MPRI 2–2 2015-2016 Exercises

Thomas Ehrhard

1) Integers in Call-by-push-value

Remember that $1 = !\top$. Let $\iota = \mathsf{Fix}\,\zeta \cdot (1 \oplus \zeta)$ be the type of strict natural (unary) numbers. We also define a type of "lazy integer" $\iota_\mathsf{l} = \mathsf{Fix}\,\zeta \cdot (1 \oplus !\zeta)$.

1.1) Explain the intuitive difference between ι and ι_l.

1.2) Write a successor succ and a predecessor pred function of type $\iota \multimap \iota$.

1.3) Write similar functions for succ'. What are the simplest types you can give to these functions?

1.4) Compute the relational semantics of the functions defined above.

1.5) Is it true that $\langle \mathsf{pred} \rangle \langle \mathsf{succ} \rangle M \rightarrow_\mathsf{w}^* M$ for any term M or type ι? Otherwise, what property must satisfy M for this property to hold?

1.6) Prove that an analogue of the property above holds for succ' and pred'.

2) Streams

Given a positive type φ, let $\rho = \mathsf{Fix}\,\zeta \cdot (\varphi \otimes !\zeta)$ be the type of streams of elements of type φ.

2.1) Taking $\varphi = \iota$, write a term M such that $\vdash M : \rho$ which represents the stream $0, 1, \dots$ of all natural numbers.

2.2) Explain why it wouldn't be a good idea to define ρ by $\rho = \mathsf{Fix}\,\zeta \cdot (\varphi \otimes \zeta)$.

2.3) Write a closed term M of type $!(\varphi \multimap \iota) \multimap \rho \multimap \varphi$ such that $\langle M \rangle F^! \, S$ returns the first element of S which is mapped to 0 by F.

3) Taken from the slides. . . Use the semantic typing system to justify the following statements and answer the following questions:

- $[\lambda x^\varphi \, x] = \{(a, a) \mid a \in [\varphi]\}$.
- $\Omega^\sigma = \mathsf{fix}\, x^{!\sigma}\, x$ satisfies $\vdash \Omega^\sigma : \sigma$. Then $[\Omega^\sigma] = \emptyset$.
- $() = (\Omega^\top)^!$, then $\vdash () : 1$ and $[()] = \{[]\}$.
- If $n \in \mathbb{N}$ one defines $\underline{n}$ such that $\vdash \underline{n} : \iota$ by $\underline{0} = \mathsf{in}_1()$ and $\underline{n+1} = \mathsf{in}_2 \underline{n}$. Then $[\underline{n}] = \{\overline{n}\}$.
- $\mathsf{succ} = \lambda x^\iota \, \mathsf{in}_2(x)$, then $\vdash \mathsf{succ} : \iota \multimap \iota$ and $\mathsf{succ} = \{(\overline{n}, \overline{n+1}) \mid n \in \mathbb{N}\}$.
- $\mathsf{add} = \lambda x^\iota \, \mathsf{fix}\, f^{!(\iota \multimap \iota)} \, \lambda y^\iota \, \mathsf{case}(y, d \cdot \underline{0}, z \cdot \langle \mathsf{succ} \rangle \langle \mathsf{der}(f) \rangle z)$ then $\vdash \mathsf{add} : \iota \multimap \iota \multimap \iota$ and one has $[\mathsf{add}] = \{(n_1, n_2, n_1 + n_2) \mid n_1, n_2 \in \mathbb{N}\}$.
- $\mathsf{maps} = \lambda f^{!(\varphi \multimap \psi)} \, \mathsf{fix}\, h^{!(\rho_\varphi \multimap \rho_\psi)} \, \lambda y^{\rho_\varphi} \, \langle \langle \mathsf{der}(f) \rangle \mathsf{pr}_1 y, (\langle \mathsf{der}(h) \rangle \mathsf{pr}_2 y)^! \rangle$. Then $\vdash \mathsf{maps} : !(\varphi \multimap \psi) \multimap \rho_\varphi \multimap \rho_\psi$ is a map functional for streams.
  Then $[\mathsf{maps}]$ is the least set of tuples $(([(a, b)] + m_1 + \dots + m_k), (a, [s_1, \dots, s_k]), (b, [t_1, \dots, t_k]))$ such that $(m_i, (s_i, t_i)) \in [\mathsf{maps}]$ for each $i \in \{1, \dots, k\}$.
- Using this we can define for instance $M = \lambda f^{!(\varphi \multimap \varphi)} \, \lambda x^\varphi \, \mathsf{fix}\, y^{!\rho_\varphi} \, \langle x, (\langle \mathsf{maps} \rangle f \, \mathsf{der}(y))^! \rangle$ such that $\vdash M : !(\varphi \multimap \varphi) \multimap \varphi \multimap \rho_\varphi$. What does this function compute? What is its relational interpretation? Execute a few step of $\rightarrow_\mathsf{w}$-reduction on $S = \langle M \rangle \mathsf{succ}^! \, \underline{0}$ and give the relational interpretation of S (observe that $\vdash S : \rho_\iota$).

