |\mu'_i|$. As $\nu_i(a_i) = \mu_i(a_i)$, we have $a_i \in |\nu_i| \cap |\mu'_i|$, thus $|\nu_i| \cap |\mu'_i| \neq \emptyset$, but then we must have $|\nu_i| \cap |\mu'_i| = \{b_i\}$, contradiction.

Let R be a factual I-valuation on X. Let $a \in |X|^I$. The inclusion $\widetilde{\overline{R}}(a) \subseteq R(a)$ is straightforward. Let $p \in R(a)$. We prove that $[a,p] \in \overline{R}$ as in the proof of lemma 8, using the uniformity of R, and the fact that R takes facts as values. ∎

Remark: In the unary case (when I is a singleton), this correspondence is particularly simple. In that case, as soon as $\perp_I$ is non-empty (that is, contains 0), any fact-valued function on $|X|$ is a factual I-valuation on X (any facts of $(P_0, \perp_I)$ must contain 0, and so the uniformity condition automatically holds), and so we have a bijective correspondence between unary prelogical relations and fact-valued functions on $|X|$. However, this unary case is too restricted for our purpose. We shall see in the sequel that the treatment of additives involves multi-ary factual valuations.

So a relational coherence space can be considered indifferently as equipped with a logical I-relation, or with a factual I-valuation. The object of the next statements is to express the constructions of the previous section from this latter viewpoint.

We shall denote by R_X the factual valuation associated to the relational coherence space X (that is $R_X = \widetilde{\mathcal{R}_X}$).

Observe first that $|\bot|^I$ has only one element (that we still denote by $*$), and that $R_\bot(*) = \bot_I$. So that $R_1(*) = \bot_I{}^\bot = \{1\}^{\bot\bot}$ where 1 is the unit of $P_0{}^I$.

As to the tensor product, observe first that there is a canonical bijection between $|X \otimes Y|^I$ and $|X|^I \times |Y|^I$. We shall implicitly use this bijection in the sequel.

Proposition 11 *Let X and Y be two relational coherence spaces. The mapping $R_{X\otimes Y}$ is given by*

$$R_{X\otimes Y}(a,b) = (R_X(a)R_Y(b))^{\bot\bot}$$

for all $a \in |X|^I$ and $b \in |Y|^I$. Equivalently,

$$R_{X\parr Y}(a,b) = (R_X(a)^\bot R_Y(b)^\bot)^\bot \ .$$

Proof: We prove that $R_{X^\bot \parr Y^\bot}(a,b) = (R_X(a)R_Y(b))^\bot$. Let $r \in R_{X^\bot\parr Y^\bot}(a,b)$. Let $\rho \in \mathcal{R}_{X^\bot\parr Y^\bot}$ be such that $r = \rho(a,b)$. Let $p \in R_X(a)$ and $q \in R_Y(b)$, we must prove that $rpq \in \bot_I$. Let $\mu \in \mathcal{R}_X$ and $\nu \in \mathcal{R}_Y$ be such that $\mu(a) = p$ and $\nu(b) = q$. We have $\rho \perp (\mu \otimes \nu)$, and we conclude by lemma 3.

Conversely, let $r \in (R_X(a)R_Y(b))^\bot$. We prove that $[(a,b),r] \in \mathcal{R}_{X^\bot\parr Y^\bot} = (\mathcal{R}_{X\otimes Y})^\bot = (\mathcal{R}_X \bar{\otimes} \mathcal{R}_Y)^\bot$. So let $\mu \in \mathcal{R}_X$ and $\nu \in \mathcal{R}_Y$. We have

$$\langle \mu \otimes \nu \mid [(a,b),r]\rangle = \mu(a)\nu(b)r$$

and we are done. ∎

Let us treat now the additive case. There is a canonical bijection between $|X \oplus Y|^I$ and

$$\sum_{J+K=I} |X|^J \times |Y|^K$$

where by $J + K = I$, we mean that J and K are disjoint, and that $J \cup K = I$. If $J + K = I$, if $a \in |X|^J$ and $b \in |Y|^K$, we denote by $a + b$ the corresponding element of $|X \oplus Y|^I$, which is defined by

$$(a+b)_i = \begin{cases} (1, a_i) & \text{if } i \in J \\ (2, b_i) & \text{if } i \in K \end{cases}$$

If $J \subseteq I$, we denote by $\varepsilon_J \in \{0,1\}^I$ the characteristic map of J. If $a \in |X|^J$ and $b \in |X|^I$ is an extension of a (that is, $b_i = a_i$ for all $i \in J$), then $\varepsilon_J R_X(b)$ depends only on a, by uniformity of R_X. For this reason, we shall abusively denote this set by $\varepsilon_J R_X(a)$. If E is a set and $e \in E^I$, we denote by $\pi_J(e)$ the restriction of e to J, so that $\pi_J(e) \in E^J$.

Lemma 12 *Let X be a relational coherence space. Let $J \subseteq I$ and let $a \in |X|^J$. Then*

$$(\varepsilon_J R_X(a))^\bot = \{p \in P_0{}^I \mid \varepsilon_J p \in \varepsilon_J R_X{}^\bot(a)\} \ .$$

Proof: Let $a' \in |X|^I$ be an extension of a. Let $p \in (\varepsilon_J R_X(a))^\bot$. Let $q \in R_X(a')$. We have to check that $\varepsilon_J pq \in \bot_I$. This is the case because $\varepsilon_J q \in \varepsilon_J R_X(a)$.

Conversely, let $p \in P_0{}^I$ be such that $\varepsilon_J p \in R_X{}^\bot(a')$. Let $q \in R_X(a')$. We have $p\varepsilon_J q = (\varepsilon_J p)q \in \bot_I$. ∎

Proposition 13 *Let X and Y be two relational coherence spaces. Let $J, K \subseteq I$ be such that $J + K = I$ and let $a \in |X|^J$ and $b \in |Y|^K$. Then*

$$R_{X \oplus Y}(a + b) = (\varepsilon_J R_X(a) \cup \varepsilon_K R_Y(b))^{\perp\perp} .$$

Equivalently,

$$R_{X \& Y}(a + b) = (\varepsilon_J R_X{}^{\perp}(a))^{\perp} \cap (\varepsilon_K R_Y{}^{\perp}(b))^{\perp} .$$

Proof: We prove the latter statement. By lemma 12, this amounts to proving that

$$R_{X \& Y}(a + b) = \{p \in P_0{}^I \mid \varepsilon_J p \in \varepsilon_J R_X(a) \text{ and } \varepsilon_K p \in \varepsilon_K R_K(b)\} .$$

By propositions 5 and 2, we have

$$R_{X \& Y}(a + b) = \{p \in P_0{}^I \mid \pi_1([a + b, p]) \in \mathcal{R}_X \text{ and } \pi_2([a + b, p]) \in \mathcal{R}_Y\} .$$

Let a' and b' be any extensions to I of a and b respectively. We have clearly $\pi_1([a + b, p]) = [a', \varepsilon_J p]$ and $\pi_2([a + b, p]) = [b', \varepsilon_K p]$ and we conclude. ∎

Last, we examine the notion of invariance introduced at the end of section 2 from this phase space viewpoint. So let X be a relational coherence space, and let x be a clique of X. The clique x is invariant if and only if, for any $a \in |X|^I$, one has

$$(x^P(a_i))_{i \in I} \in R_X(a)$$

that is, iff for any $J \subseteq I$ and any J-indexed family a of elements of x, one has

$$\varepsilon_J \in \varepsilon_J R_X(a) .$$

4 A logical system

We keep the index set I fixed.

