On strong stability and higher-order sequentiality*

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Abstract

We propose a definition by reducibility of sequentiality for the interpretations of higher-order programs and prove the equivalence between this notion and strong stability.

Introduction

Sequentiality is an abstract semantical notion which captures determinism for the interpretations of first-order programs. It has been introduced independently by Vuillemin and Milner (cf. [V] and [Mi]). The problem of extending this notion to higher-order programs, like those of Gödel system T, has led to various definitions:

- Sequential algorithms on concrete data structures (see [C1]) and more recently various game-theoretic models (see [AJ, C2, HO, L]) inspired by the work of Blass [Bl1, Bl2], which are quite intentional models where programs of functional type are not simply interpreted by functions, but by more complicated objects ("algorithms" or "strategies") which contain detailed informations about their behaviour.

- Strong stability, a notion which coincides with ordinary sequentiality at first order and admits an extension to higher orders where all programs are simply interpreted by functions. This notion has been introduced by Bucciarelli and the second author in [BE1].

We introduce here a new notion of sequentiality for functionals of finite type based on the type ι of natural numbers. This notion is defined by a reducibility-like method (by induction on the type of the functionals).

Let us give roughly the definition of this notion. We define by induction on the type σ what is sequentiality of type $\iota^k \to \sigma$ for all k.

- For $\sigma = \iota$, sequentiality is the usual Vuillemin-Milner notion.

- We say that $f : \iota^k \to (\sigma \to \tau)$ is sequential if, for any $p \in \omega$ and for any sequential function g of type $\iota^p \to \sigma$ the function
$$\lambda(x,y).f(x)(g(y)) : \iota^{k+p} \to \tau$$
is sequential.

Similar notions have been introduced by Longo and Moggi (see [LM]) and more recently by Sazonov and Voronkov (see [SV]) for extending naturally to higher order notions which make sense at first order. The authors rediscovered independently this method.

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The main result of this paper is that this new notion of sequentiality is equivalent to strong stability. This is surprising because the two definitions have very different origins, reducibility comes from proof theory whereas strong stability belongs to (almost) traditional domain theory.

For technical reasons concerning continuity, we could not obtain this result with the notion of higher order sequentiality as it is defined above without excessive complications. So we had to embed all the ι^k's into ι^ω in a natural way, and consider functions of type $\iota^\omega \to \sigma$ instead of $\iota^k \to \sigma$.

1 Preliminaries

1.1 About sets

Let E and F be two sets. If $C \subseteq E \times F$, we note C_1 or C_E the first projection of C and C_2 or C_F its second projection. We say that C is a *pairing* of E and F if $C_1 = E$ and $C_2 = F$.

The disjoint union of E and F will be noted $E + F$ and represented by $G = (E \times \{1\}) \cup (F \times \{2\})$. If $C \subseteq G$, we note $C_1 = \{a \in E \mid (a, 1) \in C\}$ its first component and $C_2 = \{b \in F \mid (b, 2) \in C\}$ its second component.

Let A and B be two sets. We say that A is a *multisection* of B and we write $A \lhd B$ if

$$\forall a \in A \exists b \in B\ a \in b \quad \text{and} \quad \forall b \in B \exists a \in A\ a \in b\ .$$

This means that $A \subseteq \bigcup B$ and that $A \cap b$ is non empty for all $b \in B$.

If E is a set, we note $\mathcal{P}^*_{\mathrm{fin}}(E)$ the set of its finite and non-empty subsets. We write $x \subseteq^*_{\mathrm{fin}} E$ when x is a finite and non-empty subset of E.

1.2 About qualitative domains and stable functions

We briefly recall the main definitions and results of [G1].

For the reader acquainted with domain theory, let us first say that a qualitative domain is a dI-domain where all primes are atoms. We give here a direct definition, borrowed from [G1].

Definition 1.1 *A* qualitative domain *is a set E satisfying the following conditions*[1] *:*

- *$\emptyset \in E$.*
- *If $x \in E$ and $y \subseteq x$ then $y \in E$.*
- *For any $\subseteq$-directed subset D of E, one has $\bigcup D \in E$.*

If E is a qualitative domain, its web *$|E|$ is defined by*

$$|E| = \{a \mid \{a\} \in E\}\ .$$

This set is usually assumed to be enumerable.
The set of all finite elements of E will be written E_{fin}.

