1) The goal of this exercise is to study the properties of the objects of the Eilenberg Moore category \( \text{Ref} \) of \( \text{Rel} \), the relational model of \( LL \).

Let \( P \) be an object of \( \text{Ref} \) (the category of coalgebras of \( ! \)). Remember that \( P = (P, h_P) \) where \( P \) is an object of \( \text{Rel} \) (a set) and \( h_P \in \text{Rel}(P, !P) \) satisfies the following commutations:

\[
\begin{array}{c}
P \xrightarrow{h_P} !P \\
\downarrow \text{der}_P \\
P \\
\end{array} \quad \begin{array}{c}
P \xrightarrow{h_P} !P \\
\downarrow \text{dig}_P \\
!P \xrightarrow{h_P} !!P \\
\end{array}
\]

1.1) Check that these commutations mean:

- for all \( a, a' \in P \), one has \( (a, [a']) \in h_P \) if and only if \( a = a' \)

- and for all \( a \in P \) and \( m_1, \ldots, m_k \in !P \), one has \( (a, m_1 + \cdots + m_k) \in h_P \) if there are \( a_i, m_i \in P \) such that \( (a_i, [a_i]) \in h_P \) and \( (a_i, m_i) \in h_P \) for \( i = 1, \ldots, k \).

Intuitively, \( (a, [a_1, \ldots, a_k]) \) means that \( a \) can be decomposed into \( a_1 + \cdots + a_k \) where the \( + \) is the decomposition operation associated with \( P \).

1.2) Prove that if \( P \) is an object of \( \text{Ref} \) such that \( P \neq \emptyset \) then there is at least one element \( e \) of \( P \) such that \( (e, [\ ] \}) \in h_P \). Explain why such an \( e \) could be called a “coneutral element of \( P \”).

If \( P \) and \( Q \) are objects of \( \text{Ref} \), remember that an \( f \in \text{Rel}(P, Q) \) (morphism of coalgebras) is an \( f \in \text{Rel}(P, Q) \) such that the following diagram commutes

\[
\begin{array}{c}
P \xrightarrow{f} Q \\
\downarrow h_P \\
!P \xrightarrow{!f} !Q \\
\end{array}
\]

1.3) Check that this commutation means that for all \( a \in P \) and \( b_1, \ldots, b_k \in Q \), the two following properties are equivalent:

- there is \( b \in Q \) such that \( (a, b) \in f \) and \( (b, [b_1, \ldots, b_k]) \in h_Q \)

- there are \( a_1, \ldots, a_k \in P \) such that \( (a, [a_1, \ldots, a_k]) \in h_P \) and \( (a_i, b_i) \in f \) for \( i = 1, \ldots, k \).

1.4) Remember that \( \{ \text{the set } \{\} \} \) can be equipped with a structure of coalgebra (still denoted \( 1 \)) with \( h_1 = \{(\{k[\]}, k) \in \mathbb{N}\} \). Prove that the elements of \( \text{Rel}(1, P) \) can be identified with the subsets \( x \) of \( P \) such that: for all \( a_1, \ldots, a_k \in P \), one has \( a_1, \ldots, a_k \in x \) if there exists \( x \in x \) such that \( (a, [a_1, \ldots, a_k]) \in h_P \). We call \textit{values} of \( P \) these subsets of \( P \) and denote as \( \text{val}(P) \) the set of these values.

Prove that an element of \( \text{val}(P) \) is never empty and that \( \text{val}(P) \), equipped with inclusion, is a complete partially ordered set (cpo), that is: the union of a set of values which is directed (with respect to \( \subseteq \)) is still a value.

1.5) Remember that if \( E \) is an object of \( \text{Rel} \) then \( (E, \text{dig}_E) \) is an object of \( \text{Ref} \) (the free coalgebra generated by \( E \), that we can identify with an object of the Kleisli category \( \text{Ref} \)). Prove that, as a partially ordered set, \( \text{val}(E, \text{dig}_E) \) is isomorphic to \( P(E) \).

