1) Let us say that a PCS is a distribution space if \( P(X) = \{ u \in \mathbb{R}^{[X]} \mid \sum_{a \in |X|} u_a \leq 1 \} \). Remember that \( X \) is a distribution space if \( P(X^+) = \{ u' \in \mathbb{R}^{[X]}_{\geq 0} \mid \forall a \in |X| \ u_a \leq 1 \} \) and that \( N \) is the distribution space such that \(|N| = N\).

1.1) Prove that if \( X \) and \( Y \) are distribution spaces then \( X \times Y \) is a distribution space.

1.2) If \( x \subseteq I \) we define \( \chi_x \in \mathbb{R}^{[x]} \) by \( (\chi_x)_i = 1 \) if \( i \in x \) and \( (\chi_x)_i = 0 \) if \( i \notin x \). Let \( X \) and \( Y \) be distribution spaces and \( f \subseteq |X| \times |Y| \). Prove that \( \chi_f \in P coh(X, Y) \) iff \( f \) is (the graph of) a partial function \(|X| \to |Y| \).

1.3) Let \( f \subseteq (|N| \times |N|) \times N \) be defined by \( f = \{ ((n, p), n + p) \mid n, p \in N \} \). Given \( u, v \in P(N) \), compute \((\chi_f \cdot (u \otimes v))_k \) for each \( k \in N \).

1.4) Let \( r \in [0, 1] \). Compute \((\chi_f \cdot (u \otimes v))_k \) when \( u = v \in P(N) \) is given by \( u_n = (1 - r)^n \) (explain why \( u \in P(N) \)).

1.5) Remember that if \( u \in P(N) \) then \( \|u\| = \sum_{n \in N} u_n \). Prove that \( \|\chi_f \cdot (u \otimes v)\| = \|u\| \|v\| \).

1.6) Let \( f \subseteq |X| \times |Y| \) be (the graph of) a partial function, where \( X \) and \( Y \) a distribution spaces. Prove that the two following properties are equivalent:

   - \( f \) is a total function
   - \( \forall u \in P(N) \|\chi_f \cdot u\| = \|u\| \).

2) Using the fact that \( P coh \) is an SMCC, prove that there is a \( C \in P coh((|N| 
\to N) \otimes (|N| \to N)), |N| \to N) \) such that, for any \( s, t \in P(|N| \to N) \), one has \( C \cdot (s \otimes t) = \frac{1}{2} (s + t) \). Given \( n_1, p_1, n_2, p_2, n, p \in N \), give the value of \( C((n_1, p_1), (n_2, p_2)), (n, p) \in \mathbb{R}_{\geq 0} \).

3) Let \( X \) and \( Y \) be probabilistic coherence spaces (PCSs) and let \( f : P(X) \to P(Y) \). Prove that if \( f \) satisfies

   - for all \( u^1, u^2 \in P(X) \) such that \( u^1 + u^2 \in P(X) \), one has \( f(u^1 + u^2) = f(u^1) + f(u^2) \),
   - for all \( u \in P(X) \) and \( \lambda \in [0, 1] \) one has \( f(\lambda u) = \lambda f(u) \),
   - for any non-decreasing sequence \((u(n))_{n \in N} \) of elements of \( P(X) \), one has \( f(\sup_{n \in N} u(n)) = \sup_{n \in N} f(u(n)) \)

then there is exactly one \( s \in P(X \to Y) \) such that \( \forall u \in P(X) \ f(u) = s \cdot u \).

   [ Hint: Observe that the first condition implies that \( f \) is monotone ]

4) Let \( S = \{0, 1\}^\omega \) (the set of finite words of 0 and 1), equipped with the prefix order: \( s \leq t \) if \( s \subseteq S \) and \( s \) is a prefix of \( t \), that is \( s = \langle a_1, \ldots, a_n \rangle \) and \( s = \langle a_1, \ldots, a_k \rangle \) with \( n \leq k \). If \( \alpha \in \{0, 1\}^\omega \) (the \( \omega \)-indexed sequences of 0 and 1), we use \( \downarrow \alpha \) for the set of all \( s \in S \) which are prefixes of \( \alpha \).