The web of a qualitative domain is the set of its prime (i.e. atomic) elements, that is the set of its "generators".

Definition 1.2 *Let E and F be two qualitative domains. A function $f : E \to F$ is said to be* stable *iff :*

- *It is Scott-continuous, that is, for any $D \subseteq E$ which is directed, one has $f(\bigcup D) = \bigcup f(D)$.*
- *It is* conditionally multiplicative*, that is, for any $x, y \in E$ such that $x \cup y \in E$, one has*

$$f(x \cap y) = f(x) \cap f(y)\ .$$

[1] Notice that E is in fact a set of sets, that $\emptyset$ corresponds to the domain-theoretic $\perp$, and that the order of the corresponding domain is the inclusion.

Let us recall that this last condition, in the case of qualitative domains, is equivalent to the following : for all $x \in E$, for all $b \in f(x)$ there exists a unique minimal $x_0 \subseteq x$ such that $b \in f(x_0)$.
This condition was originally called *stability* by Berry in [B1] and corresponds to a weak form of sequentiality.

The category of qualitative domains and stable functions is cartesian closed (see [G1]). If E and F are qualitative domains, their product $E \times F$ in this category can be described as follows :

$$E \times F = \{z \subseteq |E| + |F| \mid z_1 \in E \text{ and } z_2 \in F\} .$$

Observe also that the category has ω-products.

We shall not describe here the exponential, but we shall define the notion of *trace*, which is the key notion for this description.

Definition 1.3 *Let E and F be qualitative domains. Let $f : E \to F$ be a stable function. The* trace *of f is the set* $\mathrm{tr}\,(f)$ *defined by*

$$\mathrm{tr}\,(f) = \{(x_0, b) \in E_{\mathrm{fin}} \times |F| \mid b \in f(x_0),\ x_0 \textit{ minimal}\}$$

The interest of this notion lies mainly in the following result.

Proposition 1.1 *Let $f : E \to F$ be a stable function. Then, for any $x \in E$, one has*

$$f(x) = \{b \mid \exists x_0 \subseteq x\ (x_0, b) \in \mathrm{tr}\,(f)\} .$$

Furthermore, if $f, g : E \to F$ are two stable functions, one has $\mathrm{tr}\,(f) \subseteq \mathrm{tr}\,(g)$ iff $f \leq g$ for the stable order.

Let us just recall that $f \leq g$ for the stable order means that, for any $x, y \in E$, if $x \subseteq y$, then

$$f(x) = f(y) \cap g(x) .$$

2 Vuillemin-Milner sequentiality

The purpose of this paper is to extend the classical notion of sequentiality introduced by Vuillemin and Milner to higher order using a "reducibility" method, and to establish that this extension is identical to the extension provided by strong stability. We first recall the classical definition, for functions which are defined on and take their values in products of the "flat" domain of integers $\mathbf{N}_\perp = \{\perp, 0, 1, 2, \ldots\}$, which is ordered as follows :

$$x < y \quad \text{iff} \quad x = \perp \text{ and } y \neq \perp .$$

The products[2] $\mathbf{N}_\perp^n$ (for $n = 1, 2, \ldots, \omega$) are ordered by the standard product order.

Definition 2.1 *Let $n, m \in \{1, 2, \ldots, \omega\}$. We say that a function $f : \mathbf{N}_\perp^n \to \mathbf{N}_\perp^m$ is sequential (in the sense of Vuillemin-Milner) iff it is monotone, continuous and satisfies the following condition. For any $\alpha \in \mathbf{N}_\perp^n$, for any j such that $0 \leq j < m$, if $f(\alpha)_j = \perp$, then*

- *either $\forall \beta \in \mathbf{N}_\perp^n\ \beta \geq \alpha \Rightarrow f(\beta)_j = \perp$*
- *or there exists i such that $0 \leq i < n$ and $\alpha_i = \perp$ and, for any $\beta \in \mathbf{N}_\perp^n$, if $\beta \geq \alpha$ and $f(\beta)_j \neq \perp$ then $\beta_i \neq \perp$.*

In this definition, the integer i is called "sequentiality index of f for j at α".

