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A *coherence space* is a pair $E = (|E|, \coh_E)$ where $|E|$ is a set and $\coh_E \subseteq |E|^2$ is a reflexive and symmetric relation. Remember that $\scoh_E = \coh_E \setminus \{(a,a) \mid a \in |E|\}$.

The set of *cliques* of E is $\mathsf{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x\ a \coh_E a'\}$. Equipped with the partial order relation $\subseteq$, $\mathsf{Cl}(E)$ is closed under directed unions[1]. Observe also that a subset of a clique is a clique, that all singletons are cliques and that $\emptyset$ is a clique.

Let E and F be coherence spaces. A function $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ is *stable* is it is monotone, Scott-continuous (that is, for all directed $D \subseteq \mathsf{Cl}(E)$, one has $f(\cup D) = \cup_{x\in D} f(x)$, or, equivalently $f(\cup D) \subseteq \cup_{x\in D} f(x)$, since the converse inclusion holds by monotonicity of f) and *conditionally multiplicative*, that is

$$\forall x, y \in \mathsf{Cl}(E) \quad x \cup y \in \mathsf{Cl}(E) \Rightarrow f(x \cap y) = f(x) \cap f(y)$$

or equivalently

$$\forall x, y \in \mathsf{Cl}(E) \quad x \cup y \in \mathsf{Cl}(E) \Rightarrow f(x \cap y) \supseteq f(x) \cap f(y)$$

since the converse inclusion holds by monotonicity of f.

1) Let E be a coherence space and let $u \in \mathsf{Cl}(E)$. One defines a coherence space E_u as follows: $|E_u| = \{a \in |E| \mid \forall b \in u\ a \scoh_E b\}$ and $\coh_{E_u} = \coh_E \cap |E_u|^2$. Observe that $\mathsf{Cl}(E_u) \subseteq \mathsf{Cl}(E)$ and that, if $x \in \mathsf{Cl}(E_u)$ then $x \cap u = \emptyset$ and $x \cup u \in \mathsf{Cl}(E)$.

Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a monotone and Scott-continuous function. Given $u \in \mathsf{Cl}(E)$ one defines a function $\Delta_u f : \mathsf{Cl}(E_u) \to \mathsf{Cl}(F)$ by $\Delta_u f(x) = f(x \cup u) \setminus f(x)$.

1.1) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a stable function. Compute $\Delta_u f$ when f is constant, and when f is linear (that is $f(\emptyset) = \emptyset$ and $f(x \cup y) = f(x) \cup f(y)$ if $x, y \in \mathsf{Cl}(E)$ satisfy $x \cup y \in \mathsf{Cl}(E)$).

1.2) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a monotone and Scott-continuous function. Prove that if $\Delta_u f$ is monotone for all $u \in \mathsf{Cl}(E)$, then f is stable.

1.3) Conversely, prove that, if f is stable, then $\Delta_u f$ is stable for all $u \in \mathsf{Cl}(E)$. In particular, f is stable if and only if $\Delta_u f$ is monotone for all $u \in \mathsf{Cl}(E)$.

Let $f, g : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be stable functions. One says that f is stably less than g (notation $f \leq_{\mathsf{st}} g$) if

$$\forall x, y \in \mathsf{Cl}(E) \quad x \subseteq y \Rightarrow f(x) = f(y) \cap g(x)\,.$$

Observe that $f \leq_{\mathsf{st}} g \Rightarrow f \leq_{\mathsf{ext}} g$ (where $f \leq_{\mathsf{ext}} g$ means $\forall x \in \mathsf{Cl}(E)\ f(x) \subseteq g(x)$): take $x = y$ in the definition above.

1.4) Prove that $f \leq_{\mathsf{st}} g$ if and only if $f \leq_{\mathsf{ext}} g$ and $\forall u \in \mathsf{Cl}(E)\ \Delta_u f \leq_{\mathsf{ext}} \Delta_u g$.

Remember that, if $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ is a stable function, one defines the *trace* $\mathsf{Tr} f$ of f as the set of all pairs (x_0, b) where $b \in |Y|$ and x_0 is minimal such that $b \in f(x_0)$ (and is therefore finite by continuity of f). Remember also that, if $(x_0, b), (y_0, b) \in \mathsf{Tr} f$ satisfy $x_0 \cup y_0 \in \mathsf{Cl}(E)$, then $x_0 = y_0$.

1.5) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a stable function. Prove that

$$\mathsf{Tr}(\Delta_u f) = \{(y_0 \setminus u, b) \mid (y_0, b) \in \mathsf{Tr} f,\ y_0 \cap u \neq \emptyset \text{ and } y_0 \cup u \in \mathsf{Cl}(E)\}\,.$$

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[1] Unions filtrantes en français