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1) Let us say that a PCS is a distribution space if $\mathsf{P}(X) = \{u \in \mathbb{R}_{\geq 0}^{|X|} \mid \sum_{a \in |X|} u_a \leq 1\}$. Remember that X is a distribution space iff $\mathsf{P}(X^\perp) = \{u' \in \mathbb{R}_{\geq 0}^{|X|} \mid \forall a \in |X|\ u_a \leq 1\}$ and that N is the distribution space such that $|\mathsf{N}| = \mathbb{N}$.

1.1) Prove that if X and Y are distribution spaces then $X \otimes Y$ is a distribution space.

1.2) If $x \subseteq I$ we define $\chi_x \in \mathbb{R}_{\geq 0}^I$ by $(\chi_x)_i = 1$ if $i \in x$ and $(\chi_x)_i = 0$ if $i \notin x$. Let X and Y be distribution spaces and $f \subseteq |X| \times |Y|$. Prove that $\chi_f \in \mathbf{Pcoh}(X, Y)$ iff f is (the graph of) a partial function $|X| \to |Y|$.

1.3) Let $f \subseteq (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ be defined by $f = \{((n, p), n + p) \mid n, p \in \mathbb{N}\}$. Given $u, v \in \mathsf{P}(\mathsf{N})$, compute $(\chi_f \cdot (u \otimes v))_k$ for each $k \in \mathbb{N}$.

1.4) Let $r \in [0, 1]$. Compute $(\chi_f \cdot (u \otimes v))_k$ when $u = v \in \mathsf{P}(\mathsf{N})$ is given by $u_n = (1 - r)r^n$ (explain why $u \in \mathsf{P}(\mathsf{N})$).

1.5) Remember that if $u \in \mathsf{P}(\mathsf{N})$ then $\|u\| = \sum_{n \in \mathbb{N}} u_n$. Prove that $\|\chi_f \cdot (u \otimes v)\| = \|u\|\|v\|$.

1.6) Let $f \subseteq |X| \times |Y|$ be (the graph of) a partial function, where X and Y a distribution spaces. Prove that the two following properties are equivalent:

- f is a total function
- $\forall u \in \mathsf{P}(\mathsf{N})\ \|\chi_f \cdot u\| = \|u\|$.

2) Using the fact that **Pcoh** is an SMCC, prove that there is a $C \in \mathbf{Pcoh}(((\mathsf{N} \multimap \mathsf{N}) \otimes (\mathsf{N} \multimap \mathsf{N})), \mathsf{N} \multimap \mathsf{N})$ such that, for any $s, t \in \mathsf{P}(\mathsf{N} \multimap \mathsf{N})$, one has $C \cdot (s \otimes t) = \frac{1}{2}(s\,t + t\,s)$. Given $n_1, p_1, n_2, p_2, n, p \in \mathbb{N}$, give the value of $C_{((n_1,p_1),(n_2,p_2)),(n,p)} \in \mathbb{R}_{\geq 0}$.

3) Let X and Y be probabilistic coherence spaces (PCSs) and let $f : \mathsf{P}(X) \to \mathsf{P}(Y)$. Prove that if f satisfies

- for all $u^1, u^2 \in \mathsf{P}(X)$ such that $u^1 + u^2 \in \mathsf{P}(X)$, one has $f(u^1 + u^2) = f(u^1) + f(u^2)$,
- for all $u \in \mathsf{P}(X)$ and $\lambda \in [0, 1]$ one has $f(\lambda u) = \lambda f(u)$
- for any non-decreasing sequence $(u(n))_{n \in \mathbb{N}}$ of elements of $\mathsf{P}(X)$, one has $f(\sup_{n \in \mathbb{N}} u(n)) = \sup_{n \in \mathbb{N}} f(u(n))$

then there is exactly one $s \in \mathsf{P}(X \multimap Y)$ such that $\forall u \in \mathsf{P}(X)\ f(u) = s \cdot u$.