[2]The elements of $\mathbf{N}_\perp^n$ are sequences indexed on the set $\{i \mid 0 \leq i < n\}$

3 Hypercoherences

In a recent paper (see [E]), the second author has introduced the notion of *hypercoherence* as a simplified framework where strong stability makes sense. We recall here the basic definitions and the properties of this model that we use in the sequel.

Definition 3.1 *A hypercoherence X is a pair $(|X|, \Gamma(X))$ where $|X|$ is an enumerable set (the web) and $\Gamma(X)$ is a subset of $\mathcal{P}^*_{\text{fin}}(X)$ (the atomic coherence) such that, for any $a \in |X|$, one has $\{a\} \in \Gamma(X)$.*

If X is a hypercoherence, we note $\Gamma^*(X)$ and call *strict atomic coherence of* X the set of all elements of $\Gamma(X)$ which are not singleton (observe that X can be described by $\Gamma^*(X)$ as well as by $\Gamma(X)$).

Out of a hypercoherence, we define a *qualitative domain with coherence.* The definition and basic properties of these more general objects[3] are exposed in [BE1].

Definition 3.2 *Let X be a hypercoherence. We define* $\mathrm{qD}(X)$ *and* $\mathcal{C}(X)$ *as follows :*

$$\mathrm{qD}(X) = \{x \subseteq |X| \mid \forall u \subseteq^*_{\text{fin}} |X| \;\; u \subseteq x \Rightarrow u \in \Gamma(X)\}$$

and

$$\mathcal{C}(X) = \{A \subseteq^*_{\text{fin}} \mathrm{qD}(X) \mid \\ \forall u \subseteq^*_{\text{fin}} |X| \;\; u \lhd A \Rightarrow u \in \Gamma(X)\} \; .$$

$\mathrm{qD}(X)$ *will be called the qualitative domain generated by X and its elements are called the* states *of* $\mathrm{qD}(X)$*, and* $\mathcal{C}(X)$ *will be called the* state coherence *generated by X. The set of finite states of* $\mathrm{qD}(X)$ *will be noted* $\mathrm{qD}_{\text{fin}}(X)$.

It is clear that $\mathrm{qD}(X)$ is always a qualitative domain, and its web is $|X|$ by our only requirement about hypercoherences.

The morphisms between hypercoherences that we shall consider in this paper are the *strongly stable functions.* There is also a notion of linear morphisms between hypercoherences; their theory is developped in [E].

Definition 3.3 *Let X and Y be hypercoherences. A* strongly stable function *from X to Y (we shall also write $f : X \to Y$) is a function $f : \mathrm{qD}(X) \to \mathrm{qD}(Y)$ which is continuous and satisfies*

$$\forall A \in \mathcal{C}(X) \quad f(A) \in \mathcal{C}(Y) \;\; \textit{and} \;\; f(\bigcap A) = \bigcap f(A) \; .$$

We have chosen this notation "$f : X \to Y$" because it would not make sense to speak of a strongly stable function $f : \mathrm{qD}(X) \to \mathrm{qD}(Y)$ since strong stability involves also $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ which cannot be retrieved from $\mathrm{qD}(X)$ and $\mathrm{qD}(Y)$.

Observe that, if X is any hypercoherence, any bounded, non-empty and finite subset of $\mathrm{qD}(X)$ is in $\mathcal{C}(X)$. For this reason, any strongly stable fonction $X \to Y$ is stable from $\mathrm{qD}(X)$ to $\mathrm{qD}(Y)$, and thus we can use *traces* to represent strongly stable functions faithfully.

We note **HCohFS** the category of hypercoherences and strongly stable functions.

Let X and Y be hypercoherences. Let $X \times Y$ be the hypercoherence defined by $|X \times Y| = |X| + |Y|$ and $w \in \Gamma(X \times Y)$ if $w \subseteq^*_{\text{fin}} |X \times Y|$ and

$$(w_2 = \emptyset \Rightarrow w_1 \in \Gamma(X)) \text{ and } (w_1 = \emptyset \Rightarrow w_2 \in \Gamma(Y)) \; .$$

Let $X \to Y$ be the hypercoherence Z whose web is the set of all (x, b) where $x \in \mathrm{qD}(X)$ is finite and $b \in |Y|$, and whose atomic coherence is given by: $w \in \Gamma(Z)$ if $w \in \mathcal{P}^*_{\text{fin}}(|Z|)$ and

$$w_1 \in \mathcal{C}(X) \Rightarrow \\ (w_2 \in \Gamma(Y) \text{ and } (\#w_2 = 1 \Rightarrow \#w_1 = 1)) \; .$$

Then we have the following result:

[3] Not all qualitative domains with coherence are induced by hypercoherences in this way.

