MPRI 2–2 TD 1 du 13/11/2018 (with solutions)

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A *coherence space* is a pair $E = (|E|, \coh_E)$ where $|E|$ is a set and $\coh_E \subseteq |E|^2$ is a reflexive and symmetric relation. Remember that $\scoh_E = \coh_E \setminus \{(a,a) \mid a \in |E|\}$.

The set of *cliques* of E is $\mathsf{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x\ a \coh_E a'\}$. Equipped with the partial order relation $\subseteq$, $\mathsf{Cl}(E)$ is closed under directed unions[1]. Observe also that a subset of a clique is a clique, that all singletons are cliques and that $\emptyset$ is a clique.

Let E and F be coherence spaces. A function $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ is *stable* is it is monotone, Scott-continuous (that is, for all directed $D \subseteq \mathsf{Cl}(E)$, one has $f(\cup D) = \cup_{x \in D} f(x)$, or, equivalently $f(\cup D) \subseteq \cup_{x \in D} f(x)$, since the converse inclusion holds by monotonicity of f) and *conditionally multiplicative*, that is

$$\forall x, y \in \mathsf{Cl}(E) \quad x \cup y \in \mathsf{Cl}(E) \Rightarrow f(x \cap y) = f(x) \cap f(y)$$

or equivalently

$$\forall x, y \in \mathsf{Cl}(E) \quad x \cup y \in \mathsf{Cl}(E) \Rightarrow f(x \cap y) \supseteq f(x) \cap f(y)$$

since the converse inclusion holds by monotonicity of f.

One says that f is *linear* if, moreover, $f(\emptyset) = \emptyset$ and $\forall x, y \in \mathsf{Cl}(E)\ x \cup y \in \mathsf{Cl}(E) \Rightarrow f(x \cup y) = f(x) \cup f(y)$.

1) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$. Prove that f is linear if and only if the following property holds: for any family $(x_i)_{i \in I}$ of elements of $\mathsf{Cl}(E)$ (where I is finite or countable) such that $i \neq j \Rightarrow x_i \cap x_j = \emptyset$ and $\bigcup_{i \in I} x_i \in \mathsf{Cl}(E)$, the family $(f(x_i))_{i \in I}$ satisfies the same properties (namely $i \neq j \Rightarrow f(x_i) \cap f(x_j) = \emptyset$ and $\bigcup_{i \in I} f(x_i) \in \mathsf{Cl}(F)$), and moreover $\bigcup_{i \in I} f(x_i) = f(\bigcup_{i \in I} x_i)$.

Solution $\triangleright$ Assume first that f is linear. Let $(x_i)_{i \in I}$ be a family of elements of $\mathsf{Cl}(E)$ (where I is finite or countable) such that $i \neq j \Rightarrow x_i \cap x_j = \emptyset$ and $\bigcup_{i \in I} x_i \in \mathsf{Cl}(E)$. Let $i, j \in I$ and assume that $f(x_i) \cap f(x_j) \neq \emptyset$. Since $x_i \cup x_j \in \mathsf{Cl}(E)$ we have $f(x_i) \cap f(x_j) = f(x_i \cap x_j)$ because f is stable and hence $x_i \cap x_j \neq \emptyset$ since $f(\emptyset) = \emptyset$ by linearity. Therefore $i = j$. Since f is monotone we have $f(x_i) \subseteq f(\cup_{j \in J} x_j) \in \mathsf{Cl}(F)$ for all i and hence $\cup_{i \in I} f(x_i) \in \mathsf{Cl}(F)$. Last we must prove that $\cup_{i \in I} f(x_i) = f(\cup_{i \in I} x_i)$, that is $\cup_{i \in I} f(x_i) \supseteq f(\cup_{i \in I} x_i)$ since f is monotone. Let $b = f(\cup_{i \in I} x_i)$. Since f is continuous there is a finite clique $x_0 \subseteq \cup_{i \in I} x_i$ such that $b \in f(x_0)$. Let $I_0 \subseteq I$ be finite and such that $x_0 \subseteq \cup_{i \in I_0} x_i$. We have $b \in f(\cup_{i \in I_0} x_i)$ by monotonicity and $f(\cup_{i \in I_0} x_i) = \cup_{i \in I_0} f(x_i)$ by linearity. Therefore $b \in \cup_{i \in I} f(x_i)$.

