MPRI 2–2 Models of programming languages: domains, categories, games
TD1

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The signs **(*)** and **(**)** try to indicate more difficult and interesting questions. These are of course completely subjective indications!

1. This exercise develops a somehow degenerate model of Linear Logic which does not satisfy $*$-autonomy but satisfies all the other requirements. A pointed set is a structure $X = (\underline{X}, 0_X)$ where $\underline{X}$ is a set and $0_X \in \underline{X}$. Given pointed sets X, X_1, X_2 and Y,

   - a morphism of pointed sets from X to Y is a function $f : \underline{X} \to \underline{Y}$ such that $f(0_X) = 0_Y$
   - and a bimorphism of pointed sets from X_1, X_2 to Y is a function $f : \underline{X_1} \times \underline{X_2} \to \underline{Y}$ such that $f(0_{X_1}, x_2) = f(x_1, 0_{X_2}) = 0_Y$ for each $x_1 \in \underline{X_1}$ and $x_2 \in \underline{X_2}$.

   (a) Prove that pointed sets together with morphisms of pointed sets form a category $\mathbf{Set}_0$. What are the isos in that category?

   One sets $1 = (\{0_1, *\})$ where $*$ and 0_1 are are distinct chosen elements (for instance 0_1 is the integer 0 and $*$ is the integer 1). Given pointed sets X_1 and X_2 one defines $X_1 \otimes X_2$ as follows:

   $$\underline{X_1 \otimes X_2} = \{(x_1, x_2) \in \underline{X_1} \times \underline{X_2} \mid x_1 = 0_{X_1} \Leftrightarrow x_2 = 0_{X_2}\} \quad \text{and} \quad 0_{X_1 \otimes X_2} = (0_{X_1}, 0_{X_2})\,.$$

   Given $x_i \in \underline{X_i}$ for $i = 1, 2$, one defines

   $$x_1 \otimes x_2 = \begin{cases} (0_{X_1}, 0_{X_2}) & \text{if } x_1 = 0_{X_1} \text{ or } x_2 = 0_{X_2} \\ (x_1, x_2) & \text{otherwise.} \end{cases}$$

   (b) Prove that the function $(x_1, x_2) \mapsto x_1 \otimes x_2$ is a bimorphism from X_1, X_2 to $X_1 \otimes X_2$ which is surjective as a function $\underline{X_1} \times \underline{X_2} \to \underline{X_1 \otimes X_2}$ and that for any bimorphism f from X_1, X_2 to Y there is exactly one morphism $\widetilde{f} \in \mathbf{Set}_0(X_1 \otimes X_2, Y)$ such that $f(x_1, x_2) = \widetilde{f}(x_1 \otimes x_2)$ for all $x_1 \in \underline{X_1}$ and $x_2 \in \underline{X_2}$.

   (c) Given $f_i \in \mathbf{Set}_0(X_i, Y_i)$ for $i = 1, 2$, deduce from the above that there is exactly one morphism $f_1 \otimes f_2 \in \mathbf{Set}_0(X_1 \otimes X_2, Y_1 \otimes Y_2)$ such that

   $$\forall x_1 \in \underline{X_1}\, \forall x_2 \in \underline{X_2} \quad (f_1 \otimes f_2)(x_1 \otimes x_2) = f_1(x_1) \otimes f_2(x_2)\,.$$

   (d) Using again the universal property of Question (b) prove that the operation on morphisms defined in Question (c) is a functor.

   (e) Exhibit isomorphisms $\lambda_X \in \mathbf{Set}_0(1 \otimes X, X)$ and $\alpha_{X_1,X_2,X_3} \in \mathbf{Set}_0((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))$.

   So $\mathbf{Set}_0$ is an SMC (there is a symmetry iso $\gamma_{X_1,X_2} \in \mathbf{Set}_0(X_1 \otimes X_2, X_2 \otimes X_1)$ such that $\gamma_{X_1,X_2}(x_1 \otimes x_2) = x_2 \otimes x_1$ which is quite easy to define, and the McLane coherence diagrams commute).

(f) One defines $X \multimap Y$ by $\underline{X \multimap Y} = \mathbf{Set}_0(X, Y)$ and for $0_{X \multimap Y}$ we take the function such that $0_{X \multimap Y}(x) = 0_Y$ for all $x \in \underline{X}$. Let $e : \underline{X \multimap Y} \times \underline{X} \to \underline{Y}$ be defined by $e(f, x) = f(x)$. Prove that e is a bimorphism and that the SMC $\mathbf{Set}_0$ is closed.

(g) Prove that there is no object $\perp$ of $\mathbf{Set}_0$ which turns this symmetric monoidal closed category into a $*$-autonomous category.