4) Coalgebras in coherence spaces (from Shahin Amini's PhD thesis)

We use the following notations for coherence spaces: $|X|$ is the web of X, $\coh_X$ is the coherence relation on $|X|$, $\frown_X$ is the strict coherence relation, $\mathsf{Cl}(X)$ is the set of all cliques of X. We remind that $|!X|$ is the set of all finite cliques of X, and $u \coh_{!X} u'$ iff $u \cup u' \in \mathsf{Cl}(X)$.

We also remind that if $f \in \mathsf{Cl}(()X \multimap Y)$ then

$$!f = \{(u, v) \in |!X| \times |!Y| \mid \exists (a_1, b_1), \dots, (a_1, b_1) \in f \; u = \{a_1, \dots, a_n\} \text{ and } v = \{b_1, \dots, b_n\}\} \\ \in \mathsf{Cl}(!X \multimap !Y)$$

and that the comonad structure of the "!" functor is given by

$$\mathsf{der}_X = \{(\{a\}, a) \mid a \in |X|\} \in \mathsf{Cl}(!X \multimap X)$$
$$\mathsf{dig}_X = \{(u_1 \cup \cdots \cup u_n, \{u_1, \ldots, u_n\}) \mid \{u_1, \ldots, u_n\} \in \mathsf{Cl}(!!X)\} \in \mathsf{Cl}(!X \multimap !!X)\,.$$

We introduce now a notion of "coherent partial order": it is a pair $P = (|P|, \leq_P)$ where $|P|$ is a countable set and $\leq_P$ is a partial order relation on $|P|$ such that

- for all $a \in |P|$ the set $\downarrow a = \{a' \in |P| \mid a' \leq_P a\}$ is finite
- for any $a \in |P|$, any subset u of $\downarrow a$ has exactly one least upper bound in $\downarrow a$, that is, the set $\{b \in \downarrow a \mid u \subseteq \downarrow b\}$ has a unique least element denoted $\vee_a(u)$.

We associate with P a coherence space $\underline{P}$ as follows: $|\underline{P}| = |P|$ and $a \frown_{\underline{P}} a'$ if $\exists a'' \in |P|\ a, a' \leq_P a''$. Let $\mathsf{h}_P = \{(a, u) \in |\underline{P} \multimap !\underline{P}| \mid a = \vee_a u\}$ (in other words: $(a, u) \in \mathsf{h}_P$ exactly when u is upper-bounded by a and a is minimal with this property).

4.1) Prove that $\mathsf{h}_P \in \mathsf{Cl}(\underline{P} \multimap !\underline{P})$.

4.2) Prove that $(\underline{P}, \mathsf{h}_P)$ is a coalgebra.

4.3) Describe as simply as possible the weakening and contraction morphisms associated with P.

Let $1 = (\{*\}, =)$ (considered as a coherence space and as a coherent partial order).

4.4) Prove that a coalgebra morphism from 1 to P is exactly the same thing as a subset u of $|P|$ such that

- u is downwards-closed, that is $a' \leq_P a \in u \Rightarrow a' \in u$
- u is directed, that is, any finite subset of u is upper-bounded in u (in other words: $u \neq \emptyset$ and $\forall a_1, a_2 \in u\, \exists a \in u\ a_1, a_2 \leq a$).

Such a subset of $|P|$ is called an *ideal* and the set of these ideals is called ideal completion of P, denoted $\mathsf{Idl}(P)$. This set will always be considered as a partially ordered set, the order relation being $\subseteq$.

4.5) Prove that, if $D \subseteq \mathsf{Idl}(P)$ is directed, then $\cup D \in \mathsf{Idl}(P)$.

4.6) Prove that, if $u_1, u_2 \in \mathsf{Idl}(P)$ are upper-bounded in $\mathsf{Idl}(P)$ (that is, there is $u \in \mathsf{Idl}(P)$ such that $u_i \subseteq u$ for $i = 1, 2$), then $u_1 \cap u_2 \in \mathsf{Idl}(P)$. Explain why the upper-boundedness hypothesis is essential (the best possible answer is to give a counter-example showing that this property does not hold in general without this hypothesis).