In order to prove some form of full completeness of the previously described "phase valued" semantics (actually, for a generalization of that semantics), we introduce a version of MALL where formulae are intended to represent J-indexed families of points of the webs of standard MALL-formulae (for some $J \subseteq I$). So to each formula A of this system MALL(I) will be associated a set $d(A) \subseteq I$, that we shall sometimes call the *domain* of A. These formulae are defined as follows:

- 0 and $\top$ are two constants, both having empty domain.
- For each $J \subseteq I$, we introduce two new constant formulae: $\perp_J$ and 1_J, and both of these formulae have J as domain.
- For each $J \subseteq I$, if A and B are formulae such that $d(A) = d(B) = J$, then $A \otimes B$ and $A \parr B$ are formulae, with $d(A \otimes B) = d(A \parr B) = J$.
- If $J, K \subseteq I$, with $J \cap K = \emptyset$, if A and B are formulae with $d(A) = J$ and $d(B) = K$, then $A \oplus B$ and $A \& B$ are formulae, with $d(A \oplus B) = d(A \& B) = J + K$.

For $A \in \mathsf{MALL}(I)$ with $d(A) = J$, one *defines* $A^\perp \in \mathsf{MALL}(I)$ with $d(A^\perp) = J$ in the usual way, using the De Morgan rules.

A J-sequent is an expression of the shape $\vdash_J \Gamma$ where Γ is a (possibly empty) sequence of J-formulae. These sequences will always be denoted by capital Greek letters and will be called homogeneous sequences (of domain $d(\Gamma) = J$).

If A is a formula of $\mathsf{MALL}(I)$ with $d(A) = J$, and if $K \subseteq I$, we define the *restriction* of A by K, denoted by $A|_K$, which is a formula of $\mathsf{MALL}(I)$ of domain $J \cap K$, as follows:

- $\top|_K = \top$ and $0|_K = 0$.
- $\bot_J|_K = \bot_{J\cap K}$ and $1_J|_K = 1_{J\cap K}$.
- $(A \otimes B)|_K = A|_K \otimes B|_K$, $(A \parr B)|_K = A|_K \parr B|_K$, $(A \oplus B)|_K = A|_K \oplus B|_K$ and $(A \,\&\, B)|_K = A|_K \,\&\, B|_K$.

If $\Gamma = \langle A_1, \ldots, A_n \rangle$ is an homogeneous sequence of formulae, one defines

$$\Gamma|_K = \langle A_1|_K, \ldots, A_n|_K \rangle$$

so that again, $d(\Gamma|_K) = d(\Gamma) \cap K$. Last, observe that trivially $A^\perp|_K = (A|_K)^\perp$.

We describe now a sequent calculus for these sequents (the exchange rule is left implicit).

We have the following axioms:

$$\frac{}{\vdash_J 1_J}$$

and

$$\frac{}{\vdash_\emptyset \Gamma, \top}$$

this latter making sense only under the assumption that Γ is empty, or has empty domain.

The multiplicative rules are without surprises.

$$\frac{\vdash_J \Gamma}{\vdash_J \Gamma, \bot_J}$$

$$\frac{\vdash_J \Gamma, A \quad \vdash_J \Delta, B}{\vdash_J \Gamma, \Delta, A \otimes B}$$

$$\frac{\vdash_J \Gamma, A, B}{\vdash_J \Gamma, A \parr B}$$

In the rules for $\oplus$, observe that B has to have an empty domain.

$$\frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, A \oplus B} \qquad \frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, B \oplus A}$$

Now we give the rule for &. Assume that $d(A) = J$, $d(B) = K$ with $J \cap K = \emptyset$, and that $d(\Gamma) = J + K$.

$$\frac{\vdash_J \Gamma|_J, A \quad \vdash_K \Gamma|_K, B}{\vdash_{J+K} \Gamma, A \,\&\, B}$$

Last, the cut rule is standard.

$$\frac{\vdash_J \Gamma, A \quad \vdash_J \Delta, A^\perp}{\vdash_J \Gamma, \Delta}$$

Lemma 14 *Let A be a formula of MALL(I) of domain J. The sequent $\vdash_J A, A^\perp$ is provable.*

Proof: We just treat the additive case. So assume that $A = B \,\&\, C$, with $d(B) = K$, $d(C) = L$, $K \cap L = \emptyset$ and $K + L = J$. We have the following deduction in sequent calculus.

$$\dfrac{\dfrac{\vdash_K B, B^\perp}{\vdash_K B, B^\perp \oplus C|_\emptyset{}^\perp} \qquad \dfrac{\vdash_L C, C^\perp}{\vdash_L C, B|_\emptyset{}^\perp \oplus C^\perp}}{\vdash_{K+L} B \,\&\, C, B^\perp \oplus C^\perp}$$

as we clearly have $B|_K = B$, $C|_K = C|_\emptyset$ and similarly for L. We conclude by inductive hypothesis.

∎

Lemma 15 *If $\vdash_J \Gamma$, then $\vdash_{J\cap K} \Gamma|_K$ for any set K.*

The proof is a straightforward induction.

Proposition 16 *The sequent calculus system MALL(I) enjoys cut elimination, that is: if a sequent can be proved, it can also be proved without using the cut rule.*

It is a consequence of the forthcoming soundness and completeness theorems 25 and 27.

Observe that the formulae of MALL(I) which have empty domain are closed under all the connectives we are using, that any subformula of a formula with empty domain has empty domain and that, for the sequents of empty domain, the rules we have given are just the standard rules of MALL. So we identify MALL with the fragment of MALL(I) consisting of formulae with empty domain, and MALL(I) can be considered as a conservative extension of MALL.

To any formula A of MALL, and any family $a \in |A^*|^J$, we associate a formula $A\langle a\rangle$ of MALL(I) of domain J as follows:

- For $A = 0$ or $A = \top$, if $J \neq \emptyset$, there is nothing to say, as in that case $|A^*|^J = \emptyset$. If $J = \emptyset$, then $|A^*|^J$ has exactly one element, namely the empty family $\emptyset$, and we set $0\langle\emptyset\rangle = 0$ and $\top\langle\emptyset\rangle = \top$.
- If $A = 1$ (or $A = \perp$), then a is the constant family $*$, and we set $A\langle a\rangle = \perp_J$ (or $A\langle a\rangle = 1_J$).
- If $A = B \otimes C$ (or $A = B \parr C$), then $a = (b, c)$, with $b \in |B^*|^J$ and $c \in |C^*|^J$, and we set $A\langle a\rangle = B\langle b\rangle \otimes C\langle c\rangle$ (or $A\langle a\rangle = B\langle b\rangle \parr C\langle c\rangle$) which is a well-formed formula of MALL(I) of domain J.
- If $A = B \oplus C$ (or $A = B \,\&\, C$), then $a = b + c$ with $b \in |B^*|^K$ and $c \in |C^*|^L$ and $K + L = J$. Then we set $A\langle a\rangle = B\langle b\rangle \oplus C\langle c\rangle$ (or $A\langle a\rangle = B\langle b\rangle \,\&\, C\langle c\rangle$) which is a well-formed formula of MALL(I) of domain J.