(h) Given a family $(X_i)_{i \in I}$ of objects of $\mathbf{Set}_0$ we define an object X as follows: $\underline{X} = \prod_{i \in I} \underline{X_i}$ and $0_X = (0_{X_i})_{i \in I} \in \underline{X}$ so that the the projections $\pi_i : \underline{X} \to \underline{X_i}$ ar obviously morphisms of $\mathbf{Set}_0$. Prove that X, together with these projections, is the cartesian product of the family $(X_i)_{i \in I}$ that we denote as $\&_{i \in I} X_i$.

Notice that the terminal object (which is the product of an empty family of objects) is $\top = (\{0_\top\}, 0_\top)$.

Contrarily to **Rel**, the category $\mathbf{Set}_0$ has all (projective) limits. It seems rather difficult to build $*$-autonomous categories which are at the same type complete. A noticeable exception is the category of complete lattices.

Given an object X of $\mathbf{Set}_0$, we define $!X$ by $\underline{!X} = \{(0, 0_!)\} \cup \{1\} \times \underline{X}$ where $0_!$ is a chosen element (for instance, a given integer) and $0_{!X} = (0, 0_!)$. Notice that $(1, 0_X) \in \underline{!X}$ but $0_{!X} \neq (1, 0_X)$.

Given $f \in \mathbf{Set}_0(X, Y)$, we define $!f \in \mathbf{Set}_0(!X, !Y)$ by $!f(0_{!X}) = 0_{!Y}$ and $!f(1, x) = (1, f(x))$. This obviously defines a functor $\mathbf{Set}_0 \to \mathbf{Set}_0$.

(i) We define $\mathsf{der}_X : \mathbf{Set}_0(!X, X)$ by $\mathsf{der}_X(0_{!X}) = 0_X$ and $\mathsf{der}_X(1, x) = x$. Prove that this is a natural transformation.

(j) We define $\mathsf{dig}_X \in \mathbf{Set}_0(!X, !!X)$ by $\mathsf{dig}_X(0, 0_!) = (0, 0_!)$, that is $\mathsf{dig}_X(0_{!X}) = 0_{!!X}$, and $\mathsf{dig}_X(1, x) = (1, (1, x))$ which is easily seen to be a natural transformation. Prove that equipped with the natural transformations der and dig the functor $!\_$ is a comonad.

(k) Given two objects X and Y of $\mathbf{Set}_0$, exhibit an isomorphism between $!(X \& Y)$ and $!X \otimes !Y$.

2. In this exercise we study a model of linear logic which is based on complete sup-semilattices and linear maps. A complete sup-semilattice is a partially ordered set S (the order relation will always be denoted as $\leq$ or $\leq_S$ if required) such that any subset A of S has a least upper bound $\bigvee A \in S$. Remember that this means

- $\forall x \in A \ x \leq \bigvee A$
- $\forall x \in S \ (\forall y \in A \ y \leq x) \Rightarrow \bigvee A \leq x$.

In particular we have two elements $0 = \bigvee \emptyset$ which is the least element of S and $1 = \bigvee S$ which is the greatest element of S.

A subset A of S is *down-closed* if for all $x \in A$ and all $y \in S$, if $y \leq x$ then $y \in A$. Given $x \in S$ we set $\downarrow x = \{y \in S \mid y \leq x\}$.

A linear morphism of sup-semilattices from S to T is a function $f : S \to T$ such that for all $A \subseteq S$ $f(\bigvee A) = \bigvee f(A)$ where we define as usual $f(A) = \{f(x) \mid x \in A\}$. Notice that this implies that f is monotone: given $x \leq y$ in S we have $f(y) = f(\bigvee\{x, y\}) = f(x) \vee f(y)$, that is $f(x) \leq f(y)$. Let Slat be the category whose objects are the sup-semilattices and morphisms are the linear maps of sup-semilattices. We set $\perp = \{0 < 1\}$ for the object of Slat which has exactly two elements.

It is important to remember that any inf-semilattice, partially ordered set S where each $A \subseteq S$ has an inf (greatest lower bound) $\bigwedge A$, is also a sup-semilattice: $\bigvee A = \bigwedge\{x \in S \mid \forall y \in A \ y \leq x\}$.

It is easy to check that Slat is cartesian. The product of a family $(S_j)_{j \in J}$ of objects of Slat is the usual cartesian product $\prod_{j \in J} S_j$ equipped with the product order and projection defined in the usual way. We also use $S = \&_{j \in J} S_j$ for this product and $\pi_j \in \mathsf{Slat}(S, S_j)$ for the projections. The terminal object is $\top = \{0\}$.

(a) Show that the isomorphisms of Slat are the linear morphisms which are bijections.