Given a relation $f \subseteq A \times B$, we use $\tilde{f}$ for the function $\mathcal{P}(A) \to \mathcal{P}(B)$ defined by $\tilde{f}(u) = \{b \in B \mid \exists a \in A\ (a, b) \in f\}$.

4.7) Let P and Q be coherent partial orders and let $f \in \mathsf{Cl}(\underline{P} \multimap \underline{Q})$. Assume that f is a coalgebra morphism from $(\underline{P}, \mathsf{h}_P)$ to $(\underline{Q}, \mathsf{h}_Q)$. Prove that, if $u \in \mathsf{Idl}(P)$ then $\tilde{f}(u) \in \mathsf{Idl}(Q)$ [*Hint:* to prove that $\tilde{f}(u)$ is downwards closed, observe that if $b' \leq_Q b$ then $(b, \{b, b'\}) \in \mathsf{h}_Q$].

4.8) Prove that $\tilde{f}$ commutes with directed unions and bounded intersections (that is, if $D \subseteq \mathsf{Idl}(P)$ is directed then $\tilde{f}(\cup D) = \cup\{\tilde{f}(u) \mid u \in D\}$ and, if $u_1, u_2 \in \mathsf{Idl}(P)$ are upper-bounded, then $f(u_1) \cap f(u_2) = f(u_1 \cap u_2)$) [*Hint:* you only need the fact that f is a clique in $\underline{P} \multimap \underline{Q}$ to prove this.]. One says that $\tilde{f}$ is *stable*.

4.9) Conversely, let $F : \mathsf{Idl}(P) \to \mathsf{Idl}(Q)$ be a function which is monotone and stable. Define $\mathsf{Tr}F \subseteq |P| \times |Q|$ as the set of all pairs (a, b) such that $b \in F(\downarrow a)$ and a is minimal with this property. Prove that $\mathsf{Tr}F \in \mathsf{Cl}(\underline{P} \multimap \underline{Q})$.

4.10) Prove that $\mathsf{Tr}F$ is a coalgebra morphism from $(\underline{P}, \mathsf{h}_P)$ to $(\underline{Q}, \mathsf{h}_Q)$.

4.11) Prove that the operations $f \mapsto \tilde{f}$ and $F \mapsto \mathsf{Tr}F$ defined above are inverse of each other.

4.12) As a consequence, prove that the coalgebras $(\underline{P}, \mathsf{h}_P)$ and $(\underline{Q}, \mathsf{h}_Q)$ are isomorphic as coalgebras iff P and Q are isomorphic as partial orders. So from now on we consider freely coherent partial orders as coalgebras.

4.13) Describe as coherent partial orders the coalgebras $!X$ (when X is a coherence space), $P \otimes Q$ and $P \oplus Q$ (when P and Q are coherent partial orders).

We say that X is a sub-coherence space of Y (notation $X \sqsubseteq Y$) if $|X| \subseteq |Y|$ and $\forall a, a' \in |X|\ a \coh_X a' \Leftrightarrow a \coh_Y a'$. Then $\mathsf{i}^+_{X,Y} \in \mathsf{Cl}(X \multimap Y)$ is simply given by $\mathsf{i}^+_{X,Y} = \{(a,a) \mid a \in |X|\}$. Given coherent preorders P and Q, we stipulate accordingly that $P \sqsubseteq Q$ if $\underline{P} \sqsubseteq \underline{Q}$ and $\mathsf{i}^+_{\underline{P},\underline{Q}}$ is a coalgebra morphism from P to Q (see Section 3.5.11 of the Lecture Notes).

4.14) Prove that $P \sqsubseteq Q$ iff the following conditions are satisfied:

- $|P| \subseteq |Q|$
- for any $a \in |P|$ and $b \in |Q|$, one has $b \leq_Q a$ iff $b \in |P|$ and $b \leq_P a$
- if two elements of $|P|$ are upper-bounded in Q then they are upper-bounded in P.

4.15) Describe as simply as possible the coherent partial orders interpreting the types ι, ι_l and ρ (the interpretation of φ being given) of Exercise 1 and 2.

