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A *preorder* is a pair $S = (|S|, \leq_S)$ where $|S|$ is a set (which is usually at most countable) and $\leq_S$ is a transitive and reflexive binary relation on $|S|$ (that is, a preorder relation). A subset u of $|S|$ is an *initial segment* of S if

$$\forall a \in u \, \forall a' \in |S| \quad a' \leq_S a \Rightarrow a' \in u\,.$$

We use $\mathcal{I}(S)$ for the set of all initial segmments of S; observe that $\mathcal{I}(S)$ is closed under arbitrary unions, in particular $\emptyset \in \mathcal{I}(S)$. Observe also that $|S| \in \mathcal{I}(S)$. We will consider $\mathcal{I}(S)$ as a partially ordered set, ordered by $\subseteq$.

If $u \subseteq |S|$, we set $\downarrow u = \{a' \in |S| \mid \exists a \in u \; a' \leq_S a\}$; this is the least element of $\mathcal{I}(S)$ which contains u.

Remember also that we use S^{op} for the preorder such that $|S^{\mathsf{op}}| = |S|$ and $a \leq_{S^{\mathsf{op}}} a'$ if $a' \leq_S a$ and that $S_1 \times S_2$ is the preorder such that $|S_1 \times S_2| = |S_1| \times |S_2|$ and $(a_1, a_2) \leq_{S_1 \times S_2} (a'_1, a'_2)$ if $a_i \leq_{S_i} a'_i$ for $i = 1, 2$.

We use letters S, T and U (possibly with subscripst) to denote preorders.

1) Let $f : \mathcal{I}(S) \to \mathcal{I}(T)$ be a function. We set

$$\mathsf{tr} f = \{(a, b) \in |S| \times |T| \mid b \in f(\downarrow \{a\})\}\,.$$

Given two functions $f_1, f_2 : \mathcal{I}(S) \to \mathcal{I}(T)$, we write $f_1 \leq f_2$ if $\forall u \in \mathcal{I}(S) \; f_1(u) \subseteq f_2(u)$.

1.1) Prove that if f is monotonic then $\mathsf{tr} f \in \mathcal{I}(S^{\mathsf{op}} \times T)$.

One says that f is *linear* if it commutes with arbitrary unions, that is, for any $A \subseteq \mathcal{I}(S)$, one has $f(\cup A) = \cup_{u \in A} f(u)$. This implies that f is monotonic (because $u \subseteq u' \Leftrightarrow u \cup u' = u'$). Given $t \in \mathcal{P}(|S| \times |T|)$, one defines a function

$$\begin{aligned} \mathsf{fun}\, t : \mathcal{I}(S) &\to \mathcal{P}(T) \\ u &\mapsto \{b \in |T| \mid \exists a \in u \; (a, b) \in t\}\,. \end{aligned}$$

1.2) Prove that, if $t \in \mathcal{I}(S^{\mathsf{op}} \times T)$, then the function $\mathsf{fun}\, t$ takes its values in $\mathcal{I}(T)$ and is linear. Prove also that, if $t_1, t_2 \in \mathcal{I}(S^{\mathsf{op}} \times T)$ satisfy $t_1 \subseteq t_2$ then $\mathsf{fun}\, t_1 \subseteq \mathsf{fun}\, t_2$.

1.3) Prove that, if $f : \mathcal{I}(S) \to \mathcal{I}(T)$ is linear then $\mathsf{tr} f \in \mathcal{I}(S^{\mathsf{op}} \times T)$. Prove also that, if $f_1, f_2 : \mathcal{I}(S) \to \mathcal{I}(T)$ are linear and satisfy $f_1 \leq f_2$ then $\mathsf{tr} f_1 \subseteq \mathsf{tr} f_2$.

Let **PoL** be the category whose objects are the preorders and where $\mathbf{PoL}(S, T)$ is the set of all linear functions $\mathcal{I}(S) \to \mathcal{I}(T)$ (the identity maps and the composition operation are defined in the usual way). Remember that **PoLR** is the category which has the same objects, but where $\mathbf{PoLR}(S, T) = \mathcal{I}(S^{\mathsf{op}} \times T)$ the identity at S is $\mathsf{Id}_S = \{(a, a') \in |S|^2 \mid a' \leq_S a\}$ and composition of $s \in \mathbf{PoLR}(S, T)$ and $t \in \mathbf{PoLR}(T, U)$ is defined by

$$t\, s = \{(a, c) \in |S| \times |U| \mid \exists b \in |T| \; (a, b) \in s \text{ and } (b, c) \in t\}\,.$$

1.4) Prove that the mappings $\mathsf{fun} : \mathbf{PoLR}(S, T) \to \mathbf{PoL}(S, T)$ and $\mathsf{tr} : \mathbf{PoL}(S, T) \to \mathbf{PoLR}(S, T)$ are inverse of each other, and define therefore an order isomorphism between $(\mathbf{PoL}(S, T), \leq)$ and $(\mathcal{I}(S^{\mathsf{op}} \times T), \subseteq)$.

1.5) Prove that fun defines a functor $\mathbf{PoLR} \to \mathbf{PoL}$ and that tr defines a functor from **PoL** to **PoLR**. These functors map any preorder S to S.

2) A function $f : \mathcal{I}(S_1) \times \mathcal{I}(S_2) \to \mathcal{I}(T)$ is *bilinear* if, for any $u_2 \in \mathcal{I}(S_2)$, the function $\mathcal{I}(S_1) \to \mathcal{I}(T)$ which maps $u_1 \in \mathcal{I}(S_1)$ to $f(u_1, u_2)$ is linear and, for any $u_1 \in \mathcal{I}(S_1)$, the function $\mathcal{I}(S_2) \to \mathcal{I}(T)$ which maps $u_2 \in \mathcal{I}(S_2)$ to $f(u_1, u_2)$ is linear.