It is easily checked that the correspondence we have just described is actually a bijection: if A is a formula of MALL(I) of domain J and $A|_\emptyset$ is the corresponding MALL formula, there is a unique family $a \in |A|_\emptyset{}^*|$ such that $A = A|_\emptyset\langle a\rangle$.

If $\Gamma = \langle A_1, \ldots, A_n\rangle$ is a sequence of formulae of MALL, then $|\Gamma^*| = |A_1{}^*| \times \cdots \times |A_n{}^*|$. If $\gamma \in |\Gamma^*|^J$, then, using our usual notational conventions, one can write $\gamma = (\gamma^1, \ldots, \gamma^n)$ with $\gamma^m \in |A_m{}^*|^J$, and we set

$$\Gamma\langle\gamma\rangle = \langle A_1\langle\gamma^1\rangle, \ldots, A_n\langle\gamma^n\rangle\rangle\ .$$

Let us first give two examples for making precise the intended meaning of these constructions.

Consider first the sequent $\vdash \Gamma$ of MALL, where

$$\Gamma = \langle \bot \oplus (\bot \mathbin{\&} \bot), \bot \oplus (\bot \mathbin{\&} \bot), \bot \oplus (\bot \mathbin{\&} \bot), 1 \rangle$$

Up to isomorphism, we can take $|\bot \mathbin{\&} \bot| = \{(2,*), (3,*)\}$, and $|\bot \oplus (\bot \mathbin{\&} \bot)| = \{(1,*), (2,*), (3,*)\}$. Take $J = \{1,2,3\}$, and $\gamma \in |\Gamma^*|^J$ defined by

$$\begin{aligned} \gamma_1 &= ((1,*), (2,*), (3,*), *) \\ \gamma_2 &= ((2,*), (3,*), (1,*), *) \\ \gamma_3 &= ((3,*), (1,*), (2,*), *) \end{aligned}$$

Then we have

$$\Gamma\langle\gamma\rangle = \langle \bot_{\{1\}} \oplus (\bot_{\{2\}} \mathbin{\&} \bot_{\{3\}}), \bot_{\{3\}} \oplus (\bot_{\{1\}} \mathbin{\&} \bot_{\{2\}}), \bot_{\{2\}} \oplus (\bot_{\{3\}} \mathbin{\&} \bot_{\{1\}}), 1_{\{1,2,3\}} \rangle$$

It appears that $\vdash_J \Gamma\langle\gamma\rangle$ is not provable, as no immediate subformula of any of the $\oplus$-formulae of $\Gamma\langle\gamma\rangle$ has an empty domain and as, if $\vdash_J \Gamma\langle\gamma\rangle$ were provable, it would be cut-free provable by proposition 16, and any of its cut-free proofs should end by a $\oplus$-rule. This corresponds to the fact that the set $\{\gamma_1, \gamma_2, \gamma_3\}$ is contained in the interpretation of no proof of $\vdash \Gamma$ in MALL (it is a version of Berry's example of a stable but non sequential function).

Modifying γ by setting for instance

$$\gamma_1 = ((2,*), (2,*), (3,*), *)$$

leads to

$$\Gamma\langle\gamma\rangle = \langle \bot_{\emptyset} \oplus (\bot_{\{1,2\}} \mathbin{\&} \bot_{\{3\}}), \bot_{\{3\}} \oplus (\bot_{\{1\}} \mathbin{\&} \bot_{\{2\}}), \bot_{\{2\}} \oplus (\bot_{\{3\}} \mathbin{\&} \bot_{\{1\}}), 1_{\{1,2,3\}} \rangle$$

and the sequent $\vdash_J \Gamma\langle\gamma\rangle$ is provable in $\mathsf{MALL}(I)$. Moreover, any of its proofs, when forgetting the various indexing sets, gives rise to a proof of Γ whose interpretation (as a clique of the coherence space Γ^*) contains the points γ_1, γ_2 and γ_3.

Here is an example of a non-definable clique which, in contrast with the previous one, is accepted also by the hypercoherent semantics [Ehr93]. Consider

$$\Gamma = \langle (1 \mathbin{\&} 1) \otimes (1 \mathbin{\&} 1), \bot \oplus \bot \rangle$$

Again, take $J = \{1,2,3\}$. Let $\gamma \in |\Gamma^*|^J$ be defined by

$$\begin{aligned} \gamma_1 &= ((1,*), (1,*), (1,*)) \\ \gamma_2 &= ((2,*), (1,*), (2,*)) \\ \gamma_3 &= ((1,*), (2,*), (2,*)) \end{aligned}$$

Then we have

$$\Gamma\langle\gamma\rangle = \langle (1_{\{1,3\}} \mathbin{\&} 1_{\{2\}}) \otimes (1_{\{1,2\}} \mathbin{\&} 1_{\{3\}}), \bot_{\{1\}} \oplus \bot_{\{2,3\}} \rangle$$

The sequent $\vdash_J \Gamma\langle\gamma\rangle$ is not provable in $\mathsf{MALL}(I)$. Indeed, any of its cut-free proof should end by a $\otimes$-rule, as none of the immediate subformulae of the $\oplus$-formula has an empty domain. There are two possible ways of applying the $\otimes$-rule: either we do

$$\frac{\vdash_J 1_{\{1,3\}} \mathbin{\&} 1_{\{2\}}, \bot_{\{1\}} \oplus \bot_{\{2,3\}} \qquad \vdash_J 1_{\{1,2\}} \mathbin{\&} 1_{\{3\}}}{\vdash_J \Gamma\langle\gamma\rangle}$$

or we do

$$\frac{\vdash_J 1_{\{1,3\}} \mathbin{\&} 1_{\{2\}} \qquad \vdash_J 1_{\{1,2\}} \mathbin{\&} 1_{\{3\}}, \bot_{\{1\}} \oplus \bot_{\{2,3\}}}{\vdash_J \Gamma\langle\gamma\rangle}$$

As these two possibilities are clearly isomorphic (exchange 2 and 3), let us consider only the first one. The second premise is provable, but the first one is not. Indeed, any of its cut-free proofs should end by a &-rule, leading to (observe the use of the domain restriction operation in the premises):

$$\frac{\vdash_{\{1,3\}} 1_{\{1,3\}}, \bot_{\{1\}} \oplus \bot_{\{3\}} \qquad \vdash_{\{2\}} 1_{\{2\}}, \bot_{\emptyset} \oplus \bot_{\{2\}}}{\vdash_J 1_{\{1,3\}} \mathbin{\&} 1_{\{2\}}, \bot_{\{1\}} \oplus \bot_{\{2,3\}}}$$

The second premise is provable, as we are in position of applying a $\oplus$-rule, but the first is not, as no rule applies.

Actually, one can check that the coherent interpretation of no proof of $\vdash \Gamma$ contains simultaneously γ_1, γ_2 and γ_3.

These examples suggest a connection between the "sub-definability" of a family $\gamma \in |\Gamma^*|^J$ (for Γ in MALL), that is, the fact that there is a proof π of $\vdash \Gamma$ such that $\gamma \in (\pi^*)^J$ or, equivalently,

$$\{\gamma_j \mid j \in J\} \subseteq \pi^* \ ,$$

and the provability in $\mathsf{MALL}(I)$ of the sequent $\vdash_J \Gamma\langle\gamma\rangle$. This is what we examine now.

