# CCS for Trees

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**Abstract.** CCS can be considered as a most natural extension of finite state automata in which *interaction* is made possible thanks to parallel composition. We propose here a similar extension for top-down tree automata. We introduce a parallel composition which is parameterized by a graph at the vertices of which subprocesses are located. Communication is allowed only between subprocesses related by an edge in this graph. We define an observational equivalence based on barbs as well as weak bisimilarity equivalence and prove an adequacy theorem relating these two notions.

## Introduction

There is no need to insist on the importance of tree automata [CDG$^+$07] in modern theoretical and applied computer science: they are pervasive in logic, verification, rewriting, structured documents handling, constraint solving etc. Tree automata are similar to usual finite word automata with the difference that they recognize trees instead of words (sequences of letters). Let $\Sigma$ be a ranked signature ($\Sigma_n$ is the set of function symbols of arity $n$). A $\Sigma$-tree is just a term written with the signature $\Sigma$. A *top-down tree automaton* has a finite number of *states* and transitions labeled by elements of $\Sigma$: a transition labeled by $f \in \Sigma_n$ has a *source* and a sequence of $n$ *targets* which all are states of the automaton. A word automaton can be seen as a tree automaton over a signature $\Sigma$ such that $\Sigma_n$ is empty for all $n > 1$ and $\Sigma_0$ has a unique distinguished element $*$.

The definition of tree recognition by a top-down tree automaton $A$ is quite simple: a tree $f(t_1, \dots, t_n)$ is recognized by $A$ at state $X$ means that $A$ has an $f$-labeled transition whose source is $X$ and target is $(X_1, \dots, X_n)$ and $t_i$ is recognized by $A$ at state $X_i$ for each $i = 1, \dots, n$. There is also a notion of bottom-up tree automata, that we do not consider in this work; these two notions are equivalent in terms of the recognized languages, as long as one considers *non-deterministic* automata.

Automata feature a *dualist* vision of computation with an essential dichotomy between programs (automata) and data (words, trees), very much in the spirit of Turing machines (based on the machine/tape dichotomy). The process algebra CCS, introduced in the early 1980's by Milner [Mil80], encompasses this restriction, extending finite automata with interactive capabilities. In this framework, finite automata (labeled with letters $a, b, \dots$) can typically interact with other automata (labeled with dual letters $\overline{a}, \overline{b}, \dots$), as soon as they are combined through a new binary operation: *parallel composition*. But much more general interaction scenarii are of course possible in CCS. This fundamental invention led to very fruitful new lines of research in the theory of concurrent processes and to the introduction of new process algebra, among which the $\pi$-calculus [MPW92] is not the less remarkable, with many spectacular applications to cryptography, bioinformatics etc.

In this paper, we propose a similar "interactive closure" of tree automata, a new version of CCS which extends tree automata just as ordinary CCS extends word automata.

The natural idea is of course to add a parallel composition operation on processes, but this requires some care. Indeed when a prefixed process $f \cdot (P_1, \dots, P_n)$ — after a prefix $f \in \Sigma_n$, it is natural to have $n$ subprocesses, and not only one, as explained in [CQJ08] — interacts with a dually prefixed one $\overline{f} \cdot (Q_1, \dots, Q_n)$, we should remove the prefixes (just as in CCS) and then authorize interaction between the subprocess $P_i$ with all processes which could communicate with its father $f \cdot (P_1, \dots, P_n)$ as well as with $Q_i$, *but not with the $Q_j$'s for $j \neq i$*; neither should the $P_i$'s be allowed to communicate with each other in the resulting process. The same should hold of course for the $Q_i$'s.

One major motivation for this choice of design is that top-down tree recognition of tree automata should be implementable in our new CCS for trees, just as usual word recognition of automata is implementable in ordinary CCS. But for this purpose we have to preserve carefully the distinction between the various sons of tree nodes, thus preventing sons which are not at similar positions to interact. Indeed, with this definition, we are able to prove the interactive recognition Theorem 1.

This led us to the idea that general parallel composition should be a *graph*, at the vertices of which subprocesses (which are guarded sums) should be located; the edges of this graph specify which interactions are allowed. In Section 1, we introduce the syntax of this new process calculus CCTS, restricting ourselves to a fragment where all sums are guarded; indeed, the corresponding fragment of CCS is known to be sensible and well behaved.

In Section 2, we introduce an operational semantics for CCTS by defining a single rewriting rule. This rule generalizes the $a/\overline{a}$ reduction of CCS to the case where $a$ can be an $n$-ary function symbol and implements the idea of restricted communication capabilities explained above.

In order to define an operational equivalence on processes, we adapt the concept of *weak barbed congruence* [MS92,SW01] which is a natural way of saying that two processes behave in the same way, in all possible contexts. As usual, this notion is quite difficult to handle and we introduce therefore a notion of weak bisimilarity in Section 3 and prove that two weakly bisimilar processes are weakly barbed congruent in Section 4. For this, we define a labeled transition system on processes, and the definition of its transitions involves crucially the locations (graph vertices). The notion of bisimulation itself has to take these locations carefully into account.

In Section 2, we also argue that our version of CCS is a conservative extension of both tree automata and ordinary CCS: by this we mean that it admits restrictions which coincide with these two formalisms. Moreover, we show that tree recognition can be expressed simply in terms of interaction, using only the rewriting semantics. Though quite simple, this result uses in an essential way the restricted communication capabilities of CCTS.

These results suggest that CCTS is a sound and interesting extension of CCS. The most novel feature is that subprocesses are located at the vertices of a graph whose edges indicate which communications are possible, and the topology of this graph evolves during reduction. When no edge relates two processes, they can evolve independently, in a truly concurrent way, whereas the presence of an edge means that the corresponding processes will possibly synchronize in the future. Another interesting property of this approach is the importance of *locations* which suggests connections with the work of Castellani [Cas01], though locations are used in a different way: in this latter work, communication is possible when the involved processes are located at the same place.

