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A *coherence space* is a pair $E = (|E|, \frown\!\!\!\!\smile_E)$ where $|E|$ is a set and $\frown\!\!\!\!\smile_E \subseteq |E|^2$ is a reflexive and symmetric relation. Remember that $\frown_E = \frown\!\!\!\!\smile_E \setminus \{(a,a) \mid a \in |E|\}$.

The set of *cliques* of E is $\mathsf{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x\ a \frown\!\!\!\!\smile_E a'\}$. Equipped with the partial order relation $\subseteq$, $\mathsf{Cl}(E)$ is closed under directed unions[1]. Observe also that a subset of a clique is a clique, that all singletons are cliques and that $\emptyset$ is a clique.

Let E and F be coherence spaces. A function $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ is *stable* is it is monotone, Scott-continuous (that is, for all directed $D \subseteq \mathsf{Cl}(E)$, one has $f(\cup D) = \cup_{x \in D} f(x)$, or, equivalently $f(\cup D) \subseteq \cup_{x \in D} f(x)$, since the converse inclusion holds by monotonicity of f) and *conditionally multiplicative*, that is

$$\forall x, y \in \mathsf{Cl}(E) \quad x \cup y \in \mathsf{Cl}(E) \Rightarrow f(x \cap y) = f(x) \cap f(y)$$

or equivalently

$$\forall x, y \in \mathsf{Cl}(E) \quad x \cup y \in \mathsf{Cl}(E) \Rightarrow f(x \cap y) \supseteq f(x) \cap f(y)$$

since the converse inclusion holds by monotonicity of f.

One says that f is *linear* if, moreover, $f(\emptyset) = \emptyset$ and $\forall x, y \in \mathsf{Cl}(E)\ x \cup y \in \mathsf{Cl}(E) \Rightarrow f(x \cup y) = f(x) \cup f(y)$.

1) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$. Prove that f is linear if and only if the following property holds: for any family $(x_i)_{i \in I}$ of elements of $\mathsf{Cl}(E)$ (where I is finite or countable) such that $i \neq j \Rightarrow x_i \cap x_j = \emptyset$ and $\bigcup_{i \in I} x_i \in \mathsf{Cl}(E)$, the family $(f(x_i))_{i \in I}$ satisfies the same properties (namely $i \neq j \Rightarrow f(x_i) \cap f(x_j) = \emptyset$ and $\bigcup_{i \in I} f(x_i) \in \mathsf{Cl}(F)$), and moreover $\bigcup_{i \in I} f(x_i) = f(\bigcup_{i \in I} x_i)$.

2) Let E_1, E_2 and F be coherence spaces. A function $f : \mathsf{Cl}(E_1) \times \mathsf{Cl}(E_1) \to \mathsf{Cl}(F)$ is *bilinear* if it is separately linear, that is: for all $x_1 \in \mathsf{Cl}(E_1)$ the function $\mathsf{Cl}(E_2) \to \mathsf{Cl}(F)$ which maps x_2 to $f(x_1, x_2)$ is linear, and symmetrically (reversing the roles of E_1 and E_2).

2.1) Prove that a bilinear function $f : \mathsf{Cl}(E_1) \times \mathsf{Cl}(E_1) \to \mathsf{Cl}(F)$ is stable from $\mathsf{Cl}(E_1 \,\&\, E_2) \to \mathsf{Cl}(F)$ (identifying $\mathsf{Cl}(E_1) \times \mathsf{Cl}(E_1)$ and $\mathsf{Cl}(E_1 \,\&\, E_2)$, which are isomorphic posets). Give an example of a bilinear map which is not linear. And prove that the only linear map which is bilinear is the "empty map" (such that $f(x_1, x_2) = \emptyset$ for all x_1, x_2).

2.2) Check that the function $\tau : \mathsf{Cl}(E_1) \times \mathsf{Cl}(E_2) \to \mathsf{Cl}(E_1 \otimes E_2)$ such that $\tau(x_1, x_2) = x_1 \otimes x_2 = x_1 \times x_2$ is bilinear.

2.3) Prove that if $f : \mathsf{Cl}(E_1) \times \mathsf{Cl}(E_1) \to \mathsf{Cl}(F)$ is bilinear then there is exactly one linear morphism $\tilde{f} : \mathsf{Cl}(E_1 \otimes E_2) \to F$ such that $f = \tilde{f} \circ \tau$.

3) Let E be a coherence space and let $u \in \mathsf{Cl}(E)$. One defines a coherence space E_u as follows: $|E_u| = \{a \in |E| \mid \forall b \in u\ a \frown_E b\}$ and $\frown\!\!\!\!\smile_{E_u} = \frown\!\!\!\!\smile_E \cap |E_u|^2$. Observe that $\mathsf{Cl}(E_u) \subseteq \mathsf{Cl}(E)$ and that, if $x \in \mathsf{Cl}(E_u)$ then $x \cap u = \emptyset$ and $x \cup u \in \mathsf{Cl}(E)$.

Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a monotone and Scott-continuous function. Given $u \in \mathsf{Cl}(E)$ one defines a function $\Delta_u f : \mathsf{Cl}(E_u) \to \mathsf{Cl}(F)$ by $\Delta_u f(x) = f(x \cup u) \setminus f(x)$.

3.1) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a stable function. Compute $\Delta_u f$ when f is constant, and when f is linear (that is $f(\emptyset) = \emptyset$ and $f(x \cup y) = f(x) \cup f(y)$ if $x, y \in \mathsf{Cl}(E)$ satisfy $x \cup y \in \mathsf{Cl}(E)$).

3.2) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a monotone and Scott-continuous function. Prove that if $\Delta_u f$ is monotone for all $u \in \mathsf{Cl}(E)$, then f is stable.

3.3) Conversely, prove that, if f is stable, then $\Delta_u f$ is stable for all $u \in \mathsf{Cl}(E)$. In particular, f is stable if and only if $\Delta_u f$ is monotone for all $u \in \mathsf{Cl}(E)$.

[1] Unions filtrantes en français

Let $f, g : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be stable functions. One says that f is stably less than g (notation $f \leq_{\mathsf{st}} g$) if

$$\forall x, y \in \mathsf{Cl}(E) \quad x \subseteq y \Rightarrow f(x) = f(y) \cap g(x) \,.$$

Observe that $f \leq_{\mathsf{st}} g \Rightarrow f \leq_{\mathsf{ext}} g$ (where $f \leq_{\mathsf{ext}} g$ means $\forall x \in \mathsf{Cl}(E)\ f(x) \subseteq g(x)$): take $x = y$ in the definition above.

3.4) Prove that $f \leq_{\mathsf{st}} g$ if and only if $f \leq_{\mathsf{ext}} g$ and $\forall u \in \mathsf{Cl}(E)\ \Delta_u f \leq_{\mathsf{ext}} \Delta_u g$.

Remember that, if $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ is a stable function, one defines the *trace* $\mathsf{Tr} f$ of f as the set of all pairs (x_0, b) where $b \in |Y|$ and x_0 is minimal such that $b \in f(x_0)$ (and is therefore finite by continuity of f). Remember also that, if $(x_0, b), (y_0, b) \in \mathsf{Tr} f$ satisfy $x_0 \cup y_0 \in \mathsf{Cl}(E)$, then $x_0 = y_0$.

3.5) Let $f : \mathsf{Cl}(E) \to \mathsf{Cl}(F)$ be a stable function. Prove that

$$\mathsf{Tr}(\Delta_u f) = \{(y_0 \setminus u, b) \mid (y_0, b) \in \mathsf{Tr} f,\ y_0 \cap u \neq \emptyset \text{ and } y_0 \cup u \in \mathsf{Cl}(E)\} \,.$$

4) If E and F are coherence spaces, one says that E is a subspace of F and writes $E \subseteq F$ if $|E| \subseteq |F|$ and

$$\forall a_1, a_2 \in |E| \quad a_1 \frown_E a_2 \Leftrightarrow a_1 \frown_F a_2 \,.$$

Let $\mathbf{Coh}_{\subseteq}$ be the class of all coherence spaces, equiped with this order relation $\subseteq$.

4.1) Prove that any monotone sequence of coherence spaces $E_1 \subseteq E_2 \subseteq E_3 \cdots$ has a least upper bound (a sup) in $\mathbf{Coh}_{\subseteq}$.

4.2) Let $\Phi : \mathbf{Coh}_{\subseteq} \to \mathbf{Coh}_{\subseteq}$ be defined by $\Phi(E) = 1 \oplus {!E}$ (where 1 is the coherence space which has only one element in its web). Prove that Φ is monotone and commutes with the least upperbounds of monotone sequences of coherence spaces.

4.3) Prove that Φ has a least fixpoint in $\mathbf{Coh}_{\subseteq}$, that we denote as L and call "object of lazy integers".

4.4) Prove that one defines a function $\varphi : \mathbb{N} \to \mathsf{Cl}(\mathsf{L})$ by setting: $\varphi(0) = \{(1, *)\}$ (where $*$ is the unique element of $|1|$) and $\varphi(n+1) = \{(2, u_0) \mid u_0 \subseteq \varphi(n) \text{ and } u_0 \text{ finite}\}$. Give the values of $\varphi(0)$, $\varphi(1)$ and $\varphi(2)$.