[*Hint:* Observe that the first condition implies that f is monotone]

4) Let $S = \{0, 1\}^{<\omega}$ (the set of finite words of 0 and 1), equipped with the prefix order: $s \leq t$ if $s, t \in S$ and s is a prefix of t, that is $s = \langle a_1, \ldots, a_n \rangle$ and $s = \langle a_1, \ldots, a_k \rangle$ with $n \leq k$. If $\alpha \in \{0, 1\}^\omega$ (the ω-indexed sequences of 0 and 1), we use $\downarrow \alpha$ for the set of all $s \in S$ which are prefixes of α.

4.1) A tree is a non-empty subset T of S such that if $s \in T$ and if $t \leq s$ then $t \in T$. One says that $\alpha \in \{0, 1\}^\omega$ is an infinite branche of T if $\downarrow \alpha \subseteq T$. Prove König's Lemma: a tree which has no infinite branches is finite (as a set). [*Hint:* By contradiction.]

4.2) A subset A of S is an antichain if any two elements of A are either equal or incomparable for the prefix order. Prove that an antichain A is finite if it satisfies $\forall \alpha \in \{0, 1\}^\omega\ A \cap \downarrow \alpha \neq \emptyset$. [*Hint:* If A is non-empty, apply König's Lemma to the tree $\downarrow A = \{s \in S \mid \exists t \in A\ s \leq t\}$.].

4.3) Let $\mathcal{P}$ be the set of $u \in \mathbb{R}_{\geq 0}^S$ such that, for any antichain A, one has $\sum_{s \in A} u_s \leq 1$. Prove that $(S, \mathcal{P})$ is a PCS, that we will denote as $\mathcal{C}$.

4.4) Prove that for any $\alpha \in \{0, 1\}^\omega$, one has $\sum_{s \in \downarrow \alpha} \mathsf{e}_s \in \mathsf{P}(\mathcal{C})$ and that this defines an injection from $\{0, 1\}^\omega$ (the Cantor space) to $\mathsf{P}(\mathcal{C})$.

We say that $u \in \mathsf{P}(\mathcal{C})$ is uniform if, for all $s \in S$, one has $u_s = u_{s0} + u_{s1}$ (where, if $s = \langle a_1, \ldots, a_n \rangle$, then $sa = \langle a_1, \ldots, a_n, a \rangle$).

4.5) Give examples of $u \in \mathsf{P}(\mathcal{C})$ which are not uniform and examples of $u \in \mathsf{P}(\mathcal{C})$ which are uniform. We assume that u is uniform.

We say that $U \subseteq \{0,1\}^\omega$ is open if, for all $\alpha \in U$, there is $s \in \downarrow\alpha$ such that $\uparrow s \subseteq U$, where $\uparrow s = \{\beta \in \{0,1\}^\omega \mid s \in \downarrow\beta\}$.

4.6) Let $U \subseteq \{0,1\}^\omega$ be open. Prove that there is an antichain A such that $U = \bigcup_{s\in A} \uparrow s$. If A is such an antichain we set $\mu_A(U) = \sum_{s\in A} u_s$ so that $\mu_A(U) \in [0,1]$.

4.7) Let $s \in S$ and $A \subseteq S$ be an antichain such that $\forall t \in A\ s \leq t$ and for all $\alpha \in \uparrow s$ one has $A \cap \downarrow\alpha \neq \emptyset$. Prove that A is finite and that $u_s = \sum_{t\in A} u_t$.

4.8) Let $U \subseteq \{0,1\}^\omega$ be open and let A and B be antichains such that $U = \bigcup_{s\in A} \uparrow s = \bigcup_{s\in B} \uparrow s$. Prove that $\mu_A(U) = \mu_B(U)$. We set $\mu(U) = \mu_A(U)$. [*Hint:* Building possibly a third antichain which has the same property as A and B with respect to U, one can assume that $\forall t \in B \exists s \in A\ s \leq t$.]

4.9) Let $U, V \subseteq \{0,1\}^\omega$ be open and such that $U \cap V = \emptyset$. Prove that $\mu(U \cup V) = \mu(U) + \mu(V)$.