This paper extends non trivially [CQJ08], where parallel composition however was not dealt with. Finding the right way of formalizing this operation and of defining the relevant notions of bisimulation have been a difficult task. Beyond the interactive closure of tree automata obtained by this new formalism, we also believe that CCTS provides a new compositional framework for the study of true concurrency. Indeed, the $n$ processes forked by an $n$-ary labeled prefix behave in a truly concurrent way, and such a truly concurrent situation cannot be obtained in ordinary CCS (concurrency is modelized by interleaving).

One of our further works will deal with possible connections between CCTS and other process algebras, and in particular with the possibility of encoding CCTS within the $\pi$-calculus.

## 1 Syntax of processes

We use letters $\boldsymbol{P}, \boldsymbol{Q}, \ldots$ to denote vectors $(P_1, \ldots, P_n)$, $(Q_1, \ldots, Q_n)$ etc. Let $\mathsf{Loc}$ be a countable set whose elements are called *locations* denoted with letters $p, q \ldots$ with or without subscripts or superscripts.

### 1.1 Graphs

Let $E$ and $F$ be disjoint sets and let $p \in E$. We set $E\,[F/p] = (E \setminus \{p\}) \cup F$. In other words, $E\,[F/p]$ is the set obtained from $E$ by substituting the element $p$ with the set $F$.

By a graph we mean a pair $G = (|G|, \frown_G)$, where $|G|$ is a finite subset of $\mathsf{Loc}$ and $\frown_G$ is a symmetric and antireflexive relation on $|G|$. Let $G$ and $H$ be graphs with $|G| \cap |H| = \emptyset$ and let $p \in |G|$. We define a graph $G\,[H/p]$ as follows:

- $|G[H/p]| = |G|[|H|/p]$
- and, given $q, r \in |G[H/p]|$, we say that $q \frown_{G[H/p]} r$ if $q \frown_G r$ or $q \frown_H r$ or $q \frown_G p$ and $r \in |H|$ or $r \frown_G p$ and $q \in |H|$.

### 1.2 Processes

We assume to be given a countable set of *processes variables* $\mathcal{V}$, denoted with letters $X, Y, \dots$ with or without subscripts or superscripts.

Let $\Sigma = (\Sigma_n)_{n \in \mathbf{N}}$ be a signature. With any symbol $f \in \Sigma_n$, we associate a *co-symbol* $\bar{f}$ distinct from all the elements of $\Sigma_n$ and we set $\bar{\Sigma}_n = \Sigma_n \cup \{\bar{f} \mid f \in \Sigma_n\}$. In that way, we define an extended signature $\bar{\Sigma} = (\bar{\Sigma}_n)_{n \in \mathbf{N}}$. For $f \in \Sigma_n$, we set $\bar{\bar{f}} = f$.

We define the set of $\mathsf{CCTS}$ processes by induction.

- If $X \in \mathcal{V}$ then $X$ is a process.
- If $X \in \mathcal{V}$ and $P$ is a process, then $\mu X \cdot P$ is a process in which $X$ is bound.
- If $f \in \bar{\Sigma}_n$ and $P_1, \dots, P_n$ are processes, then $f \cdot (P_1, \dots, P_n)$ is a process.
- If $G$ is a finite $\mathsf{Loc}$-graph (that is $|G| \subseteq \mathsf{Loc}$ is finite) and $\Phi$ is a function from $|G|$ to processes, then $G\langle\Phi\rangle$ is a process, to be understood as the parallel composition of the processes $\Phi(p)$ for $p \in |G|$, with communication capabilities specified by $G$. The processes $\Phi(p)$ are called the *components* of $G\langle\Phi\rangle$.
- $0$ is a process and if $P$ and $Q$ are processes, then $P + Q$ is a process.
- If $P$ is a process and $I$ is a finite subset of $\Sigma$, then $P \setminus I$ is a process.

The notion of free and bound variable does not deserve further comments, $\mu$ being of course a binder.

### 1.3 $\alpha$-conversions of locations.

Two processes $P$ and $P'$ such that there exists a bijection $\varphi : |P| \to |P'|$ which is a graph isomorphism (that is $p \frown_P q \Leftrightarrow \varphi(p) \frown_{P'} \varphi(q)$) and $P'(\varphi(p)) = P(p)$ for all $p \in |P|$ are said to be *externally $\alpha$-equivalent*. General $\alpha$-equivalence is defined by extending this relation to sub-processses in the obvious way.

When we consider several processes $P_1, \dots, P_n$ at the same time, we always assume that the webs $|P_1|, \dots, |P_n|$ are pairwise disjoint.

### 1.4 Substitution.

If $R$ and $P$ are processes and $X \in \mathcal{V}$, then the process $R[P/X]$ is defined in the obvious way, substituting each occurrence of $X$ in $R$ with $P$. Of course, one has as usual to perform $\alpha$-conversion when needed during this process.

### 1.5 Canonical processes

We define now the notion of *canonical process*: it is a process where all sums are guarded. More precisely, we define by mutual induction three classes of objects:

– canonical processes,
– *canonical guarded sum*
– and *recursive canonical guarded sum.*

These are particular processes on which we'll focuss our attention in the sequel.

– If $X \in \mathcal{V}$ then $X$ is a canonical process.
– If $G$ is a finite $\mathsf{Loc}$-graph and $\Phi$ is a function from $|G|$ to recursive canonical guarded sums, then $G\langle\Phi\rangle$ is a canonical process.
– If $P$ is a canonical process and $I$ is a finite subset of $\Sigma$, then $P \setminus I$ is a canonical process.
– A canonical guarded sum is either 0 or a process of the shape $f\cdot(P_1, \ldots, P_n)+S$ where $f \in \bar{\Sigma}_n$, $S$ is a canonical guarded sum and $P_1, \ldots, P_n$ are canonical processes.
– A recursive canonical guarded sum is either a canonical guarded sum or a process of shape $\mu X \cdot S$ where $S$ is a recursive canonical guarded sum.