(b) Given a set X we denote as $\mathcal{P}(X)$ its powerset (that is, the set of all of its subsets) ordered under inclusion, so that $\mathcal{P}(X)$ is a sup-semilattice for $\bigvee A = \bigcup A$ for any $A \subseteq \mathcal{P}(X)$. Given $t \in \mathbf{Rel}(X, Y)$ we define $\widehat{t} : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\widehat{t}(x) = t \cdot x = \{b \in Y \mid \exists a \in x \ (a, b) \in t\}$.

Prove that $\widehat{t} \in \mathsf{Slat}(\mathcal{P}(X), \mathcal{P}(Y))$ and that, for any $f \in \mathsf{Slat}(\mathcal{P}(X), \mathcal{P}(Y))$ there is exactly one $t = \mathsf{tr} f \in \mathbf{Rel}(X, Y)$ such that $f = \widehat{t}$. In other words, the functor $L : \mathbf{Rel} \to \mathsf{Slat}$ which maps X to $\mathcal{P}(X)$ and t to $\widehat{t}$ is full and faithful.

(c) Prove that the category Slat has all equalizers, in other words: given objects S and T of Slat and $f, g \in \mathsf{Slat}(S, T)$ there is an object E of Slat and a morphism $e \in \mathsf{Slat}(E, S)$ such that $f\,e = g\,e$ and, for any object V of Slat and any morphism $h \in \mathsf{Slat}(V, S)$ such that $f\,h = g\,h$, there is exactly one morphism $h_0 \in \mathsf{Slat}(V, E)$ such that $h = e\,h_0$.

Remember that the Cantor space is the set $\{0,1\}^\omega$ of all infinites sequences α of 0 and 1 equipped with the following topology (which is the product topology of the discrete space $\{0,1\}$): a subset U of $\{0,1\}^\omega$ is open iff for any $\alpha \in U$ there is a finite prefix w of α such that, for any $\beta \in \{0,1\}^\omega$, if w is a prefix of β then $\beta \in U$. In other words, a subset F of $\{0,1\}^\omega$ is closed iff it has the following property: if $\alpha \in \{0,1\}^\omega$ is such that, for any finite prefix w of α there exists $\beta \in F$ such that w is a prefix of β, then $\alpha \in F$. As in any topological spaces, if $\mathcal{F}$ is a set of closed subsets then $\bigcap \mathcal{F}$ is closed (you are advised to check this directly using the characterization above of closed subsets).

So the set of closed subsets of $\{0,1\}^\omega$ is an inf-semilattice and hence also a sup-semilattice: the sup of a set of closed sets is the closure of its union (the intersectin of all closed sets which contain this union).

(d) **(**)** Let $W = \{0,1\}^*$ be the set of all finite sequences of 0 and 1. If $w = \langle a_1, \dots, a_n \rangle \in W$ is such a sequence and $a \in \{0,1\}$ let $wa = \langle a_1, \dots, a_n, a \rangle$. Let $\theta = \{(wa, w) \mid w \in W \text{ and } a \in \{0,1\}\} \in \mathbf{Rel}(W, W)$. Let (C, c) be the equalizer of $\mathsf{Id}, \widehat{\theta} \in \mathsf{Slat}(\mathcal{P}(W), \mathcal{P}(X))$ (so that C is a sup-semilattice and $c \in \mathsf{Slat}(C, \mathcal{P}(W))$. Exhibit an order isomorphism between C and the set of all closed subsets of the Cantor space ordered under inclusion.

Given a lattice S, we say that $x \in S$ is *prime* if

$$\forall A \subseteq S \quad x \leq \bigvee A \Rightarrow \exists y \in A \; x \leq y$$

(e) **(*)** Prove that, for a set X, the prime elements of $\mathcal{P}(X) \in \mathsf{Slat}$ are exactly the singletons. Prove that C, in sharp contrast with the previous case, has no prime elements.

[*Hint:* prove first that if F is prime, it must be a singleton $\{\alpha\}$ and then prove that no such singleton is prime. For this notice that, for a collection $\mathcal{F}$ of closed subsets of $\{0,1\}^\omega$, the closed set $\bigvee \mathcal{F}$ is the closure of $\bigcup \mathcal{F}$ (the intersection of all closed sets which contain $\bigcup \mathcal{F}$). So consider a set $\mathcal{F}$ of shape $\mathcal{F} = \{\{\alpha(n)\} \mid n \in \mathbb{N}\}$ where $\alpha(n) \to_{n\to\infty} \alpha$ and $\forall n \in \mathbb{N} \; \alpha(n) \neq \alpha$.]

This example is a concrete illustration of the fact that the category $\mathbf{Rel}$ is not complete, indeed it has no equalizer for the two maps $\theta, \mathsf{Id} \in \mathbf{Rel}(W, W)$ because the equalizer of $\widehat{\theta}$ and Id in Slat is not an object of $\mathbf{Rel}$ (one would need a further proof to make this argument completely rigorous!).