If Γ is a sequent in $\mathsf{MALL}(I)$ of domain J and if σ is a proof of this sequent (in the sequent calculus described above), we denote by $\sigma|_K$ the proof of $\Gamma|_K$ obtained by intersecting each indexing set L appearing in σ by K. It is easy to check that $\sigma|_K$ is a correct proof in the sequent calculus $\mathsf{MALL}(I)$.

Lemma 17 *Let $\vdash \Gamma$ be a sequent in MALL and let $\gamma \in |\Gamma^*|^J$. Let $K \subset I$. Let δ be the restriction of γ to $J \cap K$. Then $\Gamma\langle\delta\rangle = \Gamma\langle\gamma\rangle|_K$.*

The proof is a routine verification.

Lemma 18 *Let $\vdash \Gamma$ be a sequent in MALL and let π be a proof of $\vdash \Gamma$ in MALL. Let $\gamma \in (\pi^*)^J$ (for some $J \subseteq I$). Then the sequent $\vdash_J \Gamma\langle\gamma\rangle$ has a proof σ in $\mathsf{MALL}(I)$ such that $\sigma|_\emptyset = \pi$.*

Proof: By induction on π. As the proof is rather straightforward, we treat only one case. Assume that π ends by a &-rule:

$$\frac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Delta, A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Delta, B\end{array}}{\vdash \Delta, A \mathbin{\&} B}$$

Then we can write $\gamma = (\delta, c)$ with $\delta \in |\Delta^*|^J$ and $c \in |(A \mathbin{\&} B)^*|^J$. There exist two disjoint sets (uniquely determined by c) K and L such that $K + L = J$ and $c = a + b$ for some $a \in |A^*|^K$ and $b \in |A^*|^L$ uniquely determined by c. Moreover, denoting by δ_L and δ_K the restrictions of the J-indexed familes δ to K and L respectively, we have, since $\gamma \in (\pi^*)^J$:

$$(\delta_K, a) \in ({\pi_1}^*)^K \quad \text{and} \quad (\delta_L, b) \in ({\pi_2}^*)^L$$

Applying the inductive hypothesis, we find two proofs σ_1 and σ_2 in $\mathsf{MALL}(I)$ of $(\Delta, A)\langle(\delta_K, a)\rangle$ and $(\Delta, B)\langle(\delta_L, b)\rangle$ such that $\sigma_i|_\emptyset = \pi_i$ for $i = 1, 2$. But by lemma 17, we have

$$(\Delta, A)\langle(\delta_K, a)\rangle = (\Delta\langle\delta\rangle|_K, A\langle a\rangle) \quad \text{and} \quad (\Delta, B)\langle(\delta_L, b)\rangle = (\Delta\langle\delta\rangle|_L, B\langle b\rangle)$$

so that we have the following proof σ in $\mathsf{MALL}(I)$:

$$\dfrac{\dfrac{\vdots\ \sigma_1}{\vdash_K \Delta\langle\delta\rangle|_K, A\langle a\rangle} \qquad \dfrac{\vdots\ \sigma_2}{\vdash_L \Delta\langle\delta\rangle|_L, B\langle b\rangle}}{\vdash_J \Delta\langle\delta\rangle, A\langle a\rangle \ \&\ B\langle b\rangle}$$

and we conclude, since

$$(\Delta, A \ \&\ B)\langle(\delta, c)\rangle = (\Delta\langle\delta\rangle, A\langle a\rangle \ \&\ B\langle b\rangle)$$

and clearly $\sigma|_\emptyset = \pi$. ∎

Lemma 19 *Let $\vdash \Gamma$ be a sequent in MALL. Let $\gamma \in |\Gamma^*|^J$ (for some $J \subseteq I$), and let σ be a proof of $\vdash_J \Gamma\langle\gamma\rangle$ in MALL(I). Then $\gamma \in (\sigma|_\emptyset{}^*)^J$.*

Proof: By induction on σ. Again, it is basically a routine verification, and we treat only one case. Assume that $\Gamma\langle\gamma\rangle = (\Delta, A \ \&\ B)$, with $d(A) = K$, $d(B) = L$, $K \cap L = \emptyset$ and $K + L = J$. Assume furthermore that σ has the following shape:

$$\dfrac{\dfrac{\vdots\ \sigma_1}{\vdash_K \Delta|_K, A} \qquad \dfrac{\vdots\ \sigma_2}{\vdash_L \Delta|_L, B}}{\vdash_J \Delta, A \ \&\ B}$$

An inspection of the definition of $\Gamma\langle\gamma\rangle$ shows that necessarily $\Gamma = (\Lambda, C \ \&\ D)$ and $\gamma = (\lambda, c + d)$ with $\Lambda\langle\lambda\rangle = \Delta$ and $c \in |C^*|^K$, $d \in |D^*|^L$, $C\langle c\rangle = A$ and $D\langle d\rangle = B$. Then $\sigma|_\emptyset$ is the following proof of MALL:

$$\dfrac{\dfrac{\vdots\ \sigma_1|_\emptyset}{\vdash \Lambda, C} \qquad \dfrac{\vdots\ \sigma_2|_\emptyset}{\vdash \Lambda, D}}{\vdash \Lambda, C \ \&\ D}$$

Using lemma 17 and the inductive hypothesis, we have (with the same notations as in the previous lemma for the restriction of J-indexed families of points) $(\lambda_K, c) \in (\sigma_1|_\emptyset{}^*)^K$ and $(\lambda_L, d) \in (\sigma_2|_\emptyset{}^*)^L$, and hence $\gamma \in (\sigma|_\emptyset{}^*)^J$ as required. ∎

Summarizing these two lemmas, we get the next result.

Proposition 20 *Let Γ be a sequence of formulae of MALL, and let $\gamma \in |\Gamma^*|^J$. The two following statements are equivalent.*

i) There exists a proof π of Γ in MALL such that $\gamma \in (\pi^*)^J$.

ii) The sequent $\vdash_J \Gamma\langle\gamma\rangle$ is provable.

More precisely, we have established a bijective correspondence between the proofs π of $\vdash \Gamma$ in MALL such that $\gamma \in (\pi^*)^J$ (that is, $\forall j \in J\ \gamma_j \in \pi^*$) and the proofs in $\mathsf{MALL}(I)$ of $\vdash_J \Gamma\langle\gamma\rangle$.

This correspondence is reminiscent of what happens, for instance, in Coppo-Dezzani's system $\mathcal{D}\Omega$. Roughly speaking, in this typing system, one can associate to any type α a compact element of a model of the pure λ-calculus, and it holds that a term t is typable of type α iff the semantics of t in this model contains the element of the model corresponding to α:

$$\alpha \in t^* \quad \text{iff} \quad \vdash t : \alpha\,.$$

The main difference, from our viewpoint, is that the deduction system $\mathcal{D}\Omega$ can hardly be considered as a *logical* system, as it contains a deduction rule of the shape

$$\frac{t:\alpha \qquad t:\beta}{t:\alpha\wedge\beta}$$

which has no clear logical status (the restriction that α and β must have been obtained by proofs having the same underlying λ-term breaks the kind of modularity which characterizes logical systems).

5 The phase semantics of $\mathsf{MALL}(I)$

We generalize the notion of phase model introduced in section 3 as this latter notion is too restrictive for proving a truth-value completeness result for $\mathsf{MALL}(I)$ (this will appear clearly in section 6).