1.6) Is it always true that if \( x_1, x_2 \in \text{val}(P) \) then \( x_1 \cup x_2 \in \text{val}(P) \)?
1.7) We have seen (without proof) that \( \text{Rel}^1 \) is cartesian. Remember that the product of \( P_1 \) and \( P_2 \) is \( P_1 \otimes P_2 \), the coalgebra defined by \( P_1 \otimes P_2 = P_1 \otimes P_2 \) and \( h_{P_1 \otimes P_2} \) is the following composition of morphisms in \( \text{Rel} \):

\[
P_1 \otimes P_2 \xrightarrow{h_{P_1 \otimes P_2}} !P_1 \otimes !P_2 \xrightarrow{\mu^2_{P_1 \otimes P_2}} !(P_1 \otimes P_2)
\]

where \( \mu^2_{E_1, E_2} \in \text{Rel}(!(E_1 \otimes E_2),(E_1 \otimes E_2)) \) is the lax monoidality natural transformation of \(!._.\), remember that in \( \text{Rel} \) we have

\[
\mu^2_{E_1, E_2} = \{(([a_1], \ldots, [a_k]), [b_1, \ldots, b_k], ([a_1], b_1), \ldots, ([a_k], b_k)) \mid k \in \mathbb{N} \text{ and } (a_1, b_1, \ldots, (a_k, b_k) \in E_1 \times E_2) \}
\]

Concretely, we have simply that \( ([a_1, a_2], ([a_1], [a_2]), ([a_1], [a_2]), ([a_2], [a_2])) \in h_{P_1 \otimes P_2} \) iff \( (a_1, a_2, a_2) \subseteq h_{P_1} \) for \( i = 1, 2 \).

Prove that \( P_1 \otimes P_2 \), equipped with suitable projections, is the cartesian product of \( P_1 \) and \( P_2 \) in \( \text{Rel}^1 \). Prove also that \( 1 \) is the terminal object of \( \text{Rel}^1 \). Warning: \( \mathcal{L} \) is always cartesian when \( \mathcal{L} \) is a model of \( \text{LL} \); I’m not asking for a general proof, just for a verification that this is true in \( \text{Rel}^1 \).

1.8) Check directly that the partially ordered sets \( \text{val}(P_1 \otimes P_2) \) and \( \text{val}(P_1) \times \text{val}(P_2) \) are isomorphic.

1.9) Remember also that we have defined \( P_1 \oplus P_2 = (P_1 \oplus P_2, h_{P_1 \oplus P_2}) \) where \( h_{P_1 \oplus P_2} \) is the unique element of \( \text{Rel}(P_1 \oplus P_2, ![P_1 \oplus P_2]) \) such that, for \( i = 1, 2 \), the morphism \( h_{P_1 \oplus P_2} \pi_i \) coincides with the following composition of morphisms in \( \text{Rel} \):

\[
P_1 \xrightarrow{h_{P_1}} !P_2 \xrightarrow{\pi_i} !(P_1 \oplus P_2)
\]

Describe \( h_{P_1 \oplus P_2} \) as simply as possible and prove that, equipped with suitable injections, \( P_1 \oplus P_2 \) is the coproduct of \( P_1 \) and \( P_2 \) in \( \text{Rel}^1 \).

2) The goal of this exercise is to illustrate the fact that \( \text{Rel} \), the relational model of \( \text{LL} \), can be equipped with additional structures of various kinds without modifying the interpretation of proofs and programs. As an example we shall study the notion of **non-uniform coherence space** (NUCS). A NUCS is a triple \( X = (|X|, \bowtie_X, \sim_X) \) where

- \( |X| \) is a set (the web of \( X \))
- and \( \bowtie_X \) and \( \sim_X \) are two symmetric relations on \( |X| \) such that \( \bowtie_X \cap \sim_X = \emptyset \). In other words, for any \( a, a' \in |X| \), one never has \( a \bowtie_X a' \) and \( a \sim_X a' \).