For instance, the processes $G\langle\Phi\rangle + H\langle\Psi\rangle$ and $\mu X \cdot X$ are not canonical.

**Lemma 1.** *Let $R$ and $P$ be canonical processes. Then $R\,[P/X]$ is a canonical process. If $R$ is a recursive canonical guarded sum, then so is $R\,[P/X]$. If $R$ is a canonical guarded sum, then so is $R\,[P/X]$.*

*Proof.* Easy induction on $R$. □

With any *recursive canonical guarded sum $S$*, we associate a *canonical guarded sum* $\mathsf{cs}(S)$ as follows:

$$\mathsf{cs}(S) = \begin{cases} S & \text{if } S \text{ is a canonical guarded sum} \\ \mathsf{cs}(T\,[S/X]) & \text{if } S = \mu X \cdot T\,. \end{cases}$$

Using Lemma 1, one sees easily that this function is well defined and total.

*All the processes we consider in this paper are canonical.* By Lemma 1, processes are closed by substitution.

We denote with $\mathsf{Proc}$ the set of all canonical processes. If $P = G\langle\Phi\rangle$ is a canonical process, we use $|P| = |G|$. Also, for $p \in |P|$, we often write $P(p)$ instead of $\Phi(p)$, and we denote as $\frown_P$ the graph relation of $G$.

The empty process (the only $P$ such that $|P| = \emptyset$) is denoted as $\varepsilon$.

### 1.6 More notations

Given two graphs $G$ and $H$ with disjoint webs, and a subset $D$ of $|G| \times |H|$ we define a graph $K = G \oplus_D H$ by $|K| = |G| \cup |H|$ and, given $p, q \in |K|$, we stipulate that $p \frown_K q$ if $p \frown_G q$ or $p \frown_H q$ or $(p, q) \in D$ or $(q, p) \in D$. If $D = \emptyset$ then we set $G \oplus H = G \oplus_D H$.

Given processes $P = G\langle \Phi \rangle$ and $Q = H\langle \Psi \rangle$ and a relation $D \subseteq |P| \times |Q|$, one defines the process $P \oplus_D Q$ as $(G \oplus_D H)\langle \Phi \cup \Psi \rangle$. When $D$ is empty we simply denote this sum as $P \oplus Q$, and more generally, we denote as $\oplus \boldsymbol{P}$ the sum $P_1 \oplus \cdots \oplus P_n$ of the processes $\boldsymbol{P} = (P_1, \dots, P_n)$ (remember that we implicitly assume that the sets $|P_i|$ are pairwise disjoint). When $D = |P| \times |Q|$, the process $P \oplus_D Q$ will be denoted as $P \mid Q$ and called the *full parallel composition* of $P$ and $Q$. It corresponds to the standard parallel composition of process algebras, where all processes can freely interact with each other.

With the same notations as above, if $p \in |G|$, we denote as $P\,[Q/p]$ the process $G\,[H/p]\,\langle \Phi' \rangle$ where $\Phi'(p') = \Phi(p')$ if $p' \notin |H|$ and $\Phi'(p') = \Psi(p')$ if $p' \in |H|$.

## 2 Operational semantics

### 2.1 Internal reduction

Let $P$ and $P'$ be processes. We say that $P$ *reduces* to $P'$ if there are $p, q \in |P|$ such that $p \frown_P q$, $\mathsf{cs}(P(p)) = f \cdot (P_1, \dots, P_n) + S$, $\mathsf{cs}(P(q)) = \overline{f} \cdot (Q_1, \dots, Q_n) + T$ and $P'$ is defined as follows[3]: $|P'| = (|P| \setminus \{p, q\}) \cup \bigcup_{i=1}^n |P_i| \cup \bigcup_{i=1}^n |Q_i|$ and $\frown_{P'}$ is the least symmetric relation on $|P'|$ such that, for any, $p', q' \in |P'|$, one has $p' \frown_{P'} q'$ in one of the following cases:

1. $p' \frown_{P_i} q'$ or $p' \frown_{Q_i} q'$ for some $i = 1, \dots, n$
2. $p' \in |P_i|$ and $q' \in |Q_i|$ for some $i = 1, \dots, n$ (*the same $i$ for both*)
3. $\{p', q'\} \not\subseteq \bigcup_{i=1}^n |P_i| \cup \bigcup_{i=1}^n |Q_i|$ and $\lambda_1(p') \frown_P \lambda_1(q')$

where $\lambda_1 : |P'| \to |P|$ is the *residual function* defined by

$$\lambda_1(p') = \begin{cases} p & \text{if } p' \in \bigcup_{i=1}^n |P_i| \\ q & \text{if } p' \in \bigcup_{i=1}^n |Q_i| \\ p' & \text{otherwise.} \end{cases}$$

Observe that $\lambda_1$ is not a surjection when $n = 0$.

We finish the definition of $P'$ by saying that $P'(p') = P_i(p')$ if $p' \in |P_i|$, $P'(p') = Q_i(p')$ if $p' \in |Q_i|$ (for $i = 1, \dots, n$) and $P'(p') = P(p')$ if $p' \notin \bigcup_{i=1}^n |P_i| \cup \bigcup_{i=1}^n |Q_i|$.