Reminders

Syntax of CBPV

$$
\begin{aligned}
\varphi, \psi, \ldots &:= {!\sigma} \mid \varphi \otimes \psi \mid \varphi \oplus \psi \mid \zeta \mid \mathsf{Fix}\, \zeta \cdot \varphi \\
\sigma, \tau \ldots &:= \varphi \mid \varphi \multimap \sigma \mid \top
\end{aligned}
$$

$$
\begin{aligned}
M, N \ldots := x \mid M^! \mid \langle M, N \rangle \mid \mathsf{in}_1 M \mid \mathsf{in}_2 M \\
\mid \lambda x^\varphi\, M \mid \langle M \rangle N \mid \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) \\
\mid \mathsf{pr}_1 M \mid \mathsf{pr}_2 M \mid \mathsf{der}(M) \mid \mathsf{fix}\, x^{!\sigma}\, M
\end{aligned}
$$

Typing rules

$$
\frac{\mathcal{P} \vdash M : \sigma}{\mathcal{P} \vdash M^! : {!\sigma}}
\qquad
\frac{\mathcal{P} \vdash M_1 : \varphi_1 \qquad \mathcal{P} \vdash M_2 : \varphi_2}{\mathcal{P} \vdash \langle M_1, M_2 \rangle : \varphi_1 \otimes \varphi_2}
\qquad
\frac{\mathcal{P} \vdash M : \varphi_i}{\mathcal{P} \vdash \mathsf{in}_i M : \varphi_1 \oplus \varphi_2}
$$

$$
\frac{}{\mathcal{P}, x : \varphi \vdash x : \varphi}
\qquad
\frac{\mathcal{P}, x : \varphi \vdash M : \sigma}{\mathcal{P} \vdash \lambda x^\varphi\, M : \varphi \multimap \sigma}
\qquad
\frac{\mathcal{P} \vdash M : \varphi \multimap \sigma \qquad \mathcal{P} \vdash N : \varphi}{\mathcal{P} \vdash \langle M \rangle N : \sigma}
$$

$$
\frac{\mathcal{P} \vdash M : {!\sigma}}{\mathcal{P} \vdash \mathsf{der}(M) : \sigma}
\qquad
\frac{\mathcal{P}, x : {!\sigma} \vdash M : \sigma}{\mathcal{P} \vdash \mathsf{fix}\, x^{!\sigma}\, M : \sigma}
$$

$$
\frac{\mathcal{P} \vdash M : \varphi_1 \oplus \varphi_2 \qquad \mathcal{P}, x_1 : \varphi_1 \vdash M_1 : \sigma \qquad \mathcal{P}, x_2 : \varphi_2 \vdash M_2 : \sigma}{\mathcal{P} \vdash \mathsf{case}(M, x_1 \cdot M_1, x_2 \cdot M_2) : \sigma}
$$

$$
\frac{\mathcal{P} \vdash M : \varphi_1 \otimes \varphi_2}{\mathcal{P} \vdash \mathsf{pr}_i M : \varphi_i}
$$

Reduction rules We first define the notion of *value* as follows:

- any variable x is a value
- for any term M, the term $M^!$ is a value
- if M is a value then $\mathsf{in}_i M$ is a value for $i = 1, 2$
- if M_1 and M_2 are values then $\langle M_1, M_2 \rangle$ is a value.

Notation for values: V, $W \ldots$

$$
\frac{}{\mathsf{der}(M^!) \to_{\mathsf{w}} M}
\qquad
\frac{}{\langle \lambda x^\varphi\, M \rangle V \to_{\mathsf{w}} M\,[V/x]}
\qquad
\frac{}{\mathsf{pr}_i \langle V_1, V_2 \rangle \to_{\mathsf{w}} V_i}
$$

$$\frac{}{\mathsf{fix}\, x^{!\sigma}\, M \to_{\mathsf{w}} M\left[(\mathsf{fix}\, x^{!\sigma}\, M)^{!}/x\right]} \qquad \frac{M \to_{\mathsf{w}} M'}{\mathsf{der}(M) \to_{\mathsf{w}} \mathsf{der}(M')}$$

$$\frac{M \to_{\mathsf{w}} M'}{\langle M\rangle N \to_{\mathsf{w}} \langle M'\rangle N} \qquad \frac{N \to_{\mathsf{w}} N'}{\langle M\rangle N \to_{\mathsf{w}} \langle M\rangle N'}$$

$$\frac{M \to_{\mathsf{w}} M'}{\mathsf{pr}_i M \to_{\mathsf{w}} \mathsf{pr}_i M'} \qquad \frac{M_1 \to_{\mathsf{w}} M_1'}{\langle M_1, M_2\rangle \to_{\mathsf{w}} \langle M_1', M_2\rangle} \qquad \frac{M_2 \to_{\mathsf{w}} M_2'}{\langle M_1, M_2\rangle \to_{\mathsf{w}} \langle M_1, M_2'\rangle}$$

$$\frac{}{\mathsf{case}(\mathsf{in}_i V, x_1 \cdot M_1, x_2 \cdot M_2) \to_{\mathsf{w}} M_i\,[V/x_i]} \qquad \frac{M \to_{\mathsf{w}} M'}{\mathsf{in}_i M \to_{\mathsf{w}} \mathsf{in}_i M'}$$

$$\frac{M \to_{\mathsf{w}} M'}{\mathsf{case}(M, x_1 \cdot M_1, x_2 \cdot M_2) \to_{\mathsf{w}} \mathsf{case}(M', x_1 \cdot M_1, x_2 \cdot M_2)}$$