(f) Prove that the set of linear morphisms $S \to T$, equipped with the pointwise order (that is $f \leq g$ if $\forall x \in S \; f(x) \leq g(x)$), is a sup-semilattice. We denote it as $S \multimap T$.

(g) Given $x \in S$ define a function $x^* : S \to \bot$ by

$$x^*(y) = \begin{cases} 1 & \text{if } y \not\leq x \\ 0 & \text{if } y \leq x \end{cases}$$

Prove that $x^* \in S \multimap \bot$.

(h) Given a sup-semilattice S, we use S^{op} for the same set S equipped with the reverse order: $x \leq_{S^{\mathsf{op}}} y$ if $y \leq_S x$. Prove that the map $x \mapsto x^*$ is an order isomorphism from the poset S^{op} to $S \multimap \bot$. Warning: one must prove that it is monotone in both directions because a monotone bijection is not necessarily an order isomorphism! Call $\mathsf{k} : (S \multimap \bot) \to S^{\mathsf{op}}$ the inverse isomorphism.

(i) **(*)** Given $f \in (S \multimap T)$ define $f^* : (T \multimap \bot) \to (S \multimap \bot)$ by $f^*(y') = y'\,f$. Prove that $f^* \in \mathsf{Slat}(T \multimap \bot, S \multimap \bot)$. Let $f^\perp \in \mathsf{Slat}(T^{\mathsf{op}}, S^{\mathsf{op}})$ be the associated morphism (through the iso k

defined above, that is $f^\perp(y) = \mathsf{k}(f^*(y^*))$). Prove that

$$\forall x \in S \,\forall y \in T \quad f(x) \leq y \Leftrightarrow x \leq f^\perp(y)\,.$$

One says that f and $f^\perp$ define a Galois connection between S and T. Last prove that $f^{\perp\perp} = f$.

(j) Given sup-semilattices S and T we define $S \otimes T$ as the set of all $I \subseteq S \times T$ such that

- I is down-closed
- and, for all $A \subseteq S$ and $B \subseteq T$, if A and B satisfy $A \times B \subseteq I$ then $(\bigvee A, \bigvee B) \in I$.

Prove that $(S \otimes T, \subseteq)$ is an inf-semilattice (that is, is closed under arbitrary intersections). As a consequence, it is also a sup-semilattice: if $\mathcal{I} \subseteq S \otimes T$ then $\bigvee \mathcal{I} = \bigcap \{I \in S \otimes T \mid \bigcup \mathcal{I} \subseteq I\}$. But notice that in this sup-semilattice, the sups are not defined as unions in general.

(k) Prove that the least element of $S \otimes T$ is $0_{S \otimes T} = S \times \{0\} \cup \{0\} \times T$.

(l) We say that a map $f : S \times T \to U$ (where S, T, U are sup-semilattices) is bilinear if for all $A \subseteq S$ and $B \subseteq T$ we have $\bigvee f(A \times B) = f(\bigvee A \times B)$. Prove that this condition is equivalent to the following:

- for all $x \in S$ and $B \subseteq T$, one has $f(x, \bigvee B) = \bigvee_{y \in B} f(x, y)$
- and for all $y \in T$ and $A \subseteq S$, one has $f(\bigvee A, y) = \bigvee_{x \in A} f(x, y)$

that is, f is separately linear in both variables.

(m) **(*)** Given $x \in S$ and $y \in T$ let $x \otimes y = \downarrow(x, y) \cup 0_{S \otimes T} \subseteq S \times T$. Prove that $x \otimes y \in S \otimes T$ and that the function $\tau : (x, y) \mapsto x \otimes y$ is a bilinear map $S \times T \to S \otimes T$.

(n) Let $(S, T) \multimap U$ be the set of all bilinear maps $S \times T \to U$ ordered pointwise (that is $f \leq g$ if $\forall (x, y) \in S \times T \; f(x, y) \leq g(x, y)$). Prove that $(S, T) \multimap U \simeq (S \multimap (T \multimap U))$. Deduce from this fact that $(S, T) \multimap U$ is a sup-semilattice.

(o) Given $I \in X \otimes Y$ let $f^I : S \times T \to \perp$ be given by

$$f^I(x, y) = \begin{cases} 0 & \text{if } (x, y) \in I \\ 1 & \text{otherwise.} \end{cases}$$

Prove that f^I is bilinear. Conversely given $f \in (S, T) \multimap \perp$ prove that $\mathsf{ker}_2 f = \{(x, y) \in S \times T \mid f(x, y) = 0\}$ belongs to $S \otimes T$. Prove that these operations define an order isomorphism between $S \otimes T$ and $((S, T) \multimap \perp)^{\mathsf{op}}$.

To be continued...