Proposition 3.1 *The category* **HCohFS** *is cartesian closed. If X and Y are two hypercoherences, their cartesian product is $X \times Y$ and their exponential (object of morphisms from X to Y) is $X \to Y$.*

Furthermore, up to a natural order isomorphism, $\mathrm{qD}(X \times Y) = \mathrm{qD}(X) \times \mathrm{qD}(Y)$ (this latter product being equipped with the product order). And $\mathrm{qD}(X \to Y)$ is the poset of traces of all strongly stable functions $X \to Y$, which is naturally isomorphic to the poset of all strongly stable functions $X \to Y$ (equipped with the stable *order).*
If $t \in \mathrm{qD}(X \to Y)$, we note f^t the corresponding strongly stable function $X \to Y$.

In the sequel, we shall consider a hypercoherence which will play a central role : the hypercoherence $\mathbf{N}$ of flat integers. Its web is ω, the set of natural integers, and $\Gamma(\mathbf{N})$ is the set of all singletons of $|\mathbf{N}| = \omega$. One easily checks that, up to an order isomorphism, $\mathrm{qD}(\mathbf{N}) = \mathbf{N}_\perp$, and so more generally, for any $n \in \{1, 2, \ldots\}$, $\mathrm{qD}(\mathbf{N}^n) = \mathbf{N}_\perp^n$.

We shall also consider the ω-product $\mathbf{N}^\omega$ which can be directly defined as follows : $|\mathbf{N}^\omega| = \omega \times \omega$ and a non-empty and finite subset u of $|\mathbf{N}^\omega|$ is in $\Gamma(\mathbf{N}^\omega)$ iff :

$$\#u_1 = 1 \Rightarrow \#u_2 = 1 \ .$$

Again, up to an order isomorphism, one has $\mathrm{qD}(\mathbf{N}^\omega) = \mathbf{N}_\perp^\omega$.

One of the main reasons why strong stability has been introduced is that it allows a simple characterization of Milner-Vuillemin sequentiality :

Proposition 3.2 *Let $n, m \in \{1, 2, \ldots, \omega\}$. A function $f : \mathbf{N}_\perp^n \to \mathbf{N}_\perp^m$ is Vuillemin-Milner sequential if and only if it is strongly stable $\mathbf{N}^n \to \mathbf{N}^m$.*

Proof: See [BE1]. ∎

Now we are ready to prove two lemmas which are the main tools in the proof of the result presented in this paper.

Lemma 3.1 *Let X be a hypercoherence. Let $x_0, x_1, \ldots \in \mathrm{qD}(X)$ be an increasing ω-chain. Then there exists an increasing ω-chain $\alpha_0, \alpha_1, \ldots \in \mathrm{qD}(\mathbf{N}^\omega)$ and a strongly stable function $g : \mathbf{N}^\omega \to X$ such that $g(\alpha_i) = x_i$ for all $i \in \omega$.*

Proof: Let us define the family $(\alpha_i)_{i \in \omega}$ as follows (for instance) :

$$\alpha_i = \{(j, 0) \mid 0 \le j \le i - 1\} \ .$$

It is obviously an increasing ω-chain of $\mathrm{qD}(\mathbf{N}^\omega)$.

Let $(x_i)_{i \in \omega}$ be an increasing ω-chain in $\mathrm{qD}(X)$.