4.1) A tree is a non-empty subset \( T \) of \( S \) such that if \( s \in T \) and if \( t \leq s \) then \( t \in T \). One says that \( \alpha \in \{0, 1\}^\omega \) is an infinite branch of \( T \) if \( \downarrow \alpha \subseteq T \). Prove König's Lemma: a tree which has no infinite branches is finite (as a set). [ Hint: By contradiction. ]

4.2) A subset \( A \) of \( S \) is an antichain if any two elements of \( A \) are either equal or incomparable for the prefix order. Prove that an antichain \( A \) is finite if it satisfies \( \forall \alpha \in \{0, 1\}^\omega \ A \cap \downarrow \alpha \neq \emptyset \). [ Hint: If \( A \) is non-empty, apply König's Lemma to the tree \( \downarrow A = \{ s \in S \mid \exists t \in A : s \leq t \} \). ]

4.3) Let \( P \) be the set of \( u \in \mathbb{R}^S_{\geq 0} \) such that, for any antichain \( A \), one has \( \sum_{s \in A} u_s \leq 1 \). Prove that \((S, P) \) is a PCS, that we will denote as \( C \).

4.4) Prove that for any \( \alpha \in \{0, 1\}^\omega \), one has \( \sum_{s \downarrow \alpha} e_s \in P(C) \) and that this defines an injection from \( \{0, 1\}^\omega \) (the Cantor space) into \( P(C) \).

We say that \( u \in P(C) \) is uniform if, for all \( s \in S \), one has \( u_s = u_{a_0} + u_{a_1} \) (where, if \( s = \langle a_1, \ldots, a_n \rangle \), then \( sa = \langle a_1, \ldots, a_n, a \rangle \).
4.5) Give examples of $u \in P(C)$ which are not uniform and examples of $u \in P(C)$ which are uniform.
We assume that $u$ is uniform.

We say that $U \subseteq \{0, 1\}^\omega$ is open if, for all $\alpha \in U$, there is $s \in \downarrow \alpha$ such that $\uparrow s \subseteq U$, where $\uparrow s = \{ \beta \in \{0, 1\}^\omega \mid s \in \downarrow \beta \}$.

4.6) Let $U \subseteq \{0, 1\}^\omega$ be open. Prove that there is an antichain $A$ such that $U = \bigcup_{\alpha \in A} \uparrow s$. If $A$ is such an antichain we set $\mu_A(U) = \sum_{s \in A} u_s$ so that $\mu_A(U) \in [0, 1]$.

4.7) Let $s \in S$ and $A \subseteq S$ be an antichain such that $\forall t \in A s \leq t$ and for all $\alpha \in \uparrow s$ one has $A \cap \downarrow \alpha \neq \emptyset$. Prove that $A$ is finite and that $u_s = \sum_{\alpha \in A} u_t$.

4.8) Let $U \subseteq \{0, 1\}^\omega$ be open and let $A$ and $B$ be antichains such that $U = \bigcup_{\alpha \in A} \uparrow s = \bigcup_{\beta \in B} \uparrow s$. Prove that $\mu_A(U) = \mu_B(U)$. [Hint: Building possibly a third antichain which has the same property as $A$ and $B$ with respect to $U$, one can assume that $\forall t \in U \exists s \in A s \leq t$]

4.9) Let $U, V \subseteq \{0, 1\}^\omega$ be open and such that $U \cap V = \emptyset$. Prove that $\mu(U \cup V) = \mu(U) + \mu(V)$.

