Uniformity and the Taylor expansion of ordinary lambda-terms

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November 7, 2007

Abstract

We define the complete Taylor expansion of an ordinary lambda-term as an infinite linear combination — with rational coefficients — of terms of a resource calculus similar to Boudol's lambda-calculus with multiplicities (or with resources). In our resource calculus, all applications are (multi-)linear in the algebraic sense, *i.e.* commute with linear combinations of the function or the argument. We study the collective behaviour of the beta-reducts of the terms occurring in the Taylor expansion of any ordinary lambda-term, using, in a surprisingly crucial way, a uniformity property that they enjoy. As a corollary, we obtain (that main part of) a proof that this Taylor expansion commutes with Böhm tree computation, syntactically.

Introduction

Although the present article develops a differential approach to the lambda-calculus that we initiated in [ER03], it is self-contained and does not require any technical knowledge of [ER03]. Nevertheless, we think that the differential intuitions developped in that paper are quite helpful for understanding the present work, and therefore, we recall them shortly.

In [ER03], we introduced an extension of the lambda-calculus where terms can be differentiated with respect to their arguments. Typically (in a simply typed version of this *differential lambda-calculus*), if M is a term of type $A \to B$ and if N is a term of type A, we introduce[1] the term $\mathrm{D}M \cdot N$ of type $A \to B$, to be understood as the derivative of the function M with respect to its argument, linearly applied[2] to the value N.

Intuitively, in the term $\mathrm{D}M \cdot N$, the term M is provided with *exactly one* copy N of its argument, and this explains why A is still present as an argument type of $\mathrm{D}M \cdot N$, for the other copies that M might need in computing a result. We argued indeed in the introduction of [ER03] that the mathematical notion of linearity, which is the key concept of differentiation (computing the best possible *linear* approximation of a function), and the logical notion of linearity (a function is *linear* if it uses its argument exactly once) are deeply related, as already strongly suggested by the notations, terminology and denotational semantics of linear logic [Gir87]. The idea of extending linear logic with a differential construction, expressed as an exponential rule, is even mentioned at the end of [Gir87]. But, probably because of the fundamental incompatibility of this construction with both *coherence space* semantics and *totality*, Girard didn't

*This work has been supported by the ACI project GEOCAL.

[1] Actually, the syntax of [ER03] is more complicated since we introduced an explicit notation $\mathrm{D}_i M \cdot N$ for the derivative of M with respect to its ith argument. This has been shown useless by Lionel Vaux in his study of the differential lambda-mu calculus [Vau05].

[2] In standard mathematical notations, the derivative of M is a function M' associating to $x \in A$ a linear map $M'(x)$ from A to B, the differential of M at point x; thus M' has type $A \to (A \multimap B)$ (where $A \multimap B$ is the type of linear maps from A to B). With these notations, our $\mathrm{D}M \cdot N$ has type $A \to B$ and represents $\lambda x^A\,(M'(x)(N))$ so that "$\mathrm{D}M$" could be considered as having type $A \multimap (A \to B)$. But, on purpose, we did not introduce the syntactic construction $\mathrm{D}M$ for not having to introduce explicitely linear types in the syntax.

explore this direction further. Taking this idea seriously, we arrived to a differential extension of linear logic presented in [ER06b].

Since the differential allows to write all the derivatives of a lambda-term, it also allows to write formal *Taylor expansions* of lambda-terms, and it is quite temting to understand the *operational meaning* of such expansions. At the end of [ER03], we proved a result relating, in a special case, the Taylor expansion of a lambda-term to its *linear head reduction*[3]. More precisely, given two ordinary lambda-terms M and N such that $(M)\,N$ is β-equivalent to a variable $*$, we studied the Taylor expansion of that application, which is the following infinite linear combination of differential lambda-terms

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\mathrm{D}^n M \cdot N^n\right) 0\,,$$

where we use $\mathrm{D}^n M \cdot N^n$ for the n-th derivative of M with respect to its first parameter (it corresponds to an n-linear function) linearly applied n times to N, that is: $\mathrm{D}(\cdots \mathrm{D}M \cdot N \cdots) \cdot N$. We showed that, with our reduction rules for the differential lambda-calculus, in that sum, there is exactly one term which does not reduce to 0, and that the order n of that term corresponds to the number of times N arrives in head position during the linear head reduction of $(M)\,N$ to $*$.

Our aim here is to generalize the final result of [ER03] in two directions:

- instead of Taylor expanding only one application, we want to Taylor expand *all* the applications occurring in an ordinary lambda-term;
- instead of considering terms which reduce to a variable, we want to consider all possible situations.

We shall show that this generalized Taylor expansion makes sense and we shall give a quite simple explicit formula for the (generalized) Taylor expansion of a lambda-term.

Then we shall prove that one can beta-reduce the Taylor expansion of a lambda-term and obtain a result which generalizes the above described final theorem of [ER03]. In [ER06a], using a version of Krivine machine, we shall deduce from the main theorem of the present paper a result expressing that Taylor expansion and beta-reduction of lambda-terms (in the sense of Böhm tree computation) commute.

Outline. For defining this generalized Taylor expansion of lambda-terms, we shall introduce here a "target language" which is much simpler than the full differential lambda-calculus of [ER03], and which can be seen as a sublanguage of that calculus. Indeed, the general application of lambda-calculus will not be needed anymore, we shall only need iterated "differential applications" followed by an application to 0, corresponding to differential lambda-terms like $(\mathrm{D}^n M \cdot (N_1, \ldots, N_n))\,0$ (where $\mathrm{D}^n M \cdot (N_1, \ldots, N_n)$ is just a notation for the iterated differential application $\mathrm{D}(\ldots \mathrm{D}(\mathrm{D}M \cdot N_1) \cdot N_2) \ldots \cdot N_n)$. Keeping in mind that such a differential application is "symmetric" in the sense that its value does not change when we permute the N_is (this corresponds to the Schwarz Lemma of calculus), in our target language, we replace ordinary application by a multi-set-based notion of application: given a term s and a finite multi-set $T = t_1 \ldots t_n$ of terms[4], we allow the formation of a term $\langle s\rangle\,T$ to be understood as corresponding to the differential lambda-term $(\mathrm{D}^n s \cdot (t_1, \ldots, t_n))\,0$.

Interestingly, the calculus we arrive to by these considerations is very similar to Boudol's lambda-calculus with multiplicities or with resources (see [Bou93, BCL99]) and Kfoury's linearized lambda-calculus [Kfo00], but we insist on its standard algebraic aspects, supported by the fact that it admits the already mentioned quite natural vector space model of [Ehr05] (finiteness spaces).

This calculus has a notion of reduction, which corresponds to the differential beta-reduction of [ER03] : standard substitution is replaced by a linear version of substitution which can be seen as a partial derivative. For this reduction, the calculus enjoys confluence as well as strong normalization, even in the untyped case (from the viewpoint of linear logic, this is due to the fact that the promotion rule is absent from this calculus, see also [ER06b]).

[3] A modified beta-reduction considered explicitly for the first time by De Bruijn and called by him *mini-reduction* [DB87]; it is the reduction implemented by Krivine's abstract machine [Kri85, Kri05] and it has been extensively studied by Danos and Regnier, see for instance [DR99].

[4] Written as a product, for reasons which should be clear if one has in mind the semantics outlined in the final section of [ER03] and thoroughly presented in [Ehr05], where we insist on the fact that the space $!X$ has not only a standard co-algebraic structure which accounts for the structural rules of logic, but also an *algebraic* structure, accounting for this multi-set construction.

In this resource calculus, we are now able to define inductively the Taylor expansion M^* of an ordinary lambda-term M: it will be an infinite formal linear combination of simple[5] resource terms (with coefficients in a field), and should satisfy, in the case of an application:

$$((M)\,N)^* = \sum_{n=0}^{\infty} \frac{1}{n!} \langle M^* \rangle\, N^{*n}\,,$$

in accordance with the intended meaning, and with the denotational semantics, of application in this resource calculus. Of course we have to give meaning to the operations involved in that sum, and especially to the expression N^{*n}, where N^* will itself be an infinite linear combination of simple terms. As we shall see, this can be done using a version of the *multinomial equation* that we shall explain in Section 2.1, and one obtains in that way a direct expression of the Taylor expansion of M:

$$M^* = \sum_{t \in \mathcal{T}(M)} \frac{1}{\mathsf{m}(t)} t$$

where $\mathcal{T}(M)$ is the set of all simple resource terms which have "the same shape" as M, and $\mathsf{m}(t)$ is a positive integer called the *multiplicity coefficient* of t ("mutiplicity" because this number is larger when t has more repeated patterns). Up to some minor variations, the resource terms which are in some $\mathcal{T}(M)$ are those called *well formed* in [Kfo00]. We characterize these terms as those which are coherent with themselves for a coherence relation on simple resource terms, and call them *uniform* (not "well formed", because we are very much interested by the other terms as well, and also because this usage of the word "uniform" is reminiscent of a corresponding notion in denotational semantics, see the discussions in [BE01]).

The main purpose of the paper is then to study the behaviour of the Taylor expansion of an ordinary lambda-term M when one reduces its simple summands, which are all strongly normalizing, even if M is not. Let us denote by $\mathsf{supp}(t)$ the *support* of a resource term t, that is, the set of all *simple* terms which appear with a non-zero coefficient in t (a *resource term* will be, by definition, a possibly infinite linear combination of simple resource terms). Let us also denote by $\mathsf{NF}(t)$ the normal form of the simple resource term t, so that $\mathsf{NF}(t)$ is a finite linear combination of simple resource terms with coefficients which are positive integers.

Thanks to the uniformity and coherence of the resource terms which belong to $\mathcal{T}(M)$, the situation is quite simple:

- For two distinct simple terms t and t' in $\mathcal{T}(M)$, the supports of $\mathsf{NF}(t)$ and $\mathsf{NF}(t')$ are disjoint;
- For that reason, it makes sense to add the normal forms of all the elements t of $\mathcal{T}(M)$, getting a generally infinite sum s of simple terms with rational coefficients.
- Moreover, if $u \in \mathsf{supp}(\mathsf{NF}(t))$ for some $t \in \mathcal{T}(M)$, the coefficient of u in $\mathsf{NF}(t)$ is $\mathsf{m}(t)/\mathsf{m}(u)$, and hence the coefficient of a normal simple term u occurring in the sum s defined in the item above is just $1/\mathsf{m}(u)$.
- Last, all these normal simple terms are coherent with each other (and in particular, uniform).

So this (generally) infinite sum s of normal simple terms looks like the Taylor expansion of an ordinary lambda-term, and actually it is the Taylor expansion of the Böhm tree of M; this complementary result is explained in [ER06a], using a decorated version of Krivine machine.

1 Syntax

1.1 Notation and terminology

If X is a finite set, we use $|X|$ for its cardinality. For us the word integer means non-negative integer.

[5]We call *simple* a resource term which is not a linear combination of resource terms. Since all the operations of the resource lambda-calculus are linear, any term obtained by combining terms along the syntax of the resource lambda-calculus can be written in an unique way as a linear combination of simple terms, exactly as for polynomials in algebra: *simple terms play the role of monomials.*

In this paper we deal with some kind of power series. This notion involves two kinds of numbers: coefficients and exponents. Power series have a natural vector space (or more generally module) structure, which requires an addition and a multiplication on coefficients, more precisely, a semi-ring structure on coefficients. On the other hand, exponents have to be natural numbers.

1.1.1 I-indexed families. Let R and I be sets; we use R^I for the set of I-indexed families of elements of R, or equivalently the set of applications from I to R. An I-indexed family is denoted as $(x_u)_{u\in I}$ or as a map $x : I \mapsto R$, depending on the context.

1.1.2 Free modules. Suppose R is a commutative semi-ring: R has a commutative addition with a zero, and a commutative multiplication that is distributive over addition. Given an I-indexed family x, we use $\mathsf{supp}(x)$ for the *support* of x, that is, the set $\{u \in I, x_u \neq 0\}$.

We use $R\langle I\rangle$ for the subset of R^I consisting of families with a finite support, that is the free R-module on the set I. Concretely we view $R\langle I\rangle$ as the set of finite linear combinations of elements of I with coefficients in R. We therefore denote the family (x_u) in $R\langle I\rangle$ as the sum $\sum_{u\in I} x_u u$ which has only finitely many nonzero terms.

1.1.3 Multi-sets. In the particular case where $R = \mathbb{N}$, we may alternatively view $R\langle I\rangle$ as the free commutative monoid over I. We use $\mathcal{M}_{\mathrm{fin}}(I)$ for the set $\mathbb{N}\langle I\rangle$ and call its elements the *finite multi-sets* over I. Finite multisets are ranged over by the letters $S, T\dots$

Let S be a finite multi-set over I. We call *multiplicity* of u in S the number $S(u)$. The *cardinality* of S is the number $|S| = \sum_{u\in I} S(u)$ and its *underlying set* is $\mathsf{set}(S) = \{u \in I \mid S(u) \neq 0\}$ ($\mathsf{set}(S)$ is just another notation for $\mathsf{supp}(S)$, dedicated to multi-sets; we use sometimes the notation $u \in S$ instead of $u \in \mathsf{set}(S)$). If $n \in \mathbb{N}$, we use $\mathcal{M}_n(I)$ for the set of all $S \in \mathcal{M}_{\mathrm{fin}}(I)$ such that $|S| = n$.

Let $S, T \in \mathcal{M}_{\mathrm{fin}}(I)$. The *multi-set union* of S and T is the multi-set U defined by $U(u) = S(u)+T(u)$. This is of course the monoid operation on $\mathcal{M}_{\mathrm{fin}}(I)$ and its neutral element is the empty multi-set. Depending on the context, we use one of two notations for this operation: the *additive* notation $U = S+T$ (to be used when the multi-sets represent multi-exponents) and the *multiplicative* notation $U = ST$ (to be used when the multi-sets represent monomials).

1.1.4 Multi-sets as monomials. Multi-sets will be used for representing coefficient-free monomials. Suppose *e.g.* that I is a set of variables and pick for example two variables u and v in I; then we will write $u^p v^q$ for the multi-set where u has multiplicity p, v has multiplicity q, all the multiplicities of the other variables in I being 0. In this context, considering two multi-set $S, T \in \mathcal{M}_{\mathrm{fin}}(I)$ as monomials, it is natural to use ST to denote their multi-set union, since this operation corresponds to the product of monomials. Accordingly, in this context, we use 1 for the empty multi-set. As it is standard, given any $u \in I$, we shall identify the multi-set/monomial u^1 with u.

1.1.5 Multi-sets as multi-exponents. Let now x be a function from I to any commutative monoid R and let $S \in \mathcal{M}_{\mathrm{fin}}(I)$. Then we denote by x^S the value $\prod_{u\in I} x(u)^{S(u)} \in R$ of the monomial S under the valuation x. In this context we consider S as a multi-exponent. If T is another monomial on I then we have $x^S x^T = x^U$ where U is, again, the multi-set union of S and T so we are driven, in this context, to use an additive notation in order to get the usual equation $x^S x^T = x^{S+T}$.

We also extend to finite multi-sets (considered as multi-exponents) some notations which are standard for integers. We first define the *factorial* of S as $S! = \prod_{u\in I} S(u)!$ (this product having only finitely many factors different from 1). Observe that $S! = 1$ if S is a "set" in the sense that $\forall u \in I\, S(u) \in \{0, 1\}$. We define next the *multinomial coefficient*

$$[S] = \frac{|S|!}{S!} = \frac{\left(\sum_{u\in I} S(u)\right)!}{\prod_{u\in I} S(u)!} \in \mathbb{N}$$

which is the number of distinct enumerations of the elements of S (taking repetitions into account). For instance, if u and v are two distinct elements of I, then $[u^{n-p}v^p] = \binom{n}{p}$. More generally, if $u_1, \dots, u_k$ are pairwise distinct elements of I and $n_1, \dots, n_k \in \mathbb{N}$ with $n_1 + \cdots + n_k = n$, then $[u_1^{n_1} \dots u_k^{n_k}] = \frac{n!}{n_1! \dots n_k!} = \binom{n}{n_1,\dots,n_k}$ is the coefficient of the monomial $u_1^{n_1} \dots u_k^{n_k}$ in the expansion of $(u_1 + \cdots + u_k)^n$ in the algebra of polynomials with variables $u_1, \dots, u_k$, over any field of characteristic 0.

Given $S, T \in \mathcal{M}_{\mathrm{fin}}(I)$, one defines $S + T$ and $T \leq S$, as well as $S - T$ if $T \leq S$, in the obvious, pointwise way.

All these notations are compatible with standard mathematical practice. For instance, given $S, T \in \mathcal{M}_{\mathrm{fin}}(I)$ with $T \leq S$, we define the generalized binomial coefficient

$$\binom{S}{T} = \frac{S!}{T!(S-T)!} = \prod_{u \in I} \binom{S(u)}{T(u)} \in \mathbb{N} \tag{1}$$

where, in the last expression, the binomial coefficients are the standard ones, defined on natural numbers. Observe that $\binom{S}{T} = \binom{S}{S-T}$.

Given two valuations x and y from I to some commutative semi-ring, the binomial equation generalizes to

$$(x+y)^S = \sum_{T \leq S} \binom{S}{T} x^T y^{S-T} .$$

For instance, if $u \in I$ is such that $S(u) \geq 1$, then $U = S - u$ is the multi-set defined by $U(v) = S(v)$ if $v \neq u$ and $U(u) = S(u) - 1$. This multi-set $S - u$ corresponds to the multi-set S, from which one instance of u has been removed. One has $\binom{S}{S-u} = S(u)$.

Also, the classical Pascal formula holds under the following guise: given $S, U \in \mathcal{M}_{\mathrm{fin}}(I)$ and $u \in I$, with $U \leq S$ and $S(u) > U(u) > 0$, one has

$$\binom{S}{U} = \binom{S-u}{U} + \binom{S-u}{U-u} . \tag{2}$$

1.2 Syntax of the resource calculus

Let $\mathcal{V}$ be a countable set of variables.

1.2.1 Simple terms and simple poly-terms. They are defined by mutual induction, as follows.

Variable: if x is a variable, then x is a simple term.

Linear application: if s is a simple term and T is a simple poly-term, then $\langle s\rangle\, T$ is a simple term, the application of s to T.

Abstraction: if x is a variable and t is a simple term, then $\lambda x\, t$ is a simple term in which, as usual, the variable x is bound.

Poly-terms: any finite multi-set of simple terms is a simple poly-term viewed as a monomial of simple terms (so we use the multiplicative notations for the operations on these multi-sets). The intuition is that each of the elements of such a monomial must be used multi-linearly, that is, exactly as many times as its multiplicity in the monomial.