5) Remember that if X is a PCS, the associated norm is the function $\|\_\|_X : \mathsf{P}(X) \to [0,1]$ defined by

$$\|u\|_X = \sup_{u' \in \mathsf{P}(X^\perp)} \langle u, u' \rangle \in [0,1]\,.$$

5.1) Prove that this operation features the usual properties of a norm, namely:

- $\|u\|_X = 0 \Rightarrow u = 0$ (we recall that 0 is the element of $\mathsf{P}(X)$ which maps each element of $|X|$ to 0).
- If $u^1, u^2 \in \mathsf{P}(X)$ satisfy $u^1 + u^2 \in \mathsf{P}(X)$, then $\|u^1 + u^2\|_X \leq \|u^1\|_X + \|u^2\|_X$.
- If $u \in \mathsf{P}(X)$ and $\lambda \in [0,1]$ then $\|\lambda u\|_X = \lambda \|u\|_X$.

5.2) Prove that, if $u \leq v \in \mathsf{P}(X)$, then $\|u\|_X \leq \|v\|_X$. Prove also that the norm is Scott-continuous (that is if $(u(n))_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathsf{P}(X)$, then $\|\sup_{n\in\mathbb{N}} u(n)\|_X = \sup_{n\in\mathbb{N}} \|u(n)\|_X$).

5.3) Let $t \in \mathsf{P}(X \multimap Y)$, prove that $\|t\|_{X\multimap Y} = \sup_{u\in\mathsf{P}(X)} \|t \cdot u\|_Y$ and that $\|t^\perp\|_{Y^\perp \multimap X^\perp} = \|t\|_{X\multimap Y}$.

5.4) Prove that if $u \in \mathsf{P}(X)$ and $v \in \mathsf{P}(Y)$ then $\|u \otimes v\|_{X\otimes Y} = \|u\|_X \|v\|_Y$.

5.5) Prove that if $t \in \mathsf{P}((X \otimes Y) \multimap Z)$ then $\|t\|_{(X\otimes Y)\multimap Z} = \sup_{u\in\mathsf{P}(X), v\in\mathsf{P}(Y)} \|t \cdot (u \otimes v)\|_Z$.

6) $B = 1 \oplus 1$ is the PCS of booleans, which can be described as follows: $|B| = \{0,1\}$ and $\mathsf{P}(B) = \{u \in \mathbb{R}_{\geq 0}^{\{0,1\}} \mid u_0 + u_1 \leq 1\}$. We identify $|!B \multimap 1|$ with $\mathbb{N} \times \mathbb{N}$ (explain why this is possible). Let $s \in \mathbb{R}_{\geq 0}^{|!B\multimap 1|}$ be defined by $s_{n,p} = (1 - \delta_{n,0})\delta_{n,p} 2^n$. Prove that $s \in \mathbf{Pcoh}_!(B, 1)$.

7) We admit that there is $s \in \mathbf{Pcoh}_!((1 \Rightarrow 1) \,\&\, 1, 1)$ such that, for all $t \in \mathsf{P}(1 \Rightarrow 1)$ and $u \in \mathsf{P}(1)$ (so that we can consider that $u \in [0,1]$ and that $(t, u) \in \mathsf{P}((1 \Rightarrow 1) \,\&\, 1)$):

$$\widehat{s}(t, u) = \frac{1}{2} + \frac{1}{2} u\, \widehat{t}(u)^2\,.$$

The existence of such an s is essentially a consequence of the cartesian closeness of $\mathbf{Pcoh}_!$. So we have $\mathsf{Cur}(s) \in \mathbf{Pcoh}_!(1 \Rightarrow 1, 1 \Rightarrow 1)$ and hence a Scott-continuous function $\widehat{\mathsf{Cur}(s)} : \mathsf{P}(1 \Rightarrow 1) \to \mathsf{P}(1 \Rightarrow 1)$, let $t \in \mathbf{Pcoh}_!(1, 1)$ be the least fixed point of $\widehat{\mathsf{Cur}(s)}$.

7.1) Prove that necessarily the function $\widehat{t} : [0,1] \to [0,1]$ is given by

$$\widehat{t}(u) = \begin{cases} \frac{1-\sqrt{1-u}}{u} & \text{if } 0 < u \leq 1 \\ \frac{1}{2} & \text{if } u = 0 \end{cases}$$

7.2) Identifying $|!1 \multimap 1|$ with $\mathbb{N}$ and using the expression above as well as the Taylor expansion of $\sqrt{1-u}$, give the value of t_n for each $n \in \mathbb{N}$.