We define the function g by its trace t :

$$t = \{((j, 0), a) \mid j \in \omega \text{ and } a \in x_j \setminus x_{j-1}\} \ .$$

(We set by convention $x_{-1} = \emptyset$.)
It is obvious that $t \in \mathrm{qD}(\mathbf{N}^\omega \to X)$, and so $g = f^t$ is a strongly stable function $\mathbf{N}^\omega \to X$. Furthermore we clearly have $g(\alpha_j) = x_j$ for all j and we are done. ∎

Lemma 3.2 *Let X be a hypercoherence. Let $A \in \mathcal{C}(X)$. Then there exists $G \in \mathcal{C}(\mathbf{N}^\omega)$ and a strongly stable function $g : \mathbf{N}^\omega \to X$ such that $g(G) = A$.*

Proof: For any integer $k \ge 1$, we define a family $\{\gamma_j^k\}_{j=1\ldots k}$ of elements of $\mathrm{qD}(\mathbf{N}^\omega)$ as follows :

$$\gamma_j^k = \{(0, k-j+1), \ldots, (j-2, k-1)\} \cup \\ \{(j, 1), \ldots, (k-1, k-j)\} \ .$$

It is easily checked that the set $\{\gamma_j^k\}_{j=1\ldots k}$ is in $\mathcal{C}(\mathbf{N}^\omega)$, but that no proper subset of this set of cardinality strictly greater than 1 is in $\mathcal{C}(\mathbf{N}^\omega)$.

Let $A = \{x_1, \ldots, x_k\}$ be any element of $\mathcal{C}(X)$. Let $x_0 = \bigcap_{i=1}^n x_i$. Let us define the following set t :

$$\{(\emptyset, a) \mid a \in x_0\} \cup \{(\gamma_i^k, a) \mid a \in x_i \setminus x_0\}_{i=1,\ldots,k} \ .$$

Then we claim that $t \in \mathrm{qD}(\mathbf{N}^\omega \to X)$. Actually, let u be any non-empty subset of t, and assume that $u_1 \in \mathcal{C}(\mathbf{N}^\omega)$. Observe first that, by construction of t, one has : $\forall a \in u_2 \ \exists x \in A \quad a \in x$. There are three cases :

- $\emptyset \in u_1$. Let $a \in u_2$ be such that $(\emptyset, a) \in u$. Then we have $a \in x_0$ and hence $\forall x \in A \quad a \in x$, hence $u_2 \lhd A$, hence $u \in \Gamma(X)$. If furthermore u_2 is the singleton $\{a\}$, then u_1 must obviously be the singleton $\{\emptyset\}$.
- $\emptyset \notin u_1$ and u_1 is a singleton $\{\gamma_i^k\}$ (for one $i \in \{1, \ldots, k\}$). Then $u_2 \subseteq x_i$ and hence $u_2 \in \Gamma(X)$.
- $\emptyset \notin u_1$ and $\# u_1 \geq 2$. Then we know that $u_1 = \{\gamma_i^k\}_{i=1,\ldots,k}$. Let $i \in \{1, \ldots, k\}$. Let $a \in u_2$ be such that $(\gamma_i^k, a) \in u$. Then we have $a \in x_i$ by construction of t and hence $u_2 \lhd A$, thus $u_2 \in \Gamma(X)$. Furthermore, if u_2 is a singleton $\{a\}$, then we must have, for any $i \in \{1, \ldots, k\}$, $a \in x_i \setminus x_0$ which is clearly impossible.

Let $f = f^t$. We have, for any $i \in \{1, \ldots, k\}$,

$$f(\gamma_i^k) = (x_i \setminus x_0) \cup x_0 = x_i$$

and we are done. ∎

Proposition 3.3 *Let X and Y be two hypercoherences. A function $f : \mathrm{qD}(X) \to \mathrm{qD}(Y)$ is strongly stable from X to Y iff, for any strongly stable function $g : \mathbf{N}^\omega \to X$, the function $f \circ g$ is strongly stable from $\mathbf{N}^\omega$ to Y.*

Proof: The direction $\Rightarrow$ is obvious. Let us prove the converse, so let $f : \mathrm{qD}(X) \to \mathrm{qD}(Y)$ be any function satisfying the condition mentioned above. Let us first check that f is continuous. Let $(x_i)_{i \in \iota}$ be an increasing ω-chain in $\mathrm{qD}(X)$ (we can restrict our attention to increasing ω-chain instead of general directed sets because we have assumed that the web of a hypercoherence is enumerable). Using lemma 3.1, we can find a strongly stable function $g : \mathbf{N}^\omega \to X$ and an increasing ω-chain $\alpha_0, \alpha_1, \ldots \in \mathrm{qD}(\mathbf{N}^\omega)$ such that $g(\alpha_i) = x_i$ for all $i \in \iota$. By hypothesis, we know that $f \circ g$ is strongly stable, and hence continuous, so