So we can consider an ordinary coherence space (in the sense of the first part of this series of lectures) as a NUCS \( X \) which satisfies moreover:

\[
\forall a, a' \in |X| \quad (a \bowtie_X a' \lor a \sim_X a') \iff a \neq a'.
\]

It is then possible to introduce three other natural symmetric relations on the elements of \( |X| \):

- \( a \equiv_X a' \) if it is not true that \( a \bowtie_X a' \lor a \sim_X a' \).
- \( a \bowtie_X a' \) if \( a \bowtie_X a' \lor a \equiv_X a' \).
- \( a \equiv_X a' \) if \( a \sim_X a' \lor a \equiv_X a' \).

A **clique** of a NUCS \( X \) is a subset \( x \) of \( |X| \) such that \( \forall a, a' \in |X| \ a \bowtie_X a' \), we use \( \text{Cl}(X) \) for the set of cliques of \( X \).

We say that a NUCS \( X \) satisfies the Boudes’ Condition\(^1\) (or simply that \( X \) is Boudes) if

\[
\forall a, a' \in |X| \quad a \equiv_X a' \Rightarrow a = a'.
\]

\(^1\)From Pierre Boudes who discovered this condition and the nice properties of these objects.
We shall show that the class of NUCS’s can be turned into a categorical model of LL in such a way that all the operations on objects coincide with the corresponding operations on objects in Rel. For instance, we shall define |X| in such a way that |X| = |X| = Mfin(|X|). Moreover, all the “structure morphisms” of this model will be defined exactly as in Rel. For instance, the digging morphism from |X| to !!X will simply be dig|X|. Important: such definitions are impossible with ordinary coherence spaces. When defining |E| in ordinary coherence spaces one needs to restrict to the finite multisets (or finite sets) of elements of |E| which are cliques of E. It is exactly for that reason that, in NUCS’s, the relation ≡X is not required to coincide with equality. Nevertheless, the weaker Boudes’ condition will be preserved by all of our constructions.

2.1) Check that a NUCS can be specified by |X| together with any of the following seven pairs of relations.

- Two symmetric relations ◦X and ≃X on |X| such that ◦X ⊆ X. Then setting ◦X = (|X| × |X|) \ ≃X, the relation ◦X is the one canonically associated with the NUCS (|X|, ◦X, ≃X).
- Two symmetric relations ≃X and ◦X on |X| such that ◦X ⊆ ≃X. How should we define ◦X in that case?
- Two symmetric relations ◦X and ≡X on |X| such that ≡X ⊆ ◦X. How should we define ◦X in that case?
- Two symmetric relations ≃X and ≡X on |X| such that ≡X ⊆ ≃X. How should we define ◦X and ≃X in that case?
- Two symmetric relations ◦X and ≃X on |X| such that ≡X ⊆ ◦X. How should we define ◦X and ≃X in that case?
- Two symmetric relations ◦X and ≃X on |X| such that ≡X ⊆ ≃X. How should we define ◦X and ≃X in that case?
- Two symmetric relations ◦X and ≃X such that ◦X ∪ ≃X = |X| × |X|. How should we define ◦X and ≃X in that case?

2.2) Given NUCS’s X and Y, we define a NUCS X → Y by |X → Y| = |X| × |Y| and

- (a, b) ≡X→Y (a', b') if a ≡X a' and b ≡Y b'
- and (a, b) ◦X→Y (a', b') if a ◦X a' or b ◦Y b'.

Check that we have defined in that way a NUCS. Prove that Id|X| = {(a, a) | a ∈ |X|} ∈ Cl(X → X). Prove that if X and Y are Boudes then X → Y is Boudes.

2.3) Prove that, if s ∈ Cl(X → Y) and t ∈ Cl(Y → Z) then ts ∈ Cl(X → Z). So we define a category Nucs by taking the NUCS’s as object and by setting Nucs(X, Y) = Cl(X → Y).

2.4) We define X ⊗ by |X ⊗| = |X|, ◦X ⊗ = ◦X and ◦X ⊗ = ◦X. Then we set X ⊗ Y = (X → Y ⊗)⊥. Define as simply as possible the NUCS structure of X ⊗ Y. We set 1 = ({{}, 0, 0}) (in other words * = 1 *).

Prove that if X and Y are Boudes then X ⊗ and X ⊗ Y is Boudes.