Conversely assume that $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ satisfies the stated property. Let $x, x' \in \mathsf{Cl}(E)$ be such that $x \subseteq x'$. By our assumtion we have $f(x' \setminus x) \cap f(x) = \emptyset$ and $f(x') = f(x' \setminus x) \cup f(x)$, hence $f(x) \subseteq f(x')$. Let $x \in \mathsf{Cl}(E)$, by our assumption we have $f(x) = f(\cup_{a \in x} \{a\}) = \cup_{a \in x} f(\{a\})$. So if $b \in f(x)$ there exists $a \in x$ such that $b \in f(\{a\})$ and hence f is continuous. Moreover there is only one such a (if a' is another one we have $b \in f(\{a\}) \cap f(\{a'\})$ which is impossible since $\{a\} \cap \{a'\} = \emptyset$). This shows that f is stable. Last let $x, x' \in \mathsf{Cl}(E)$ be such that $x \cup x' \in \mathsf{Cl}(E)$, we have to prove that $f(x) \cup f(x') \subseteq f(x \cup x')$ so let $b \in f(x) \cup f(x') = \cup_{a \in x \cup x'} f(\{a\})$ so that there is $a \in x \cup x'$ such that $b \in f(\{a\})$, hence $b \in f(x) \cup f(x')$ by monotonicity of f. $\triangleleft$

2) Let E_1, E_2 and F be coherence spaces. A function $f : \mathsf{Cl}(E_1) \times \mathsf{Cl}(E_1) \to \mathsf{Cl}(F)$ is *bilinear* if it is separately linear, that is: for all $x_1 \in \mathsf{Cl}(E_1)$ the function $\mathsf{Cl}(E_2) \to \mathsf{Cl}(F)$ which maps x_2 to $f(x_1, x_2)$ is linear, and symmetrically (reversing the roles of E_1 and E_2).

2.1) Prove that a bilinear function $f : \mathsf{Cl}(E_1) \times \mathsf{Cl}(E_1) \to \mathsf{Cl}(F)$ is stable from $\mathsf{Cl}(E_1 \,\&\, E_2) \to \mathsf{Cl}(F)$ (identifying $\mathsf{Cl}(E_1) \times \mathsf{Cl}(E_1)$ and $\mathsf{Cl}(E_1 \,\&\, E_2)$, which are isomorphic posets). Give an example of a bilinear map which is not linear. And prove that the only linear map which is bilinear is the "empty map" (such that $f(x_1, x_2) = \emptyset$ for all x_1, x_2).

[1] Unions filtrantes en français

Solution ▷ If $z \in \mathsf{Cl}(E_1 \mathbin{\&} E_2)$, we use z_1 and z_2 for its two projections so that $z = \{1\} \times z_1 \cup \{2\} \times z_2 = (z_1, z_2)$ up to the identification of $\mathsf{Cl}(E_1 \mathbin{\&} E_2)$ with $\mathsf{Cl}(E_1) \times \mathsf{Cl}(E_2)$. Let $f : \mathsf{Cl}(E_1 \mathbin{\&} E_2) \to \mathsf{Cl}(F)$ be bilinear. Let $z, z' \in \mathsf{Cl}(E_1 \mathbin{\&} E_2)$ be such that $z \subseteq z'$. We have $f(z) = f(z_1, z_2) \subseteq f(z'_1, z_2) \subseteq f(z'_1, z'_2) = f(z')$ so f is monotonic. If $D \subseteq \mathsf{Cl}(E_1 \mathbin{\&} E_2)$ is directed then the two projections $D_i \subseteq \mathsf{Cl}(E_i)$ are directed and $\cup D = (\cup D_1, \cup D_2)$. By bilinearity we have $f(\cup D) = f(\cup D_1, \cup D_2) = \cup_{x_1 \in D_1} f(x_1, \cup D_2) = \cup_{(x_1,x_2) \in D_1 \times D_2} f(x_1, x_2) = \cup_{z \in D} f(z)$, this latter equation results from the fact that for any $(x_1, x_2) \in D_1 \times D_2$ there is $z \in D$ such that $x_i \subseteq z_i$ for $i = 1, 2$ because D is directed.