This crucial definition clearly deserves some explainations. The process $P$ to be reduced has two subprocesses located at $p$ and $q$, with dual prefixes: $f \cdot \boldsymbol{P}$ and $\overline{f} \cdot \boldsymbol{Q}$. The fact that $p$ and $q$ are connected in $P$ ($p \frown_P q$) means that these

[3] We heavily use the implicit hypothesis that, when several processes $P_1, \dots, P_n$ are considered at the same time, the sets $|P_i|$ are pairwise disjoint.

processes can interact. This interaction consists in suppressing both prefixes and in replacing the vertice $p$ of the graph $G$ of $P$ by the graph $G_1 \oplus \cdots \oplus G_n$ (where $G_i$ is the graph of $P_i$) and the vertice $q$ by the graph $H_1 \oplus \cdots \oplus H_n$ (where $H_i$ is the graph of $Q_i$) within the graph $G$ of $P$. The connection between $p$ and $q$ in $P$ is inherited by the vertices of $G_i$ and $H_i$ in $P'$, but a process located on $G_i$ (one of the components of $P_i$) cannot communicate with a process located on $H_j$ with $j \neq i$. The connections between $p$ and other vertices of $P$, distinct from $q$, are also inherited by the vertices of all $G_i$'s and similarly for the $H_i$'s.

We denote with $\to$ the internal reduction relation and with $\to^*$ its reflexive and transitive closure.

*Example 1.* Let $a \in \Sigma_0$ and $f \in \Sigma_2$. Consider the process $P = \overline{a} \mid a \mid f \cdot (a, \overline{a}) \mid \overline{f} \cdot (a, \overline{a})$ (we write simply "$a$" instead of $a \cdot ()$). In other words, the graph of $P$ is a complete graph with 4 vertices, say $1, 2, 3, 4$, and we have $P(1) = a$, $P(2) = \overline{a}$, $P(3) = f \cdot (a, \overline{a})$ and $P(4) = \overline{f} \cdot (a, \overline{a})$. Since 3 and 4 are connected in that graph and the corresponding prefixes $f$ and $\overline{f}$ are dual, we can reduce $P$ to a process $P'$ such that $|P'| = \{1, 2, 5, 6, 7, 8\}$ (remember that we work up to $\alpha$-equivalence, so the names of locations are irrelevant) with $P'(1) = a$, $P'(2) = \overline{a}$, $P'(5) = a$, $P'(6) = \overline{a}$, $P'(7) = a$, and $P'(8) = \overline{a}$, and the edges of $P'$ are all $\{i, j\}$ with $i \in \{1, 2\}$ and $j \neq i$, $\{5, 7\}$ and $\{6, 8\}$. So, in $P'$, the interaction of $a$ located at 5 with $\overline{a}$ located at 8 is not possible, but of course $a$ located at 5 can interact with $\overline{a}$ located at 2. Performing that reduction, we get $P''$ with $|P''| = \{1, 6, 7, 8\}$ and the edges of $P''$ are all $\{1, j\}$ with $j \neq 1$ and $\{6, 8\}$, with $P''(1) = a$, $P''(6) = \overline{a}$, $P''(7) = a$ and $P''(8) = \overline{a}$. In $P''$, the only possible reductions are between $a$ located at 1 and $\overline{a}$ located at 6 or 8. Both lead to the process $a \oplus \overline{a}$ where no reduction is possible.

### 2.2 Top-down tree automata as a particular case

A *top-down tree automaton* is a pair $A = (\mathcal{Q}, \mathcal{T})$ where $\mathcal{Q}$ is a finite subset of $\mathcal{V}$, whose elements are called *states*, and $\mathcal{T}$ is a finite set of triples $(X, f, (X_1, \ldots, X_n))$ where $f \in \Sigma_n$ and $X_1, \ldots, X_n \in \mathcal{Q}$ and whose elements are called *transitions*. The *language recognized by $A$ at state $X \in \mathcal{Q}$*, denoted as $\mathsf{L}(A, X)$, is the least set of $\Sigma$-trees such that $f(t_1, \ldots, t_n) \in \mathsf{L}(A, X)$ as soon as there are $X_1, \ldots, X_n \in \mathcal{Q}$ such that $(X, f, (X_1, \ldots, X_n)) \in \mathcal{T}$ and $t_i \in \mathsf{L}(A, X_i)$ for $i = 1, \ldots, n$.

We associate a process $\langle A \rangle_X$ with any pair $(A, X)$ where $A = (\mathcal{Q}, \mathcal{T})$ is a tree automaton and $X \in \mathcal{Q}$. More generally we define $\langle A \rangle_X^{\mathcal{X}}$ where $\mathcal{X}$ is a finite subset of $\mathcal{V}$ (intuitively, $\mathcal{X}$ is the set of already defined processes), and then we set $\langle A \rangle_X = \langle A \rangle_X^{\emptyset}$.

- If $X \notin \mathcal{X}$, then $\langle A \rangle_X^{\mathcal{X}} = \mu X \cdot S$ where $S$ is the sum of all prefixed processes $f \cdot (\langle A \rangle_{X_1}^{\mathcal{X} \cup \{X\}}, \ldots, \langle A \rangle_{X_n}^{\mathcal{X} \cup \{X\}})$ where $(X, f, (X_1, \ldots, X_n)) \in \mathcal{T}$,
- and if $X \in \mathcal{X}$, then $\langle A \rangle_X^{\mathcal{X}} = X$.

This inductive definition is well founded because the parameter $\mathcal{X}$ increases strictly at each inductive step, and remains included in $\mathcal{Q}$. Moreover, the invariant that all the free variables of $\langle A \rangle_X^{\mathcal{X}}$ belong to $\mathcal{X}$ is preserved by the inductive step, and hence $\langle A \rangle_X$ is closed.

