MPRI 2–2 TD 1

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1) The goal of this exercise is to study the properties of the objects of the Eilenberg Moore category $\mathbf{Rel}^!$ of $\mathbf{Rel}$, the relational model of LL.

Let P be an object of $\mathbf{Rel}^!$ (the category of coalgebras of $!\_$). Remember that $P = (\underline{P}, \mathsf{h}_P)$ where $\underline{P}$ is an object of $\mathbf{Rel}$ (a set) and $\mathsf{h}_P \in \mathbf{Rel}(\underline{P}, !\underline{P})$ satisfies the following commutations:

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\mathsf{h}_P} & !\underline{P} \\ & \underline{P}\searrow & \downarrow \mathsf{der}_{\underline{P}} \\ & & \underline{P} \end{array} \qquad\qquad \begin{array}{ccc} \underline{P} & \xrightarrow{\mathsf{h}_P} & !\underline{P} \\ \mathsf{h}_P \downarrow & & \downarrow \mathsf{dig}_{\underline{P}} \\ !\underline{P} & \xrightarrow{!\mathsf{h}_P} & !!\underline{P} \end{array}$$

1.1) Check that these commutations mean:

- for all $a, a' \in \underline{P}$, one has $(a, [a']) \in \mathsf{h}_P$ iff $a = a'$
- and for all $a \in \underline{P}$ and $m_1, \dots, m_k \in !\underline{P}$, one has $(a, m_1 + \cdots + m_k) \in \mathsf{h}_P$ iff there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, m_i) \in \mathsf{h}_P$ for $i = 1, \dots, k$.

Intuitively, $(a, [a_1, \dots, a_k])$ means that a can be decomposed into "$a_1 + \cdots + a_k$" where the "$+$" is the decomposition operation associated with P.

1.2) Prove that if P is an object of $\mathbf{Rel}^!$ such that $\underline{P} \neq \emptyset$ then there is at least one element e of $\underline{P}$ such that $(e, [\,]) \in \mathsf{h}_P$. Explain why such an e could be called a "coneutral element of P".

If P and Q are objects of $\mathbf{Rel}^!$, remember that an $f \in \mathbf{Rel}^!(P, Q)$ (morphism of coalgebras) is an $f \in \mathbf{Rel}(\underline{P}, \underline{Q})$ such that the following diagram commutes

$$\begin{array}{ccc} \underline{P} & \xrightarrow{f} & \underline{Q} \\ \mathsf{h}_P \downarrow & & \downarrow \mathsf{h}_Q \\ !\underline{P} & \xrightarrow{!f} & !\underline{Q} \end{array}$$

1.3) Check that this commutation means that for all $a \in \underline{P}$ and $b_1, \dots, b_k \in \underline{Q}$, the two following properties are equivalent

- there is $b \in \underline{Q}$ such that $(a, b) \in f$ and $(b, [b_1, \dots, b_k]) \in \mathsf{h}_Q$
- there are $a_1, \dots, a_k \in \underline{P}$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$ and $(a_i, b_i) \in f$ for $i = 1, \dots, k$.

1.4) Remember that 1 (the set $\{*\}$) can be equipped with a structure of coalgebra (still denoted 1) with $\mathsf{h}_1 = \{(*, k[*]) \mid k \in \mathbb{N}\}$. Prove that the elements of $\mathbf{Rel}^!(1, P)$ can be identified with the subsets x of $\underline{P}$ such that: for all $a_1, \dots, a_k \in \underline{P}$, one has $a_1, \dots, a_k \in x$ iff there exists $a \in x$ such that $(a, [a_1, \dots, a_k]) \in \mathsf{h}_P$. We call *values* of P these subsets of $\underline{P}$ and denote as $\mathsf{val}(P)$ the set of these values.

Prove that an element of $\mathsf{val}(P)$ is never empty and that $\mathsf{val}(P)$, equipped with inclusion, is a complete partially ordered set (cpo), that is: the union of a set of values which is directed (with respect to $\subseteq$) is still a value.

1.5) Remember that if E is an object of $\mathbf{Rel}$ then (E, dig_E) is an object of $\mathbf{Rel}^!$ (the free coalgebra generated by E, that we can identify with an object of the Kleisli category $\mathbf{Rel}_!$). Prove that, as a partially ordered set, $\mathsf{val}(E, \mathsf{dig}_E)$ is isomorphic to $\mathcal{P}(E)$.

1.6) Is it always true that if $x_1, x_2 \in \mathsf{val}(P)$ then $x_1 \cup x_2 \in \mathsf{val}(P)$?