A phase model of $\mathsf{MALL}(I)$ is a triple $(M, \bot, (e_J)_{J\subseteq I})$ where M is a commutative monoid (we use a multiplicative notation for the operation of M, and the neutral element of M is denoted by e), $\bot$ is a subset of M, and for any $J \subseteq I$, e_J is an element of M. We require the following properties to hold:

(i) $e_I = e$ and if $J, K \subseteq I$, then $e_{J\cap K} = e_J e_K$

(ii) For any $J \subseteq I$, one has $e_J \bot \subseteq \bot$.

The phase model $(M, \bot, (e_J)_{J\subseteq I})$ will abusively be denoted by M.[5]

Observe that $(M, \bot)$ is just a standard phase model for MALL and the notions of orthogonality and of fact we consider for M are just the standard corresponding notions for $(M, \bot)$: if $U \subseteq M$, we set

$$U^\bot = \{m \in M \mid mU \subseteq \bot\}$$

and we say that U is a fact if $U^{\bot\bot} = U$. We denote by $\mathcal{F}(M)$ the set of facts of M. If U is any subset of M, then $U^\bot$ is always a fact.

Condition (ii) above can be rephrased as follows: $e_J \in 1$, where $1 = \bot^\bot$. More generally, for any $F \in \mathcal{F}(M)$ and any $J \subseteq I$, one has $e_J F \subseteq F$, as easily checked.

The next lemma summarizes some useful properties of the orthogonal operation in phase spaces.

Lemma 21 *Let $U, V \subseteq M$.*

- *If $U \subseteq V$ then $V^\bot \subseteq U^\bot$.*
- $U^{\bot\bot\bot} = U^\bot$.
- $(U \cup V)^\bot = U^\bot \cap V^\bot$.
- $(UV^{\bot\bot})^\bot = (UV)^\bot$.

The support set of the monoid M is a fact of M, that we denote by $\top$, and we set $0 = \top^\bot = \emptyset^{\bot\bot}$, which is clearly the least fact (w.r.t. inclusion).

If $F, G \in \mathcal{F}(M)$, one defines as usual $F \otimes G$, $F \parr G$, $F \oplus G$ and $F \mathbin{\&} G$ as facts of M as follows:

[5] It is clear that this notion generalizes the kind of phase model we introduced in section 3; with the notations of that section, take $M = {P_0}^I$ and $e_J = \varepsilon_J$. In the models of section 3, $e_\emptyset M$ is a singleton, whereas with the generalized notion of model, $e_\emptyset M$ can be a non trivial (and even complete) model of MALL, in accordance with the identification of MALL with the fragments of $\mathsf{MALL}(I)$ containing all the formulae of empty domain.

- $F \otimes G = (FG)^{\perp\perp}$
- $F \parr G = (F^{\perp}G^{\perp})^{\perp}$
- $F \oplus G = (F \cup G)^{\perp\perp}$
- $F \,\&\, G = F \cap G$

If $J \subseteq I$, one sets

$$\bot_J = \{e_J\}^{\perp} \quad \text{and} \quad 1_J = \{e_J\}^{\perp\perp} \ .$$

Observe that $1_J \otimes F = (e_J F)^{\perp\perp}$ and that

$$\bot_J \parr F = (e_J F^{\perp})^{\perp} = \{m \in M \mid e_J m \in F\}$$

If F and G are two facts of M, and if $J, K \subseteq I$, we define

$$F \underset{J,K}{\oplus} G = (1_J \otimes F) \oplus (1_K \otimes G)$$

and dually

$$\begin{aligned} F \underset{J,K}{\&} G &= (F^{\perp} \underset{J,K}{\oplus} G^{\perp})^{\perp} \\ &= (\bot_J \parr F) \,\&\, (\bot_K \parr G) \\ &= \{m \in M \mid e_J m \in F \text{ and } e_K m \in G\} \end{aligned}$$

For any subset J of I, we say that two subsets U and V of M are J-equivalent, and we write $U =_J V$, if $e_J U = e_J V$.

A subset U of M is said to be J-closed if $e_J U \subseteq U$. Observe that if U and V are J-closed subsets of M, so are UV and $U \cup V$.

Lemma 22 *Let $J \subseteq I$ and let $U \subseteq M$ be J-closed. Then*

$$e_J U^{\perp} = e_J (e_J U)^{\perp} \ .$$

Proof: Since U is J-closed, we just have to prove that $e_J(e_J U)^{\perp} \subseteq e_J U^{\perp}$. So let $m \in (e_J U)^{\perp}$, we have to prove that $e_J m \in U^{\perp}$, so let $m' \in U$. We have $(e_J m)m' = m(e_J m')$ and $e_J m' \in e_J U$, so $(e_J m)m' \in \bot$ and we conclude. ∎

So if $U, V \subseteq M$ are J-closed, and if U and V are J-equivalent, then $U^{\perp}$ and $V^{\perp}$ are J-equivalent.

Observe also that if U, U', V, V' are J-closed and if $U =_J U'$ and $V =_J V'$, then $UV =_J U'V'$ and $U \cup V =_J U' \cup V'$.

The next lemma is a consequence of these observations.

Lemma 23 *Let $F, F', G, G' \in \mathcal{F}(M)$ be such that $F =_J F'$ and $G =_J G'$. Then $F \otimes G =_J F' \otimes G'$, $F \parr G =_J F' \parr G'$, $F \oplus G =_J F' \oplus G'$ and $F \,\&\, G =_J F' \,\&\, G'$.*

To each formula A of $\mathsf{MALL}(I)$, we associate a fact $A^{\bullet}$ of M as follows.

- $\top^{\bullet} = \top$
- $0^{\bullet} = 0$

- $(\bot_J)^\bullet = \bot_J$
- $(1_J)^\bullet = 1_J$
- $(A \otimes B)^\bullet = A^\bullet \otimes B^\bullet$
- $(A \parr B)^\bullet = A^\bullet \parr B^\bullet$
- $(A \oplus B)^\bullet = A^\bullet \underset{d(A),d(B)}{\oplus} B^\bullet$
- $(A \,\&\, B)^\bullet = A^\bullet \underset{d(A),d(B)}{\&} B^\bullet$

Lemma 24 *Let A be a formula of MALL(I) and let $J \subseteq I$. Then $A^\bullet =_J A|_J{}^\bullet$.*

The proof is a straightforward application of lemma 23.

If $\Gamma = \langle A_1, \ldots, A_n \rangle$ is a sequence of formulae, one sets $\Gamma^\bullet = A_1{}^\bullet \parr \ldots \parr A_n{}^\bullet$. If Γ and Δ are two such sequences, with the same domain J, observe that $e_J \in (\Gamma, \Delta)^\bullet$ iff

$$e_J \Gamma^{\bullet\bot} \subseteq \Delta^\bullet \ .$$

Theorem 25 (Soundness) *If a sequent $\vdash_J \Gamma$ of domain J of MALL(I) is provable, then $e_J \in \Gamma^\bullet$.*

Proof: By induction on the proof in MALL(I).

If the proof is the axiom $\vdash_J 1_J$, we conclude since $1_J{}^\bullet = (\{e_J\})^{\bot\bot} \ni e_J$.