$$\bigcup_{i \in \iota} f(g(\alpha_i)) = f(g(\bigcup_{i \in \iota} \alpha_i))$$

that is, since g is continuous :

$$\bigcup_{i \in \iota} f(x_i) = f(\bigcup_{i \in \iota} x_i) \ .$$

Now let us check that f preserves coherence and commutes to the intersections of coherent sets. Let $A = \{x_1, \ldots, x_k\}$ be any element of $\mathcal{C}(X)$. Using lemma 3.2, we can find a $G \in \mathcal{C}(\mathbf{N}^\omega)$ and a strongly stable function $g : \mathbf{N}^\omega \to X$ such that $g(G) = A$. But by hypothesis, $f \circ g$ is strongly stable, so $f(A) = f(g(G))$ is in $\mathcal{C}(Y)$. Furthermore, we have

$$\begin{aligned} f(\bigcap A) &= f(g(\bigcap G)) \text{ since } g \text{ is strongly stable} \\ &= \bigcap f(g(G)) \text{ since } f \circ g \text{ is strongly stable} \\ &= \bigcap f(A) \end{aligned}$$

and we conclude that f is strongly stable. ∎

4 A functional version of the hypercoherent model

To any type σ of our functional language, we associate its interpretation H_σ in the hypercoherent model. This interpretation is of course a hypercoherence. The definition is straightforward :

- $H_\iota = \mathbf{N}$
- $H_{\sigma \to \tau} = H_\sigma \to H_\tau$

For the purpose of what follows, we want to consider the interpretation of a functional type like $\iota \to \iota$ as a set of functions. So to any type σ we associate a set D_σ and a bijection $\Phi_\sigma : D_\sigma \to \mathrm{qD}(H_\sigma)$. We procede as follows :

- we set $D_\iota = \mathbf{N}_\perp$ (the flat domain of integers), and Φ_ι is the obvious bijection defined by :

$$\Phi_\iota(\perp) = \emptyset \quad \text{and} \quad \Phi_\iota(n) = \{n\} \text{ for } n \neq \perp.$$

- And we set

$$D_{\sigma\to\tau} = \{f : D_\sigma \to D_\tau \mid \\ \Phi_\tau \circ f \circ \Phi_\sigma^{-1} \text{ is strongly stable } H_\sigma \to H_\tau\}$$

  and $\Phi_{\sigma\to\tau}(f) = \mathrm{tr}\left(\Phi_\tau \circ f \circ \Phi_\sigma^{-1}\right)$. So $\Phi_{\sigma\to\tau}$ is actually a bijection, and its reciproque is given by : $\Phi_{\sigma\to\tau}^{-1}(t) = \Phi_\tau^{-1} \circ f^t \circ \Phi_\sigma$.

Furthermore, to any type σ, we associate the set D_σ^ω of all functions $f : \mathbf{N}_\perp^\omega \to D_\sigma$ such that the function $\Phi_\sigma \circ f$ be strongly stable $\mathbf{N}^\omega \to H_\sigma$. (In this definition, we identify the posets $\mathbf{N}_\perp^\omega$ and $\mathrm{qD}\left(\mathbf{N}^\omega\right)$, which are isomorphic.)

5 The reducibility model

We define two "projections" $\pi_o, \pi_e : \mathbf{N}_\perp^\omega \to \mathbf{N}_\perp^\omega$ as follows :

$$\pi_o(\alpha) = \langle \alpha_1, \alpha_3, \ldots \rangle \quad \text{and} \quad \pi_e(\alpha) = \langle \alpha_0, \alpha_2, \ldots \rangle \ .$$

We define also a "pairing" function $p : {\mathbf{N}_\perp^\omega}^2 \to \mathbf{N}_\perp^\omega$ by

$$p(\alpha, \beta) = \alpha_0 \beta_0 \alpha_1 \beta_1 \ldots$$

To any type σ, we associate inductively a set L_σ, and a set L_σ^ω of functions from $\mathbf{N}_\perp^\omega$ to L_σ.

The intuition is that L_σ is the domain of interpretation of *closed* terms and L_σ^ω is the domain of interpretation of terms whose free variables are of basic type ι (these terms are called *almost closed* in [Co1]).

