MPRI 2–2 TD 3 du 22/01/2019

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1) The goal of this exercise is to understand the structure of the category $\mathbf{PoLR}^!$ (the category of coalgebras of the comonad ! on the category $\mathbf{PoLR}$). We refer to the lecture notes for all basic definitions and notations.

1.1) Given a preorder S, we set $h_S = \{(a, u^0) \in |S| \times |!S| \mid \forall a' \in u^0\ a' \leq_S a\}$. Prove that $h_S \in \mathbf{PoLR}(S, !S)$.

1.2) Is the family of morphisms $(h_S)_S$ natural in S? That is, is it true that $h_T\, t = !t\, h_S$ for all $t \in \mathbf{PoLR}(S, T)$?

1.3) Prove that $\mathsf{der}_S\ h_S = \mathsf{Id}_S$.

1.4) Prove that $\mathsf{dig}_S\ h_S = !h_S\, h_S$. So we have shown that (S, h_S) is an object of $\mathbf{PoLR}^!$: any preorder has a canonical structure of coalgebra. We prove now that this structure is unique.

1.5) Let $h \in \mathbf{PoLR}(S, !S)$ be a coalgebra structure. Using the fact that $\mathsf{der}_S\ h \subseteq \mathsf{Id}_S$ prove that $h \subseteq h_S$ (take $(a, u^0) \in h$ and then for any $a' \in u^0$ observe that $(u^0, a') \in \mathsf{der}_S$).

1.6) Using the fact that $\mathsf{Id}_S \subseteq \mathsf{der}_S\ h$, prove that $(a, \{a\}) \in h$ for all $a \in |S|$ (do not forget that $h \in \mathbf{PoLR}(S, !S)!$).

1.7) Prove that $h = h_S$.

Strangely enough we have not used the equation $\mathsf{dig}_S\ h = !h\, h$. We have shown that any object S of $\mathbf{PoLR}$ has exactly one structure of !-coalgebra. Observe that one has accordingly $\mathsf{dig}_S = h_{!S}$, for instance, since $(!S, \mathsf{dig}_S)$ is a typical !-coalgebra, the free one generated by S.

A natural question is whether such a phenomenon occurs in all models of LL.

1.8) (Open question) Look for a counter-example in the model of coherence spaces, that is: a coherence space which has no coalgebra structures or which has several coalgebra structures), for the usual "!" comonad on coherence spaces.

1.9) Let S and T be preorders and let $s \in \mathbf{PoLR}(S, T)$, remember that $s \in \mathbf{PoLR}^!((S, h_S), (T, h_T)))$ iff $h_T\, s = !s\, h_S$. Prove that this condition is equivalent to: for all $a \in |S|$ and $b_1, \dots, b_n \in |T|$ (with $n \in \mathbb{N}$), there is $b \in |T|$ such that $(a, b) \in s$ and $b_i \leq_T b$ for all i iff there are $a_1, \dots, a_n \in |S|$ such that $a_i \leq_S a$ and $(a_i, b_i) \in s$ for all i. What does this condition mean when $n = 0$?

1.10) An *ideal* of S is a downwards-closed directed subset of $|S|$, that is, a subset u of $|S|$ such that

- $u \neq \emptyset$
- $\forall a_1, a_2 \in u\, \exists a \in u\ a_1, a_2 \leq_S a$
- $\forall a \in u\, \forall a' \in |S|\ a' \leq_S a \Rightarrow a' \in u$.

We use $\widehat{\mathcal{I}}(S)$ for the set of all ideals of $|S|$ (sometimes called the *ideal completion* of S), ordered under inclusion. Prove that $\widehat{\mathcal{I}}(S)$ is a cpo (which has not necessarily a least element however). Prove that, for any $a \in |S|$, one has $\downarrow a \in \widehat{\mathcal{I}}(S)$ and that $\downarrow a$ is isolated in $\widehat{\mathcal{I}}(S)$ (see Chapter 5 in the lecture notes). Last prove that $\widehat{\mathcal{I}}(S)$ is algebraic (actually any algebraic cpo D is of shape $\widehat{\mathcal{I}}(S)$ for S the set of isolated elements of D equiped with the induced order relation).

1.11) Exhibit a canonical bijection between $\widehat{\mathcal{I}}(S)$ and $\mathbf{PoLR}^!((1, h_1), (S, h_S))$ (remember that $1 = (\{*\}, =)$ so that simply $h_1 = \{(*, *)\}$). Using it prove that, if $s \in \mathbf{PoLR}^!((S, h_S), (T, h_T))$ and $u \in \widehat{\mathcal{I}}(S)$ one has $s\, u \in \widehat{\mathcal{I}}(T)$ (you can also prove this directly). We use $\mathsf{fun}^!(s)$ for this function $\widehat{\mathcal{I}}(S) \to \widehat{\mathcal{I}}(T)$.