If the proof is the axiom $\vdash_\emptyset \Gamma, \top$, we conclude since

$$(\Gamma, \top)^\bullet = (\Gamma^{\bullet\bot}\emptyset^{\bot\bot})^\bot = (\Gamma^{\bullet\bot}\emptyset)^\bot = \emptyset^\bot = \top \ni e_\emptyset \ .$$

If the last rule of the proof is

$$\frac{\vdash_J \Gamma}{\vdash_J \Gamma, \bot_J}$$

we know by inductive hypothesis that $e_J \in \Gamma^\bullet$, hence $\Gamma^{\bullet\bot} \subseteq (\{e_J\})^\bot = \bot_J{}^\bullet$ and we conclude since $\Gamma^{\bullet\bot}$ is J-closed.

If the last rule of the proof is

$$\frac{\vdash_J \Gamma, A \qquad \vdash_J \Delta, B}{\vdash_J \Gamma, \Delta, A \otimes B}$$

we know by inductive hypothesis that $e_J\Gamma^{\bullet\bot} \subseteq A^\bullet$ and that $e_J\Delta^{\bullet\bot} \subseteq B^\bullet$, hence

$$e_J\Gamma^{\bullet\bot}\Delta^{\bullet\bot} \subseteq A^\bullet B^\bullet$$

thus

$$(e_J\Gamma^{\bullet\bot}\Delta^{\bullet\bot})^{\bot\bot} \subseteq (A \otimes B)^\bullet$$

that is

$$(e_J(\Gamma, \Delta)^{\bullet\bot})^{\bot\bot} \subseteq (A \otimes B)^\bullet \ .$$

If the last rule of the proof is

$$\frac{\vdash_J \Gamma, A, B}{\vdash_J \Gamma, A \parr B}$$

one concludes straightforwardly.

If the last rule of the proof is

$$\frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, A \oplus B}$$

by inductive hypothesis we have $e_J \Gamma^{\bullet\perp} \subseteq A^\bullet$, hence

$$e_J \Gamma^{\bullet\perp} \subseteq e_J A^\bullet \subseteq (e_J A^\bullet \cup e_\emptyset B^\bullet)^{\perp\perp} = (A \oplus B)^\bullet \ .$$

If the last rule of the proof is

$$\frac{\vdash_J \Gamma|_J, A \quad \vdash_K \Gamma|_K, B}{\vdash_{J+K} \Gamma, A \,\&\, B}$$

we know by inductive hypothesis that $e_J \Gamma|_J{}^{\bullet\perp} \subseteq A^\bullet$, and hence, by lemma 24, we have

$$e_J \Gamma^{\bullet\perp} \subseteq A^\bullet$$

and similarly for B. We have to prove that

$$e_{J+K} \Gamma^{\bullet\perp} \subseteq (A \,\&\, B)^\bullet \ ,$$

that is

$$e_{J+K} \Gamma^{\bullet\perp} \subseteq \perp_J \mathbin{⅋} A^\bullet \ ,$$

and similarly for B. This amounts to proving that

$$e_J e_{J+K} \Gamma^{\bullet\perp} \subseteq A^\bullet$$

and we conclude since $e_J e_{J+K} = e_J$.

If the last rule of the proof is a cut rule

$$\frac{\vdash_J \Gamma, A \quad \vdash_J \Delta, A^\perp}{\vdash_J \Gamma, \Delta}$$

then we have by inductive hypothesis $e_J \Gamma^{\bullet\perp} \subseteq A^\bullet$ and $e_J (A^{\perp\bullet})^\perp \subseteq \Delta^\bullet$. From the former inclusion, we derive $e_J \Gamma^{\bullet\perp} \subseteq e_J A^\bullet$ and we conclude, since by definition of $A^\perp$ and of the phase interpretation of formulae, one clearly has $(A^{\perp\bullet})^\perp = A^\bullet$. ∎

As usual, for proving completeness of this phase semantics, one builds a syntactical model. For M, we take the sets of all pairs of the shape (J, Γ) where Γ is a multiset of formulae of $\mathsf{MALL}(I)$ having all the same domain J (the presence of J in the pair is useful only when Γ is the empty multiset of formulae, otherwise it is redundant). Denoting by simple juxtaposition the addition of multisets, the multiplication of this monoid is given by

$$(J, \Gamma)(K, \Delta) = (J \cap K, \Gamma|_K \Delta|_J) \ .$$

This multiplication is associative, commutative, and its neutral element is $(I, [])$, where $[]$ denotes the empty multiset, as easily checked.

As to the restricting element of the model, we set, for any $J \subseteq I$,

$$e_J = (J, [])$$

so that without surprise

$$e_J(K, \Gamma) = (J \cap K, \Gamma|_J) \ .$$

The subset $\bot$ of M is the set of all pairs (J, Γ) such that $\vdash_J \Gamma$ is provable in $\mathsf{MALL}(I)$ (we identify sequences and multisets, which is harmless as we are dealing with commutative logic).

The next two results hold in this particular syntactical model. For proving completeness, we follow Okada's method [Oka99].

Lemma 26 *Let A be a formula of $\mathsf{MALL}(I)$ and let J be its domain. Then*

$$A^\bullet \subseteq \{(J, A)\}^\bot$$

Proof: By induction on the structure of A.

Assume that $A = 0$. Then $A^\bullet \subseteq \{(J, A)\}^\bot$ holds since $A^\bullet = \emptyset^{\bot\bot}$ is the least fact, and $\{(J, A)\}^\bot$ is a fact.

Assume that $A = \top$. Then $(J, A) = (\emptyset, \top)$. Let (K, Γ) be any element of M. We have $(K, \Gamma)(\emptyset, \top) = (\emptyset, (\Gamma|_\emptyset, \top))$ and the sequent $\vdash_\emptyset \Gamma|_\emptyset, \top$ is provable (it is an axiom sequent), so that $(K, \Gamma)(\emptyset, \top) \in \bot$. Hence $\{(J, A)\}^\bot = M = A^\bullet$.

Assume $A = 1_J$. We have $(J, 1_J) \in \bot$ thus $e_J(J, 1_J) \in \bot$ hence $e_J \in \{(J, 1_J)\}^\bot$ and we conclude since ${1_J}^\bullet = \{e_J\}^{\bot\bot}$.

Assume $A = \bot_J$. Let $(K, \Gamma) \in {\bot_J}^\bullet = \{e_J\}^\bot$. This means that $\vdash_{J\cap K} \Gamma|_J$ is provable. Hence $\vdash_{J\cap K} \Gamma|_J, \bot_{J\cap K}$ is provable, that is $(K, \Gamma)(J, \bot_J) \in \bot$ as required.

Assume $A = B \otimes C$, of domain J. We have

$$(B \otimes C)^\bullet = (B^\bullet C^\bullet)^{\bot\bot} \subseteq (\{(J, B)\}^\bot \{(J, C)\}^\bot)^{\bot\bot}$$

by inductive hypothesis. So it will be sufficient to prove that

$$(J, B \otimes C) \in (\{(J, B)\}^\bot \{(J, C)\}^\bot)^\bot .$$

Let $(K, \Gamma) \in \{(J, B)\}^\bot$ and let $(L, \Delta) \in \{(J, C)\}^\bot$, and thus the sequents

$$\vdash_{K\cap J\cap L} \Gamma|_{J\cap L}, B|_{K\cap L} \quad \text{and} \quad \vdash_{L\cap J\cap K} \Delta|_{J\cap K}, C|_{L\cap K}$$

are provable. Applying a tensor rule, we get

$$(K, \Gamma)(L, \Delta)(J, B \otimes C) \in \bot$$

as required.