Let Δ be the set of all simple terms; they will be ranged over by the letters $s, t, \ldots$. Let $\Delta^! = \mathcal{M}_{\mathrm{fin}}(\Delta)$ be the collection of all simple poly-terms, which will be ranged over by the letters $S, T, \ldots$. Then, according to the notations introduced in 1.1.3, remember that $\mathcal{M}_n(\Delta)$ is the set of all the elements S of $\Delta^!$ of the shape $S = s_1 \ldots s_n$, with $s_i \in \Delta$ for $i = 1, \ldots, n$. We use $\Delta^{(!)}$ for Δ or $\Delta^!$ when we do not want to be specific and then we use the letters $\sigma, \tau \ldots$ to range over individuals.

When we write $\langle s\rangle\, t_1 \ldots t_n$ (where $s, t_1, \ldots, t_n$ are simple terms), we mean the linear application of s to the poly-term $t_1 \ldots t_n$. When we want to denote iterated applications, we keep the brackets explicit in order to avoid confusions: we write in that case *e.g.* $\langle \cdots \langle s\rangle\, T_1 \cdots \rangle\, T_p$ and not $\langle s\rangle\, T_1 \cdots T_p$ which would be ambiguous, though compatible with standard lambda-calculus practice.

As in lambda-calculus, we have bound and free variables in simple (poly-)terms. Standard lambda-calculus technics may be applied to this system to define α-equivalence and substitution of a term for a variable into a term.

A (poly-)term σ can have various subterms which are equivalent up to α-equivalence, but nevertheless syntactically distinct. We say that σ is *α-canonical* if this is not the case. Clearly, any (poly-)term admits an α-equivalent α-canonical (poly-)term. We assume all the (poly-)terms we deal with to be in α-canonical form. For instance, an α-canonical form of the simple poly-term $(\lambda x\, x)(\lambda y\, y)$ is $(\lambda x\, x)^2$.

If σ is a simple (poly-)term, we use $\mathsf{fv}(\sigma)$ for the set of all free variables of σ.

In 2.2.2, we shall associate a (generally infinite) set $\mathcal{T}(M)$ of resource terms with any ordinary lambda-term M. The interested reader can already have a look at the definition of $\mathcal{T}(M)$ to get more intuition on the syntax of the resource lambda-calculus and its connection with the syntax of the ordinary lambda-calculus.

1.2.2 Size of a simple (poly-)term. We define the *size* of a simple (poly-)term by the following induction:

- $\mathsf{size}(x) = 1$;
- $\mathsf{size}(\lambda x\, t) = 1 + \mathsf{size}(t)$;
- $\mathsf{size}(\langle t\rangle\, T) = 1 + \mathsf{size}(t) + \mathsf{size}(T)$;
- $\mathsf{size}(t_1 \dots t_n) = n + \sum_{i=1}^{n} \mathsf{size}(t_i)$.

Concerning the last clause, observe that one has $\mathsf{size}(T) = 0$ iff $T = 1$ (the empty simple poly-term).

1.2.3 Finite terms and finite poly-terms. Let R be a semiring with multiplicative unit[6] 1 and let I be a set. Recall that we use $R\langle I\rangle$ for the free R-module generated by I, the set of *finite* linear combinations with coefficients in R of elements of I. If f is a function from I to some R-module E, we use $\widetilde{f}$ for the function $R\langle I\rangle \to E$ which is defined in the obvious way, extending f by linearity.

We call *finite terms* and *finite poly-terms* the elements of $R\langle\Delta\rangle$ and $R\langle\Delta^!\rangle$ respectively, and we extend to these terms our notational conventions: we use letters like s, t, u,... for denoting finite terms and letters like S, T, U,... for denoting finite poly-terms. Also, we use Greek letters to cover both cases. Of course, simple (poly-)terms are considered as particular finite (poly-)terms. Finite combinations of (poly-)terms are mandatory for being able to define partial derivatives of (poly-)terms, see 1.2.4. More general (infinite) linear combinations will be used later for writing Taylor expansions, see Section 2.1.

A possible intuition behind linear combinations is to consider them as non deterministic superimposition of (poly-)terms. The (poly-)term 0 can be considered as a kind of "error" or "failure" expressing that no further computation is possible. It has probably some similarities with the *daemon* of Girard's ludics [Gir01].

We extend by multi-linearity all the constructions of the syntax of 1.2.1 to finite terms and finite poly-terms. For instance, if $U = \sum_{S\in\Delta^!} a_S S$ and $V = \sum_{T\in\Delta^!} b_T T$ are elements of $R\langle\Delta^!\rangle$, the product $UV \in R\langle\Delta^!\rangle$ is defined as $UV = \sum_{S,T\in\Delta^!} a_S b_T ST = \sum_{W\in\Delta^!} c_W W$ where $c_W = \sum_{ST=W} a_S b_T \in R$ vanishes for almost all values of W.

Similarly $\lambda x\, u$ is defined by linearity in u and $\langle u\rangle\, U$ is defined by bilinearity in u and U. In particular, we have $\lambda x\, 0 = 0$ and $\langle 0\rangle\, U = \langle u\rangle\, 0 = 0$. This bilinearity of application justifies the terminology "linear application" for this construction. Standard lambda-calculus application is definitely not linear in the argument (see the introduction of [ER03]). The point of the Taylor formula is precisely to provide an analysis of this non-linearity.

1.2.4 Partial derivatives. We define now formally the finite (poly-)term $\frac{\partial\sigma}{\partial x}\cdot t$ where σ is a finite (poly-)term, x is a variable and t is a finite term. This will be called the *partial derivative* of σ with respect to x in the direction t. The intuition is that $\frac{\partial\sigma}{\partial x}\cdot t$ is the (poly-)term σ where *exactly one occurrence* of x is replaced by the simple term t. Of course, since σ can contain several occurrences of x, there are several ways to perform this substitution, whence the sums which appear in this definition.

[6] At some point, we shall require that each element of the shape $n \cdot 1$ (with $n \in \mathbb{N}^+$) has an inverse, as for instance in the semiring of positive rational numbers.

We first give the definition for σ simple and t finite:

$$\begin{aligned}
\frac{\partial y}{\partial x}\cdot t &= \begin{cases} t & \text{if } y = x \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \lambda y\, s}{\partial x}\cdot t &= \lambda y \left(\frac{\partial s}{\partial x}\cdot t\right) \quad \text{with the usual proviso that } x \neq y \text{ and } y \text{ is not free in } t \\
\frac{\partial \langle s\rangle\, T}{\partial x}\cdot t &= \left\langle \frac{\partial s}{\partial x}\cdot t\right\rangle T + \langle s\rangle \left(\frac{\partial T}{\partial x}\cdot t\right) \\
\frac{\partial s_1 \dots s_n}{\partial x}\cdot t &= \sum_{i=1}^{n} s_1 \dots s_{i-1}\left(\frac{\partial s_i}{\partial x}\cdot t\right) s_{i+1}\dots s_n\,.
\end{aligned}$$

Observe that, due to the last two rules, even when $t \in \Delta$ is simple, $\frac{\partial \sigma}{\partial x}\cdot t$ is generally a non-trivial sum, that is, $\frac{\partial \sigma}{\partial x}\cdot t$ is a finite (poly-)terms which is generally not simple.

The following properties follow from the above definition:

$$\begin{aligned}
\frac{\partial 1}{\partial x}\cdot t &= 0 \\
\frac{\partial ST}{\partial x}\cdot t &= \left(\frac{\partial S}{\partial x}\cdot t\right)T + S\left(\frac{\partial T}{\partial x}\cdot t\right) \\
\frac{\partial sT}{\partial x}\cdot t &= \left(\frac{\partial s}{\partial x}\cdot t\right)T + s\left(\frac{\partial T}{\partial x}\cdot t\right).
\end{aligned}$$

For instance, if s and t are two simple terms, one has $\frac{\partial s^2}{\partial x}\cdot t = 2s\left(\frac{\partial s}{\partial x}\cdot t\right)$.

Lemma 1 *Let σ be a simple (poly-)term, x be a variable and t be a simple term. Then, for any $\tau \in \mathsf{supp}(\frac{\partial \sigma}{\partial x}\cdot t)$, one has $\mathsf{size}(\tau) = \mathsf{size}(\sigma) + \mathsf{size}(t) - 1$.*

The proof is a straightforward induction on σ. The "-1" corresponds to the fact that exactly one occurrence of x disappears in this process.

Finally, we extend the definition of the partial derivative $\frac{\partial \sigma}{\partial x}\cdot t$ to the case where σ is a finite (poly-)term by linearity. Partial derivation should be understood as a linear substitution operation. Indeed one shows easily that $\frac{\partial \sigma}{\partial x}\cdot t$ it is linear in t. Moreover, it is clear that $\frac{\partial \sigma}{\partial x}\cdot t = 0$ as soon as x does not occur free in σ.

1.2.5 Iterated partial derivatives. The following lemma expresses that partial derivatives commute with each others. It corresponds to Schwarz Lemma in analysis. Here of course the lemma boils down to a simple formal verification.

Lemma 2 *Let σ be a finite (poly-)term and let s and t be finite terms. Let x and y be variables such that x does not occur free in t. Then we have*

$$\frac{\partial}{\partial y}\left(\frac{\partial \sigma}{\partial x}\cdot s\right)\cdot t = \frac{\partial}{\partial x}\left(\frac{\partial \sigma}{\partial y}\cdot t\right)\cdot s + \frac{\partial \sigma}{\partial x}\cdot\left(\frac{\partial s}{\partial y}\cdot t\right)$$

and in particular, when y does not occur free in s,

$$\frac{\partial}{\partial y}\left(\frac{\partial \sigma}{\partial x}\cdot s\right)\cdot t = \frac{\partial}{\partial x}\left(\frac{\partial \sigma}{\partial y}\cdot t\right)\cdot s\,.$$

Proof. The second equation follows easily from the first one, which is proved by induction on the size of the simple (poly-)term σ. We just check the case where $\sigma = \langle u\rangle\, U$. One has

$$\begin{aligned}
\frac{\partial}{\partial y}\Big(\frac{\partial \sigma}{\partial x}\cdot s\Big)\cdot t &= \frac{\partial}{\partial y}\Big(\Big\langle \frac{\partial u}{\partial x}\cdot s\Big\rangle U + \langle u\rangle\, \frac{\partial U}{\partial x}\cdot s\Big)\cdot t\\
&= \frac{\partial}{\partial y}\Big(\Big\langle \frac{\partial u}{\partial x}\cdot s\Big\rangle U\Big)\cdot t + \frac{\partial}{\partial y}\Big(\langle u\rangle\, \frac{\partial U}{\partial x}\cdot s\Big)\cdot t\\
&= \Big\langle \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}\cdot s\Big)\cdot t\Big\rangle U + \Big\langle \frac{\partial u}{\partial x}\cdot s\Big\rangle \Big(\frac{\partial U}{\partial y}\cdot t\Big)\\
&\quad + \Big\langle \frac{\partial u}{\partial y}\cdot t\Big\rangle \Big(\frac{\partial U}{\partial x}\cdot s\Big) + \langle u\rangle \Big(\frac{\partial}{\partial y}\Big(\frac{\partial U}{\partial x}\cdot s\Big)\cdot t\Big)
\end{aligned}$$

so that, applying the inductive hypothesis, we get

$$\begin{aligned}
\frac{\partial}{\partial y}\Big(\frac{\partial \sigma}{\partial x}\cdot s\Big)\cdot t &= \Big\langle \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\cdot t\Big)\cdot s\Big\rangle U + \Big\langle \frac{\partial u}{\partial x}\cdot \Big(\frac{\partial s}{\partial y}\cdot t\Big)\Big\rangle U + \Big\langle \frac{\partial u}{\partial x}\cdot s\Big\rangle \Big(\frac{\partial U}{\partial y}\cdot t\Big)\\
&\quad + \Big\langle \frac{\partial u}{\partial y}\cdot t\Big\rangle \Big(\frac{\partial U}{\partial x}\cdot s\Big) + \langle u\rangle \Big(\frac{\partial}{\partial x}\Big(\frac{\partial U}{\partial y}\cdot t\Big)\cdot s\Big) + \langle u\rangle \Big(\frac{\partial U}{\partial x}\cdot\Big(\frac{\partial s}{\partial y}\cdot t\Big)\Big)\\
&= \Big\langle \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\cdot t\Big)\cdot s\Big\rangle U + \Big\langle \frac{\partial u}{\partial x}\cdot s\Big\rangle \Big(\frac{\partial U}{\partial y}\cdot t\Big)\\
&\quad + \Big\langle \frac{\partial u}{\partial y}\cdot t\Big\rangle \Big(\frac{\partial U}{\partial x}\cdot s\Big) + \langle u\rangle \Big(\frac{\partial}{\partial x}\Big(\frac{\partial U}{\partial y}\cdot t\Big)\cdot s\Big)\\
&\quad + \Big\langle \frac{\partial u}{\partial x}\cdot \Big(\frac{\partial s}{\partial y}\cdot t\Big)\Big\rangle U + \langle u\rangle \Big(\frac{\partial U}{\partial x}\cdot\Big(\frac{\partial s}{\partial y}\cdot t\Big)\Big)\\
&= \frac{\partial}{\partial x}\Big(\frac{\partial \sigma}{\partial y}\cdot t\Big)\cdot s + \frac{\partial \sigma}{\partial x}\cdot\Big(\frac{\partial s}{\partial y}\cdot t\Big)
\end{aligned}$$

as expected. □

So we introduce the standard notation

$$\frac{\partial^n \sigma}{\partial x_1\cdots\partial x_n}\cdot(t_1,\dots,t_n) = \frac{\partial}{\partial x_n}\Big(\cdots\frac{\partial \sigma}{\partial x_1}\cdot t_1\cdots\Big)\cdot t_n$$

when no x_i occurs free in any of the simple terms t_j. For any permutation f of $\{1,\dots,n\}$, we have

$$\frac{\partial^n \sigma}{\partial x_1\cdots\partial x_n}\cdot(t_1,\dots,t_n) = \frac{\partial^n \sigma}{\partial x_{f(1)}\cdots\partial x_{f(n)}}\cdot\big(t_{f(1)},\dots,t_{f(n)}\big) \tag{3}$$

1.2.6 Ordinary substitution. As already mentioned, one can also define a substitution operation of a finite term t for a variable x in a simple (poly)-term σ, yielding a finite (poly-)term that we denote as $\sigma\,[t/x]$. This operation is then extended by linearity on σ to arbitrary (poly-)terms σ. However, just as ordinary lambda-calculus application is not linear in the argument, this notion of substitution is not linear in t, in sharp contrast with the partial derivative operation defined above.

This operation will be used essentially when t is the finite term 0, in which case it is a simple *occur-check* of x in σ: $\sigma\,[0/x]$ is equal to 0 if x occurs free in σ and to σ otherwise, see Lemma 3.

It will also be used for substituting variables for other variables. In that case, we write $\sigma\,[x\,/x_1,\dots,x_n]$ for the (poly-)term σ where the variables $x_1,\dots,x_n$ are replaced by x.

1.2.7 Degree of a simple (poly-)term in a variable. If σ is a *simple* (poly-)term and x is a variable, the *degree* of σ in x is the number of free occurrences of x in σ, taking multiplicities into account. This number is denoted by $\mathsf{deg}_x(\sigma)$. For instance, the degree of the simple term $\langle x\rangle\,(\langle x\rangle\, y^2)^3$ in x is 4 and its degree in y is 6. Due to the fact that all the syntactic constructions of this calculus are linear, this notion of degree coincides with the standard algebraic one.

Typically, if σ is a simple (poly-)term and if $a \in R$, we have $\sigma\,[ax/x] = a^{\mathsf{deg}_x(\sigma)}\sigma$. Also, $\mathsf{deg}_x(ST) = \mathsf{deg}_x(S) + \mathsf{deg}_x(T)$ when S and T are simple poly-terms, and $\mathsf{deg}_x(t_1\dots t_n) = \sum_{i=1}^n \mathsf{deg}_x t_i$ when the t_is are simple terms.

Lemma 3 *Let σ be a simple (poly-)term and let t be a* simple *term. Let x be a variable and let $n = \mathsf{deg}_x\sigma$. then $\frac{\partial\sigma}{\partial x}\cdot t$ is a sum $\sigma_1+\cdots+\sigma_n$ of n simple (poly-)terms and one has $\mathsf{deg}_x\sigma_i = \mathsf{deg}_x\sigma + \mathsf{deg}_x t - 1$ for each $i = 1,\ldots,p$. In particular, when $n = \mathsf{deg}_x\sigma = 0$, one has $\frac{\partial\sigma}{\partial x}\cdot t = 0$.*

Last

$$\sigma\,[0/x] = \begin{cases}\sigma & \textit{if } \mathsf{deg}_x\sigma = 0\\ 0 & \textit{if } \mathsf{deg}_x\sigma > 0\end{cases}$$

and

$$\mathsf{deg}_x(\sigma\,[x\,/x_1,\ldots,x_m]) = \sum_{i=1}^{m}\mathsf{deg}_{x_i}\sigma\,.$$

The proof is by induction on σ. As an example, let us check the first statement, in the case where $\sigma = \langle s\rangle\,T$, s being a simple term and T being a simple poly-term. Then by inductive hypothesis, setting $p = \mathsf{deg}_x s$ and $q = \mathsf{deg}_x T$, one has $\frac{\partial s}{\partial x}\cdot t = s_1+\cdots+s_p$ where each s_i is a simple term which satisfies $\mathsf{deg}_x s_i = \mathsf{deg}_x s + \mathsf{deg}_x t - 1$, and $\frac{\partial T}{\partial x}\cdot t = T_1+\cdots+T_q$ where each T_j is a simple poly-term which satisfies $\mathsf{deg}_x T_j = \mathsf{deg}_x T + \mathsf{deg}_x t - 1$. But $\frac{\partial\sigma}{\partial x}\cdot t = \left\langle\frac{\partial s}{\partial x}\cdot t\right\rangle T + \langle s\rangle\left(\frac{\partial T}{\partial x}\cdot t\right) = \sum_{i=1}^{p}\langle s_i\rangle\,T + \sum_{j=1}^{q}\langle s\rangle\,T_j$, and this expression is a sum of $p+q = \mathsf{deg}_x\sigma$ simple terms. Moreover, for $i = 1,\ldots,p$, we have $\mathsf{deg}_x(\langle s_i\rangle\,T) = \mathsf{deg}_x s + \mathsf{deg}_x t - 1 + \mathsf{deg}_x T = \mathsf{deg}_x(\langle s\rangle\,T) + \mathsf{deg}_x t - 1$ and similarly for the other summands, as announced.