Now let $z, z' \in \mathsf{Cl}(E_1 \mathbin{\&} E_2)$ be such that $z \subseteq z'$, we have $f(z) \supseteq f(z_1, z'_2) \cap f(z'_1, z_2)$ (a property that we call (*) in the sequel). By separate linearity (using the first exercise of this sheet) we have $f(z_1, z'_2) \cap f(z'_1, z_2) = f(z_1, z_2 \cup (z'_2 \setminus z_2)) \cap f(z_1 \cup (z'_1 \setminus z_1), z_2) = (f(z_1, z_2) \cup f(z_1, z'_2 \setminus z_2)) \cap (f(z_1, z_2) \cup f(z'_1 \setminus z_1, z_2)) = f(z_1, z_2) \cup (f(z_1, z'_2 \setminus z_2) \cap f(z'_1 \setminus z_1, z_2))$ (since $f(z_1, z_2) \cap f(z_1, z'_2 \setminus z_2) = \emptyset$ by separate linearity). We have $f(z_1, z'_2 \setminus z_2) \cap f(z'_1 \setminus z_1, z_2) \subseteq (z'_1, z'_2 \setminus z_2) \cap f(z'_1, z_2) = \emptyset$ by separate linearity again. Consider now $z, z' \in \mathsf{Cl}(E_1 \mathbin{\&} E_2)$ such that $z \cup z' \in \mathsf{Cl}(E_1 \mathbin{\&} E_2)$. Observe first that $f(z) = f(z_1 \cup z'_1, z_2) \cap f(z_1, z_2 \cup z'_2)$ by Property (*). We have $f(z \cap z') = f(z_1 \cap z'_1, z_2 \cap z'_2) = f(z_1, z_2 \cap z'_2) \cap f(z'_1, z_2 \cap z'_2) = f(z_1, z_2) \cap f(z_1, z'_2) \cap f(z'_1, z_2) \cap f(z'_1, z'_2) = f(z_1 \cup z'_1, z_2) \cap f(z_1, z_2 \cup z'_2) \cap f(z_1 \cup z'_1, z'_2) \cap f(z_1, z_2 \cup z'_2) \cap f(z_1 \cup z'_1, z_2) \cap f(z'_1, z_2 \cup z'_2) \cap f(z'_1 \cup z_1, z'_2) \cap f(z'_1, z_2 \cup z'_2) = f(z_1, z_2 \cup z'_2) \cap f(z_1 \cup z'_1, z_2) \cap f(z'_1, z_2 \cup z'_2) \cap f(z_1 \cup z'_1, z'_2) = f(z) \cap f(z')$ by Property (*) again.

Erratum: Contrarily to what I have claimed during the Nov. 13th session, **it is no true** that a Scott continuous $f : \mathsf{Cl}(E_1 \mathbin{\&} E_2) \to \mathsf{Cl}(F)$ which is separately stable is stable. Take indeed $E_1 = E_2 = F = 1$ where 1 is the coherence space whose web is a singleton $\{*\}$. Take $f : \mathsf{Cl}(1) \times \mathsf{Cl}(1) \to \mathsf{Cl}(1)$ defined by $f(z) = \emptyset$ if $z = \emptyset$ and $f(z) = \{*\}$ otherwise. Then f is separately stable but not stable because $\{*\} = f(\{*\}, \emptyset) \cap f(\emptyset, \{*\})$ and $f((\{*\}, \emptyset) \cap (\emptyset, \{*\})) = f(\emptyset, \emptyset) = \emptyset$. The function f is a simplified version of the "parallel or" non stable function.