1.12) Prove that $\mathsf{fun}^!(s)$ is Scott-continuous. Conversely, given a Scott-continuous function $f : \widehat{\mathcal{I}}(S) \to \widehat{\mathcal{I}}(T)$, define $\mathsf{tr}^!(f) = \{(a, b) \in |S| \times |T| \mid b \in f(\downarrow a)\}$. Prove that $\mathsf{tr}^!(s) \in \mathbf{PoLR}^!((S, h_S), (T, h_T))$.

1.13) Prove that the operations $\mathsf{fun}^!(\_)$ and $\mathsf{tr}^!(\_)$ are inverse of each other.

1.14) Prove that $\mathbf{PoLR}^!$ is cartesian (with cartesian product defined using $\otimes$ and not &) and also co-cartesian (with co-product defined using $\oplus$). Describe the corresponding operations on cpos. Compare with what happens in $\mathbf{PoLR}$ for & and $\oplus$.

1.15) Prove that $\widehat{\mathcal{I}}(!S) = \mathcal{I}(S)$. Using this observation explain how the canonical inclusion functor $\mathbf{PoLR}_! \to \mathbf{PoLR}^!$ (from free coalgebras into general ones), which maps S to $!S$ and $s \in \mathbf{PoLR}_!(S,T)$ to $s^! = \,!s\,\mathsf{dig}_S$ can simply be described as an inclusion of categories in that special case (using the characterization of $\mathbf{PoLR}_!(S,T)$ as the set of Scott-contuous functions $\mathcal{I}(S) \to \mathcal{I}(T)$).

2) Remember that $\mathcal{Z} \in \mathbf{PoLR}_!((S \Rightarrow S) \Rightarrow S, (S \Rightarrow S) \Rightarrow S)$ has been defined during a lesson as a morphism such that, setting $F = \mathsf{Fun}\,\mathcal{Z}$, one has $\mathsf{Fun}\,(F(Y))(s) = \mathsf{Fun}\,t(\mathsf{Fun}\,Y(s))$ for all $s \in \mathbf{PoLR}_!(S,S)$.

2.1) Given $t \in \mathbf{PoLR}_!(T,T)$, we set $\varphi(t) = \bigcup_{n=0}^{\infty}(\mathsf{Fun}\,t)^n(\emptyset) \in \mathcal{I}(T)$, the least fixed point of $\mathsf{Fun}\,t$. Prove that $\varphi(t)$ is the least element of $\mathcal{I}(T)$ such that for all $b \in \varphi(t)$, there exists u^0 such that $(u^0, b) \in t$ and $u^0 \subseteq \varphi(t)$.

2.2) We set $Y_0 = \varphi(\mathcal{Z}) \in \mathcal{I}((S \Rightarrow S) \Rightarrow S)$. Prove that $\mathsf{Fun}\,Y_0(s) = \varphi(\mathsf{Fun}\,s)$ for all $s \in \mathcal{I}(S \Rightarrow S)$. To this end, prove that $\mathsf{Fun}\,(F^n(\emptyset))(s) = (\mathsf{Fun}\,s)^n(\emptyset)$ by induction on n. Use also the fact that $\mathsf{Fun}\,\_$ is an order isomorphism (between $\mathbf{PoLR}_!(T,U)$ ordered by inclusion and $\mathbf{PoC}(\mathcal{I}(T), \mathcal{I}(U))$ ordered by the pointwise ordering on functions).

2.3) Prove that $(V^0, b) \in Y_0$ iff there exists u^0 such that $(u_0, b) \in \downarrow V^0$ and $\forall b' \in u^0\ (V^0, b') \in Y_0$.

3) Using the semantic typing system of LPCF, compute the Scott semantics of the following terms (given with their types).

- $\vdash \Omega^\iota : \iota$ where $\Omega^A = \mathsf{fix}\, x^A \cdot x$.
- $\vdash \mathsf{fix}\, x^\iota \cdot \underline{\mathsf{succ}}(x) : \iota$ (give a recursive description of the interpretation of this term).
- $\vdash \lambda x^\iota\, \mathsf{if}(x, \Omega^\iota, z \cdot \underline{0}) : \iota \to \iota$.
- $\vdash \lambda x^\iota\, \mathsf{fix}\, a^{\iota \to \iota} \cdot \lambda y^\iota\, \mathsf{if}(y, x, z \cdot \underline{\mathsf{succ}}((a)\, z)) : \iota \to \iota \to \iota$