1.2.8 Big step differentiation. Given a simple term σ, a variable x and a simple poly-term $T = t_1\ldots t_n$ where the variable x does not appear free, we define

$$\partial_x(\sigma,T) = \left(\frac{\partial^n\sigma}{\partial x^n}\cdot(t_1,\ldots,t_n)\right)[0/x] \in R\langle\Delta^{(!)}\rangle \tag{4}$$

which does not depend on the enumeration $t_1,\ldots,t_n$ of T thanks to Equation (3).

By Lemma 3, this expression is non zero iff $n = \mathsf{deg}_x(\sigma)$.

By the same lemma, if x does not occur free in any of the t_is, then x does not occur free in (any of the summands of) $\frac{\partial^n\sigma}{\partial x^n}\cdot(t_1,\ldots,t_n)$.

Lemma 4 *Let σ be a simple (poly-)term and let T be a simple poly-term, and assume that $|T| = \mathsf{deg}_x\sigma = n$. Then, for any $\tau\in\mathsf{supp}(\partial_x(\sigma,T))$, one has $\mathsf{size}(\tau) = \mathsf{size}(\sigma) + \mathsf{size}(T) - n$.*

The proof is by induction on n, applying Lemma 1 at the inductive step.

1.2.9 Extensions of big step differentiations. Observe that Formula (4) still makes sense if $\sigma\in R\langle\Delta^{(!)}\rangle$ and $t_1,\ldots,t_n\in R\langle\Delta\rangle$, and then $\partial_x(\sigma,T)$ is $(n+1)$-linear in $\sigma,t_1,\ldots,t_n$ and symmetric in $t_1,\ldots,t_n$. Therefore, for each $n\in\mathbb{N}$, we can consider $\partial_x(\sigma,T)$ as a bilinear operation $R\langle\Delta^{(!)}\rangle\times R\langle\mathcal{M}_n(\Delta)\rangle\to R\langle\Delta^{(!)}\rangle$.

Next, this operation can canonically be extended as a bilinear map $R\langle\Delta^{(!)}\rangle\times R\langle\Delta^!\rangle\to R\langle\Delta^{(!)}\rangle$, since $R\langle\Delta^!\rangle = \bigoplus_{n=0}^{\infty}R\langle\mathcal{M}_n(\Delta)\rangle$.

We use $\partial_{x_1,\ldots,x_m}(\sigma,T_1,\ldots,T_m)$ for the iterated big step differentiation

$$\partial_{x_m}(\cdots\partial_{x_1}(\sigma,T_1),\cdots,T_m)\,.$$

The value of this expression does not depend on the order we put on the pairwise distinct variables $x_1,\ldots,x_m$. More precisely, if f is any permutation on $\{1,\ldots,m\}$, one has

$$\partial_{x_1,\ldots,x_m}(\sigma,T_1,\ldots,T_m) = \partial_{x_{f(1)},\ldots,x_{f(m)}}(\sigma,T_{f(1)},\ldots,T_{f(m)})\,.$$

1.2.10 Partial derivative *vs.* substitution. The partial derivative can be understood as a linear substitution. Let σ be a simple (poly-)term and let x be a variable. Let $n = \mathsf{deg}_x(\sigma)$ and let $x_1,\ldots,x_n$ be pairwise distinct variables which do not occur free in σ or in t. Let σ' be a simple (poly-)term obtained by replacing the n occurrences of x in σ by the pairwise distinct variables $x_1,\ldots,x_n$. Such a σ' will be called an *x-linearization* of σ in $x_1,\ldots,x_n$. For any simple term t, we have

$$\frac{\partial\sigma}{\partial x}\cdot t = \sum_{i=1}^{n}\sigma'\,[t/x_i]\,[x\,/x_1,\ldots,x_n]\,. \tag{5}$$

This formula extends by linearity to the case where t is not simple, but we shall not use this fact.

Iterating this result, we get the following crucial formula.

Lemma 5 *Let σ be a simple (poly-)term, let x be a variable and let $n = \mathsf{deg}_x \sigma$. Let $T = t_1 \dots t_n$ be a simple poly-term of cardinality n and assume that x is not free in T. Then*

$$\partial_x(\sigma, T) = \sum_{f \in \mathfrak{S}_n} \sigma' \left[t_{f(1)}/x_1, \dots, t_{f(n)}/x_n \right] \tag{6}$$

where $\mathfrak{S}_n$ is the group of all permutations of $\{1, \dots, n\}$.

This formula could also be generalized to situations where σ and T are not necessarily simple, but we shall never need such generalizations.

The meaning of the lemma is that $\partial_x(\sigma, T)$ is obtained by substituting in σ all the n occurrences of x by $t_1, \dots, t_n$, in all possible ways, the result being the sum of these $n!$ possibilities.

1.2.11 Leibniz law and partial derivative. Let σ be a simple (poly-)term and let t be a simple term. Let x, x_1 and x_2 be variables, with $x_1 \neq x_2$ and x not free in σ. Assume moreover that x_1 and x_2 do not occur free in t.

The Leibniz law concerns the interaction between differentiation and contraction, and can be written as follows:

$$\frac{\partial \sigma \left[x / x_1, x_2 \right]}{\partial x} \cdot t = \left(\frac{\partial \sigma}{\partial x_1} \cdot t \right) \left[x / x_1, x_2 \right] + \left(\frac{\partial \sigma}{\partial x_2} \cdot t \right) \left[x / x_1, x_2 \right] . \tag{7}$$

The hypothesis that $x_1, x_2 \notin \mathsf{fv}(t)$ is of course essential: take for instance $\sigma = t = x_1$, then the left-hand side of the equation is x_1 whereas the right-hand side is x.

The proof is a simple induction on σ. Iterating, we obtain the following formula.

Lemma 6 *Let σ be a simple (poly-)term and let T be a simple poly-term. Let x, x_1 and x_2 be variables, with $x_1 \neq x_2$, $x \notin \mathsf{fv}(\sigma)$ and $x, x_1, x_2 \notin \mathsf{fv}(T)$. Then*

$$\partial_x(\sigma \left[x / x_1, x_2 \right], T) = \sum_{UV = T} \binom{T}{U} \partial_{x_1, x_2}(\sigma, U, V) .$$

Proof. Let $n = \mathsf{deg}_x(\sigma \left[x / x_1, x_2 \right]) = \mathsf{deg}_{x_1}(\sigma) + \mathsf{deg}_{x_2}(\sigma)$. If $|T| \neq n$, the equation holds because both expressions vanish. So assume that $|T| = n$ and let us prove the equation by induction on n.

The case $n = 0$ is trivial, so assume $n = |T| > 0$, we can write $T = tS$ for some simple term t and we have

$$
\begin{aligned}
\partial_x(\sigma\,[x\,/x_1,x_2]\,,tS) &= \partial_x\Big(\frac{\partial\sigma\,[x\,/x_1,x_2]}{\partial x}\cdot t,S\Big) \quad\text{by definition of } \partial_x(\_,\_)\\
&= \partial_x\Big(\Big(\frac{\partial\sigma}{\partial x_1}\cdot t\Big)\,[x\,/x_1,x_2]\,,S\Big)+\partial_x\Big(\Big(\frac{\partial\sigma}{\partial x_2}\cdot t\Big)\,[x\,/x_1,x_2]\,,S\Big) \quad\text{by Equation (7)}\\
&= \sum_{UV=S}\binom{S}{U}\Big(\partial_{x_1,x_2}\Big(\frac{\partial\sigma}{\partial x_1}\cdot t,U,V\Big)+\partial_{x_1,x_2}\Big(\frac{\partial\sigma}{\partial x_2}\cdot t,U,V\Big)\Big)\\
&\qquad\qquad\text{by inductive hypothesis}\\
&= \sum_{UV=S}\binom{S}{U}(\partial_{x_1,x_2}(\sigma,tU,V)+\partial_{x_1,x_2}(\sigma,U,tV))\\
&\qquad\qquad\text{by definition of } \partial_{x_1,x_2}(\_,\_)\\
&= \sum_{\substack{U'V=T\\ t\in U'}}\binom{T-t}{U'-t}\partial_{x_1,x_2}(\sigma,U',V)+\sum_{\substack{UV'=T\\ t\in V'}}\binom{T-t}{U}\partial_{x_1,x_2}(\sigma,U,V')\\
&\qquad\qquad\text{setting } U'=tU \text{ and } V'=tV\\
&= \sum_{\substack{U'V'=T\\ t\in U',\,t\in V'}}\Big(\binom{T-t}{U'-t}+\binom{T-t}{U'}\Big)\partial_{x_1,x_2}(\sigma,U',V')\\
&\quad+\sum_{\substack{U'V'=T\\ t\in U',\,t\notin V'}}\binom{T-t}{U'-t}\partial_{x_1,x_2}(\sigma,U',V')+\sum_{\substack{U'V'=T\\ t\notin U',\,t\in V'}}\binom{T-t}{U'}\partial_{x_1,x_2}(\sigma,U',V')\,.
\end{aligned}
$$

We conclude, applying Pascal's formula (2) for the first of these three sums, and observing that, in the two last sums, the binomial coefficients are equal to $\binom{T}{U'}$. Indeed, when U' and V' are such that $U'V'=T$, $t\in U'$ and $t\notin V'$, we have $U'(t)=T(t)$, and hence also $(T-t)(t)=(U'-t)(t)$, so applying Formula (1), we get $\binom{T-t}{U'-t}=\binom{T}{U'}$. When U' and V' are such that $U'V'=T$, $t\notin U'$ and $t\in V'$, one has $\binom{T-t}{U'}=\binom{T}{U'}$ simply because $U'(t)=0$. □

1.3 Reduction and normal forms

1.3.1 Linear relations. If E and F are two R-modules, we say that a relation $\rho\subseteq E\times F$ is *linear* if it is a linear subspace of the direct product $E\times F$ (in other words, if $u\ \rho\ u'$ and $v\ \rho\ v'$ then $au+bv\ \rho\ au'+bv'$ for any $a,b\in R$).

Let I be a set. Given a relation $\rho\subseteq I\times R\langle I\rangle$, we define a linear relation $R\langle\rho\rangle\subseteq R\langle I\rangle\times R\langle I\rangle$ as the linear span of ρ in this product space and call $R\langle\rho\rangle$ the *linear extension* of ρ. Spelling out this definition, we have $u\ R\langle\rho\rangle\ v$ iff we can find $u_1,\dots,u_n\in I$, $a_1,\dots,a_n\in R$ and $v_1,\dots,v_n\in R\langle I\rangle$ such that $u=\sum_{i=1}^n a_iu_i$, $v=\sum_{i=1}^n a_iv_i$ and $u_i\ \rho\ v_i$ for each i.

1.3.2 Small step (non-deterministic) reduction. A *redex* is a simple term of the shape $\langle\lambda x\,s\rangle\,S$ *where we always assume that x is not free in S*. As usual, this condition can always be fulfilled by simply α-converting the abstraction $\lambda x\,s$.

The reduction of such a redex is defined by cases, according to whether S is empty or not. The second case is non-deterministic as it consists in choosing an element u in S and then in computing a partial derivative of s in the direction u. The result of such a reduction is a linear combination of simple terms, with integer coefficients.

$$
\begin{aligned}
\langle\lambda x\,s\rangle\,1 &\quad \beta^1_\Delta \quad s\,[0/x]\in R\langle\Delta\rangle\\
\langle\lambda x\,s\rangle\,uT &\quad \beta^1_\Delta \quad \Big\langle\lambda x\Big(\frac{\partial s}{\partial x}\cdot u\Big)\Big\rangle T\in R\langle\Delta\rangle\,,
\end{aligned}
$$

so that β^1_Δ is a relation from Δ to $R\langle\Delta\rangle$, that is $\beta^1_\Delta\subseteq\Delta\times R\langle\Delta\rangle$.

The following is a straightforward, but essential observation.

Lemma 7 *Let t and u be simple terms such that, for some finite term t', one has $t\ \beta^1_\Delta\ t'$ and $u \in \mathsf{supp}(t')$. Then $\mathsf{size}(u) < \mathsf{size}(t)$.*

Proof. If we are in the first case of the definition of β^1_Δ, then $\mathsf{size}(u) = \mathsf{size}(t) - 2$ (the abstraction and the application disappear). If we are in the second case, $\mathsf{size}(u) = \mathsf{size}(t) - 2$ as well, by Lemma 1. □

1.3.3 Extending β^1_Δ to all simple contexts.

By extending this reduction to all simple contexts, we define the *one step reduction relation* on simple terms and on simple poly-terms, $\bar\beta^1_\Delta \subseteq (\Delta \times R\langle\Delta\rangle) \cup (\Delta^! \times R\langle\Delta^!\rangle)$. More precisely, we say that $\sigma\ \bar\beta^1_\Delta\ \sigma'$ in one of the following situations:

(Redex) $\sigma\ \beta^1_\Delta\ \sigma'$;

(Abs) $\sigma = \lambda x\, t$ and $\sigma' = \lambda x\, t'$ with $t\ \bar\beta^1_\Delta\ t'$;

(App) $\sigma = \langle t\rangle\, S$ and

- $\sigma' = \langle t'\rangle\, S$ with $t\ \bar\beta^1_\Delta\ t'$ or
- $\sigma' = \langle t\rangle\, S'$ with $S\ \bar\beta^1_\Delta\ S'$;

(Prod) σ is the poly-term uS and $\sigma' = u'S$ with $u\ \bar\beta^1_\Delta\ u'$.

Lemma 8 *Let t and u be simple terms such that, for some finite term t', one has $t\ \bar\beta^1_\Delta\ t'$ and $u \in \mathsf{supp}(t')$. Then $\mathsf{size}(u) < \mathsf{size}(t)$.*

Immediate consequence of Lemma 7.

1.3.4 Linear extension of $\bar\beta^1_\Delta$.

We use β_Δ for the reflexive and transitive closure of $R\langle\bar\beta^1_\Delta\rangle \subseteq (R\langle\Delta\rangle \times R\langle\Delta\rangle) \cup (R\langle\Delta^!\rangle \times R\langle\Delta^!\rangle)$ (the linear extension of $\bar\beta^1_\Delta$, see 1.3.1). This relation $\beta_\Delta \subseteq (R\langle\Delta\rangle \times R\langle\Delta\rangle) \cup (R\langle\Delta^!\rangle \times R\langle\Delta^!\rangle)$ is contextual (in the obvious sense) by construction.

Theorem 9 *The relation $\beta_\Delta \subseteq (R\langle\Delta\rangle \times R\langle\Delta\rangle) \cup (R\langle\Delta^!\rangle \times R\langle\Delta^!\rangle)$ has the following properties:*

- *it is confluent on $R\langle\Delta\rangle$ and on $R\langle\Delta^!\rangle$,*
- *and if $R = \mathbb{N}$, it is strongly normalizing*[7].

Proof. The confluence property is proved as in [ER03] (and is simpler in the present context). The normalization property results from Lemma 8. □

Remark: This untyped calculus is (essentially) strongly normalizing, and so cannot represent general recursive computations as the lambda-calculus does. Later we shall introduce infinite sums which will allow us to encode ordinary lambda-terms, making explicit the potential infiniteness of the lambda-calculus.

If $\sigma \in \Delta^{(!)}$, we use $\mathsf{NF}(\sigma)$ for the unique normal form of σ, which is an element of $\mathbb{N}\langle\Delta^{(!)}\rangle$ (and so can be considered as an element of any $R\langle\Delta^{(!)}\rangle$).

1.3.5 Big step (deterministic) reduction.

We define now a big step reduction relation $\bar\beta^{\mathrm{b}}_\Delta$ which is more convenient for dealing with the problems at hand. The definition is the same as the definition of $\bar\beta^1_\Delta$, replacing the small step redex reduction β^1_Δ by the following one:

$$\langle\lambda x\, s\rangle\, T \quad \beta^{\mathrm{b}}_\Delta \quad \partial_x(s, T)\,,$$

where, as usual, one assumes that x is not free in T. Remember from 1.2.8 that the finite term $\partial_x(s,T)$ is 0, unless $|T| = \mathsf{deg}_x s$.

This reduction is very similar to the β-reduction of the ordinary λ-calculus — $(\lambda x\, M)\, N\ \beta\ M\,[N/x]$ — and for that reason, it is the good notion of reduction on simple terms for studying the Taylor expansion of ordinary lambda-terms. Observe that this reduction is deterministic, in the sense that the reduction of a redex is uniquely determined by the shape of that redex.

[7] This very strong hypothesis can be weakened a little bit as explained in [ER03], but not really significantly.

The relation $\bar{\beta}^{\mathrm{b}}_{\Delta} \subseteq \Delta \times R\langle\Delta\rangle$ is included in β_Δ, and a simple (poly-)term is normal (that is, redex-free) for one of these reductions iff it is normal for the other one. Therefore, for any $\sigma \in \Delta^{(!)}$, we can compute $\mathsf{NF}(\sigma)$ by iteratively applying the reduction $\bar{\beta}^{\mathrm{b}}_{\Delta}$ to σ.

1.3.6 An explicit formula for normal forms.

As in the ordinary lambda-calculus, any simple term s can be written (in a unique way) as follows:

$$s = \lambda x_1 \ldots \lambda x_n \, \langle \cdots \langle t \rangle \, T_1 \cdots \rangle \, T_k$$

where t is a simple term which is either a variable possibly equal to one of the x_is, and in that case we say that s is in *head normal form*, or a redex, and in that case we say that t (or rather, this particular occurrence of t in s) is the *head redex* of s.

We use Δ_0 for the set of normal simple terms. We introduce similarly the notations $\Delta^!_0$ and $\Delta^{(!)}_0$ for normal simple poly-terms and for the union of these two sets.

Lemma 10 *Let σ be a simple (poly-)term. Then $\mathsf{NF}(\sigma) \in \mathbb{N}\langle\Delta^{(!)}_0\rangle$ satisfies the following property.*

- *If $\sigma = \lambda x_1 \ldots \lambda x_n \, \langle \cdots \langle \langle \lambda y \, s \rangle \, S \rangle \, T_1 \cdots \rangle \, T_k$ then*

$$\begin{aligned} \mathsf{NF}(\sigma) &= \widetilde{\mathsf{NF}}(\lambda x_1 \ldots \lambda x_n \, \langle \cdots \langle \partial_y(s, S) \rangle \, T_1 \cdots \rangle \, T_k) \\ &= \sum_{u \in \Delta} \partial_y(s, S)_u (\lambda x_1 \ldots \lambda x_n \, \mathsf{NF}(\langle \cdots \langle u \rangle \, T_1 \cdots \rangle \, T_k)) \end{aligned} \tag{8}$$

  (Remember that we use $\widetilde{\mathsf{NF}}$ for the linear extension of NF to arbitrary finite (poly-)terms and that $\partial_y(s, S)_u$, the coefficient of u in the linear combination of simple terms $\partial_y(s, S)$, is an integer.)