**Lemma 2.** *With the notations above,* $\mathsf{cs}(\langle A\rangle_Y)$ *is the sum of all prefixed processes* $f\cdot(\langle A\rangle_{Y_1},\ldots,\langle A\rangle_{Y_n})$ *where* $(Y,f,(Y_1,\ldots,Y_n))\in\mathcal{T}$*.*

*Proof.* More generally, $\mathsf{cs}(\langle A\rangle_X^{\{X_1,\ldots,X_p\}}\left[\langle A\rangle_{X_1}/X_1,\ldots,\langle A\rangle_{X_p}/X_p\right])$ is equal to the sum above, for any subset $\{X_1,\ldots,X_p\}$ of $\mathcal{Q}$ (with the $X_i$'s pairwise distinct). The proof is a simple induction on $q-p$, where $q$ is the cardinality of $\mathcal{Q}$. □

We represent dually any $\Sigma$-tree $t=f(t_1,\ldots,t_n)$ as a process $\overline{t}$ by setting $\overline{t}=\overline{f}\cdot(\overline{t_1},\ldots,\overline{t_n})$. The following results expresses that our process algebra, together with its internal reduction, is a conservative extension of tree automata by showing that tree recognition boils down to a (very) particular case of interaction between processes.

**Theorem 1.** *Let* $A=(\mathcal{Q},\mathcal{T})$ *be a tree automaton, let* $X\in\mathcal{Q}$ *and let* $t$ *be a* $\Sigma$*-tree. Then* $t\in\mathsf{L}(A,X)$ *iff* $(\langle A\rangle_X\mid\overline{t})\to^*\varepsilon$*.*

*Proof.* This is straightforward, once observed that, if $t=f(t_1,\ldots,t_n)$ and if $(X,f,(X_1,\ldots,X_n))\in\mathcal{T}$, one has $\langle A\rangle_X\mid\overline{t}\to(\langle A\rangle_{X_1}\mid\overline{t_1})\oplus\cdots\oplus(\langle A\rangle_{X_n}\mid\overline{t_n})$, thanks to Lemma 2. Observe then that $(\langle A\rangle_{X_1}\mid\overline{t_1})\oplus\cdots\oplus(\langle A\rangle_{X_n}\mid\overline{t_n})$ reduces to $\varepsilon$ iff each process $\langle A\rangle_{X_i}\mid\overline{t_i}$ reduces to $\varepsilon$ since these processes cannot interact with each other. If $\mathcal{T}$ has no element of the shape $(X,f,(X_1,\ldots,X_n))$, then the process $\langle A\rangle_X\mid\overline{t}$ does not reduce. □

### 2.3 CCS for words as a particular case

We assume here that $\Sigma_n=\emptyset$ for all $n>1$ and that $\Sigma_0=\{*\}$. Then a $\Sigma$-tree is the same thing as a $\Sigma_1$-word, written $a_1\ldots a_p*$. We restrict our attention to processes in which all the graphs parameterizing parallel compositions are complete, so that any process is of the shape $S_1\mid\cdots\mid S_p$ where each $S_i$ is a recursive canonical guarded sum $\mu\boldsymbol{X}\cdot(a_1\cdot P_1+\cdots+a_m\cdot P_m)$: this restriction of our process algebra coincides with *guarded CCS*. Observe also that, if $P$ is a process in this restricted setting (arities $\leq 1$ and all parallel compositions are complete graphs), and if $P\to P'$, then $P'$ belongs to the same restriction and the reduction $P\to P'$ is a standard $\tau$-reduction of CCS. In that way we see that our process algebra is also a conservative extension of ordinary guarded CCS.

There is a slight, innocuous, variation in this way of representing ordinary CCS within CCTS. It consists in taking $\Sigma_n=\emptyset$ for $n\neq 1$ and $\Sigma_1$ as word alphabet. Then one can use $\varepsilon$ (the empty process) instead of the $*$ symbol of arity 0. For simplicity, it is this coding that we'll use in Section 5. The drawback of this representation is that it does not scale down to automata considered as particular processes as explained in Section 2.2.

### 2.4 Weak barbed bisimilarity

Let $f\in\bar{\Sigma}$ and let $P$ be a process. We say that $f$ is a *barb* of $P$, and write $P\downarrow_f$, if there exists $p\in|P|$ such that $\mathsf{cs}(P(p))$ is of shape $f\cdot(P_1,\ldots,P_n)+S$.

A relation $\mathcal{B} \subseteq \mathsf{Proc}^2$ is a *weak barbed bisimulation* if it is symmetric and satisfies the following conditions. For any $P, Q \in \mathsf{Proc}$ such that $P \mathcal{B} Q$,

- for any $P' \in \mathsf{Proc}$, if $P \to^* P'$, then there exists $Q' \in \mathsf{Proc}$ such that $Q \to^* Q'$ and $P' \mathcal{B} Q'$ (one says that $\mathcal{B}$ is a *weak reduction bisimulation*);
- for any $P' \in \mathsf{Proc}$ and any $f \in \bar{\Sigma}$, if $P \to^* P'$ and $P' \downarrow_f$, then there exists $Q' \in \mathsf{Proc}$ such that $Q \to^* Q'$ and $Q' \downarrow_f$ (one says that $\mathcal{B}$ is *weak barb preserving*; observe that one does not require that $P' \mathcal{B} Q'$).

The diagonal relation $\{(P, P) \mid P \in \mathsf{Proc}\}$ is a weak barbed bisimulation, and if $\mathcal{B}$ and $\mathcal{B}'$ are weak barbed bisimulations, then so are $\mathcal{B}' \circ \mathcal{B}$ and $\mathcal{B} \cup \mathcal{B}'$. We say that $P, Q \in \mathsf{Proc}$ are *weakly barbed bisimilar* if there exists a weak barbed bisimulation $\mathcal{B}$ such that $P \mathcal{B} Q$. Notation: $P \stackrel{\bullet}{\approx} Q$.

**Lemma 3.** *Weak barbed bisimilarity is an equivalence relation.*

*Proof.* Straightforward, using the above closure properties of weak barbed bisimulations. □

### 2.5 Weak barbed congruence

Let $Y$ be a variable; a *$Y$-context* is a process $R$ which contains exactly one free occurrence of $Y$, which does not occur in a subprocess of $R$ of the shape $\mu X \cdot R'$ (in other words, $Y$ must really occur only once in $R$). If $R$ and $S$ are $Y$-contexts, so is $R[S/Y]$.