2.5) Given s, ∈ Nucs(Xi, Yi) for i = 1, 2, prove that s ⊗ s2 ∈ Rel(|X1| ⊗ |X2|, |Y1| ⊗ |Y2|) (defined as in Rel) does actually belong to Nucs(X1 ⊗ X2, Y1 ⊗ Y2).

2.6) Check quickly that Nucs (equipped with the ⊗ defined above and 1 as tensor unit, and ⊥ = 1 as dualizing object) is a *-autonomous category.

2.7) Prove that the category Nucs is cartesian and cocartesian, with X = ⊔j∈I Xj given by |X| = ⊔j∈I |Xj|, and

- (i, a) ≡X (i', a') if i = i' and a ≡X a'
- (i, a) ◦X (i', a') if i = i' and a ◦X a'.

and the associated operations (projections, tupling of morphisms) defined as in Rel.

Prove that if all Xj’s are Boudes then ⊔j∈I Xj is Boudes.
2.8) We define $!X$ as follows. We take $!X = M_{\text{fin}}([X])$ and, given $m, m' \in !X$:
- we have $m \bowtie_X m'$ if for all $a \in \text{supp}(m)$ and $a' \in \text{supp}(m')$ one has $a \bowtie_X a'$
- and $m \equiv_X m'$ if $m \bowtie_X m'$ and $m = [a_1, \ldots, a_k]$, $m' = [a'_1, \ldots, a'_k]$ with $a_i \equiv_X a'_i$ for each $i \in \{1, \ldots, k\}$.

Notice that $m \bowtie_X m'$ iff there is $a \in \text{supp}(m)$ and $a' \in \text{supp}(m')$ such that $a \bowtie_X a'$. Remember that $\text{supp}(m) = \{a \in [X] | m(a) \neq 0\}$.

Let $s \in \text{Nucs}(X,Y)$. Prove that $!s \in \text{Rel}(!X, !Y)$ actually belongs to $\text{Nucs}(!X, !Y)$.

2.9) Prove that $\text{der}(!X) = \{([a], a) \in [X] \} \text{ belongs to } \text{Nucs}(!X, X)$.

2.10) Prove that $\text{dig}_X = \{(m_1 + \cdots + m_k, [m_1, \ldots, m_k]) \mid m_1, \ldots, m_k \in M_{\text{fin}}([X])\}$ is an element of $\text{Nucs}(!X, !!X)$.

2.11) Prove that if $X$ is Boudes then $!X$ is Boudes.

2.12) Let $X = 1 \oplus 1$, and let $t, f$ be the two elements of $[X]$ ($X$ is the “type of booleans”). Let $s \in \text{Rel}([X] \otimes [X], [X])$ by $s = \{(t,f), t, (f,t), f\}$. Prove that $s \in \text{Nucs}(X \otimes X, X)$. Let then $t \in \text{Nucs}(!X, X)$ be defined by the following composition of morphisms in $\text{Nucs}$:

$$!X \xrightarrow{c_X} !X \otimes !X \xrightarrow{\text{der}_X \otimes \text{der}_X} X \otimes X \xrightarrow{s} X$$

We recall that contraction $c_X \in \text{Nucs}(!X, !X \otimes !X)$ is given by $c_X = \{m_1 + m_2, (m_1, m_2) \mid m_1, m_2 \in !X\}$ and dereliction $\text{der}_X \in \text{Nucs}(!X, X)$ is given by $\text{der}_X = \{([a], a) \mid a \in [X]\}$.

Prove that $([t,f], t), ([t,f], f) \in t$. So any notion of coherence on $![X]$ must satisfy $[t,f] \bowtie_X [t,f]$ since we have $t \bowtie_X f$ by the definition of the NUCS $1 \oplus 1$ since we must have $([t,f], t) \bowtie_{!X-\bowtie} ([t,f], f)$ because $t$ is a clique. In particular it is impossible to endow $![X]$ with a notion of Girard’s coherence space since in such a coherence space we would have $[t,f] \bowtie_X [t,f]$ and hence $([t,f], t) \bowtie_{!X-\bowtie} ([t,f], f)$. 