2.2) Check that the function $\tau : \mathsf{Cl}(E_1) \times \mathsf{Cl}(E_2) \to \mathsf{Cl}(E_1 \otimes E_2)$ such that $\tau(x_1, x_2) = x_1 \otimes x_2 = x_1 \times x_2$ is bilinear.

Solution ▷ This is straightforward. Observe that $\mathsf{Tr}(\tau) = \{(\{(1, a_1), (2, a_2)\}, (a_1, a_2) \mid a_i \in |E_i| \text{ for } i = 1, 2\}$. ◁

2.3) Prove that if $f : \mathsf{Cl}(E_1) \times \mathsf{Cl}(E_1) \to \mathsf{Cl}(F)$ is bilinear then there is exactly one linear morphism $\tilde{f} : \mathsf{Cl}(E_1 \otimes E_2) \to F$ such that $f = \tilde{f} \circ \tau$.

Solution ▷ The trace $\mathsf{Tr}(f) \in \mathsf{Cl}(E_1 \mathbin{\&} E_2 \multimap F)$ of f is the set of all $(z^0, b) \in \mathsf{Cl}_{\mathsf{fin}}(E_1 \mathbin{\&} E_2) \times |F|$ such that $b \in f(z^0)$ and z^0 is minimal with this property. Necessarily z^0 has shape $\{(1, a_1), (2, a_2)\}$ with $a_i \in |E_i|$: by bilinearity we have $f(z^0) = \cup_{a_1 \in z^0_1} f(\{a_1\}, z^0_2) = \cup_{a_1 \in z^0_1, a_2 \in z^0_2} f(\{(1, a_1), (2, a_2)\})$ so if $b \in f(z^0)$ there is some $\{(1, a_1), (2, a_2)\} \subseteq z^0$ such that $b \in f(\{(1, a_1), (2, a_2)\})$ hence z^0 must be $\subseteq$ in one of these $\{(1, a_1), (2, a_2)\}$. Written as a couple, a strict subset of $\{(1, a_1), (2, a_2)\}$ is of shape $(\emptyset, z_2)$ or $(z_1, \emptyset)$ and therefore is mapped to $\emptyset$ by f, by bilinearity. So if $(z^0, b) \in \mathsf{Tr}(f)$, z^0 has shape $\{(1, a_1), (2, a_2)\}$ (this shows btw. that there is no f which is at the same time linear and bilinear, apart from the completely undefined f such that $\mathsf{Tr}(f) = \emptyset$). Now we define $\tilde{f}$ by its linear trace $\{((a_1, a_2), b) \mid (\{(1, a_1), (2, a_2)\}, b) \in \mathsf{Tr} f\} \in \mathsf{Cl}(E_1 \otimes E_2 \multimap F)$. ◁

3) Let E be a coherence space and let $u \in \mathsf{Cl}(E)$. One defines a coherence space E_u as follows: $|E_u| = \{a \in |E| \mid \forall b \in u \ a \frown_E b\}$ and $\coh_{E_u} = \coh_E \cap |E_u|^2$. Observe that $\mathsf{Cl}(E_u) \subseteq \mathsf{Cl}(E)$ and that, if $x \in \mathsf{Cl}(E_u)$ then $x \cap u = \emptyset$ and $x \cup u \in \mathsf{Cl}(E)$, which defines a linear map $\mathsf{Cl}(E_1 \otimes E_2) \to \mathsf{Cl}(F)$ that we also denote as $\tilde{f}$.

Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a monotone and Scott-continuous function. Given $u \in \mathsf{Cl}(E)$ one defines a function $\Delta_u f : \mathsf{Cl}(E_u) \to \mathsf{Cl}(F)$ by $\Delta_u f(x) = f(x \cup u) \setminus f(x)$.