Assume that $A = B \parr C$, of domain J. We have

$$(B \parr C)^\bullet = (B^{\bullet\bot} C^{\bullet\bot})^\bot \subseteq (\{(J, B)\}\{(J, C)\})^\bot = \{(J, [B, C])\}^\bot$$

by inductive hypothesis, and so it is sufficient to prove that

$$\{(J, [B, C])\}^\bot \subseteq \{(J, B \parr C)\}^\bot$$

which is obtained by a simple application of the par rule.

Assume that $A = B \oplus C$, with B of domain J and C of domain K, with $J \cap K = \emptyset$. We have

$$(B \oplus C)^\bullet = (e_J B^\bullet \cup e_K C^\bullet)^{\bot\bot} \subseteq (e_J\{(J, B)\}^\bot \cup e_K\{(K, C)\}^\bot)^{\bot\bot}$$

by inductive hypothesis and so it is sufficient to prove that

$$e_J\{(J, B)\}^\bot \subseteq \{(J + K, B \oplus C)\}^\bot .$$

So let $(L, \Gamma) \in \{(J, B)\}^\perp$, which means that $\vdash_{L \cap J} \Gamma|_J, B|_L$ is provable. We have to prove that $e_J(L, \Gamma)(J + K, B \oplus C) \in \perp$, that is:

$$\vdash_{J \cap L} \Gamma|_J, B|_{J \cap L} \oplus C|_\emptyset \quad \text{is provable.}$$

This is obtained by an application of the plus rule.

Last assume that $A = B \mathbin{\&} C$, with B of domain J and C of domain K, with $J \cap K = \emptyset$. We have

$$\begin{array}{rcll} (B \mathbin{\&} C)^\bullet & = & (\perp_J \parr B^\bullet) \mathbin{\&} (\perp_K \parr C^\bullet) & \\ & \subseteq & (\perp_J \parr \{(J, B)\}^\perp) \cap (\perp_K \parr \{(K, C)\}^\perp) & \text{by inductive hypothesis} \\ & = & (e_J\{(J, B)\})^\perp \cap (e_K\{(K, C)\})^\perp & \\ & = & \{(J, B), (K, C)\}^\perp & \end{array}$$

Let $(L, \Gamma) \in \{(J, B), (K, C)\}^\perp$. This means that the sequents

$$\vdash_{L \cap J} \Gamma|_J, B|_L \quad \text{and} \quad \vdash_{L \cap K} \Gamma|_K, C|_L$$

are provable. Applying a with rule, we get that

$$\vdash_{L \cap (J+K)} \Gamma|_{J+K}, (B \mathbin{\&} C)|_L$$

is provable and we conclude. ∎

Theorem 27 (Completeness) *Let A be a formula of MALL(I) such that $e_J \in A^\bullet$ in all phase models of MALL(I). Then the sequent $\vdash_J A$ is provable in MALL(I).*

Proof: Applying the lemma above, we get $e_J \in \{(J, A)\}^\perp$, that is $e_J(J, A) \in \perp$, that is $(J, A) \in \perp$, that is $\vdash_J A$ is provable in MALL(I). ∎

Observe that in the proof of lemma 26, we have not used the cut rule. So the completeness theorem above also holds if, in the definition of M, we take for $\perp$ the set of all pairs (J, Γ) such that the sequent $\vdash_\Gamma J$ is *cut-free provable*. But the soundness theorem 25 holds in particular for that new model. In this way, we get a proof of the cut-elimination theorem 16 for MALL(I)[6].

6 Product phase models

The goal of this short section is to show that the particular class of phase models of MALL(I) which are of the shape $(P_0^I, \perp_I, (\varepsilon_J)_{J \subseteq I})$, and that we shall call in the sequel *product phase models* is essentially incomplete for MALL(I).

A first observation is that these models admit a “partiality” principle. We can extend MALL by adding the following axioms, or *paralogisms* in Girard’s terminology [Gir99b] (for any sequence of formulae Γ):

$$\frac{}{\vdash \Gamma}$$

The proof of Γ consisting of that axiom will be called Ω_Γ.

[6]The idea of proving an inclusion and not an equality in lemma 26 is due to Okada [Oka99]. The equality holds, but proving it would involve the use of the cut rule and hence would prevent us from deriving cut-elimination from completeness.

A proof of a sequent containing some occurrences of that axiom should be considered as a partial proof, to be completed by proofs of the sequents introduced by that axiom (if possible), so that adding these axioms to MALL is similar to an operation which is standard in λ-calculus, and which consists in extending the language by a new constant, usually denoted by Ω, and which denotes a term on which no information is available (Böhm trees for instance are defined in that way). Let us call MALL_Ω the corresponding system.

The denotational semantics of a proof π of a sequent Γ in that system is still a subset of $|\Gamma^*|$, and is defined as prescribed in section 1, with the following interpretation for the new axioms:

$$\Omega_\Gamma{}^* = \emptyset$$

which, by the way, is the only "polymorphic" possible choice.

We define accordingly an extension $\mathsf{MALL}_\Omega(I)$ of $\mathsf{MALL}(I)$ by adding, for each multiset Γ of formulae *of empty domain*, the following axiom

$$\overline{\vdash_\emptyset \Gamma}$$

so that $\mathsf{MALL}_\Omega(I)$ is a conservative extension of MALL_Ω, in the sense explained in section 4.

The analogue of proposition 20 is easily seen to hold for $\mathsf{MALL}_\Omega(I)$ (the proof is the same, one has just to consider an additional base case corresponding to the new axiom scheme).

A reasonable conjecture would be that the class of product phase models is complete for $\mathsf{MALL}_\Omega(I)$, and this would imply that any clique which is invariant by all the logical relations introduced in section 2 would be contained in the interpretation of some proof of $\mathsf{MALL}_\Omega(I)$. However, surprisingly enough, this is not the case, as shown by the following counter-example.

Let A and B be two formulae of $\mathsf{MALL}(I)$ of disjoint domains L and R respectively. Then the formula

$$C = (A \mathbin{\&} B) \multimap ((A \mathbin{\&} 1_R) \otimes (1_L \mathbin{\&} B))$$

of $\mathsf{MALL}(I)$, which has domain $L + R$, is valid in any product phase model of $\mathsf{MALL}(I)$, that is, ε_{L+R} always belongs to the interpretation of C in such a model[7], as easily checked: let $(P_0^I, \perp_I)$ be a product phase model of $\mathsf{MALL}(I)$, and let $p \in (A \mathbin{\&} B)^\bullet$. This means that p is an element of P_0^I such that

$$\varepsilon_L p \in A^\bullet \quad \text{and} \quad \varepsilon_R p \in B^\bullet \ .$$

Let $l, r \in P_0^I$ be defined by

$$l_i = \begin{cases} p_i & \text{if } i \in L \\ 1 & \text{if } i \in R \\ 0 & \text{if } i \notin L + R \end{cases}$$

and

$$r_i = \begin{cases} p_i & \text{if } i \in R \\ 1 & \text{if } i \in L \\ 0 & \text{if } i \notin L + R \end{cases}$$

so that clearly $\varepsilon_{L+R} p = lr$. We have

$$l \in (A \mathbin{\&} 1_R)^\bullet \quad \text{and} \quad r \in (1_L \mathbin{\&} B)^\bullet$$

and so $\varepsilon_{L+R} p \in ((A \mathbin{\&} 1_R) \otimes (1_L \mathbin{\&} B))^\bullet$ as announced. But it is easily checked that the formula C is not provable in $\mathsf{MALL}(I)$ (using the cut elimination theorem), and is not provable in $\mathsf{MALL}_\Omega(I)$ either[8].