A relation $\mathcal{R} \subseteq \mathsf{Proc}^2$ is a *congruence* if it is reflexive and such that, for any $Y$-context $R$, one has $P \mathcal{R} Q \Rightarrow R[P/Y] \mathcal{R} R[Q/Y]$.

**Proposition 1.** *For any reflexive relation $\mathcal{R} \subseteq \mathsf{Proc}^2$, there exists a largest congruence $\overline{\mathcal{R}}$ contained in $\mathcal{R}$. This relation is characterized by: $P \overline{\mathcal{R}} Q$ iff for any $Y$-context $R$ one has $R[P/Y] \mathcal{R} R[Q/Y]$. If $\mathcal{R}$ is an equivalence relation, so is $\overline{\mathcal{R}}$.*

*Proof.* The first statement results from the fact that congruences are closed under arbitrary unions and that $\mathcal{R}$ contains the identity relation which is a congruence. As to the second statement, let $\mathcal{E}$ be the relation defined by $P \mathcal{E} Q$ iff for any $Y$-context $R$ one has $R[P/Y] \mathcal{R} R[Q/Y]$. Then $\mathcal{E}$ is a congruence which is contained in $\mathcal{R}$ (since we can take $R = Y$) and hence $\mathcal{E} \subseteq \overline{\mathcal{R}}$. Conversely, assume that $P \overline{\mathcal{R}} Q$ and let $R$ be a $Y$-context. Since $\overline{\mathcal{R}}$ is a congruence, we have $R[P/Y] \overline{\mathcal{R}} R[Q/Y]$ and hence $R[P/Y] \mathcal{R} R[Q/Y]$ since $\overline{\mathcal{R}} \subseteq \mathcal{R}$ by definition of $\overline{\mathcal{R}}$ and hence $P \mathcal{E} Q$. The last statement results from the second one since $\mathcal{E}$ is an equivalence relation when $\mathcal{R}$ is an equivalence relation. □

The largest congruence contained in $\stackrel{\bullet}{\approx}$ is denoted as $\cong$ and is called *weak barbed congruence*: it is our main notion of operational equivalence on processes. It is an equivalence relation by the proposition above and by Lemma 3. Moreover, we have

$$P \cong Q \text{ iff for any } Y\text{-context } R, \text{ we have } R[P/Y] \stackrel{\bullet}{\approx} R[Q/Y] .$$

## 3 Localized transition systems of processes

Just as in ordinary CCS, it is very difficult to prove that two processes are weak barbed congruent, because of the universal quantification on contexts used in the definition of this equivalence relation. In order to prove weak barbed congruence of processes, one needs therefore more convenient tools.

The most canonical of these tools is *weak bisimilarity*, an equivalence relation which expresses that two processes manifest the same communication capabilities along their internal reductions. This equivalence relation is defined as the union of all *weak bisimulations*.

The main feature of weak bisimilarity is that it is a congruence: this fact is the main ingredient in the proof that two weakly bisimilar processes are weakly barbed congruent. To prove this result, one needs to associate with each weak bisimulation $\mathcal{R}$ a new weak bisimulation $\mathcal{R}'$ called its *parallel extension*. In ordinary CCS, the definition is as follows: one says that $U \mathcal{R}' V$ if $U = P \mid S$ and $V = Q \mid S$ with $P \mathcal{R} Q$ and $S$ is a process. The main step is of course to show that $\mathcal{R}'$ is a weak bisimulation.

In CCTS however, we cannot simply speak of "the parallel composition" $U$ of $P$ and $S$, we have to specify a relation $C \subseteq |P| \times |S|$, and then we can set $U = P \oplus_C S$. Similarly we have to say that $V = Q \oplus_D S$ for some relation $D \subseteq |Q| \times |S|$, and that $P \mathcal{R} Q$. Not surprisingly, we shall see that these relations $C$ and $D$ must fulfill some requirement.

Moreover our bisimulations cannot be simple relations between processes, because, when two processes $P = G\langle\Phi\rangle$ and $Q = H\langle\Psi\rangle$ are bisimilar, we have to say which subprocesse $\Phi(p)$ of $P$ should be in bisimulation with which subprocesses $\Psi(q)$ of $Q$.

For instance, if $P = f \cdot (P_1, P_2)$ and $Q = f \cdot (Q_1, Q_2)$ (with $|P| = |Q| = \{1\}$) are related by a bisimulation $\mathcal{R}$, then (after performing the action $f$ on both sides), the processes $P_1 \oplus P_2$ and $Q_1 \oplus Q_2$ (with $|P_1 \oplus P_2| = |Q_1 \oplus Q_2| = \{1, 2\}$, and $P_i$ and $Q_i$ located at $i$ for $i = 1, 2$) should be related by $\mathcal{R}$. But this cannot be achieved by saying that $P_1 \mathcal{R} Q_2$ for instance: if $P_1$ manifests some communication capability $a$, we should insist that the same capability $a$ be manifested by $Q_1$.

A convenient way to enforce this discipline is to say that a bisimulation is a set of triples $(P, E, Q)$ where $P$ and $Q$ are processes and $E \subseteq |P| \times |Q|$. In the example above, we start with $(P, \{(1, 1)\}, Q) \in \mathcal{R}$ (where 1 is the location of $f \cdot (P_1, P_2)$ in $P$ and similarly for $Q$), and then, after having performed the action $f$ on both sides, we arrive to $(P_1 \oplus Q_1, \{(1, 1), (2, 2)\}, P_2 \oplus Q_2) \in \mathcal{R}$.