Semantic typing rules A semantic typing judgment is an expression $\Phi = (x_1 : a_1 : \varphi_1, \dots, x_k : a_k : \varphi_k)$ where the variables x_i are pairwise distinct, the φ_i's are positive types and $a_i \in [\varphi_i]$. Given such a semantic judgment Φ, we define its underlying typing judgment $\underline{\Phi} = (x_1 : \varphi_1, \dots, x_k : \varphi_k)$ and the tuple of points $\widehat{\Phi} = (a_1, \dots, a_k) \in [\underline{\Phi}]$.

$$\frac{(\widehat{\Phi}, []) \in \mathsf{h}_{\underline{\Phi}}}{\Phi, x : a : \varphi \vdash x : a : \varphi}$$

$$\frac{\Phi_i \vdash M : a_i : \sigma \text{ for } i = 1, \dots, k \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \dots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash M^{!} : [a_1, \dots, a_k] : !\sigma}$$

Remember that we assume that $\underline{\Phi} = \underline{\Phi_i}$ for each i.

$$\frac{\Phi_1 \vdash M_1 : a_1 : \varphi_1 \qquad \Phi_2 \vdash M_2 : a_2 : \varphi_2 \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \langle M_1, M_2\rangle : (a_1, a_2) : \varphi_1 \otimes \varphi_2}$$

$$\frac{\Phi \vdash M : a : \varphi_i}{\Phi \vdash \mathsf{in}_i M : (i, a) : \varphi_1 \oplus \varphi_2} \qquad \frac{\Phi, x : a : \varphi \vdash M : b : \sigma}{\Phi \vdash \lambda x^{\varphi}\, M : (a, b) : \varphi \multimap \sigma}$$

$$\frac{\Phi_1 \vdash M : (a, b) : \varphi \multimap \sigma \qquad \Phi_2 \vdash N : a : \varphi \qquad (\widehat{\Phi}, [\widehat{\Phi_1}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \langle M\rangle N : b : \sigma}$$

$$\frac{\Phi \vdash M : [a] : !\sigma}{\Phi \vdash \mathsf{der}(M) : a : \sigma} \qquad \frac{\Phi \vdash M : (a_1, a_2) : \varphi_1 \otimes \varphi_2 \qquad (a_2, []) \in \mathsf{h}_{\varphi_2}}{\Phi \vdash \mathsf{pr}_1 M : a_1 : \varphi_1}$$

$$\frac{\Phi \vdash M : (a_1, a_2) : \varphi_1 \otimes \varphi_2 \qquad (a_1, []) \in \mathsf{h}_{\varphi_1}}{\Phi \vdash \mathsf{pr}_2 M : a_2 : \varphi_2}$$

$$\frac{\Phi_0 \vdash M : (1, a_1) : \varphi_1 \oplus \varphi_2 \quad \Phi_1, x : a_1; \varphi_1 \vdash N_1 : b : \sigma \quad \underline{\Phi}, x_2 : \varphi_2 \vdash N_2 : \varphi_2 \quad (\widehat{\Phi}, [\widehat{\Phi_0}, \widehat{\Phi_1}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) : b : \sigma}$$

$$\frac{\Phi_0 \vdash M : (2, a_2) : \varphi_1 \oplus \varphi_2 \quad \Phi_2, x : a_2; \varphi_2 \vdash N_2 : b : \sigma \quad \underline{\Phi}, x_1 : \varphi_1 \vdash N_1 : \varphi_1 \quad (\widehat{\Phi}, [\widehat{\Phi_0}, \widehat{\Phi_2}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2) : b : \sigma}$$

$$\frac{\Phi_0, x : [a_1, \dots, a_k] : !\sigma \vdash M : a : \sigma \qquad \forall i\ \Phi_i \vdash \mathsf{fix}\, x^{!\sigma}\, M : a_i : \sigma \qquad (\widehat{\Phi}, [\widehat{\Phi_0}, \dots, \widehat{\Phi_k}]) \in \mathsf{h}_{\underline{\Phi}}}{\Phi \vdash \mathsf{fix}\, x^{!\sigma}\, M : a : \sigma}$$