Given $f : \mathcal{I}(S_1) \times \mathcal{I}(S_2) \to \mathcal{I}(T)$, we define

$$\mathsf{tr}_2 f = \{((a_1, a_2), b) \in |S_1 \otimes S_2| \times |T| \mid b \in f(\downarrow\{a_1\}, \downarrow\{a_2\})\} .$$

Remember that $S_1 \otimes S_2 = S_1 \times S_2$ (defined in the previous exercise).

2.1) Let $\tau : \mathcal{I}(S_1) \times \mathcal{I}(S_2) \to \mathcal{I}(S_1 \otimes S_2)$ be defined by $\tau(u_1, u_2) = u_1 \times u_2$. Prove that τ is bilinear and compute $\mathsf{tr}_2\tau$.

2.2) Let $f : \mathcal{I}(S_1) \times \mathcal{I}(S_2) \to \mathcal{I}(T)$ be bilinear. Prove that $\mathsf{tr}_2 f \in \mathbf{PoLR}(S_1 \otimes S_2, T)$. Let $\tilde{f} = \mathsf{fun}\,(\mathsf{tr}_2 f) \in \mathbf{PoL}(S_1 \otimes S_2, T)$. Prove that $\tilde{f}$ satisfies $\tilde{f}\,\tau = f$, and that it is the unique element of $\mathbf{PoL}(S_1 \otimes S_2, T)$ with that property.

2.3) Assume that $S_1 = S_2$ and let us say that a bilinear function $f : \mathcal{I}(S) \times \mathcal{I}(S) \to \mathcal{I}(T)$ is *symmetric* if $f(u_1, u_2) = f(u_2, u_1)$ for all $u_1, u_2 \in \mathcal{I}(S)$. Define a preorder S^2 and a bilinear and symmetric function $\beta : \mathcal{I}(S) \times \mathcal{I}(S) \to \mathcal{I}(S^2)$ such that, for any preorder T and any bilinear and symmetric $f : \mathcal{I}(S) \times \mathcal{I}(S) \to \mathcal{I}(T)$, there is exactly one $\tilde{f} \in \mathbf{PoL}(S^2, T)$ such that $f = \tilde{f}\,\beta$.

3) Let S and T be preorders. We say that a function $f : \mathcal{I}(S) \to \mathcal{I}(T)$ is Scott-continuous if f is monotonic and, for any *directed* subset D of $\mathcal{I}(S)$, one has $f(\cup D) = \cup_{u \in D} f(u)$ (or equivalently $f(\cup D) \subseteq \cup_{u \in D} f(u)$).

Given a function $f : \mathcal{I}(S) \to \mathcal{I}(T)$, one defines

$$\mathsf{Tr} f = \{(u, b) \in \mathcal{P}_{\mathrm{fin}}(|S|) \times |T| \mid b \in f(\downarrow u)\} .$$

and one defines the preorder $!_{\mathsf{s}} S = (\mathcal{P}_{\mathrm{fin}}(|S|), \leq_{!_{\mathsf{s}} S})$ as follows: $u \leq_{!_{\mathsf{s}} S} u'$ iff $\downarrow u \subseteq \downarrow u'$ (that is: forall $a \in u$ there exists $a' \in u'$ such that $a \leq_S a'$).

3.1) Prove that if f is monotonuic then $\mathsf{Tr} f \in \mathbf{PoLR}(!_{\mathsf{s}} S, T)$.

Given $t \in \mathbf{PoLR}(!_{\mathsf{s}} S, T)$ we define

$$\begin{aligned} \mathsf{Fun}\, t : \mathcal{I}(S) &\to \mathcal{I}(T) \\ u &\mapsto \{b \in |T| \mid \exists u_0 \subseteq u\ (u_0, b) \in t\} . \end{aligned}$$

3.2) Prove that $\mathsf{Fun}\, t$ is Scott-continuous.

Let $\mathbf{PoC}$ be the category whose objects are the preorders and where $\mathbf{PoC}(S, T)$ is the set of all Scott-continuous functions $\mathcal{I}(S) \to \mathcal{I}(T)$ (identity functions and composition being defined in the obvious way).

3.3) Prove that the operations Tr and Fun define an order isomorphism between $\mathbf{PoLR}(!_{\mathsf{s}} S, T)$ (ordered by $\subseteq$) and $\mathbf{PoC}(S, T)$ (equiped with the order relation $\leq$ defined by $f \leq g$ if $\forall u \in \mathcal{I}(S)\ f(u) \subseteq g(u)$).

Remember that an element x of a domain (cpo) X is *compact* (or *isolated*) if, for any directed subset D of X, if $\bigvee D \geq x$ then there exists $x' \in D$ such that $x' \geq x$.

3.4) Prove that an element $u \in \mathcal{I}(S)$ is compact iff there exists a finite set $u_0 \subseteq |S|$ such that $u = \downarrow u_0$.

3.5) Let N be the preorder such that $|\mathsf{N}| = \mathbb{N}$ and $n \leq_{\mathsf{N}} n'$ iff $n = n'$. Show that, in $(!_{\mathsf{s}}\mathsf{N})^{\mathsf{op}} \times \mathsf{N}$ (which will represent the type $\iota \Rightarrow \iota$), there are compact elements which have infinitely many lower bounds, which are not necessarily compact.

3.6) Show that, in a coherence space, a compact clique has only finitely many lower bounds, which are all compact.