1.7) We have seen (without proof) that $\mathbf{Rel}^!$ is cartesian. Remember that the product of P_1 and P_2 is $P_1 \otimes P_2$, the coalgebra defined by $\underline{P_1 \otimes P_2} = \underline{P_1} \otimes \underline{P_2}$ and $\mathsf{h}_{P_1 \otimes P_2}$ is the following composition of morphisms in $\mathbf{Rel}$:

$$\underline{P_1} \otimes \underline{P_2} \xrightarrow{\mathsf{h}_{P_1} \otimes \mathsf{h}_{P_2}} !\underline{P_1} \otimes !\underline{P_2} \xrightarrow{\mu^2_{\underline{P_1},\underline{P_2}}} !(\underline{P_1} \otimes \underline{P_2})$$

where $\mu^2_{E_1,E_2} \in \mathbf{Rel}(!E_1 \otimes !E_2, !(E_1 \otimes E_2))$ is the lax monoidality natural transformation of $!\_$, remember that in $\mathbf{Rel}$ we have

$$\mu^2_{E_1,E_2} = \{(([a_1,\dots,a_k],[b_1,\dots,b_k]),[(a_1,b_1),\dots,(a_k,b_k)]) \mid k \in \mathbb{N} \text{ and } (a_1,b_1),\dots,(a_k,b_k) \in E_1 \times E_2\}\ .$$

Concretely, we have simply that $((a_1,a_2),[(a_1^1,a_2^1),\dots,(a_1^k,a_2^k)]) \in \mathsf{h}_{P_1 \otimes P_2}$ iff $(a_i,[a_i^1,\dots,a_i^k]) \in \mathsf{h}_{P_i}$ for $i = 1,2$.

Prove that $P_1 \otimes P_2$, equipped with suitable projections, is the cartesian product of P_1 and P_2 in $\mathbf{Rel}^!$. Prove also that 1 is the terminal object of $\mathbf{Rel}^!$. Warning: $\mathcal{L}^!$ is always cartesian when $\mathcal{L}$ is a model of LL; I'm not asking for a general proof, just for a verification that this is true in $\mathbf{Rel}^!$.

1.8) Check directly that the partially ordered sets $\mathsf{val}(P_1 \otimes P_2)$ and $\mathsf{val}(P_1) \times \mathsf{val}(P_2)$ are isomorphic.

1.9) Remember also that we have defined $P_1 \oplus P_2 = (\underline{P_1} \oplus \underline{P_2}, \mathsf{h}_{P_1 \oplus P_2})$ where $\mathsf{h}_{P_1 \oplus P_2}$ is the unique element of $\mathbf{Rel}(\underline{P_1} \oplus \underline{P_2}, !(\underline{P_1} \oplus \underline{P_2}))$ such that, for $i = 1,2$, the morphism $\mathsf{h}_{P_1 \oplus P_2}\, \overline{\pi}_i$ coincides with the following composition of morphisms in $\mathbf{Rel}$:

$$\underline{P_i} \xrightarrow{\mathsf{h}_{P_i}} !\underline{P_i} \xrightarrow{!\overline{\pi}_i} !(\underline{P_1} \oplus \underline{P_2})$$

Describe $\mathsf{h}_{P_1 \oplus P_2}$ as simply as possible and prove that, equipped with suitable injections, $P_1 \oplus P_2$ is the coproduct of P_1 and P_2 in $\mathbf{Rel}^!$.

2) The goal of this exercise is to illustrate the fact that $\mathbf{Rel}$, the relational model of LL, can be equipped with additional structures of various kinds *without modifying the interpretation of proofs and programs*. As an example we shall study the notion of *non-uniform coherence space* (NUCS). A NUCS is a triple $X = (|X|, \frown_X, \smile_X)$ where

- $|X|$ is a set (the web of X)
- and $\frown_X$ and $\smile_X$ are two symmetric relations on $|X|$ such that $\frown_X \cap \smile_X = \emptyset$. In other words, for any $a, a' \in |X|$, one never has $a \frown_X a'$ and $a \smile_X a'$.

So we can consider an ordinary coherence space (in the sense of the first part of thise series of lectures) as a NUCS X which satisfies moreover:

$$\forall a, a' \in |X| \quad (a \frown_X a' \text{ or } a \smile_X a') \Leftrightarrow a \neq a'\ .$$

It is then possible to introduce three other natural symmetric relations on the elements of $|X|$:

- $a \equiv_X a'$ if it is not true that $a \frown_X a'$ or $a \smile_X a'$.
- $a \coh_X a'$ if $a \frown_X a'$ or $a \equiv_X a'$.
- $a \asymp_X a'$ if $a \smile_X a'$ or $a \equiv_X a'$.

A *clique* of a NUCS X is a subset x of $|X|$ such that $\forall a, a' \in |X|\ a \coh_X a'$, we use $\mathsf{Cl}(X)$ for the set of cliques of X.