- The set L_ι is $\mathbf{N}_\perp$, and the set L_ι^ω is the set of all Vuillemin-Milner sequential functions from $\mathbf{N}_\perp^\omega$ to $\mathbf{N}_\perp$.
- The set $L_{\sigma\to\tau}$ is the set of all functions $f : L_\sigma \to L_\tau$ such that :

$$\forall g \quad g \in L_\sigma^\omega \Rightarrow f \circ g \in L_\tau^\omega$$

  and the set $L_{\sigma\to\tau}^\omega$ is the set of all functions $f : \mathbf{N}_\perp^\omega \to L_{\sigma\to\tau}$ such that, for any $g \in L_\sigma^\omega$, the function

$$\begin{array}{rcl} h : \mathbf{N}_\perp^\omega & \to & L_\tau \\ \alpha & \mapsto & f(\pi_e(\alpha))(g(\pi_o(\alpha))) \end{array}$$

  is in L_τ^ω.

The following lemma states the connection between L_σ and L_σ^ω. It is not used in the sequel, and its proof is left to the reader.

Lemma 5.1 *For any type σ and for any $\alpha \in \mathbf{N}_\perp^\omega$ one has*

$$L_\sigma = \{f(\alpha) \mid f \in L_\sigma^\omega\} \ .$$

Now we can state the main result of the paper.

Theorem 5.1 *For any type σ :*

$$L_\sigma = D_\sigma \quad \textit{and} \quad L_\sigma^\omega = D_\sigma^\omega$$

Proof: We prove this result by induction on types (we have to prove the two equations in the same induction).

For $\sigma = \iota$, the first equation is obvious, and the second comes from the fact that a function $\mathrm{qD}(\mathbf{N}^\omega) \to \mathrm{qD}(\mathbf{N})$ is strongly stable iff it is sequential.

Now assume that the equations hold for σ and τ, and let us prove them for $\sigma \to \tau$. Let $f \in L_{\sigma\to\tau}$. We have to prove that $f \in D_{\sigma\to\tau}$, that is we have to prove that $\Phi_\tau \circ f \circ \Phi_\sigma^{-1} : \mathrm{qD}(H_\sigma) \to \mathrm{qD}(H_\tau)$ is strongly stable. We use the characterization given by proposition 3.3. So let $g : \mathbf{N}_\perp^\omega \to \mathrm{qD}(H_\sigma)$ be strongly stable. We just have to prove that $\Phi_\tau \circ f \circ \Phi_\sigma^{-1} \circ g$ is strongly stable. Since $L_\sigma^\omega = D_\sigma^\omega$, we know that $\Phi_\sigma^{-1} \circ g \in L_\sigma^\omega$. So, by definition of $L_{\sigma\to\tau}$, we have $f \circ \Phi_\sigma^{-1} \circ g \in L_\tau^\omega$, that is, since $L_\tau^\omega = D_\tau^\omega$, the function $\Phi_\tau \circ f \circ \Phi_\sigma^{-1} \circ g$ is strongly stable from $\mathbf{N}^\omega$ to H_τ, and we are done. Now let $f \in D_{\sigma\to\tau}$, we prove that $f \in L_{\sigma\to\tau}$. So let $g \in L_\sigma^\omega = D_\sigma^\omega$. The function $\Phi_\sigma \circ g$ is strongly stable. But, by definition of $D_{\sigma\to\tau}$, the function $\Phi_\tau \circ f \circ \Phi_\sigma^{-1}$ is strongly stable. So $\Phi_\tau \circ f \circ g$ is strongly stable, that is $f \circ g \in L_\tau^\omega$ by inductive hypothesis.