- *If $\sigma = \lambda x_1 \ldots \lambda x_n \, \langle \cdots \langle y \rangle \, T_1 \cdots \rangle \, T_k$ then $\mathsf{NF}(\sigma) = \lambda x_1 \ldots \lambda x_n \, \langle \cdots \langle y \rangle \, \mathsf{NF}(T_1) \cdots \rangle \, \mathsf{NF}(T_k)$.*

- *If $\sigma = t_1 \ldots t_n$ then $\mathsf{NF}(\sigma) = \prod_{i=1}^n \mathsf{NF}(t_i)$.*

The proof is based on the fact that, for each $u \in \mathsf{supp}(\partial_y(s, S))$, one has $\mathsf{size}(u) < \mathsf{size}(\langle \lambda y \, s \rangle \, S)$ by Lemma 4. For that reason also, and by the confluence property, the lemma above can be considered as an inductive definition of NF and will be used as such.

Let us conclude by a simple example of computation of a normal form, using the process presented in Lemma 10.

$$\begin{aligned} \mathsf{NF}(\langle \langle \lambda f \, \lambda x \, \langle f \rangle \, \langle f \rangle \, x \rangle \, (\lambda y \, y)^2 \rangle \, z) &= 2 \, \mathsf{NF}(\langle \lambda x \, \langle \lambda y \, y \rangle \, \langle \lambda y \, y \rangle \, x \rangle \, z) \\ &= 2 \, \mathsf{NF}(\langle \lambda y \, y \rangle \, \langle \lambda y \, y \rangle \, z) \\ &= 2 \, \mathsf{NF}(\langle \lambda y \, y \rangle \, z) \\ &= 2z \, . \end{aligned}$$

2 The Taylor expansion of ordinary lambda-terms

We show now how to represent ordinary lambda-terms in this calculus by recursively Taylor expanding all ordinary applications. As remarked above, this requires dealing with infinite linear combinations of (poly-)terms.

2.1 Infinite terms and poly-terms.

2.1.1 Infinite dimensional product spaces.

If M is a set, we use $R\langle M\rangle_\infty$ for the R-module of all formal linear combinations $x = \sum_{u \in M} x_u u$ where (x_u) is an arbitrary M-indexed family of scalars taken in R (so that $R\langle M\rangle_\infty = R^M$). Let J be a countable set. We say that a family $(x_j)_{j \in J}$ of elements of $R\langle M\rangle_\infty$ is *summable* if, for each $u \in M$, the family $((x_j)_u)_{j \in J}$ vanishes for almost all values of j. We then define its sum $x = \sum_{j \in J} x_j$ by setting $x_u = \sum_{j \in J} (x_j)_u$, a finite sum in R by assumption. This is just usual convergence for the product topology, R being endowed with the discrete topology. If $J = \mathbb{N}$, observe that for this topology, the convergence of a series is equivalent to the convergence to 0 of its general term. Observe also that all the module operations on $R\langle M\rangle_\infty$ are continuous (R being endowed with the discrete topology).

If M has a structure of commutative monoid (with multiplicative notation) with the property that for each $u \in M$ there are only finitely many pairs $(v,w) \in M^2$ such that $u = vw$, then $R\langle M\rangle_\infty$ is an algebra, with multiplication given by

$$xy = \sum_{u\in M}\Big(\sum_{vw=u} x_v y_w\Big)u\,.$$

Moreover, it is easily checked that this multiplication is continuous with respect to the product topology on $R\langle M\rangle_\infty \times R\langle M\rangle_\infty$. In particular, we have the following summability property for "product families".

Lemma 11 *If $x = (x_i)_{i\in I} \in R\langle M\rangle_\infty^I$ and $y = (y_j)_{j\in J} \in R\langle M\rangle_\infty^J$ are summable, then the family $x \otimes y = (x_i y_j)_{(i,j)\in I\times J} \in R\langle M\rangle_\infty^{I\times J}$ is summable, with a sum equal to $(\sum_{i\in I} x_i)(\sum_{j\in J} y_j)$.*

2.1.2 Products of infinite sums.

Consider the particular case where M is $\Delta^!$, the free commutative monoid over Δ (what we say now would hold actually for an arbitrary free commutative monoid M). As we have just seen, $R\langle\Delta^!\rangle_\infty$ has a canonical structure of commutative algebra, with continuous multiplication given by

$$ST = \sum_{U\in\Delta^!}\Big(\sum_{VW=U} S_V T_W\Big)U \tag{9}$$

for each $S, T \in R\langle\Delta^!\rangle_\infty$.

We shall always consider the module $R\langle\Delta\rangle_\infty$ as a submodule of $R\langle\Delta^!\rangle_\infty$, by identifying the element $t = \sum_{s\in\Delta} t_s s$ of $R\langle\Delta\rangle_\infty$ with the element $\sum_{s\in\Delta} t_s s$ of $R\langle\Delta^!\rangle_\infty$ (in this sum, "s" stands for the multiset which has s as unique element), this inclusion being continuous and admitting a continuous left inverse (which maps $T \in R\langle\Delta^!\rangle_\infty$ to $\sum_{s\in\Delta} T_s s$).

If $\mathcal{T} = (T_j)_{j\in J}$ is a family of elements of $R\langle\Delta^!\rangle_\infty$ and if $\mu \in \mathcal{M}_{\text{fin}}(J)$, remember from 1.1.5 that we write $\mathcal{T}^\mu = \prod_{j\in J} T_j^{\mu(j)} \in R\langle\Delta^!\rangle_\infty$ (this is a finite product since μ is a finite multi-set, so it makes sense in the algebra $R\langle\Delta^!\rangle_\infty$).

Let $n \in \mathbb{N}$. Remember from 1.1.3 that we use $\mathcal{M}_n(J)$ for the set of all multi-sets over J whose cardinality is n and if $\mu \in \mathcal{M}_n(J)$, remember from 1.1.5 that we have defined a multinomial coefficient as follows: $[\mu] = n!/\prod_{j\in J}\mu(j)! \in \mathbb{N}$.

Lemma 12 *Let $n \in \mathbb{N}$. Let $\mathcal{T} = (T_j)_{j\in J}$ be a summable family in $R\langle\Delta^!\rangle_\infty$. Then the family $([\mu]\,\mathcal{T}^\mu)_{\mu\in\mathcal{M}_n(J)}$ is summable in $R\langle\Delta^!\rangle_\infty$ and the following "multinomial equation" holds:*

$$\Big(\sum_{j\in J} T_j\Big)^n = \sum_{\mu\in\mathcal{M}_n(J)} [\mu]\,\mathcal{T}^\mu\,. \tag{10}$$

Proof. The proof is an easy induction on n, applying Lemma 11 at the inductive step. □

A particularly simple case where we shall apply this formula is when each T_j is a singleton multiplied by a scalar, in other words, the sum $\sum_{j\in J} T_j$ is an element $t = \sum_{s\in\Delta} t_s s$ of $R\langle\Delta\rangle_\infty \subset R\langle\Delta^!\rangle_\infty$ (as explained at the beginning of this paragraph). Then Formula (10) reads

$$t^n = \sum_{S\in\Delta^!} [S]\,t^S S \tag{11}$$

where we recall that t^S stands for the finite product $\prod_{s\in\Delta} t_s^{S(s)}$.

Let $\mathcal{T} = (T_j)_{j\in J}$ be a summable family in $R\langle\Delta^!\rangle_\infty$ and assume moreover that $(T_j)_1 = 0$ for each $j \in J$, where we recall that $1 \in \Delta^!$ stands for the empty multi-set. Then it is clear that, for each $\mu \in \mathcal{M}_{\text{fin}}(J)$, one has

$$\forall S \in \mathsf{supp}(\mathcal{T}^\mu) \quad |S| \geq |\mu|\,.$$

From this simple observation, we can derive the following property.

Lemma 13 *Let $\mathcal{T} = (T_j)_{j\in J}$ be a summable family in $R\langle\Delta^!\rangle_\infty$ such that $(T_j)_1 = 0$ for each $j \in J$. Then the family $((\sum_{j\in J} T_j)^n)_{n\in\mathbb{N}}$ is summable in $R\langle\Delta^!\rangle_\infty$.*

2.1.3 Extension of the syntax to infinite terms and poly-terms. The constructions of the syntax of our resource calculus can now be extended to these infinite linear combinations of simple (poly-)terms in an obvious way, by linearity (and "continuity" since we require the constructs to commute to arbitrary linear combinations, not only to finite ones). For instance, if $t = \sum_{s\in\Delta} t_s s$ and $T = \sum_{S\in\Delta^!} T_S S$ are arbitrary elements of $R\langle\Delta\rangle_\infty$ and $R\langle\Delta^!\rangle_\infty$ respectively, $\langle t\rangle T$ is defined as $\sum_{s\in\Delta, S\in\Delta^!} t_s T_S \langle s\rangle S$, which is a perfectly well defined element of $R\langle\Delta\rangle_\infty$.

But we need to check carefully that partial derivatives still make sense in that extended setting. Given $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $t \in R\langle\Delta\rangle_\infty$, generalizing the linearity properties of partial derivatives stated in 1.2.4, one would like to write

$$\frac{\partial\sigma}{\partial x}\cdot t = \sum_{\tau\in\Delta^{(!)},u\in\Delta} \sigma_\tau t_u \Big(\frac{\partial\tau}{\partial x}\cdot u\Big)$$

where the partial derivatives $\frac{\partial\tau}{\partial x}\cdot u$ are partial derivatives of simple (poly-)terms, as defined inductively in 1.2.4. It is not clear however that the infinite sum above makes sense, that is, it is not clear that the family $\left(\frac{\partial\tau}{\partial x}\cdot u\right)_{\tau\in\mathsf{supp}(\sigma),u\in\mathsf{supp}(t)}$ is summable. This is exactly what expresses the forthcoming Lemma 17.

2.1.4 Finiteness properties of the partial differential of simple (poly-)terms. So we want to make sense of the expression $\frac{\partial\sigma}{\partial x}\cdot t$ when $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$, $t \in R\langle\Delta\rangle_\infty$ and x is not free in t.

We need first some basic combinatorial properties of differentiation in the case where the involved (poly-)terms are simple: Lemma 15 expresses that, a simple term t being chosen, it is not possible to find infinitely many pairwise distinct simple (poly-)terms σ_i $(i \in I)$ such that all the sets $\mathsf{supp}(\frac{\partial\sigma_i}{\partial x}\cdot t)$ have a common element. In other words, the family $(\frac{\partial\sigma_i}{\partial x}\cdot t)_{i\in I}$ is summable, whatever be the family $(\sigma_i)_{i\in I}$ of pairwise distinct simple (poly-)terms.

Lemma 14 *Let $n \geq 1$, let $\sigma_1,\dots,\sigma_n \in \Delta^{(!)}$ be* pairwise distinct *and let x be a variable such that $\mathsf{deg}_x\sigma_i = 1$ for $i = 1,\dots,n$. Let $t \in \Delta$ and assume that*

$$\sigma_1\,[t/x] = \cdots = \sigma_n\,[t/x] \tag{12}$$

Then, for any sequence $y_1,\dots,y_n$ of pairwise distinct variables, which are not free in t and in the σ_is, there exists a simple (poly-)term σ such that $\mathsf{deg}_{y_i}\sigma = 1$ and $\sigma_i = \sigma\,[t/y_1,\dots,t/y_{i-1},x/y_i,t/y_{i+1},\dots,t/y_n]$ for each $i = 1,\dots,n$.

Of course, the dependency of σ on $y_1,\dots,y_n$ is trivial: if $z_1,\dots,z_n$ is another sequence of variables satisfying the required conditions, the corresponding (poly-)term τ is obtained by substituting z_i for y_i in σ for each i.

Proof. We proceed by induction on the common size m of the σ_is: these sizes are equal to $\mathsf{size}(\sigma_1\,[t/x]) - \mathsf{size}(t)$ by (12).

If $m = 0$, then all the σ_is must be equal to the empty poly-term 1, hence we must have $n = 1$ and we conclude straightforwardly.

Assume next that $m = 1$ so that σ_1 is a variable. Since $\mathsf{deg}_x\sigma_1 = 1$, we must have $\sigma_1 = x$. For $i > 1$, we have $\mathsf{size}(\sigma_i\,[t/x]) = \mathsf{size}(\sigma_i) + \mathsf{size}(t) - 1$, and we must have $\sigma_i\,[t/x] = \sigma_1\,[t/x] = t$. This implies $\mathsf{size}\,\sigma_i = 1$ and hence σ_i must be a variable, and thus must be equal to x, in contradiction with our hypothesis that the σ_is are pairwise distinct. Hence we must have $n = 1$ and one concludes easily (take $\sigma = y_1$).

Suppose now that $m \geq 2$ and that $\sigma_1 = \langle s_1\rangle S_1$. If, for some $i > 1$, σ_i is not a linear application, then $\sigma_i = x$ and t is a linear application. But this is impossible because $\mathsf{size}(\sigma_1\,[t/x]) = \mathsf{size}(s_1) + \mathsf{size}(S_1) + \mathsf{size}(t) > \mathsf{size}(t)$ since $\mathsf{size}(s_1) > 0$. So for each $i = 2,\dots,n$, the simple (poly-)term σ_i must be a linear application: $\sigma_i = \langle s_i\rangle S_i$. Since each σ_i has degree 1 in x, we can assume without loss of generality that there is p such that $1 \leq p \leq n$ and

- $\mathsf{deg}_x\sigma_i = 1$ and $\mathsf{deg}_x S_i = 0$ for $1 \leq i \leq p$
- $\mathsf{deg}_x\sigma_i = 0$ and $\mathsf{deg}_x S_i = 1$ for $p+1 \leq i \leq n$.

Due to the hypothesis (12), the S_is have a common value $S_0 \in \Delta^!$ for $1 \leq i \leq p$ and the s_is have a common value $s_0 \in \Delta$ for $p+1 \leq i \leq n$. Moreover, the s_is are pairwise distinct for $1 \leq i \leq p$ and the S_is are pairwise distinct for $p+1 \leq i \leq n$. Let $y_1, \dots, y_n$ be a sequence of pairwise distinct variables. By inductive hypothesis, we can find $s \in \Delta$, $S \in \Delta^!$, such that

- for each $i = 1, \dots, p$, the simple term s has degree 1 in y_i and $s_i = s\,[t/y_1, \dots, t/y_{i-1}, x/y_i, t/y_{i+1}, \dots, t/y_p]$
- for each $i = p+1, \dots, n$, the simple poly-term S has degree 1 in y_i and $S_i = S\,[t/y_{p+1}, \dots, t/y_{i-1}, x/y_i, t/y_{i+1}, \dots, t/y_n]$.

By (12), we have $s_i\,[t/x] = s_0$ for $1 \leq i \leq p$ and $S_i\,[t/x] = S_0$ for $p+1 \leq i \leq n$. Let $\sigma = \langle s \rangle\, S$. For all $i = 1, \dots, p$, we have $\mathsf{deg}_{y_i}\sigma = 1$ and $\sigma_i = \sigma\,[t/y_1, \dots, t/y_{i-1}, x/y_i, t/y_{i+1}, \dots, t/y_n]$.

The case where $m \geq 2$ and σ_1 is an abstraction is trivial, so let us assume that $m \geq 2$ and that σ_1 is a poly-term: $\sigma_1 = s_1 S_1$. By the same reasoning as above, all the σ_is are of the same shape: $\sigma_i = s_i S_i$. Moreover, since each σ_i is of degree 1 in x, we can assume to have chosen the s_is in such a way that $\mathsf{deg}_x s_i = 1$ and $\mathsf{deg}_x S_i = 0$ for each i. Then we conclude straightforwardly, applying the inductive hyopthesis to $s_1, \dots, s_n$ (we must have $S_1 = \cdots = S_n$ by (12) so the s_is must be pairwise distinct). □

Lemma 15 *Let $\tau \in \Delta^{(!)}$, let x be a variable and let $t \in \Delta$. There are only finitely many $\sigma \in \Delta^{(!)}$ such that $\tau \in \mathsf{supp}(\frac{\partial \sigma}{\partial x} \cdot t)$.*

Proof. Assume that $\tau \in \cap_{i=1}^n \mathsf{supp}(\frac{\partial \sigma_i}{\partial x} \cdot t)$ for a finite family $(\sigma_i)_{i=1,\dots,n}$ of pairwise distinct simple (poly-)terms. So for each $i = 1, \dots, n$, one obtains the simple (poly-)term τ by replacing in the simple (poly-)term σ_i exactly one of the occurrences of x by the simple term t, see 1.2.10.

Since x is not free in t, we must have $\mathsf{deg}_x\sigma_i = \mathsf{deg}_x\tau + 1$ by Lemma 3. Let d be the common degree of the σ_is in the variable x. Let us choose d pairwise distinct variables $x_1, \dots, x_d$, distinct from x and free in the σ_is and in t, and, for each i, let σ'_i be a simple (poly-)terms such that

- x is not free in σ'_i,
- $\mathsf{deg}_{x_j}\sigma'_i = 1$ for $j = 1, \dots, d$
- and $\sigma_i = \sigma'_i\,[x/x_1, \dots, x_d]$.

In other words, σ'_i is an x-linearization of σ_i, in the sense of 1.2.10. For each $i = 1, \dots, n$, we can find $f(i) \in \{1, \dots, d\}$ such that

$$\tau = \sigma'_i \left[x/x_1, \dots, x/x_{f(i)-1}, t/x_{f(i)}, x/x_{f(i)+1}, \dots, x/x_d\right] .$$

Up to permutation of the x_js in the σ'_is, we can assume that $f(i) = 1$ for each $i = 1, \dots, n$ and, up to permutations of the $x_2, \dots, x_d$ in the simple (poly-)terms $\sigma'_i\,[t/x_1]$, we can say that these terms are pairwise equal:

$$\sigma'_1\,[t/x_1] = \cdots = \sigma'_n\,[t/x_1] .$$

But the σ_is are pairwise distinct, so the σ'_is must be pairwise distinct as well. Let $y_1, \dots, y_n$ be pairwise distinct variables, not free in t nor in the σ'_is. By Lemma 14 applied to $\sigma'_1, \dots, \sigma'_n$, there is a simple (poly-)term σ' such that, for $i = 1, \dots, n$,

- $\mathsf{deg}_{y_i}(\sigma') = 1$
- and $\sigma'_i = \sigma'\,[t/y_1, \dots, t/y_{i-1}, x_1/y_i, t/y_{i+1}, \dots, t/y_n]$.

From this one clearly sees that n is upper bounded by the size of τ. □

Lemma 16 generalizes Lemma 15 to the case where t can vary as well.