Let us come back to the concept of parallel extension of a bisimulation $\mathcal{R}$. The bisimulation $\mathcal{R}$ is a set of triples $(P, E, Q)$ as explained above. We shall say that $(U, F, V) \in \mathcal{R}'$ when we can find a process $S$ and two relations $C \subseteq |P| \times |S|$ and $D \subseteq |Q| \times |S|$ with $U = P \oplus_C S$ and $V = Q \oplus_D S$. We require moreover the existence of a relation $E$ such that $(P, E, Q) \in \mathcal{R}$ and $F = E \cup \mathrm{Id}_{|S|}$ (in other words, $(u, v) \in F$ if $(u, v) \in E$, or $u = v \in |S|$), and we also require $C$ and $D$ to be "equivalent up to $E$", meaning that, when $(p, q) \in E$, we have $(p, s) \in C$ iff

$(q, s) \in D$, which seems to be the correct assumption in the proof that $\mathcal{R}'$ is a bisimulation.

Bisimulations are usually defined in terms of a *transition system*, a very general and flexible concept which is essential in the study of concurrency. Due to our more complex definition of bisimulations involving triples $(P, E, Q)$ instead of pairs $(P, Q)$, it is not clear anymore how to use transition systems in our framework; at least should we generalize them so as to take localization of subprocesses into account. An abstract notion of *localized transition system* might be of general interest, but we prefer to focus here on CCTS and to define one particular localized transition system of processes. Its states are processes. As usual in CCS-like formalisms, there are $\tau$-transitions between processes $P \xrightarrow[\rho]{\tau} P'$ corresponding to one internal reduction.

The additional information $\rho$ is a function $|P'| \to |P|$ which allows to trace the "locative history" of the reduction. Labeled transition have shape $P \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'$ where $p \in |P|$, $\boldsymbol{L} = (L_1, \ldots, L_n)$ with $L_i \subseteq |P'|$ and $\lambda_1 : |P'| \to |P|$ are again informations which allow to keep track of the locative history of the reduction. These additional informations about locations are sufficient to define an adequate notion of bisimulation.

### 3.1 Localized transitions

We define now this localized transition system[4].

Let $P$ and $P'$ be processes. We write $P \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'$ if $p \in |P|$, $\mathsf{cs}(P(p)) = f \cdot (P_1, \ldots, P_n) + S$ with $P' = P[\oplus \boldsymbol{P}/p]$, $L_1 = |P_1|, \ldots, L_n = |P_n|$ and $\lambda_1 : |P'| \to |P|$ is the residual function defined by $\lambda_1(p') = p$ if $p' \in \bigcup_{i=1}^{n} L_i$ and $\lambda_1(p') = p'$ otherwise[5].

We write $P \xrightarrow[\lambda_1]{\tau} P'$ if $P \to P'$ in the sense of 2.1 and, with the notations of that section, $\lambda_1 : |P'| \to |P|$ is the residual function defined by $\lambda_1(p') = p$ if $p' \in \bigcup_i |P_i|$, $\lambda_1(p') = q$ if $p' \in \bigcup_i |Q_i|$, and $\lambda_1(p') = p'$ otherwise.

We define the reflexive-transitive closure $\xrightarrow[\lambda]{\tau*}$ as follows. We say that $P \xrightarrow[\lambda]{\tau*} P'$ if there are $n \geq 1$, processes $P_1, \ldots, P_n$ and functions $\lambda_1, \ldots, \lambda_{n-1}$ such that $P = P_1$, $P_n = P'$ and $P_i \xrightarrow[\lambda_i]{\tau} P_{i+1}$ for $i = 1, \ldots, n-1$, and $\lambda = \lambda_1 \circ \cdots \circ \lambda_{n-1}$.

We write $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda_1,\lambda'}{\Longrightarrow}} P'$ if there are processes $P_1$ and $P_1'$ such that $P \xrightarrow[\lambda]{\tau*} P_1 \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P_1' \xrightarrow[\lambda']{\tau*} P'$.

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[4] Again, we don't try to provide a general definition of this concept; this could be the object of further work

[5] There are redundancies in these notations, for instance $\lambda_1$ is completely determined by the data $p$, $\boldsymbol{L}$. This redundancy will be useful in the sequel.

### 3.2 Localized weak bisimilarity

We introduce now our notion of weak bisimilarity which will be shown to imply weak barbed congruence of processes. The definition is coalgebraic and is based on a concept of bisimulation which, due to the importance of the graph structure in the operational semantics of CCTS, strongly uses locations.

A *localized relation* (on processes) is a set $\mathcal{R} \subseteq \mathsf{Proc} \times \mathcal{P}(\mathsf{Loc}^2) \times \mathsf{Proc}$ such that, if $(P, E, Q) \in \mathcal{R}$ then $E \subseteq |P| \times |Q|$. Such a relation $\mathcal{R}$ is *symmetric* if $(P, E, Q) \in \mathcal{R} \Rightarrow (Q, {}^{\mathrm{t}}E, P) \in \mathcal{R}$ where ${}^{\mathrm{t}}E = \{(q, p) \mid (p, q) \in E\}$.

A *(localized) weak bisimulation* is a symmetric localized relation such that

- if $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\lambda_1]{\tau} P'$ then $Q \xrightarrow[\rho]{\tau *} Q'$ with $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that, if $(p', q') \in E'$ then $(\lambda_1(p'), \rho(q')) \in E$ (this latter condition will be called *condition on residuals*)
- if $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'$ then $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ with $(p, \rho(q)) \in E$ and $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that if $(p', q') \in E'$ then $(\lambda_1(p'), \rho\rho_1\rho'(q')) \in E$, and, moreover, if $n \geq 2$, then either $(p', \rho'(q')) \in \bigcup_{i=1}^{n}(L_i \times M_i)$ or $p' \notin \bigcup_{i=1}^{n} L_i$ and $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$ (this condition is called *condition on residuals*).

This latter dichotomy, according to whether $n = 1$ or $n \geq 2$ (where $n$ is the arity of $f$) is essential in order to obtain three effects which seem impossible to conciliate otherwise:

- weak bisimilarity must be transitive
- it must imply weak barbed congruence
- and it should be an extension of the standard weak bisimilarity of CCS (considering CCS as a subsystem of CCTS as explained in Section 2.3).