[7] Observe that the converse linear implication is provable in $\mathsf{MALL}(I)$.

[8] Cut elimination also holds for $\mathsf{MALL}_\Omega(I)$.

7 Back to denotational semantics

After this excursion through the truth value semantics of $\mathsf{MALL}(I)$, we come back to our initial concern, which after all is the denotational semantics of MALL.

Let $(M, \bot, (e_J)_{J\subseteq I})$ be a fixed phase model of $\mathsf{MALL}(I)$, that we abusively denote by M. We shall build a denotational model of MALL.

If E is a set, we define

$$\mathrm{Fam}_I(E) = \sum_{J\subseteq I} E^J \ ,$$

it is the set of families of E indexed by some subset of I.

Definition 28 A M-phase valued space is a pair $X = (|X|, [X])$ where $|X|$ is a set (the web) and $[X]$ is a function from $\mathrm{Fam}_I(|X|)$ to $\mathcal{F}(M)$ satisfying the following uniformity condition: if $K \subseteq J \subseteq I$ and $a \in |X|^J$, then

$$e_K[X](a|_K) = e_K[X](a)$$

where $a|_K$ denotes the restriction of a to K.

Given two M-phase valued spaces X and Y, one defines $X \otimes Y$, $X \parr Y$, $X \oplus Y$ and $X \,\&\, Y$ as we did in section 3: take propositions 11 and 13 as definitions. Explicitly, this means that we take $|X \otimes Y| = |X \parr Y| = |X| \times |Y|$, $|X \oplus Y| = |X \,\&\, Y| = |X| + |Y|$ and that we set

$$[X \otimes Y](a, b) = [X](a) \otimes [Y](b) \quad \text{and} \quad [X \parr Y](a, b) = [X](a) \parr [Y](b)$$

and that, for $a \in |X|^J$ and $b \in |Y|^K$ with $J \cap K = \emptyset$, we set

$$[X \oplus Y](a + b) = [X](a) \underset{J,K}{\oplus} [Y](b) \quad \text{and} \quad [X \,\&\, Y](a + b) = [X](a) \underset{J,K}{\&} [Y](b) \ .$$

Of course, $X^\perp$ is given by $|X^\perp| = |X|$ and $[X^\perp](a) = [X](a)^\perp$. The additive and multiplicative constants are interpreted in a similar way. Checking that these spaces satisfy the uniformity condition is just an application of lemma 23.

A morphism from X to Y is just a subset of $|X| \times |Y|$. Composition is defined as the composition of relations, identity from X to X as the diagonal subset of $|X| \times |X|$. The various operations on morphisms (tensorisation, pairing ...) are defined in the standard way, like in the coherence space semantics.

To any formula A of MALL we can associate a M-phase valued space using the constructions above, we denote this space by A^*_M. We obviously have $|A^*_M| = |A^*|$. Let A be a formula of MALL, let $J \subseteq I$ and let $a \in |A^*|^J$. Then

$$[A^*_M](a) = A\langle a\rangle^\bullet \ .$$

Combining theorems 25, 27 and proposition 20, we get the following "weak" denotational completeness result.

Theorem 29 *Assume that I is infinite (and denumerable). Let A be a formula of MALL and let $x \subseteq |A^*|$. Then the following three conditions are equivalent:*

- *There exists a proof π of A in MALL such that $x \subseteq \pi^*$.*
- *There exists an enumeration a of x by some subset J of I such that $e_J \in [A^*_M](a)$ for all phase models M of $\mathsf{MALL}(I)$.*

- *For any enumeration a of x by some subset J of I, one has $e_J \in [A^*_M](a)$ for all phase models M of* $\mathsf{MALL}(I)$.

Actually, the requirement that I is infinite is not crucial: it suffices to require the cardinality of I to be greater than the cardinality of x.

We now show how coherence semantics can be retrieved as particular cases of this kind of model, like in [Lam95]. Let P be the monoid $\{1, \tau\}$, where 1 is the unit, and whose multiplication is defined by the following equation: $\tau\tau = \tau$. Let I be some fixed finite set, and take

$$\bot_I = \{p \in {P_0}^I \mid (\forall i \in I\ p_i \neq 0) \Rightarrow (\forall i \in I\ p_i = 1)\}$$

or, in an equivalent but rather pedantic way,

$$\bot_I = \{p \in {P_0}^I \mid \prod_{i\in I} p_i \in \{0, 1\}\}\,.$$

This defines a particular product phase model of $\mathsf{MALL}(I)$.

One checks easily that, as soon as I has more than one element (and we assume from now on that it is the case), this phase model has exactly three facts, namely

- $C = {P_0}^I$,
- $\overline{C} = \{p \in {P_0}^I \mid \exists i \in I\ p_i = 0\}$
- and $E = \overline{C} \cup \{e\} = \bot_I$ (where e is the neutral element of ${P_0}^I$, $e_i = 1$ for all $i \in I$)

with obviously $\overline{C} \subset E \subset C$.

In this model, linear negation leaves E unchanged and exchanges C and $\overline{C}$. The multiplicatives are given by the following tables.

$\otimes$	$\overline{C}$	E	C
$\overline{C}$	$\overline{C}$	$\overline{C}$	$\overline{C}$
E	$\overline{C}$	E	C
C	$\overline{C}$	C	C

and

$\parr$	$\overline{C}$	E	C
$\overline{C}$	$\overline{C}$	$\overline{C}$	C
E	$\overline{C}$	E	C
C	C	C	C

Observe also that if J is a *strict* subset of I, then ε_J belongs to all of the facts, so that the only relevant families in the denotational model induced by this phase model are those indexed by I.

Let A be a formula of MALL, and let us say that a I-indexed family of elements of $|A^*|$ is *coherent* if $[A^*_M](a)$ contains the unit of M, that is if $[A^*_M](a) \in \{E, C\}$, and let us say that a is *strictly coherent* if $[A^*_M](a) = C$. One checks easily, by induction on A, that a is coherent and not strictly coherent iff a is a constant family.

Moreover:

- If $A = B \otimes C$, then a can be written $a = (b, c)$ with $b \in |B^*|^I$ and $c \in |C^*|^I$, and then a is coherent in A iff b is coherent in B and c is coherent in C.
- If $A = B \parr C$, then a can be written $a = (b, c)$ with $b \in |B^*|^I$ and $c \in |C^*|^I$, and then a is strictly coherent in A iff b is strictly coherent in B or c is strictly coherent in C.
- If $A = B \oplus C$, then a can be written $a = b + c$ with $b \in |B^*|^J$ and $c \in |C^*|^K$ with some disjoint J and K such that $J + K = I$. Then a is coherent in A iff $K = \emptyset$ and b is coherent in B, or $J = \emptyset$ and c is coherent in C.

- If $A = B \mathbin{\&} C$, then a can be written $a = b + c$ with $b \in |B^*|^J$ and $c \in |C^*|^K$ with some disjoint J and K such that $J + K = I$. Then a is coherent in A iff $K = \emptyset$ implies that b is coherent in B, and $J = \emptyset$ implies that c is coherent in C.

Observe that in the case where I has two elements, this is exactly the definition of the interpretation of MALL in coherence spaces. Endowing A with all these notions of coherence simultaneously amounts to interpreting A in the hypercoherence model (see [Ehr93]).

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