Lemma 16 *Let x be a variable and let $\tau \in \Delta^{(!)}$. There are only finitely many $\sigma \in \Delta^{(!)}$ and $t \in \Delta$ such that $\tau \in \mathsf{supp}(\frac{\partial \sigma}{\partial x} \cdot t)$.*

Proof. If $(\sigma_i, t_i)_{i\in I}$ is a family of pairwise distinct pairs of simple (poly-)terms and simple terms and if $\tau \in \cap_{i\in I} \mathsf{supp}(\frac{\partial \sigma_i}{\partial x}\cdot t_i)$ then each simple term t_i must appear as a sub-term of τ and therefore there can be only a finite number of distinct t_is. If I is infinite, this leads to a contradiction with Lemma 15. Therefore I is finite and the lemma is proved. □

2.1.5 Differentiation of infinite (poly-)terms.

Lemma 16 means precisely that the whole family of finite (poly-)terms $\left(\frac{\partial \sigma}{\partial x}\cdot t\right)_{\sigma\in\Delta^{(!)},t\in\Delta}$ is summable. So, for $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $t \in R\langle\Delta\rangle_\infty$, it makes sense to define the partial derivative $\frac{\partial \sigma}{\partial x}\cdot t$ as follows:

$$\frac{\partial \sigma}{\partial x}\cdot t = \sum_{\tau\in\Delta^{(!)},\, u\in\Delta} \sigma_\tau t_u \frac{\partial \tau}{\partial x}\cdot u \in R\langle\Delta\rangle_\infty\,.$$

And this generalized partial differential is bilinear in $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $t \in R\langle\Delta\rangle_\infty$. We can derive a bit more from Lemma 16.

Lemma 17 *The map $(\sigma, t) \mapsto \frac{\partial \sigma}{\partial x}\cdot t$ from $R\langle\Delta^{(!)}\rangle_\infty \times R\langle\Delta\rangle_\infty$ to $R\langle\Delta^{(!)}\rangle_\infty$ is continuous (these spaces being endowed with the product topology). In particular, if $(\sigma_i)_{i\in I}$ and $(t_j)_{j\in J}$ are summable families in $R\langle\Delta^{(!)}\rangle_\infty$ and $R\langle\Delta\rangle_\infty$ respectively (with respective sums σ and t), then the family $(\frac{\partial \sigma_i}{\partial x}\cdot t_j)_{i\in I,j\in J}$ is summable, with sum equal to $\frac{\partial \sigma}{\partial x}\cdot t$.*

Proof. By linearity, it suffices to prove continuity at the origin $(0,0)$ of $R\langle\Delta^{(!)}\rangle_\infty \times R\langle\Delta\rangle_\infty$. We take a neighborhood of 0 in $R\langle\Delta^{(!)}\rangle_\infty$: it is induced by a finite subset W of $\Delta^{(!)}$ (the corresponding neighborhood of 0 in $R\langle\Delta^{(!)}\rangle_\infty$ is the collection $\mathcal{V}_W(0)$ of all $\theta \in R\langle\Delta^{(!)}\rangle_\infty$ such that $W \cap \mathsf{supp}(\theta) = \emptyset$). Then by Lemma 16, for each $\varphi \in W$, we can find two finite sets $U_\varphi \subseteq \Delta^{(!)}$ and $V_\varphi \subseteq \Delta$ such that $\varphi \notin \mathsf{supp}(\frac{\partial \sigma}{\partial x}\cdot t)$ for each $(\sigma, t) \notin U_\varphi \times V_\varphi$. Then taking $U = \bigcup_{\varphi\in W} U_\varphi$ and $V = \bigcup_{\varphi\in W} V_\varphi$, we have $\frac{\partial \sigma}{\partial x}\cdot t \in \mathcal{V}_W(0)$ for each $\sigma \in \mathcal{V}_U(0)$ and $t \in \mathcal{V}_V(0)$. □

So $\frac{\partial \sigma}{\partial x}\cdot t \in R\langle\Delta^{(!)}\rangle_\infty$ is well defined for all $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $t \in R\langle\Delta\rangle_\infty$ and has all the required linearity and continuity properties.

2.1.6 Big step differentiation of infinite (poly-)terms.

We can of course iterate this construction and define $\frac{\partial^n \sigma}{\partial x_1\cdots\partial x_n}\cdot(t_1,\dots,t_n)$ for arbitrary $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $t_1,\dots,t_n$ of $R\langle\Delta\rangle_\infty$. Again, this operation is linear in each of its parameters $\sigma, t_1,\dots,t_n$, and is continuous in these parameters (for the product topology).

For that reason, for each given $n \in \mathbb{N}$, we can extend the construction $\partial_x(\sigma, T)$ to $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $T \in R\langle\mathcal{M}_n(\Delta)\rangle_\infty$, and this operation is bilinear and continuous in σ and T (this generalizes to infinite sums the linear extension of $\partial_x(\_,\_)$ to $R\langle\Delta^{(!)}\rangle \times R\langle\mathcal{M}_n(\Delta)\rangle$, explained in 1.2.9).

The second linear extension of $\partial_x(\_,\_)$ explained in 1.2.9, to $R\langle\Delta^{(!)}\rangle \times R\langle\Delta^!\rangle$, can also be generalized to infinite sums. Observing indeed that, for $\sigma \in \Delta^{(!)}$ and $T \in \mathcal{M}_n(\Delta)$, the size of any element of the support of $\partial_x(\sigma, T)$ must be greater than n, we see that, for any $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and any $T \in R\langle\Delta^!\rangle_\infty$, the sequence $(\partial_x(\sigma, T^{(n)}))_{n\in\mathbb{N}}$ converges to 0 in $R\langle\Delta^{(!)}\rangle_\infty$ (where we use $T^{(n)}$ for the restriction of T to $\mathcal{M}_n(\Delta)$, that is $T^{(n)} = \sum_{S\in\mathcal{M}_n(\Delta)} T_S\, S$). So the series $\sum_{n=0}^{\infty} \partial_x(\sigma, T^{(n)})$ converges. Its sum is denoted by $\partial_x(\sigma, T)$; this operation is bilinear and continuous in (σ, T).

So all the differentiation operations we have considered for finite (poly-)terms make sense in the infinite case as well, without any restriction on the infinite linear combinations we consider. This fact will be used at the end of the present paper, when we shall give a "substitution-oriented" version of Taylor's formula in Theorem 32.

2.1.7 The exponential and the promotion.

From now on, we assume that R possesses inverses for all integers $\neq 0$.

As explained at the beginning of 2.1.2, any $t \in R\langle\Delta\rangle_\infty$ can canonically be seen as an element of $R\langle\Delta^!\rangle_\infty$ (identifying $u \in \Delta$ with $u \in \Delta^!$, the multi-set whose only element is u, with multiplicity 1). It is clear that $t^n \to 0$ when $n \to \infty$ so that the following sum converges (this can also be seen as a trivial application of Lemma 13):

$$\exp t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \in R\langle \Delta^! \rangle_\infty$$

where the exponents correspond to multiplication in the algebra $R\langle \Delta^! \rangle_\infty$. Using Formula (11) of 2.1.2, one can check that actually

$$\exp t = \sum_{T \in \Delta^!} \frac{t^T}{T!} T$$

(remember that, with our notations, $T! = \prod_{u\in\Delta} T(u)! \in \mathbb{N}^+$ and that $t^T = \prod_{u\in\Delta} t_u^{T(u)} \in R$).

Without surprises, we have $\exp 0 = 1$ and $\exp(s+t) = \exp s\ \exp t$. This operation $t \mapsto \exp t$ corresponds to *promotion* in linear logic. We could then recover the ordinary application of the lambda-calculus by setting:

$$(s)\,t = \langle s\rangle \exp t\,. \tag{13}$$

This formula can also be seen as defining an encoding of the ordinary lambda-calculus in infinite resource terms.

The purpose of the sequel is precisely to analyze the properties of this encoding.

Remark: applying Lemma 13, this exponential operation could be defined not only for $t \in R\langle\Delta\rangle_\infty$ but for arbitrary $S \in R\langle \Delta^! \rangle_\infty$, as soon as $S_1 = 0$. When $S_1 \neq 0$, computing $\exp S$ involves an infinite sum of scalars, or maybe the use of an "exponential map" e_R on the semi-ring R, setting $\exp S = e_R(S_1)\exp(S - S_1 \cdot 1)$. This idea might lead to an interesting generalization of the promotion of linear logic.

2.2 Complete Taylor expansion of an ordinary lambda-term

2.2.1 Multiplicity coefficients. Given a simple term t, we define a positive integer $\mathsf{m}(t)$, the *multiplicity coefficient* of t by the following inductive definition.

$$\begin{aligned}
\mathsf{m}(x) &= 1\\
\mathsf{m}(\lambda x\, s) &= \mathsf{m}(s)\\
\mathsf{m}(\langle s\rangle\, T) &= \mathsf{m}(s) \prod_{t\in\Delta} T(t)!\,\mathsf{m}(t)^{T(t)} = \mathsf{m}(s)\ T!\ \mathsf{m}^T
\end{aligned}$$

with our concise notations for arithmetic operations on multi-sets. This definition of m is not circular, because, when defining $\mathsf{m}(\langle s\rangle\, T)$, in the expression $\mathsf{m}^T = \prod_{t\in\Delta} \mathsf{m}(t)^{T(t)}$, the only simple terms t for which the value of $\mathsf{m}(t)$ is needed are subterms of $\langle s\rangle\, T$.

For a poly-term T, we define accordingly $\mathsf{m}(T) = T!\ \mathsf{m}^T$, so that $\mathsf{m}(\langle s\rangle\, T) = \mathsf{m}(s)\mathsf{m}(T)$. So if $T = t_1^{n_1} \cdots t_p^{n_p}$, with the t_is pairwise distinct (up to α-conversion), we have

$$\mathsf{m}(T) = \prod_{i=1}^{p} n_i!\mathsf{m}(t_i)^{n_i}\,.$$

In Section 4, paragraph 4.2.3, we shall give a precise combinatorial interpretation of these coefficients. We shall see that $\mathsf{m}(t)$ is the number of permutations of variable occurrences of t which preserve the names of the variables (one cannot swap an occurrence of x with an occurrence of y, if x and y are distinct variables) and leave t unchanged (taking into account the fact that poly-term multiplication is a *commutative* operation).

As an example, we have $\mathsf{m}(\langle x\rangle\,(\langle x\rangle\, y^3)^2) = 2!(3!)^2 = 72$.

2.2.2 The expansion. Given an ordinary lambda-term M, we define a subset $\mathcal{T}(M)$ of Δ which is the collection of all simple terms having *the same shape as* M. This set is defined as follows, by induction on M.

$$\begin{aligned}
\mathcal{T}(x) &= \{x\}\\
\mathcal{T}(\lambda x\, M) &= \{\lambda x\, t \mid t \in \mathcal{T}(M)\}\\
\mathcal{T}((M)\, N) &= \{\langle t\rangle\, T \mid t \in \mathcal{T}(M) \text{ and } T \in \mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))\}\,.
\end{aligned}$$

Observe that, as soon as the lambda-term M contains an application, the set $\mathcal{T}(M)$ is infinite. To give an example, the set $\mathcal{T}(\lambda x\,(x)\,(x)\,y)$ contains, among infinitely many other simple terms, *e.g.* $\lambda x\,\langle x\rangle\,1$, $\lambda x\,\langle x\rangle\,\langle x\rangle\,y$, $\lambda x\,\langle x\rangle\,((\langle x\rangle\,1)^2\,\langle x\rangle\,y^3),\dots$

Observe also that $\mathcal{T}(M)$ contains a simple term $l(M)$ which which looks very much like M, and is defined by: $l(x) = x$, $l(\lambda x\,M) = \lambda x\,l(M)$ and $l((M)\,N) = \langle l(M)\rangle\,l(N)$. For instance, $l(\lambda x\,(x)\,x) = \lambda x\,\langle x\rangle\,x$. But this simple term $l(M)$, which is a "linearization" of M, has not the same properties as M with respect to β-reduction (even if M is unsolvable, $l(M)$ is strongly normalizing: in that case, the normal form of $l(M)$ is 0).

We define the *complete Taylor expansion* of an ordinary lambda-term M:

$$M^* = \sum_{t\in\mathcal{T}(M)} \frac{1}{\mathsf{m}(t)}\,t \;\in R\langle\Delta\rangle_\infty\,. \tag{14}$$

This expansion satisfies the following lemma, whose last statement means that M^* can be obtained by recursively Taylor expanding all applications in M. This motivates our terminology for this operation.

Lemma 18 *If x is a variable and if M and N are terms of the standard lambda-calculus, one has*

- $x^* = x$,
- $(\lambda x\,M)^* = \lambda x\,M^*$ *and*
- $((M)\,N)^* = \langle M^*\rangle \exp N^* = \sum_{n=0}^\infty \frac{1}{n!}\,\langle M^*\rangle\,N^{*n}$.

Proof. The only interesting case is the last one. We have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n!}\,\langle M^*\rangle\,N^{*n} &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big\langle \sum_{s\in\mathcal{T}(M)}\frac{1}{\mathsf{m}(s)}s\Big\rangle\Big(\sum_{t\in\mathcal{T}(N)}\frac{1}{\mathsf{m}(t)}t\Big)^n\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\Big\langle \sum_{s\in\mathcal{T}(M)}\frac{1}{\mathsf{m}(s)}s\Big\rangle\Big(\sum_{T\in\mathcal{M}_n(\mathcal{T}(N))}[T]\,\frac{1}{\mathsf{m}^T}T\Big)\\
&= \sum_{\substack{s\in\mathcal{T}(M)\\ T\in\mathcal{M}_{\rm fin}(\mathcal{T}(N))}}\frac{1}{|T|!}\,[T]\,\frac{1}{\mathsf{m}(s)\mathsf{m}^T}\,\langle s\rangle\,T\\
&= \sum_{\substack{s\in\mathcal{T}(M)\\ T\in\mathcal{M}_{\rm fin}(\mathcal{T}(N))}}\frac{1}{T!\mathsf{m}(s)\mathsf{m}^T}\,\langle s\rangle\,T \quad \text{since } [T]=\frac{|T|!}{T!}\\
&= ((M)\,N)^*\,.
\end{aligned}$$

□

It must be observed that the coefficient of t in Formula (14) *does not depend on M*. This remarkable property is lost if we want to define similarly a complete Taylor expansion for an extension of the ordinary lambda-calculus where finite linear combinations of terms are allowed.

2.2.3 Outline of the sequel. As explained in the introduction, our aim is to understand the behaviour of this Taylor expansion with respect to beta-reduction. The first thing to observe is that the resource terms occurring in the Taylor expansion of an ordinary lambda-term are coherent with each other and with themselves (a simple term which is coherent with itself will be said to be "uniform"), for a binary coherence relation we define below, on simple terms. Then we shall see that the normal form operator is *stable* (in the sense of [Ber78] and [Gir86]) with respect to this coherence relation. This is a *qualitative* property whose main consequence will be a "non-interference" effect: the supports of the normal forms of two distinct terms of the Taylor expansion are disjoint.

Last, we shall see that the multiplicity coefficients of uniform terms evolve very simply during big step differential reduction —a *quantitative* property—.

These two main results will lead to our final Corollary 34.

3 Qualitative properties: the coherence relation on simple terms and poly-terms

We define a binary *coherence* relation $\coh$ on simple terms and on simple poly-terms, which is easily seen to be symmetric (but neither reflexive nor anti-reflexive). We use the notation $\frown$ for the largest anti-reflexive sub-relation of $\coh$. The definition is by induction on simple terms.

- $x \coh t'$ if $t' = x$;
- $\lambda x\, s \coh t'$ if $t' = \lambda x\, s'$ with $s \coh s'$;
- $\langle s\rangle\, T \coh t'$ if $t' = \langle s'\rangle\, T'$ with $s \coh s'$ and $T \coh T'$.
- And, for two simple poly-terms T and T', one has $T \coh T'$ if, for all $t, t' \in TT'$, one has $t \coh t'$.

Observe first that, if s and s' are simple terms, one has $s \coh s'$ (considering s and s' as simple terms) iff $s \coh s'$ (considering them as singleton poly-terms).

This coherence relation is not reflexive: if x and y are distinct variable, then $xy \coh xy$ does not hold (we shall say that xy is a non-uniform poly-term). It is not transitive either, since, considering x and y as poly-terms, one has $x \coh 1 \coh y$, but it is no true that $x \coh y$.

We say that a simple (poly-)term σ is *uniform* if $\sigma \coh \sigma$. This corresponds to the notion of well-formed term in [Kfo00] (however, in that paper, the relation corresponding to $\coh$ is a partial equivalence relation because empty multi-sets are not accepted as arguments). Observe that, for two simple poly-terms T and T', one has $T \coh T'$ iff $TT' \coh 1$ iff TT' is uniform.

A clique for this coherence relation is a subset $\mathcal{U}$ of $\Delta^{(!)}$ such that $\tau \coh \tau'$ whenever $\tau, \tau' \in \mathcal{U}$. In particular, each element of a clique must be uniform. Observe by the way that it results from the definition that if $\sigma \coh \sigma'$ for two simple (poly-)terms σ and σ', then automatically σ and σ' are uniform.

Lemma 19 *If M is a lambda-term, then $\mathcal{T}(M)$ is a maximal clique in $(\Delta, \coh)$.*

The proof is straightforward. However, not all maximal cliques of Δ are of the shape $\mathcal{T}(M)$ for some lambda-term M. For instance, a maximal extension of the clique $\{\langle x\rangle\, 1, \langle x\rangle\, \langle x\rangle\, 1, \dots\}$ cannot be of that shape. Such maximal cliques could probably be seen as some kind of infinitary generalized lambda-terms.

3.1 Coherence and differentiation.

Coherence is not preserved by partial differentiation. For instance, the poly-term x^2 is uniform and y is a uniform term, but $\frac{\partial x^2}{\partial x} \cdot y = 2xy$ is not uniform if x and y are distinct variables.

3.1.1 Stability of big-step differentiation. However, big step differentiation — or, more preciesely, the map $\mathsf{supp} \circ \partial_x$ — satisfies a "stability" property with respect to the coherence relation we have defined on (poly-)terms, similar to the characterization of the trace of stable linear functions between coherence spaces in [Gir87, GLT89]. More precisely, Theorem 20 expresses that the set $\{((\sigma, S), \varphi) \mid \varphi \in \mathsf{supp}(\partial_x(\sigma, S))\}$ is a clique in the coherence space $(\Delta^{(!)} \otimes \Delta^!) \multimap \Delta^{(!)}$. That is, the map $f : \mathcal{P}(\Delta^{(!)}) \times \mathcal{P}(\Delta^!) \to \mathcal{P}(\Delta^{(!)})$ defined by $f(U, V) = \cup_{\sigma \in U, S \in V} \mathsf{supp}(\partial_x(\sigma, S))$ maps pairs of cliques to cliques, and is a stable function on pairs of cliques. The precise statement is given in Theorem 20.