3.1) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a stable function. Compute $\Delta_u f$ when f is constant, and when f is linear (that is $f(\emptyset) = \emptyset$ and $f(x \cup y) = f(x) \cup f(y)$ if $x, y \in \mathsf{Cl}(E)$ satisfy $x \cup y \in \mathsf{Cl}(E)$).

Solution ▷ Let $x \in \mathsf{Cl}(E_u)$. If f is constant then $\Delta_u f(x) = \emptyset$. If f is linear then $\Delta_u f(x) = f(x \cup u) \setminus f(x) = (f(x) \cup f(u)) \setminus f(x) = f(u)$ because $f(x) \cap f(u) = f(x \cap u) = f(\emptyset) = \emptyset$. ◁

3.2) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a monotone and Scott-continuous function. Prove that if $\Delta_u f$ is monotone for all $u \in \mathsf{Cl}(E)$, then f is stable.

Solution $\triangleright$ Let $x, x' \in \mathsf{Cl}(E)$ be such that $x \cup x' \in \mathsf{Cl}(E)$, we must prove that $f(x) \cap f(x') \subseteq f(x \cap x')$. Let $b \in f(x) \cap f(x')$ and assume that $b \notin f(x \cap x')$. Let $u = x \setminus (x \cap x')$, then $x' \cap u = \emptyset$ and hence $x' \in \mathsf{Cl}(E_u)$, so we have $\Delta_u f(x \cap x') \subseteq \Delta_u f(x')$. By our assumption $b \in \Delta_u f(x \cap x')$ since $(x \cap x') \cup u = x$ and hence $b \in \Delta_u f(x') = f(x' \cup u) \setminus f(x')$ which implies $b \notin f(x')$, contradiction. $\triangleleft$

3.3) Conversely, prove that, if f is stable, then $\Delta_u f$ is stable for all $u \in \mathsf{Cl}(E)$. In particular, f is stable if and only if $\Delta_u f$ is monotone for all $u \in \mathsf{Cl}(E)$.

Solution $\triangleright$ Let $u \in \mathsf{Cl}(E)$. Let us first prove that $\Delta_u f$ is monotone so let $x, x' \in \mathsf{Cl}(E_u)$ be such that $x \subseteq x'$. Let $b \in \Delta_u f(x) = f(x \cup u) \setminus f(x)$. By monotonicity of f we have $b \in f(x' \cup u)$. Il $b \in f(x')$ then $b \in f(x \cup u) \cap f(x') = f((x \cup u) \cap x')$ by stability (observe indeed that $x \cup u \cup x' \subseteq u \cup x' \in \mathsf{Cl}(E)$) and this is impossible because $(x \cup u) \cap x' = x$ and $b \in \Delta_u f(x)$. So $b \in f(x' \cup u) \setminus f(x') = \Delta_u f(x')$.

Now we prove that $\Delta_u f$ is continuous, so let $x \in \mathsf{Cl}(E_u)$ and let $b \in \Delta_u f(x) = f(x \cup u) \setminus f(x)$. Since f is continuous there is a finite clique $x_1 \subseteq x \cup u$ such that $b \in f(x_1)$. Let $x_0 = x \cap x_1 \in \mathsf{Cl}(E_u)$. We have $b \in f(x_1) \subseteq f(x_0 \cup u)$ by monotonicity of f, and for the same reason $b \notin f(x_0)$ since we know that $b \notin f(x)$. Hence $b \in \Delta_u f(x_0)$.