We say that a NUCS X satisfies the Boudes' Condition[1] (or simply that X is Boudes) if

$$\forall a, a' \in |X|\ a \equiv_X a' \Rightarrow a = a'\ .$$

[1] From Pierre Boudes who discovered this condition and the nice properties of these objects.

We shall show that the class of NUCS's can be turned into a categorical model of LL in such a way that all the operations on objects coincide with the corresponding operations on objects in **Rel**. For instance we shall define $!X$ in such a way that $|!X| = !|X| = \mathcal{M}_{\mathrm{fin}}(|X|)$. Moreover, all the "structure morphisms" of this model *will be defined exactly as in* **Rel**. For instance, the digging morphism from $!X$ to $!!X$ will simply be $\mathsf{dig}_{|X|}$. Important: such definitions are impossible with ordinary coherence spaces. When defining $|!E|$ in ordinary coherence spaces one *needs* to restrict to the finite multisets (or finite sets) of elements of $|E|$ which *are cliques of* E. It is exactly for that reason that, in *NUCS*'s, the relation $\equiv_X$ is not required to coincide with equality. Nevertheless, the weaker Boudes' condition will be preserved by all of our constructions.

2.1) Check that a NUCS can be specified by $|X|$ together with any of the following seven pairs of relations.

- Two symmetric relations $\frown_X$ and $\frown_X$ on $|X|$ such that $\frown_X \subseteq \frown_X$. Then setting $\smile_X = (|X| \times |X|) \setminus \frown_X$, the relation $\frown_X$ is the one canonically associated with the NUCS $(|X|, \frown_X, \smile_X)$.
- Two symmetric relations $\asymp_X$ and $\smile_X$ on $|X|$ such that $\smile_X \subseteq \asymp_X$. How should we define $\frown_X$ in that case?
- Two symmetric relations $\frown_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \subseteq \frown_X$. How should we define $\frown_X$ and $\smile_X$ in that case?
- Two symmetric relations $\asymp_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \subseteq \asymp_X$. How should we define $\frown_X$ and $\smile_X$ in that case?
- Two symmetric relations $\frown_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \cap \frown_X = \emptyset$. How should we define $\smile_X$ in that case?
- Two symmetric relations $\smile_X$ and $\equiv_X$ on $|X|$ such that $\equiv_X \cap \smile_X = \emptyset$. How should we define $\frown_X$ in that case?
- Two symmetric relation $\frown_X$ and $\asymp_X$ such that $\frown_X \cup \asymp_X = |X| \times |X|$. How should we define $\frown_X$ and $\smile_X$ in that case?

2.2) Given *NUCS*'s X and Y, we define a NUCS $X \multimap Y$ by $|X \multimap Y| = |X| \times |Y|$ and

- $(a, b) \equiv_{X \multimap Y} (a', b')$ if $a \equiv_X a'$ and $b \equiv_Y b'$
- and $(a, b) \frown_{X \multimap Y} (a', b')$ if $a \smile_X a'$ or $b \frown_Y b'$.

Check that we have defined in that way a NUCS. Prove that $\mathsf{Id}_{|X|} = \{(a, a) \mid a \in |X|\} \in \mathsf{Cl}(X \multimap X)$. Prove that if X and Y are Boudes then $X \multimap Y$ is Boudes.

2.3) Prove that, if $s \in \mathsf{Cl}(X \multimap Y)$ and $t \in \mathsf{Cl}(Y \multimap Z)$ then $t\,s \in \mathsf{Cl}(X \multimap Z)$. So we define a category **Nucs** by taking the NUCS's as object and by setting $\mathbf{Nucs}(X, Y) = \mathsf{Cl}(X \multimap Y)$.

2.4) We define $X^\perp$ by $|X^\perp| = |X|$, $\frown_{X^\perp} = \smile_X$ and $\smile_{X^\perp} = \frown_X$. Then we set $X \otimes Y = \left(X \multimap Y^\perp\right)^\perp$. Define as simply as possible the NUCS structure of $X \otimes Y$. We set $1 = (\{*\}, \emptyset, \emptyset)$ (in other words $* \equiv_1 *$). Prove that if X and Y are Boudes then $X^\perp$ and $X \otimes Y$ is Boudes.

2.5) Given $s_i \in \mathbf{Nucs}(X_i, Y_i)$ for $i = 1, 2$, prove that $s_1 \otimes s_2 \in \mathbf{Rel}(|X_1| \otimes |X_2|, |Y_1| \otimes |Y_2|)$ (defined as in **Rel**) does actually belong to $\mathbf{Nucs}(X_1 \otimes X_2, Y_1 \otimes Y_2)$.

2.6) Check quickly that **Nucs** (equipped with the $\otimes$ defined above and 1 as tensor unit, and $\perp = 1$ as dualizing object) is a $*$-autonomous category.