Given $U, U' \subseteq \Delta^{(!)}$, let us write $U \coh U'$ when $\forall \sigma \in U, \sigma' \in U'\ \sigma \coh \sigma'$. Then $U \coh U$ means that U is a clique.

Theorem 20 *Let x be a variable. Let $\sigma, \sigma' \in \Delta^{(!)}$ and $S, S' \in \Delta^!$.*

- *If $\sigma \coh \sigma'$ and $S \coh S'$, then* $\mathsf{supp}(\partial_x(\sigma, S)) \coh \mathsf{supp}(\partial_x(\sigma', S'))$
- *and if, moreover, $\sigma \neq \sigma'$ or $S \neq S'$, then* $\mathsf{supp}(\partial_x(\sigma, S)) \cap \mathsf{supp}(\partial_x(\sigma', S')) = \emptyset$.

Proof. We assume that $\sigma \frown \sigma'$ and $S \frown S'$. Let $\varphi \in \mathsf{supp}(\partial_x(\sigma, S))$ and $\varphi' \in \mathsf{supp}(\partial_x(\sigma', S'))$. We prove that $\varphi \frown \varphi'$ and that, if moreover $\varphi = \varphi'$, then $\sigma = \sigma'$ and $S = S'$. We proceed by induction on the sum of the sizes of σ and σ', for σ and σ' in $\Delta^{(!)}$.

Assume that σ is a variable y. Then $\sigma' = y$. If $y \neq x$, we must have $S = S' = 1$ since $\varphi \in \mathsf{supp}(\partial_x(\sigma, S))$ and $\varphi' \in \mathsf{supp}(\partial_x(\sigma', S'))$ (otherwise at least one of these sets would be empty). So $\varphi = \varphi' = y$ and we conclude trivially. If $y = x$ then S and S' must be singleton multi-sets (otherwise again at least one of the two sets $\mathsf{supp}(\partial_x(\sigma', S'))$ and $\mathsf{supp}(\partial_x(\sigma, S))$ would be empty). Say $S = t$ and $S' = t'$ (with $t, t' \in \Delta$, $t \frown t'$). Then we have $\varphi = t$ and $\varphi' = t'$ and we conclude straightforwardly.

The case where σ is an abstraction is trivial.

Assume that $\sigma = \langle t \rangle T$ (with $t \in \Delta$ and $T \in \Delta^!$). Then by definition of coherence we must have $\sigma' = \langle t' \rangle T'$ with $t \frown t'$ and $T \frown T'$. Since $\varphi \in \mathsf{supp}(\partial_x(\sigma, S))$, we must have $\varphi = \langle u \rangle U$ and there must exist $S_1, S_2 \in \Delta^!$ such that $S = S_1 S_2$, $u \in \mathsf{supp}(\partial_x(t, S_1))$, $U \in \mathsf{supp}(\partial_x(T, S_2))$. Similarly, $\varphi' = \langle u' \rangle U'$ and there exist $S'_1, S'_2 \in \Delta^!$ such that $S' = S'_1 S'_2$, $u' \in \mathsf{supp}(\partial_x(t', S'_1))$, $U' \in \mathsf{supp}(\partial_x(T', S'_2))$. But by definition of coherence we have $S_1 \frown S'_1$ and $S_2 \frown S'_2$ and hence by inductive hypothesis $u \frown u'$ and $U \frown U'$, so $\varphi \frown \varphi'$. If furthermore $\varphi = \varphi'$, then $u = u'$ and $U = U'$ and the inductive hypothesis yields $t = t'$, $S_1 = S'_1$ and $S_2 = S'_2$ and we conclude.

Assume last that σ and σ' are poly-terms. If $\sigma = 1$, we must have $S = 1$ (as otherwise $\mathsf{supp}(\partial_x(\sigma, S))$ would be empty) and there are two sub-cases: the case $\sigma' = 1$ is straightforward. Let us assume that $\sigma' \neq 1$ so that we can write $\sigma' = u'U'$. In that case we have $\varphi = 1$ and $\varphi' = v'V'$ with $v' \in \mathsf{supp}(\partial_x(u', S'_1))$ and $V' \in \mathsf{supp}(\partial_x(U', S'_2))$ for some $S'_1, S'_2 \in \mathcal{M}_{\mathrm{fin}}(\Delta)$ satisfying $S'_1 S'_2 = S'$. We have to show that $1 \frown v'V'$, or equivalently that $\{v'\} \cup \mathsf{set}(V')$ is a clique. That $\mathsf{set}(V')$ is a clique results from the inductive hypothesis. So let $w' \in \mathsf{set}(V')$ and let us show that $v' \frown w'$. We have $w' \in \mathsf{supp}(\partial_x(w'_0, S'_3))$ where $w'_0 \in \mathsf{set}(U')$ and S'_3 is a factor of S'_2. We have $u' \frown w'_0$ and $S'_1 \frown S'_3$, hence the inductive hypothesis yields $v' \frown w'$ as desired. In the present case we know that $\varphi \neq \varphi'$ so there is nothing more to prove.

The last sub-case to consider is the case where σ and σ' are simple poly-terms both distinct from 1. Then we can write $\varphi = vV$ and $\varphi' = v'V'$ where $v \in \mathsf{supp}(\partial_x(t, S_1))$, $V \in \mathsf{supp}(\partial_x(U, S_2))$, $v' \in \mathsf{supp}(\partial_x(t', S'_1))$ and $V' \in \mathsf{supp}(\partial_x(U', S'_2))$ with $tU = \sigma$ and $t'U' = \sigma'$, for some $S_1, S_2, S'_1, S'_2 \in \Delta^{(!)}$ satisfying $S_1 S_2 = S$ and $S'_1 S'_2 = S'$. One shows exactly as above that $\varphi \frown \varphi'$. If moreover $\varphi = \varphi'$, then we can take $v = v'$ and $V = V'$ and again we conclude straightforwardly by inductive hypothesis, since we know that $t \frown t'$ and $S_1 \frown S'_1$ (and hence $t = t'$ and $S_1 = S'_1$) on one hand, and $U \frown U'$ and $S_2 \frown S'_2$ (and hence $U = U'$ and $S_2 = S'_2$) on the other hand. This concludes the proof. □

Corollary 21 *Let $\sigma \in \Delta^{(!)}$ and $S \in \Delta^!$ be uniform. Then $\mathsf{supp}(\partial_x(\sigma, S))$ is a clique.*

3.1.2 Stability of the normal form operator. As a consequence of Theorem 20 and Lemma 10, the NF operator — or, more precisely, the map $\mathsf{supp} \circ \mathsf{NF}$ — satisfies also a stability property with respect to the coherence relation we have defined on (poly-)terms.

Theorem 22 *Let $\sigma, \sigma' \in \Delta^{(!)}$.*

- *If $\sigma \frown \sigma'$, then $\mathsf{supp}(\mathsf{NF}(\sigma)) \frown \mathsf{supp}(\mathsf{NF}(\sigma'))$*
- *and if, moreover, $\sigma \neq \sigma'$, then $\mathsf{supp}(\mathsf{NF}(\sigma)) \cap \mathsf{supp}(\mathsf{NF}(\sigma')) = \emptyset$.*

Proof. Let $\sigma, \sigma' \in \Delta^{(!)}$ and assume that $\sigma \frown \sigma'$. Let $\varphi \in \mathsf{supp}(\mathsf{NF}(\sigma))$ and $\varphi' \in \mathsf{supp}(\mathsf{NF}(\sigma'))$. By induction on the sum of the sizes of the simple (poly-)terms σ and σ', we show that $\varphi \frown \varphi'$ and that, if $\varphi = \varphi'$, then $\sigma = \sigma'$.

For this purpose, we use Lemma 10.

If $\mathsf{size}(\sigma) + \mathsf{size}(\sigma') = 0$ then σ and σ' are poly-terms and $\sigma = \sigma' = 1$; one concludes straightforwardly. Otherwise, assume first that σ is a simple term, we consider the following cases.

- If $\sigma = \lambda \bar{x} \, \langle \cdots \langle x \rangle \, S_1 \cdots \rangle \, S_n$, then $\sigma' = \lambda \bar{x} \, \langle \cdots \langle x \rangle \, S'_1 \cdots \rangle \, S'_n$ with $S_i \frown S'_i$ for $i = 1, \ldots, n$. Since $\varphi \in \mathsf{supp}(\mathsf{NF}(\sigma))$ and $\varphi' \in \mathsf{supp}(\mathsf{NF}(\sigma'))$, these simple terms are of the shape $\varphi = \lambda \bar{x} \, \langle \cdots \langle x \rangle \, T_1 \cdots \rangle \, T_n$ and $\varphi' = \lambda \bar{x} \, \langle \cdots \langle x \rangle \, T'_1 \cdots \rangle \, T'_n$ with $T_i \in \mathsf{supp}(\mathsf{NF}(S_i))$ and $T'_i \in \mathsf{supp}(\mathsf{NF}(S'_i))$ for each i. Then we apply the inductive hypothesis for each i (since $S_i \frown S'_i$) and we conclude.

- If $\sigma = \lambda\bar{x}\,\langle\cdots\langle\langle\lambda x\,t\rangle\,U\rangle\,S_1\cdots\rangle\,S_n$ then σ' must be of the shape $\sigma' = \lambda\bar{x}\,\langle\cdots\langle\langle\lambda x\,t'\rangle\,U'\rangle\,S'_1\cdots\rangle\,S'_n$ with of course $t \frown t'$, $U \frown U'$ and $S_i \frown S'_i$ for each i. There exists $u \in \mathsf{supp}(\partial_x(t, U))$ and $u' \in \mathsf{supp}(\partial_x(t', U'))$ such that $\varphi \in \mathsf{supp}(\mathsf{NF}(\lambda\bar{x}\,\langle\cdots\langle u\rangle\,S_1\cdots\rangle\,S_n))$ and $\varphi' \in \mathsf{supp}(\mathsf{NF}(\lambda\bar{x}\,\langle\cdots\langle u'\rangle\,S'_1\cdots\rangle\,S'_n))$. By Theorem 20 we have $u \frown u'$ and hence, since the size of $\lambda\bar{x}\,\langle\cdots\langle u\rangle\,S_1\cdots\rangle\,S_n$ is strictly smaller than the size of σ (and similarly for $\lambda\bar{x}\,\langle\cdots\langle u'\rangle\,S'_1\cdots\rangle\,S'_n$), we have $\varphi \frown \varphi'$ by inductive hypothesis. If moreover $\varphi = \varphi'$, then the inductive hypothesis implies that $u = u'$ and $S_i = S'_i$ for each i and hence (applying again Theorem 20), we obtain that $\sigma = \sigma'$.

Assume last that $\sigma = S$ and $\sigma' = S'$ are poly-terms. Let $T \in \mathsf{supp}(\mathsf{NF}(S))$ and $T' \in \mathsf{supp}(\mathsf{NF}(S'))$, we must show that $T \frown T'$, so let $t, t' \in \mathsf{set}(T) \cup \mathsf{set}(T')$. We are reduced to showing that $t \frown t'$. There exists $s, s' \in \mathsf{set}(S) \cup \mathsf{set}(S')$ such that $t \in \mathsf{NF}(s)$ and $t' \in \mathsf{NF}(s')$. We know that $s \frown s'$ (by definition of coherence for poly-terms) and moreover, with our definition of the size, we have $\mathsf{size}(s) + \mathsf{size}(s') < \mathsf{size}(S) + \mathsf{size}(S')$. Therefore the inductive hypothesis applies and yields $t \frown t'$ and hence $T \frown T'$. Assume moreover that $T = T' = t_1 \dots t_k$. Then S and S' must be of the shape $S = s_1 \dots s_k$ and $S' = s'_1 \dots s'_k$ with $t_i \in \mathsf{supp}(\mathsf{NF}(s_i)) \cap \mathsf{supp}(\mathsf{NF}(s'_i))$ for each i, and hence $s_i = s'_i$ for each i (by inductive hypothesis again). Hence $S = S'$. □

Corollary 23 *Let $\sigma \in \Delta^{(!)}$ be uniform. Then $\mathsf{supp}(\mathsf{NF}(\sigma))$ is a clique.*

4 Quantitative properties: combinatorial considerations

We shall now study the behaviour of the mutiplicity coefficients of a simple (poly-)term along its big step reduction. In the present paper, we want to solve this question when the simple (poly-)term under consideration appears in the complete Taylor expansion of an ordinary lambda-term, and hence is uniform. This hypothesis will be extremely useful.

For this purpose, we shall first observe in Lemma 25 that $\mathsf{m}(\sigma)$ is the number of permutations of the *free or bound* variable *occurrences* in σ which respect the variables associated with these occurrences and leave σ unchanged. These permutations form a subgroup of a symmetric group, called the *isotropy* group of σ. This group is generally non trivial because the multi-set construction used in the syntax of poly-terms is commutative. For instance, the term $\lambda x\,\langle\langle z\rangle\,x^3\rangle\,y^2$ has multiplicity coefficient $3! \times 2!$.

Doing that, we shall transform our problem into a combinatorial group-theoretic one: relate the isotropy group of a term to the isotropy group of the same term where a big step differential substitution has been performed. This will be the main purpose of the present section with, as a result, a proof of the *Uniform Plugging Equation*.

4.1 A group equation

Let G be a finite group and let L and R be subgroups of G. Then $LR = \{lr \mid l \in L \text{ and } r \in R\} \subseteq G$ is not a subgroup of G in general. Nevertheless, the cardinality of this set satisfies the following well known equation which is essential in the forthcoming considerations.

Lemma 24 *If L and R are subgroups of a finite group G, then*

$$|LR| = \frac{|L|\,|R|}{|L \cap R|}\,.$$

Proof. The set LR is the union of the left cosets lR (for $l \in L$), and these cosets are either disjoint or equal and have $|R|$ as cardinality. Given $l, l' \in L$, the left cosets lR and $l'R$ are equal subsets of G iff $l^{-1}l'$ belongs to the subgroup $L \cap R$ of G. Therefore, LR is the disjoint union of exactly $|L| / |L \cap R|$ disjoint sets of cardinality $|R|$, whence the equation. □

We shall also use the fact that if $h : G \to H$ is a group homomorphism and G is finite, then $|h(G)| = |G| / |\ker h|$.

4.2 The uniform plugging equation

In order to give a precise definition of the group of permutations of variable occurrences in a simple (poly-)term σ which leave σ unchanged, we need to separate the various occurrences of all the variables appearing, free or bound, in σ. This is exactly the purpose of the notion of "multilinear-visible" (poly-)term we introduce now. The idea is to separate the occurrences in σ by using pairwise distinct variables, producing a term φ, and then recovering the original names of variables through a "naming function" (we will use letters p, q... for these functions from variables to variables). Such a pair (φ, p) will be called a multilinear-visible representation of σ. Because the permutations we consider should act also on the *bound* occurrences of σ, all the variables occurring in φ will be required to be free.

For instance, we shall represent the simple term $\lambda x \langle y \rangle x^2$ by means of the multilinear-visible term $\lambda x \langle y \rangle x_1 x_2$ where y, x_1, x_2, x are pairwise distinct, together with the function p such that $p(y) = y$ and $p(x_1) = p(x_2) = x$. Observe that the bound variable x is not modified, but it does not appear free in the multilinear-visible term; the role of the function p is precisely to record the information that the variables x_1 and x_2 stand for the two occurrences of x.

Remark: These multilinear-visible (poly-)terms, that we present as particular (poly-)terms, are just combinatorial artifacts, introduced for defining cleanly the isotropy group of (poly-)terms, they should not be considered as "real", computationally meaningful, (poly-)terms. We could have introduced an additional syntax for these objects, where, for instance, the various occurrences of a variable x would have been replaced by pairs (x, i) where i is *e.g.* an integer attached to this particular occurrence of x (if x has n occurrences in the (poly-)term, n distinct values of i would have been used in the corresponding multilinear-visible (poly-)term, for distinguishing the various occurrences of x). We prefered not to do so for avoiding additional bureaucracy.

4.2.1 Multilinear-visible representation of a (poly-)term.

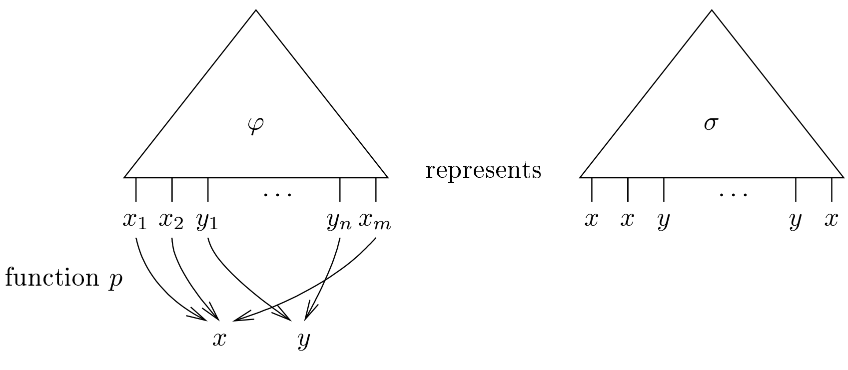
Let us say that a simple (poly-)term φ is *multilinear-visible* if each variable occurring in φ *occurs exactly once*, and *occurs free* in φ.

Let us say that a partial function (substitution) Φ from $\mathcal{V}$ to multilinear-visible terms is a *multilinear-visible substitution* if $\mathsf{fv}(\Phi(x)) \cap \mathsf{fv}(\Phi(x')) = \emptyset$ when x and x' are two distinct elements of $\mathsf{Dom}\,\Phi$ (the domain of Φ). We use $\mathsf{fv}(\Phi)$ for the disjoint union $\bigcup_{x \in \mathsf{Dom}\,\Phi} \mathsf{fv}(\Phi(x))$.

Given a multilinear-visible (poly-)term φ and a multilinear-visible substitution Φ, we say that the pair (φ, Φ) is *adapted* if $\mathsf{fv}(\varphi) \subseteq \mathsf{Dom}\,\Phi$, and no element of $\mathsf{fv}(\Phi)$ is bound in φ. In that situation, we can apply the substitution Φ to the term φ, getting a (poly-)term $\varphi[\Phi]$ which is clearly also multilinear-visible.