Last we prove that $\Delta_u f$ is conditionally multiplicative, so let $x, x' \in \mathsf{Cl}(E_u)$ be such that $x \cup x' \in \mathsf{Cl}(E_u)$ (equivalently $x \cup x' \in \mathsf{Cl}(E)$ by definition of the coherence space E_u). We must prove that $\Delta_u f(x) \cap \Delta_u f(x') \subseteq \Delta_u f(x \cap x')$, so let $b \in \Delta_u f(x) \cap \Delta_u f(x')$. This implies $b \in f(x \cup u) \cap f(x' \cup u)$. But we have $(x \cup u) \cup (x' \cup u) = x \cup x' \cup u \in \mathsf{Cl}(E)$ by our assumption on x and x', and hence $b \in f((x \cup u) \cap (x' \cup u)) = f((x \cap x') \cup u)$ by stability of f. Since $b \in \Delta_u f(x)$, we know moreover that $b \notin f(x)$ and hence $b \notin f(x \cap x')$ by monotonicity of f, hence $b \notin f(x \cap x')$. So we have $b \in \Delta_u f(x \cap x')$. $\triangleleft$

Let $f, g : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be stable functions. One says that f is stably less than g (notation $f \leq_{\mathsf{st}} g$) if

$$\forall x, y \in \mathsf{Cl}(E) \quad x \subseteq y \Rightarrow f(x) = f(y) \cap g(x)\,.$$

Observe that $f \leq_{\mathsf{st}} g \Rightarrow f \leq_{\mathsf{ext}} g$ (where $f \leq_{\mathsf{ext}} g$ means $\forall x \in \mathsf{Cl}(E)\ f(x) \subseteq g(x)$): take $x = y$ in the definition above.

3.4) Prove that $f \leq_{\mathsf{st}} g$ if and only if $f \leq_{\mathsf{ext}} g$ and $\forall u \in \mathsf{Cl}(E)\ \Delta_u f \leq_{\mathsf{ext}} \Delta_u g$.

Solution $\triangleright$ Assume first that $f \leq_{\mathsf{st}} g$ and let us prove that $\Delta_u f \leq_{\mathsf{ext}} \Delta_u g$ (where $u \in \mathsf{Cl}(E)$). Let $x \in \mathsf{Cl}(E_u)$ and assume that $b \in \Delta_u f(x) = f(x \cup u) \setminus f(x)$. Since $f \leq_{\mathsf{ext}} g$ we have $b \in g(x \cup u)$. Assume that $b \in g(x)$. Since $f \leq_{\mathsf{st}} g$ we have $f(x) = f(x \cup u) \cap g(x)$ and hence $b \in f(x)$, contradiction. Hence $b \in \Delta_u g(x)$, which shows that $\Delta_u f \leq_{\mathsf{ext}} \Delta_u g$.

Assume conversely that $f \leq_{\mathsf{ext}} g$ and $\forall u \in \mathsf{Cl}(E)\ \Delta_u f \leq_{\mathsf{ext}} \Delta_u g$ and let us prove that $f \leq_{\mathsf{st}} g$. So let $x, x' \in \mathsf{Cl}(E)$ be such that $x \subseteq x'$, we must prove that $f(x') \cap g(x) \subseteq f(x)$ (the other inclusion results from our assumption that $f \leq_{\mathsf{ext}} g$). Let $b \in f(x') \cap g(x)$ and assume towards a contradiction that $b \notin f(x)$. Let $u = x' \setminus x$, so that $x \in \mathsf{Cl}(E_u)$. By our assumption $b \in f(x') \setminus f(x) = \Delta_u f(x) \subseteq \Delta_u g(x)$ (since $\Delta_u f \leq_{\mathsf{ext}} \Delta_u g$) and hence $b \notin g(x)$, contradiction. $\triangleleft$

Remember that, if $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ is a stable function, one defines the *trace* $\mathsf{Tr} f$ of f as the set of all pairs (x_0, b) where $b \in |Y|$ and x_0 is minimal such that $b \in f(x_0)$ (and is therefore finite by continuity of f). Remember also that, if $(x_0, b), (y_0, b) \in \mathsf{Tr} f$ satisfy $x_0 \cup y_0 \in \mathsf{Cl}(E)$, then $x_0 = y_0$.