Now we prove that $L_{\sigma\to\tau}^\omega = D_{\sigma\to\tau}^\omega$. Let $f \in L_{\sigma\to\tau}^\omega$. By the proof above, we know that f is a function $\mathbf{N}_\perp^\omega \to D_{\sigma\to\tau}$, and its exponential transpose f' is thus a function $\mathbf{N}_\perp^\omega \times D_\sigma \to D_\tau$. We have to prove that the function

$$h = \Phi_\tau \circ f' \circ (\mathrm{Id} \times \Phi_\sigma^{-1}) : \mathbf{N}_\perp^\omega \times \mathrm{qD}(H_\sigma) \to \mathrm{qD}(H_\tau)$$

is strongly stable $\mathbf{N}^\omega \times H_\sigma \to H_\tau$. For this purpose, we use proposition 3.3. Let $g : \mathbf{N}^\omega \to \mathbf{N}^\omega \times H_\sigma$ be any strongly stable function, we want to prove that $h \circ g : \mathrm{qD}(\mathbf{N}^\omega) \to \mathrm{qD}(H_\tau)$ is strongly stable. Let $g_1 = \pi_1 \circ g : \mathbf{N}^\omega \to \mathbf{N}^\omega$ and $g_2 = \pi_2 \circ g : \mathbf{N}^\omega \to H_\sigma$. By inductive hypothesis, we know that $\Phi_\sigma^{-1} \circ g_2 \in L_\sigma^\omega$, and hence, by definition of $L_{\sigma\to\tau}^\omega$ we know that the function

$$\begin{array}{rcl} k : \mathbf{N}_\perp^\omega & \to & L_\tau \\ \alpha & \mapsto & f(\pi_e(\alpha))((\Phi_\sigma^{-1} \circ g_2)(\pi_o(\alpha))) \end{array}$$

is in L_τ^ω. Hence, by inductive hypothesis, $k \in D_\tau^\omega$, thus $\Phi_\tau \circ k$ is strongly stable $\mathbf{N}^\omega \to H_\tau$. Let $l : \mathbf{N}^\omega \to \mathbf{N}^\omega$ be the strongly stable function defined by $l(\alpha) = p(g_1(\alpha), \alpha)$. One clearly has $h \circ g = \Phi_\tau \circ k \circ l$ and we conclude that $h \circ g$ is strongly stable.

Conversely, let $f \in D_{\sigma\to\tau}^\omega$. We check that $f \in L_{\sigma\to\tau}^\omega$. So let g be any element of L_σ^ω, we have to prove that the function

$$\begin{array}{rcl} h : \mathbf{N}_\perp^\omega & \to & L_\tau \\ \alpha & \mapsto & f(\pi_e(\alpha))(g(\pi_o(\alpha))) \end{array}$$

is in L_τ^ω. But let $f' : \mathbf{N}_\perp^\omega \times D_\sigma \to D_\tau$ be the exponential transpose of f. We know that the function

$$k = \Phi_\tau \circ f' \circ (\mathrm{Id} \times \Phi_\sigma^{-1}) : \mathbf{N}_\perp^\omega \times \mathrm{qD}(H_\sigma) \to \mathrm{qD}(H_\tau)$$

is strongly stable. But $g \in D_\sigma^\omega$ by inductive hypothesis, and hence $\Phi_\sigma \circ g$ is strongly stable $\mathbf{N}^\omega \to H_\sigma$. Hence the function

$$k \circ (\mathrm{Id} \times (\Phi_\sigma \circ g)) = \Phi_\tau \circ f' \circ (\mathrm{Id} \times g) : \mathbf{N}_\perp^\omega \times \mathbf{N}_\perp^\omega \to \mathrm{qD}(H_\tau)$$

is strongly stable, so the function

$$\begin{array}{rcl} h' : \mathbf{N}_\perp^\omega & \to & \mathrm{qD}(H_\tau) \\ \alpha & \mapsto & (\Phi_\tau \circ f')(\pi_e(\alpha), g(\pi_o(\alpha))) \end{array}$$

is strongly stable, that is $\Phi_\tau^{-1} \circ h' \in D_\tau^\omega$, that is, by inductive hypothesis, $\Phi_\tau^{-1} \circ h' \in L_\tau^\omega$. But clearly $h = \Phi_\tau^{-1} \circ h'$ and we are done. ∎

6 Conclusion

We consider that the definition (higher order sequentiality defined by induction on types) and the result presented in this paper are important for two reasons. First, the notion of sequentiality presented here gives the possibility to prove that functional languages are sequential by mean of a simple induction on types, without using any abstract denotational model. For instance, sequentiality of Gödel system T is easily proved by this method. Second, it provides, at least in the hierarchy of simple types, a new characterization of strong stability which is completely different from the original definition of this notion.

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