Let φ be a multilinear-visible (poly-)term and let $p : \mathsf{fv}(\varphi) \to \mathcal{V}$ be a function. We use φ^p for the (poly-)term obtained by substituting each variable y occurring in φ with $p(y)$, in the most naive way (that is, *without renaming captured variables*).

Let σ be a (poly-)term, we say that (φ, p) *represents* σ if $\varphi^p = \sigma$, a situation which can be pictured as follows:



Example. The simple term $\sigma = \langle z \rangle (z(\lambda y\, y)^2)$ is represented by the pair (φ, p) where

$$\varphi = \langle z_1 \rangle (z_2(\lambda y\, y_1)(\lambda y\, y_2)) \quad \text{and} \quad \begin{cases} p(z_1) = p(z_2) = z \\ p(y_1) = p(y_2) = y \end{cases}.$$

Clearly, if both (φ, p) and (ψ, q) represent σ, there is a (generally not unique) bijection $f : \mathsf{fv}(\varphi) \to \mathsf{fv}(\psi)$ such that $qf = p$ and $\varphi[f] = \psi$ (observe that f is a multilinear-visible substitution, which is adapted to φ since φ and ψ have the same lambda-abstracted variables, which are the lambda-abstracted variables of σ, and none of the elements of $\mathsf{fv}(\psi)$ is lambda-abstracted in ψ, so the notation $\varphi[f]$ makes sense). This can be proved by induction on σ. If σ is the simple term of the example above, there are two such bijections f.

4.2.2 Isotropy group of a multilinear-visible (poly-)term.

Let us introduce two important notations.

- If $p : \mathcal{V} \to \mathcal{V}$ is a finite partial function, we use $\mathfrak{S}_p$ for the subgroup of $\mathfrak{S}_{\mathsf{Dom}\, p}$ of all bijections f on $\mathsf{Dom}\, p$ such that $pf = p$: it is a finite product of symmetric groups.
- If φ is a multilinear-visible (poly-)term and $p : \mathsf{fv}(\varphi) \to \mathcal{V}$, we use $\mathsf{Iso}(\varphi, p)$ for the subgroup of $\mathfrak{S}_p$ whose elements f satisfy $\varphi[f] = \varphi$, since it is the isotropy group of φ for the action of $\mathfrak{S}_p$ on the multilinear-visible simple (poly-)terms having the same free variables as φ.

Example. Consider the following closed simple term:

$$\sigma = \lambda x \, \langle x \rangle \, (\lambda y \, \langle x \rangle \, y^2)^2 \, .$$

We represent this terms by the pair (φ, p) where

$$\varphi = \lambda x \, \langle x_1 \rangle \, (\lambda y \, \langle x_2 \rangle \, y_1 y_2)(\lambda y \, \langle x_3 \rangle \, y_3 y_4) \quad \text{and} \quad \begin{cases} p(x_1) = p(x_2) = p(x_3) = x \\ p(y_1) = \cdots = p(y_4) = y \end{cases} .$$

Remember that the poly-terms $y_1 y_2$, $y_3 y_4$ and $(\lambda y \, \langle x_2 \rangle \, y_1 y_2)(\lambda y \, \langle x_3 \rangle \, y_3 y_4)$ are multisets which have two elements each, so they are respectively equal to $y_2 y_1$, $y_4 y_3$ and $(\lambda y \, \langle x_2 \rangle \, y_4 y_3)(\lambda y \, \langle x_3 \rangle \, y_1 y_2)$, for instance.

We have $\mathfrak{S}_p \simeq \mathfrak{S}_{\{x_1,x_2,x_3\}} \times \mathfrak{S}_{\{y_1,y_2,y_3,y_4\}}$ (a group with 144 elements). Then $\mathsf{Iso}(\varphi, p)$ is the subgroup generated by the two transpositions which swap respectively y_1, y_2 and y_3, y_4, and by the permutation f given by $f(x_1) = x_1$, $f(x_2) = x_3$, $f(x_3) = x_2$, $f(y_1) = y_3$, $f(y_2) = y_4$, $f(y_3) = y_1$ and $f(y_4) = y_2$. This subgroup has 8 elements, as easily checked. Observe by the way that $\mathsf{m}(\sigma) = 2 \times 2^2 = 8$.

4.2.3 Combinatorial interpretation.

Here is the announced combinatorial interpretation of the multiplicity coefficients.

Lemma 25 *Let σ be a (poly-)term, let φ be a multilinear-visible (poly-)term and $p : \mathsf{fv}(\varphi) \to \mathcal{V}$ be a function such that (φ, p) represents σ. Then $|\mathsf{Iso}(\varphi, p)| = \mathsf{m}(\sigma)$.*

The proof is by induction on σ.

4.2.4 Isotropy group of a multilinear-visible substitution.

More generally, if Φ is a multilinear-visible substitution and if $p : \mathsf{Dom}\, \Phi \to \mathcal{V}$ and $q : \mathsf{fv}(\Phi) \to \mathcal{V}$ are functions, we define the group

$$\mathsf{Iso}(p, \Phi, q) = \{ g \in \mathfrak{S}_q \mid \exists f \in \mathfrak{S}_p \; \Phi[g] = \Phi f \} \, ,$$

where $\Phi[g]$ stands for the multilinear-visible substitution which has the same domain as Φ and is given by $\Phi[g](x) = \Phi(x)[g]$.

Due to the injectivity of Φ as a function from variables to multilinear-visible terms, the bijection f associated with g in the definition above is uniquely determined, and clearly the map $g \mapsto f$ is a group homomorphism. In other words, $\mathsf{Iso}(p, \Phi, q)$ comes equipped with a group homomorphism $\mathsf{Iso}(p, \Phi, q) \to \mathfrak{S}_q$, that we shall always denote as π, and which is uniquely determined by the following property:

$$\forall g \in \mathsf{Iso}(p, \Phi, q) \quad \Phi[g] = \Phi \pi(g) \, . \tag{15}$$

Let Φ, p and q be as above. For each $x \in \mathcal{V}$, $p^{-1}(x)$ is a finite set which is empty for almost all xs since p is finite. Let T_x be the poly-term which is the multiset of simple terms $[\Phi(y_1)^q, \ldots, \Phi(y_n)^q]$ where $\{y_1, \ldots, y_n\} = p^{-1}(x)$. Then, by Lemma 25, we have

$$|\mathsf{Iso}(p, \Phi, q)| = \prod_{x \in \mathcal{V}} \mathsf{m}(T_x) \tag{16}$$

as easily checked.

4.2.5 A combined isotropy set. Assume that we are given φ, Φ, p and q as above, with (φ, Φ) adapted. Then there is yet another set of permutations which will play an important role in the sequel, and this set *is not a group in general*, namely:

$$\mathsf{Iso}(\varphi, p, \Phi, q) = \{f \in \mathfrak{S}_p \mid \exists g \in \mathfrak{S}_q \ (\varphi[\Phi])[g] = \varphi[\Phi f]\} \,.$$

Lemma 26 *Let φ be a multilinear-visible (poly-)term. Let Φ be a multilinear-visible substitution such that (φ, Φ) is adapted. Let $p : \mathsf{Dom}\,\Phi \to \mathcal{V}$ and $q : \mathsf{fv}(\Phi) \to \mathcal{V}$. Then*

$$\pi(\mathsf{Iso}(p, \Phi, q))\,\mathsf{Iso}(\varphi, p) \subseteq \mathsf{Iso}(\varphi, p, \Phi, q)\,,$$

where we recall that π is defined by equation (15).

Proof. Let $g \in \mathsf{Iso}(p, \Phi, q)$ and let $f \in \mathsf{Iso}(\varphi, p)$. Then $\varphi[\Phi\pi(g)f] = (\varphi[f])[\Phi\pi(g)] = \varphi[\Phi\pi(g)]$ since $f \in \mathsf{Iso}(\varphi, p)$ and hence $\varphi[\Phi\pi(g)f] = \varphi[\Phi[g]]$ since $g \in \mathsf{Iso}(p, \Phi, q)$. But we have $\varphi[\Phi[g]] = (\varphi[\Phi])[g]$ and so $\pi(g)f \in \mathsf{Iso}(\varphi, p, \Phi, q)$. □

We shall see that, under some uniformity condition on the pair (φ, p), the converse inclusion holds as well. The crucial step for proving this is the forthcoming factorization property, Lemma 27.

4.2.6 Uniform pairs. We define when a pair (F, p) is *uniform*, F being a multilinear-visible poly-term and $p : \mathsf{fv}(F) \to \mathcal{V}$ a naming function. We shall see in Lemma 28 that this notion is equivalent to the concept of uniformity we have already defined in Section 3, using the coherence relation on poly-terms, but we give first the following self-contained definition, very suitable to our present combinatorial considerations. The definition is by induction. The pair (F, p) is uniform in one of the following situations:

- $F = x_1 \dots x_n$ where the x_is are variables and $p(x_i) = p(x_j)$ for all i, j;
- $F = (\lambda y\, \varphi_1) \dots (\lambda y\, \varphi_n)$ and $(\varphi_1 \dots \varphi_n, p)$ is uniform;
- $F = (\langle\varphi_1\rangle\, G_1) \dots (\langle\varphi_n\rangle\, G_n)$ and $(\varphi_1 \dots \varphi_n, l)$ and $(G_1 \dots G_n, r)$ are uniform, where l and r are the obvious restrictions of p.

When u is a multilinear-visible simple term, we say that (u, p) is uniform if (F, p) is uniform, where F is the multilinear-visible poly-term which has u as single element.

4.2.7 The factorization property of uniform pairs. The main property of uniform pairs is the following factorization lemma.

Lemma 27 (factorization) *Let (φ, p) be a uniform pair and let Φ and Φ' be two multilinear-visible substitutions of domain $\mathsf{fv}(\varphi)$. If $\varphi[\Phi] = \varphi[\Phi']$, then there exists $f \in \mathsf{Iso}(\varphi, p)$ such that $\Phi' = \Phi f$.*

Proof. We can restrict our attention to the case where φ is a poly-term, and the only interesting case in the inductive definition 4.2.6 of uniformity is obviously the last one. With the notations of that definition, we can find, by inductive hypothesis, $g \in \mathsf{Iso}(\varphi_1 \dots \varphi_n, l)$ such that $\Lambda' = \Lambda g$ and $h \in \mathsf{Iso}(G_1 \dots G_n, r)$ such that $P' = Ph$ where Λ, Λ' and P, P' are the restrictions of Φ, Φ' to $\mathsf{fv}(\varphi_1 \dots \varphi_n)$ and $\mathsf{fv}(G_1 \dots G_n)$ respectively. Taking the union f of these two bijections g and h, we obtain an element f of $\mathfrak{S}_p$, and it remains to show that $F[f] = F$.

For this, it will be sufficient to show that there is an index i such that $\varphi_1[g] = \varphi_i$ and $G_1[h] = G_i$. We know that there is an i such that $\varphi_1[g] = \varphi_i$ since $g \in \mathsf{Iso}(\varphi_1 \dots \varphi_n, l))$ (and this i is unique since each φ_j contains at least one variable, and all these variables are distinct).

We know moreover that $(\langle\varphi_1\rangle\, G_1 \dots \langle\varphi_n\rangle\, G_n)[\Phi] = (\langle\varphi_1\rangle\, G_1 \dots \langle\varphi_n\rangle\, G_n)[\Phi']$ and hence there is a (uniquely determined) j such that $(\langle\varphi_1\rangle\, G_1)[\Phi'] = (\langle\varphi_j\rangle\, G_j)[\Phi]$, hence $\varphi_1[\Lambda'] = \varphi_j[\Lambda]$, that is $\varphi_1[\Lambda g] = \varphi_j[\Lambda]$. This implies that $\varphi_1[g] = \varphi_j$ (because Λ is an injective partial function from variables to simple terms), hence $\varphi_i = \varphi_j$ and so we must have $j = i$. Therefore $(\langle\varphi_1\rangle\, G_1)[\Phi'] = (\langle\varphi_i\rangle\, G_i)[\Phi]$, hence $G_1[P'] = G_i[P]$, that is $G_1[Ph] = G_i[P]$. If $G_1 = 1$ then $G_i = 1$ and $hG_1 = G_i$ holds trivially. Otherwise we conclude again using the injectivity of P. □

The uniformity hypothesis is essential: take for φ the poly-term xy, for p the identity map on $\{x, y\}$, and define Φ and Φ' by $\Phi(x) = x$, $\Phi(y) = y$ and $\Phi'(x) = y$, $\Phi'(y) = x$. Then $\varphi[\Phi] = \varphi[\Phi'] = \varphi$ but

$\Phi \neq \Phi'$ and the only element of $\mathsf{Iso}(\varphi, p)$ is the identity. The problem is of course that the pair (φ, p) is not uniform.

Here is another, maybe more illuminating, example: take $\varphi = x_1 \langle x_2 \rangle 1$ (which is a multilinear-visible poly-term) and let p be defined by $p(x_1) = p(x_2) = x$. Let Φ and Φ' be given by: $\Phi(x_1) = \langle x_1 \rangle 1$, $\Phi(x_2) = x_2$, $\Phi'(x_1) = \langle x_2 \rangle 1$ and $\Phi'(x_2) = x_1$. Then we have $\varphi[\Phi] = \varphi[\Phi'] = \langle x_1 \rangle 1 \langle x_2 \rangle 1$ but there is no permutation f such that $\Phi' = \Phi f$. Again, the point is that the pair (φ, p) is not uniform.

We state now the equivalence between the two notions of uniformity introduced so far.

Lemma 28 *Let σ be a (poly-)term. Let φ be a multilinear-visible (poly-)term and $p : \mathsf{fv}(\varphi) \to \mathcal{V}$ be a function such that $\sigma = \varphi^p$. Then σ is uniform (that is $\sigma \bigcirc \sigma$) iff the pair (φ, p) is uniform.*

The proof is a straightforward induction on σ.

4.2.8 The equation. Let φ be a multilinear-visible simple term, Φ be a multilinear-visible substitution with $\mathsf{Dom}\,\Phi = \mathsf{fv}(\varphi)$, $p : \mathsf{fv}(\varphi) \to \mathcal{V}$ and $q : \mathsf{fv}(\Phi) \to \mathcal{V}$ be functions. Assume that the pair (φ, Φ) is adapted and that the pair (φ, p) is uniform.

Let us first check that

$$\pi(\mathsf{Iso}(p, \Phi, q))\,\mathsf{Iso}(\varphi, p) = \mathsf{Iso}(\varphi, p, \Phi, q)\,.$$

Let $f \in \mathsf{Iso}(\varphi, p, \Phi, q)$, that is $f \in \mathfrak{S}_p$ and there exists $g \in \mathfrak{S}_q$ such that $(\varphi[\Phi])[g] = \varphi[\Phi f]$, that is (replacing g by its inverse), there exists $g \in \mathfrak{S}_q$ such that $\varphi[\Phi] = (\varphi[\Phi f])[g] = \varphi[\Phi[g] f]$.

Since the pair (φ, p) is uniform, we can apply Lemma 27 and hence there exists $f' \in \mathsf{Iso}(\varphi, p)$ such that $\Phi[g] f = \Phi f'$. This means that $g \in \mathsf{Iso}(p, \Phi, q)$ and $\pi(g) = f' f^{-1}$. Hence $f = \pi(g^{-1}) f' \in \pi(\mathsf{Iso}(p, \Phi, q))\,\mathsf{Iso}(\varphi, p)$. The converse inclusion holds by Lemma 26.

Since $|\pi(\mathsf{Iso}(p, \Phi, q))| = |\mathsf{Iso}(p, \Phi, q)| \,/\, |\ker \pi|$, applying Lemma 24 we obtain

$$|\mathsf{Iso}(\varphi, p, \Phi, q)| = \frac{|\mathsf{Iso}(p, \Phi, q)|\,|\mathsf{Iso}(\varphi, p)|}{|\ker \pi|\,|\pi(\mathsf{Iso}(p, \Phi, q)) \cap \mathsf{Iso}(\varphi, p)|}\,.$$

To conclude, we show that $|\pi(\mathsf{Iso}(p, \Phi, q)) \cap \mathsf{Iso}(\varphi, p)| = |\pi(\mathsf{Iso}(\varphi[\Phi], q))|$.

Let $g \in \mathsf{Iso}(\varphi[\Phi], q)$. Since the pair (φ, p) is uniform, by Lemma 27 again, there exists $f \in \mathsf{Iso}(\varphi, p)$ such that $\Phi[g] = \Phi f$. In other words $\mathsf{Iso}(\varphi[\Phi], q) \subseteq \mathsf{Iso}(p, \Phi, q)$ and also $\pi(\mathsf{Iso}(\varphi[\Phi], q)) \subseteq \mathsf{Iso}(\varphi, p)$. So $\pi(\mathsf{Iso}(\varphi[\Phi], q)) \subseteq \pi(\mathsf{Iso}(p, \Phi, q)) \cap \mathsf{Iso}(\varphi, p)$. But the converse implication holds as well. Indeed, let $g \in \mathsf{Iso}(p, \Phi, q)$ be such that $\pi(g) \in \mathsf{Iso}(\varphi, p)$. Then $(\varphi[\Phi])[g] = \varphi[\Phi[g]] = \varphi[\Phi \pi(g)] = \varphi[\Phi]$ and hence $g \in \mathsf{Iso}(\varphi[\Phi], q)$.

Last observe that obviously $\ker \pi \subseteq \mathsf{Iso}(\varphi[\Phi], q)$. So

$$|\pi(\mathsf{Iso}(p, \Phi, q)) \cap \mathsf{Iso}(\varphi, p)| = |\pi(\mathsf{Iso}(\varphi[\Phi], q))| = \frac{|\mathsf{Iso}(\varphi[\Phi], q)|}{|\ker \pi|}\,.$$

So we have proved the following result which will be essential in the sequel.

Theorem 29 (Uniform plugging equation) *If φ is a multilinear-visible simple term, Φ a multilinear-visible substitution with (φ, Φ) adapted, if $p : \mathsf{fv}(\varphi) \to \mathcal{V}$ and $q : \mathsf{fv}(\Phi) \to \mathcal{V}$ are functions and if the pair (φ, p) is uniform, then the following equation holds:*

$$|\mathsf{Iso}(\varphi, p, \Phi, q)| = \frac{|\mathsf{Iso}(p, \Phi, q)|\,|\mathsf{Iso}(\varphi, p)|}{|\mathsf{Iso}(\varphi[\Phi], q)|}\,.$$

The uniformity hypothesis is necessary. Take indeed for φ the non uniform poly-term $\varphi = x_1(\langle x_2 \rangle 1)$ (p being the constant function $x_i \mapsto x$ where x is a fixed element of $\mathcal{V}$). Then $|\mathsf{Iso}(\varphi, p)| = 1$. Define Φ by $\Phi(x_1) = \langle y_1 \rangle 1$ and $\Phi(x_2) = y_2$ and take for q a constant function $q(y_j) = y$. Then $|\mathsf{Iso}(p, \Phi, q)| = 1$, but $\varphi[\Phi] = (\langle y_1 \rangle 1)(\langle y_2 \rangle 1)$ so that $|\mathsf{Iso}(\varphi[\Phi], q)| = 2$ and the equation above cannot hold since its left hand member must be an integer.