3.5) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a stable function. Prove that

$$\mathsf{Tr}(\Delta_u f) = \{(y_0 \setminus u, b) \mid (y_0, b) \in \mathsf{Tr} f,\ y_0 \cap u \neq \emptyset \text{ and } y_0 \cup u \in \mathsf{Cl}(E)\}\,.$$

Solution $\triangleright$ Let $(x_0, b) \in \mathsf{Tr}(\Delta_u f)$ so that $x_0 \in \mathsf{Cl}_{\mathsf{fin}}(E_u)$, $b \in f(x_0 \cup u) \setminus f(x_0)$ and x_0 minimal with these properties. Since $b \in f(x_0 \cup u)$ there is a uniquely defined $y_0 \subseteq x_0 \cup u$ such that $(y_0, b) \in \mathsf{Tr} f$. We cannot have $y_0 \subseteq x_0$ since $b \notin f(x_0)$ and hence $y_0 \cap u \neq \emptyset$. Last $y_0 \cup u \subseteq x_0 \cup u \in \mathsf{Cl}(E)$ since $x_0 \in \mathsf{Cl}(E_u)$.

Conversely let $(y_0, b) \in \mathsf{Tr} f$ be such that $y_0 \cap u \neq \emptyset$ and $y_0 \cup u \in \mathsf{Cl}(E)$. Let $x_0 = y_0 \setminus u$, we have $x_0 \in \mathsf{Cl}(E_u)$ and $b \in f(y_0) \setminus f(x_0)$ by minimality of y_0. Hence $b \in \Delta_u f(x_0)$ since $y_0 \subseteq x_0 \cup u$. We prove that x_0 is minimal with that property so let $x'_0 \subseteq x_0$ be such that $b \in \Delta_u f(x'_0)$. We have $b \in f(y_0) \cap f(x'_0 \cup u)$ and $y_0 \cup x'_0 \cup u \subseteq x_0 \cup u \in \mathsf{Cl}(E)$ hence, by stability, $b \in f(y_0 \cap (x'_0 \cup u))$. By minimality of y_0 we must have $y_0 \subseteq x'_0 \cup u$ and hence $x_0 = y_0 \setminus u \subseteq (x'_0 \cup u) \setminus u = x'_0$ since $x'_0 \cap u = \emptyset$, so $x'_0 = x_0$ which proves the minimality of x_0. $\triangleleft$

4) If E and F are coherence spaces, one says that E is a subspace of F and writes $E \subseteq F$ if $|E| \subseteq |F|$ and

$$\forall a_1, a_2 \in |E| \quad a_1 \coh_E a_2 \Leftrightarrow a_1 \coh_F a_2 \, .$$

Let $\mathbf{Coh}_\subseteq$ be the class of all coherence spaces, equiped with this order relation $\subseteq$.

4.1) Prove that any monotone sequence of coherence spaces $E_1 \subseteq E_2 \subseteq E_3 \cdots$ has a least upper bound (a sup) in $\mathbf{Coh}_\subseteq$.

4.2) Let $\Phi : \mathbf{Coh}_\subseteq \to \mathbf{Coh}_\subseteq$ be defined by $\Phi(E) = 1 \oplus !E$ (where 1 is the coherence space which has only one element in its web). Prove that Φ is monotone and commutes with the least upper bounds of monotone sequences of coherence spaces.

4.3) Prove that Φ has a least fixpoint in $\mathbf{Coh}_\subseteq$, that we denote as L and call "object of lazy integers".

4.4) Prove that one defines a function $\varphi : \mathbb{N} \to \mathsf{Cl}(\mathsf{L})$ by setting: $\varphi(0) = \{(1, *)\}$ (where $*$ is the unique element of $|1|$) and $\varphi(n+1) = \{(2, u_0) \mid u_0 \subseteq \varphi(n) \text{ and } u_0 \text{ finite}\}$. Give the values of $\varphi(0)$, $\varphi(1)$ and $\varphi(2)$.