2.7) Prove that the category **Nucs** is cartesian and cocartesian, with $X = \&_{i \in I} X_i$ given by $|X| = \bigcup_{i \in I} \{i\} \times |X_i|$, and

- $(i, a) \equiv_X (i', a')$ if $i = i'$ and $a \equiv_{X_i} a'$
- $(i, a) \smile_X (i', a')$ if $i = i'$ and $a \smile_{X_i} a'$.

and the associated operations (projections, tupling of morphisms) defined as in **Rel**.

Prove that if all X_i's are Boudes then $\&_{i \in I} X_i$ is Boudes.

2.8) We define $!X$ as follows. We take $|!X| = \mathcal{M}_{\text{fin}}(|X|)$ and, given $m, m' \in |!X|$

- we have $m \frown_{!X} m'$ if for all $a \in \mathsf{supp}(m)$ and $a' \in \mathsf{supp}(m')$ one has $a \frown_X a'$
- and $m \equiv_{!X} m'$ if $m \frown_{!X} m'$ and $m = [a_1, \dots, a_k]$, $m' = [a'_1, \dots, a'_k]$ with $a_i \equiv_X a'_i$ for each $i \in \{1, \dots, k\}$.

Notice that $m \smile_{!X} m'$ iff there is $a \in \mathsf{supp}(m)$ and $a' \in \mathsf{supp}(m')$ such that $a \smile_X a'$. Remember that $\mathsf{supp}(m) = \{a \in |X| \mid m(a) \neq 0\}$.

Let $s \in \mathbf{Nucs}(X, Y)$. Prove that $!s \in \mathbf{Rel}(!|X|, !|Y|)$ actually belongs to $\mathbf{Nucs}(!X, !Y)$.

2.9) Prove that $\mathsf{der}_{|X|} = \{([a], a) \mid a \in |X|\}$ belongs to $\mathbf{Nucs}(!X, X)$.

2.10) Prove that $\mathsf{dig}_X = \{(m_1 + \cdots + m_k, [m_1, \dots, m_k]) \mid m_1, \dots, m_k \in \mathcal{M}_{\text{fin}}(|X|)\}$ is an element of $\mathbf{Nucs}(!X, !!X)$.

2.11) Prove that if X is Boudes then $!X$ is Boudes.

2.12) Let $X = 1 \oplus 1$, and let $\mathbf{t}, \mathbf{f}$ be the two elements of $|X|$ (X is the "type of booleans"). Let $s \in \mathbf{Rel}(|X| \otimes |X|, |X|)$ by $s = \{((\mathbf{t}, \mathbf{f}), \mathbf{t}), ((\mathbf{f}, \mathbf{t}), \mathbf{f})\}$. Prove that $s \in \mathbf{Nucs}(X \otimes X, X)$. Let then $t \in \mathbf{Nucs}(!X, X)$ be defined by the following composition of morphisms in $\mathbf{Nucs}$:

$$!X \xrightarrow{\mathsf{c}_X} !X \otimes !X \xrightarrow{\mathsf{der}_X \otimes \mathsf{der}_X} X \otimes X \xrightarrow{s} X$$

We recall that contraction $\mathsf{c}_X \in \mathbf{Nucs}(!X, !X \otimes !X)$ is given by $\mathsf{c}_X = \{m_1 + m_2, (m_1, m_2) \mid m_1, m_2 \in !|X|\}$ and dereliction $\mathsf{der}_X \in \mathbf{Nucs}(!X, X)$ is given by $\mathsf{der}_X = \{([a], a) \mid a \in |X|\}$.

Prove that $([\mathbf{t}, \mathbf{f}], \mathbf{t}), ([\mathbf{t}, \mathbf{f}], \mathbf{f}) \in t$. So any notion of coherence on $!|X|$ *must* satisfy $[\mathbf{t}, \mathbf{f}] \smile_{!X} [\mathbf{t}, \mathbf{f}]$ since we have $\mathbf{t} \smile_X \mathbf{f}$ by the definition of the NUCS $1 \oplus 1$ since we *must have* $([\mathbf{t}, \mathbf{f}], \mathbf{t}) \frown_{!X \multimap X} ([\mathbf{t}, \mathbf{f}], \mathbf{f})$ because t is a clique. In particular it is impossible to endow $!|X|$ with a notion of Girard's coherence space since in such a coherence space we would have $[\mathbf{t}, \mathbf{f}] \frown_{!X} [\mathbf{t}, \mathbf{f}]$ and hence $([\mathbf{t}, \mathbf{f}], \mathbf{t}) \smile_{!X \multimap X} ([\mathbf{t}, \mathbf{f}], \mathbf{f})$.