5 Reducing the Taylor expansion of an ordinary lambda-term

With the qualitative Theorems 20 and 22 and the quantitative Theorem 29, we have the main tools for studying the beta-reduction of the Taylor expansion of an ordinary lambda-term.

Extension of NF to infinite, coherent (poly-)terms. We need first to consider the case of a single big step differentiation: for dealing with this case, we apply the uniform plugging equation straightforwardly.

Lemma 30 *Let $\sigma \in \Delta^{(!)}$ be uniform, let x be a variable and let $T \in \Delta^!$. Let $\theta \in \mathsf{supp}(\partial_x(\sigma, T))$. Then the coefficient $\partial_x(\sigma, T)_\theta$ of θ in $\partial_x(\sigma, T)$, which is a positive integer, is given by*

$$\partial_x(\sigma, T)_\theta = \frac{\mathsf{m}(\sigma)\mathsf{m}(T)}{\mathsf{m}(\theta)}\,.$$

Proof. Observe first that our hypotheses imply that $|T| = \mathsf{deg}_x\sigma$ since otherwise the set $\mathsf{supp}(\partial_x(\sigma, T))$ would be empty. Let φ be a multilinear-visible (poly-)term and let $p : \mathsf{fv}(\varphi) \to \mathcal{V}$ be a function such that $\varphi^p = \sigma$. Then, by Lemma 28, the pair (φ, p) is uniform since σ is. By Formula (6), we can choose a multilinear-visible substitution Φ and a function $q : \mathsf{fv}(\Phi) \to \mathcal{V}$ in such a way that the following requirements be fulfilled:

- the pair (φ, Φ) is adapted;
- $(\prod_{p(x')=x} \Phi(x'))^q = T$ (that is, (Φ, q), when restricted to $p^{-1}(\{x\})$, represents T);
- if $p(x') \neq x$ then $\Phi(x') = x'$ and $q(x') = p(x')$ (that is, the substitution Φ acts trivially on all occurrences of variables distinct from x);
- $\theta = (\varphi[\Phi])^q$.

By Formula (6), the coefficient $\partial_x(\sigma, T)_\theta$ is the number of permutations $f \in \mathfrak{S}_n$ such that

$$\sigma' \left[t_{f(1)}/x_1, \ldots, t_{f(n)}/x_n\right] = \theta\,,$$

where $t_1 \ldots t_n = T$, the variables $x_1, \ldots, x_n$ are fresh and σ' is an x-linearization in $x_1, \ldots, x_n$ of σ. This x-linearization can be chosen such that $\sigma' [t_1/x_1, \ldots, t_n/x_n] = \theta$ and in that case the above mentioned set of permutations contains the identity permutation and is in canonical bijective correspondance with $\mathsf{Iso}(\varphi, p, \Phi, q)$ (remember that this set is not a group in general) because Φ acts trivially on the variables of φ which do not correspond to x. Therefore we have $\partial_x(\sigma, T)_\theta = |\mathsf{Iso}(\varphi, p, \Phi, q)|$.

By Theorem 29, since (φ, p) is uniform, we have

$$|\mathsf{Iso}(\varphi, p, \Phi, q)| = \frac{|\mathsf{Iso}(p, \Phi, q)|\,|\mathsf{Iso}(\varphi, p)|}{|\mathsf{Iso}(\varphi[\Phi], q)|}\,.$$

and we conclude because, by Lemma 25, we have $|\mathsf{Iso}(\varphi, p)| = \mathsf{m}(\sigma)$ and $|\mathsf{Iso}(\varphi[\Phi], q)| = \mathsf{m}(\theta)$, and we have $|\mathsf{Iso}(p, \Phi, q)| = \mathsf{m}(T)$ by Equation (16). □

Again, the uniformity condition is absolutely essential.

Two corollaries. We derive two easy corollaries of this formula, before applying it to our main concern, which is the study of the normal forms of the terms occurring in the Taylor expansion of an ordinary lambda-term.

First, we generalize the formula to iterated big step differentiation.

Proposition 31 *Let $\sigma \in \Delta^{(!)}$ be uniform, let $x_1, \ldots, x_n$ be pairwise distinct variables and let $T_1, \ldots, T_n \in \Delta^!$ be uniform. Let $\theta \in \mathsf{supp}(\partial_{x_1,\ldots,x_n}(\sigma, T_1, \ldots, T_n))$. Then*

$$\partial_{x_1,\ldots,x_n}(\sigma, T_1, \ldots, T_n)_\theta = \frac{\mathsf{m}(\sigma)\mathsf{m}(T_1)\cdots\mathsf{m}(T_n)}{\mathsf{m}(\theta)}\,.$$

Proof. It will be enough to deal with the case $n = 2$. We have

$$\begin{aligned} \partial_{x_1,x_2}(\sigma, T_1, T_2)_\theta &= \partial_{x_2}(\partial_{x_1}(\sigma, T_1), T_2)_\theta \\ &= \sum_{\rho\in\Delta^{(!)}} \partial_{x_1}(\sigma, T_1)_\rho \partial_{x_2}(\rho, T_2)_\theta\,, \end{aligned}$$

but since σ and T_1 are uniform, $\mathsf{supp}(\partial_{x_1}(\sigma, T_1))$ is a clique by Theorem 20 and hence there is at most one $\rho \in \mathsf{supp}(\partial_{x_1}(\sigma, T_1))$ such that $\theta \in \mathsf{supp}(\partial_{x_2}(\rho, T_2))$. Hence, since we have assumed that $\theta \in \mathsf{supp}(\partial_{x_1,x_2}(\sigma, T_1, T_2))$, there is exactly one such ρ and we know that this ρ is uniform, so we get, applying twice Lemma 30,

$$\partial_{x_1,x_2}(\sigma, T_1, T_2)_\theta = \frac{\mathsf{m}(\sigma)\mathsf{m}(T_1)}{\mathsf{m}(\rho)} \cdot \frac{\mathsf{m}(\rho)\mathsf{m}(T_2)}{\mathsf{m}(\theta)} = \frac{\mathsf{m}(\sigma)\mathsf{m}(T_1)\mathsf{m}(T_2)}{\mathsf{m}(\theta)}\,.$$

□

The second corollary is another version of the Taylor formula, which is now substitution-oriented instead of being application-oriented as in Lemma 18.

Theorem 32 *Let M and N be ordinary lambda-terms and let x be a variable. One has $\partial_x(M^*, N^{*n}) \to 0$ as $n \to \infty$, and the following equation holds:*

$$(M\,[N/x])^* = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_x(M^*, N^{*n})\,.$$

Proof. The convergence statement results from the fact that $M^{*n} \to 0$ and from the continuity of ∂_x. Just as in the proof of Lemma 18, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \partial_x(M^*, N^{*n}) = \sum_{\substack{s\in\mathcal{T}(M) \\ T\in\mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))}} \frac{1}{\mathsf{m}(s)\mathsf{m}(T)} \partial_x(s, T)\,.$$

To conclude, observe that the family of sets $(\mathsf{supp}(\partial_x(s, T)))_{(s,T)\in\mathcal{T}(M)\times\mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))}$ is a partition of $\mathcal{T}(M\,[N/x])$ (disjointness results from Theorem 20, and the equality of sets is proved by an easy induction on M, using the Leibniz law in the case where M is an application), and then apply Lemma 30.

□

Proposition 33 *Let $\sigma \in \Delta^{(!)}$ be uniform and let $\theta \in \mathsf{supp}(\mathsf{NF}(\sigma))$. Then $\mathsf{m}(\theta)$ divides $\mathsf{m}(\sigma)$, and more precisely*

$$\frac{\mathsf{m}(\sigma)}{\mathsf{m}(\theta)} = \mathsf{NF}(\sigma)_\theta\,.$$

Proof. We proceed by induction on the size of the simple (poly-)term σ, using Lemma 10. Indeed observe that when σ is uniform, the terms to which NF is applied in the "recursive calls" of that lemma are themselves uniform (the only non-trivial case is the first one, and in that case our claim results from Theorem 20 and from the fact that any (poly-)subterm of a uniform (poly-)term is uniform).

If $\sigma = \lambda x_1 \dots x_n \langle \cdots \langle x\rangle\, T_1 \cdots\rangle\, T_k$ then $\theta = \lambda x_1 \dots x_n \langle \cdots \langle x\rangle\, U_1 \cdots\rangle\, U_k$ with $U_j \in \mathsf{supp}(\mathsf{NF}(T_j))$ for $j = 1, \dots, k$. By inductive hypothesis, $\mathsf{m}(T_j)/\mathsf{m}(U_j) = \mathsf{NF}(T_j)_{U_j}$, but $\mathsf{m}(\sigma) = \mathsf{m}(T_1) \cdots \mathsf{m}(T_k)$ and $\mathsf{m}(\theta) = \mathsf{m}(U_1) \cdots \mathsf{m}(U_k)$ and we conclude because, by multilinearity of application,

$$\mathsf{NF}(\sigma) = \sum_{V_1,\dots,V_k} \mathsf{NF}(T_1)_{V_1} \cdots \mathsf{NF}(T_k)_{V_k} \lambda x_1 \dots x_n \langle \cdots \langle x\rangle\, V_1 \cdots\rangle\, V_k\,.$$

Assume now that $\sigma = \lambda x_1 \dots x_n \langle \cdots \langle r\rangle\, T_1 \cdots\rangle\, T_k$ where $r = \langle \lambda x\, s\rangle\, T$. Then there exists $s' \in \mathsf{supp}(\partial_x(s, T))$ such that $\theta \in \mathsf{supp}(\mathsf{NF}(\lambda x_1 \dots x_n \langle \cdots \langle s'\rangle\, T_1 \cdots\rangle\, T_k))$, and this simple term s' is unique by Theorem 22, since $\mathsf{supp}(\partial_x(s, T))$ is a clique by Theorem 20. By inductive hypothesis,

$$\frac{\mathsf{m}(\lambda x_1 \dots x_n \langle s'\rangle\, T_1 \dots T_k)}{\mathsf{m}(\theta)} = \mathsf{NF}(\lambda x_1 \dots x_n \langle \cdots \langle s'\rangle\, T_1 \cdots\rangle\, T_k)_\theta\,.$$

But $\mathsf{NF}(\sigma) = \widetilde{\mathsf{NF}}(\lambda x_1 \dots x_n \, \langle \cdots \langle \partial_x(s,T) \rangle \, T_1 \cdots \rangle \, T_k)$ and so $\mathsf{NF}(\sigma)_\theta = \partial_x(s,T)_{s'} \, \mathsf{NF}(\lambda x_1 \dots x_n \, \langle \cdots \langle s' \rangle \, T_1 \cdots \rangle \, T_k)_\theta$ (see Equation (8)). Therefore by Lemma 30 we get

$$\begin{aligned}
\mathsf{NF}(\sigma)_\theta &= \frac{\mathsf{m}(s)\mathsf{m}(T)\mathsf{m}(\lambda x_1 \dots x_n \, \langle \cdots \langle s' \rangle \, T_1 \cdots \rangle \, T_k)}{\mathsf{m}(s')\mathsf{m}(\theta)} \\
&= \frac{\mathsf{m}(s)\mathsf{m}(T)\mathsf{m}(T_1) \cdots \mathsf{m}(T_k)}{\mathsf{m}(\theta)} \\
&= \frac{\mathsf{m}(\sigma)}{\mathsf{m}(\theta)} \, .
\end{aligned}$$

As a last case, consider the situation where $\sigma = s_1^{p_1} \dots s_k^{p_k}$ is a uniform poly-term, with $s_i \frown s_j$ for all i, j, and s_i and s_j not α-equivalent when $i \neq j$, so that

$$\mathsf{m}(\sigma) = \prod_{j=1}^{k} p_j! \; \mathsf{m}(s_j)^{p_j} \, .$$

Then, by Theorem 22, $\mathsf{supp}(\mathsf{NF}(s_1)), \dots, \mathsf{supp}(\mathsf{NF}(s_k))$ are *pairwise disjoint* cliques and θ is of the shape $\theta = U_1 \dots U_k$ with $U_j \in \mathsf{supp}(\mathsf{NF}(s_j)^{p_j})$ for $j = 1, \dots, k$, and so the multi-sets U_j are pairwise disjoint, so that

$$\mathsf{m}(\theta) = \mathsf{m}(U_1) \cdots \mathsf{m}(U_k) \, .$$

Let $j \in \{1, \dots, k\}$, we have $\mathsf{m}(U_j) = U_j! \; \mathsf{m}^{U_j}$ so that

$$\begin{aligned}
\frac{\mathsf{m}(\sigma)}{\mathsf{m}(\theta)} &= \prod_{j=1}^{k} \frac{p_j! \; \mathsf{m}(s_j)^{p_j}}{U_j! \; \mathsf{m}^{U_j}} \\
&= \prod_{j=1}^{k} [U_j] \, \frac{\mathsf{m}(s_j)^{p_j}}{\mathsf{m}^{U_j}}
\end{aligned}$$

but for each j,

$$\begin{aligned}
\mathsf{NF}(s_j)^{p_j} &= \left(\sum_{u \in \Delta_0} \mathsf{NF}(s_j)_u u \right)^{p_j} \\
&= \left(\sum_{u \in \Delta_0} \frac{\mathsf{m}(s_j)}{\mathsf{m}(u)} u \right)^{p_j} && \text{by inductive hypothesis} \\
&= \sum_{U \in \mathcal{M}_{p_j}(\Delta_0)} [U] \, \frac{\mathsf{m}(s_j)^{p_j}}{\mathsf{m}^U} \, U && \text{by the multinomial identity,}
\end{aligned}$$

so

$$\begin{aligned}
\mathsf{NF}(\sigma)_\theta &= \prod_{j=1}^{k} \mathsf{NF}(s_j)^{p_j}_{U_j} \\
&= \prod_{j=1}^{k} [U_j] \, \frac{\mathsf{m}(s_j)^{p_j}}{\mathsf{m}^{U_j}} \\
&= \frac{\mathsf{m}(\sigma)}{\mathsf{m}(\theta)}
\end{aligned}$$

and we are done. □

Given an element τ of $R\langle \Delta^{(!)} \rangle_\infty$, the sum $\mathsf{NF}(\tau) = \sum_{\theta \in \Delta^{(!)}} \tau_\theta \, \mathsf{NF}(\theta)$ does not always converge (in the sense of 2.1.1): it can involve infinite sums of coefficients. But in the case where τ is the Taylor expansion of a lambda-term, it does converge.

Corollary 34 *Let M be an ordinary lambda-term and let $u \in \mathsf{supp}(\mathsf{NF}(M^*))$. Then the sum $\mathsf{NF}(M^*)$ converges and, for any simple term u occurring in that sum, one has $\mathsf{NF}(M^*)_u = 1/\mathsf{m}(u)$. Moreover, there is exactly one simple term $s \in \mathcal{T}(M)$ such that $u \in \mathsf{supp}(\mathsf{NF}(s))$.*

Proof. Remember that $M^* = \sum_{s\in\mathcal{T}(M)} \frac{1}{\mathsf{m}(s)} s$ and that $\mathcal{T}(M)$ is a clique (Lemma 19). Therefore the supports of the terms $\mathsf{NF}(s)$, for $s \in \mathcal{T}(M)$, are pairwise disjoint, by Theorem 22. Hence, the sum $\mathsf{NF}(M^*) = \sum_{s\in\mathcal{T}(M)} \frac{1}{\mathsf{m}(s)} \mathsf{NF}(s)$ converges, and, for any simple term u which occurs with a non-zero coefficient in that sum, there is exactly one $s \in \mathcal{T}(M)$ such that $u \in \mathsf{supp}(\mathsf{NF}(s))$, by Theorem 22 again. The coefficient of u in $\mathsf{NF}(M^*)$ is $\mathsf{NF}(s)_u/\mathsf{m}(s) = 1/\mathsf{m}(u)$ by Proposition 33. □

Corollary 35 *The sum* $\mathsf{NF}(M^*)$ *has the following shape*

$$\mathsf{NF}(M^*) = \sum_{u\in\mathcal{U}} \frac{1}{\mathsf{m}(u)} u$$

where $\mathcal{U}$ *is a set of normal simple terms, which is a clique (by Theorem 22, since* $\mathcal{T}(M)$ *is a clique).*

In [ER06a], it is shown, using Krivine machine, that actually $\mathcal{U} = \mathcal{T}(M_0)$, where $\mathsf{BT}(M)$ is the *Böhm tree* of M. Therefore, we have

$$\mathsf{NF}(M^*) = (\mathsf{BT}(M))^* \,. \tag{17}$$

In other words, Taylor expansion commutes with (infinite) normalization. The analysis developped in [ER06a] shows that the simple term s associated with u (in the statement of Corollary 34) represents the part of M which is necessay for computing the part u of M_0 in Krivine machine, taking multiplicities into account.

Example. Let M be the ordinary lambda-term

$$M = (\lambda f\,(f)\,\lambda x\,(f)\,\lambda d\,x)\,\lambda z\,(z)\,(z)\star$$

where $\star$ is a distinguished variable. It is easily seen that M reduces to $\star$. By the theorem above, there is at most one simple term $s \in \mathcal{T}(M)$ such that $\star \in \mathsf{supp}(\mathsf{NF}(s))$. One checks easily that

$$s = \left\langle \lambda f\,\langle f\rangle\,(\lambda x\,\langle f\rangle\,\lambda d\,x)^2\right\rangle (\lambda z\,\langle z\rangle\,\langle z\rangle\,\star)(\lambda z\,\langle z\rangle\,1)^2$$

is such a term, and more precisely that s reduces to $4\star$, in accordance with the fact that $\mathsf{m}(s) = 4$. This simple term can be seen as a "decoration" of M giving an exact quantitative account of how much each subterm of M is used during the run of the Krivine's machine starting with term M (empty environment and empty stack) and leading to the final value $\star$.

Conclusion

The main result of this paper, Corollary 35 and its consequence, Formula (17), show that the situation is as simple and natural as one could expect. The striking fact, maybe, is not the result itself but its proof, which is based on Theorems 22 and 29, and so uses uniformity twice, and each time in a crucial way. So an essential step in the understanding of the differential extension of the functional paradigm proposed in [ER03] will be to examine the behaviour of Taylor expansions in this more general and non uniform setting.

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