5) Remember that if $X$ is a PCS, the associated norm is the function $\|\| : P(X) \rightarrow [0, 1]$ defined by

$$\|u\|_X = \sup_{u' \in P(X)} \langle u, u' \rangle \in [0, 1].$$

5.1) Prove that this operation features the usual properties of a norm, namely:

- $\|u\|_X = 0 \Rightarrow u = 0$ (we recall that 0 is the element of $P(X)$ which maps each element of $|X|$ to 0).
- If $u, u^1, u^2 \in P(X)$ satisfy $u^1 + u^2 \in P(X)$, then $\|u^1 + u^2\|_X \leq \|u^1\|_X + \|u^2\|_X$.
- If $u \in P(X)$ and $\lambda \in [0, 1]$ then $\|\lambda u\|_X = \lambda \|u\|_X$.

5.2) Prove that, if $u \leq v \in P(X)$, then $\|u\|_X \leq \|v\|_X$. Prove also that the norm is Scott-continuous (that is if $(u(n))_{n \in \mathbb{N}}$ is a non-decreasing sequence in $P(X)$, then $\sup_{n \in \mathbb{N}} u(n)) = \sup_{n \in \mathbb{N}} \|u(n)\|_X$).

5.3) Let $t \in P(X \rightarrow Y)$, prove that $\|t\|_{X \rightarrow Y} = \sup_{u \in P(X)} \|t \cdot u\|_Y$ and that $\|t^\perp\|_{Y^\perp \rightarrow X^\perp} = \|t\|_{X \rightarrow Y}$.

5.4) Prove that if $u \in P(X)$ and $v \in P(Y) \cap 0$ then $\|u \otimes v\|_{X \otimes Y} = \|u\|_X \|v\|_Y$.

5.5) Prove that if $t \in P((X \otimes Y) \rightarrow Z)$ then $\|t\|_{(X \otimes Y) \rightarrow Z} = \sup_{u \in P(X), v \in P(Y)} \|t \cdot (u \otimes v)\|_Z$.

6) $B = 1 \top 1$ is the PCS of booleans, which can be described as follows: $|B| = \{0, 1\}$ and $P(B) = \{ u \in \mathbb{R}_{\geq 0}^{[0, 1]} \mid u_0 + u_1 \leq 1 \}$. We identify $|B \rightarrow 1|$ with $\mathbb{N} \times \mathbb{N}$ (explain why this is possible). Let $s \in \mathbb{R}_{\geq 0}^{[0, 1]}$ be defined by $s_{n,p} = (1 - \delta_{n,0})\delta_{n,p} 2^n$. Prove that $s \in \text{Pcoh}(B, 1)$.

7) We admit that there is $s \in \text{Pcoh}(1 \rightarrow 1 \top 1, 1, 1)$ such that, for all $t \in P(1 \rightarrow 1)$ and $u \in P(1)$ (so that we can consider that $u \in [0, 1]$ and that $(t, u) \in P((1 \rightarrow 1) \top 1)$):

$$\tilde{s}(t, u) = \frac{1}{2} + \frac{1}{2} u \tilde{t}(u)^2.$$

The existence of such an $s$ is essentially a consequence of the cartesian closeness of $\text{Pcoh}$. So we have $\text{Cur}(s) \in \text{Pcoh}(1 \rightarrow 1 \top 1)$ and hence a Scott-continuous function $\text{Cur}(s) : P(1 \rightarrow 1) \rightarrow P(1 \rightarrow 1)$, let $t \in \text{Pcoh}(1, 1)$ be the least fixed point of $\text{Cur}(s)$.

7.1) Prove that necessarily the function $\tilde{t} : [0, 1] \rightarrow [0, 1]$ is given by

$$\tilde{t}(u) = \begin{cases} \frac{1 - \sqrt{1 - u}}{u} & \text{if } 0 < u \leq 1 \\ \frac{1}{2} & \text{if } u = 0 \end{cases}.$$

7.2) Identifying $|1 \rightarrow 1|$ with $\mathbb{N}$ and using the expression above as well as the Taylor expansion of $\sqrt{1 - u}$, give the value of $t_n$ for each $n \in \mathbb{N}$. 