**Lemma 4.** *Let $\mathcal{R}$ be a weak bisimulation. If $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\lambda]{\tau *} P'$, then $Q \xrightarrow[\rho]{\tau *} Q'$ with $(P', E', Q') \in \mathcal{R}$ for some $E' \subseteq |P'| \times |Q'|$ such that if $(p', q') \in E'$ then $(\lambda'(p'), \rho'(q')) \in E$.*

*Proof.* Simple induction on the length of the sequence of reductions $P \xrightarrow[\lambda]{\tau *} P'$. □

**Lemma 5.** *If $P \xrightarrow[\lambda]{\tau *} P_1$, $P_1 \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda_1,\lambda_2,\lambda_1'}{\Longrightarrow}} P_1'$ and $P_1' \xrightarrow[\lambda']{\tau *} P'$ then $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda\lambda_1,\lambda_2,\lambda_1'\lambda'}{\Longrightarrow}} P'$.*

*Proof.* Results immediately from the definitions. □

Now we provide a characterization of weak bisimulation which is more symmetric than the definition above of these relations.

**Lemma 6.** *A symmetric localized relation $\mathcal{R} \subseteq \mathsf{Proc} \times \mathcal{P}(\mathsf{Loc}^2) \times \mathsf{Proc}$ is a weak bisimulation iff the following properties hold.*

– *If* $(P, E, Q) \in \mathcal{R}$ *and* $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda_1,\lambda'}{\Longrightarrow}} P'$*, then* $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ *with* $(\lambda(p), \rho(q)) \in E$ *and* $(P', E', Q') \in \mathcal{R}$ *for some* $E' \subseteq |P'| \times |Q'|$ *such that if* $(p', q') \in E'$ *then* $(\lambda\lambda_1\lambda'(p'), \rho\rho_1\rho'(q')) \in E)$ *and, moreover, if* $n \geq 2$*, either* $(\lambda'(p'), \rho'(q')) \in \bigcup_{i=1}^{n}(L_i \times M_i)$ *or* $\lambda'(p') \notin \bigcup_{i=1}^{n} L_i$ *and* $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$*.*
– *If* $(P, E, Q) \in \mathcal{R}$ *and* $P \xrightarrow[\lambda]{\tau*} P'$*, then* $Q \xrightarrow[\rho]{\tau*} Q'$ *with* $(P', E', Q') \in \mathcal{R}$ *for some* $E' \subseteq |P'| \times |Q'|$ *such that if* $(p', q') \in E'$ *then* $(\lambda(p'), \rho(q')) \in E$*.*

*Proof.* The stated property are obviously sufficient, we prove that the first one is necessary (necessity of the second one is Lemma 4). Assume that $(P, E, Q) \in \mathcal{R}$ and $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda_1,\lambda'}{\Longrightarrow}} P'$, that is $P \xrightarrow[\lambda]{\tau*} P_1 \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P_1' \xrightarrow[\lambda']{\tau*} P'$. By Lemma 4 one has $Q \xrightarrow[\rho]{\tau*} Q_1$ with $(P_1, E_1, Q_1) \in \mathcal{R}$ where $E_1$ is such that $(p_1, q_1) \in E_1 \Rightarrow (\lambda(p_1), \rho(q_1)) \in E$.

Since $P_1 \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P_1'$ and $(P_1, E_1, Q_1) \in \mathcal{R}$, one has $Q_1 \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho_1,\rho_2,\rho_1'}{\Longrightarrow}} Q_1'$ with $(p, \rho_1(q)) \in E_1$ and $(P_1', E_1', Q_1') \in \mathcal{R}$ where $E_1'$ is such that if $(p_1', q_1') \in E_1'$ then $(\lambda_1(p_1'), \rho_1\rho_2\rho_1'(q_1')) \in E_1$ and, if $n \geq 2$, then either $(p_1', \rho_1'(q_1')) \in \bigcup_{i=1}^{n}(L_i \times M_i)$, or $p_1' \notin \bigcup_{i=1}^{n} L_i$ and $\rho_1'(q_1') \notin \bigcup_{i=1}^{n} M_i$. Since $P_1' \xrightarrow[\lambda']{\tau*} P'$ and $(P_1', E_1', Q_1') \in \mathcal{R}$, we can apply Lemma 4 again which shows that $Q_1' \xrightarrow[\rho']{\tau*} Q'$ with $(P', E', Q') \in \mathcal{R}$ where $E'$ is such that $(p', q') \in E' \Rightarrow (\lambda'(p'), \rho'(q')) \in E_1'$. By Lemma 5, we have $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho\rho_1,\rho_2,\rho_1'\rho'}{\Longrightarrow}} Q'$ and remember that $(P', E', Q') \in \mathcal{R}$. We have $(p, \rho_1(q)) \in E_1$ and hence $(\lambda(p), \rho\rho_1(q)) \in E$ by definition of $E_1$. Last, the condition on residuals obviously holds. □

**Lemma 7.** *Let* $\mathcal{I}$ *be the localized relation defined by:* $(P, E, Q) \in \mathcal{I}$ *if* $P = Q$ *and* $E = \mathrm{Id}_{|P|}$*. Then* $\mathcal{I}$ *is a weak bisimulation.*

*Proof.* Straightforward. □

If $\mathcal{R}$ and $\mathcal{R}'$ are weak bisimulations, so is $\mathcal{R} \cup \mathcal{R}'$: this results immediately from the definition. We say that $P$ and $Q$ are weakly bisimilar (notation $P \approx Q$) if there exists a weak bisimulation $\mathcal{R}$ and a set $E \subseteq |P| \times |Q|$ such that $(P, E, Q) \in \mathcal{R}$.

Let $\mathcal{R}$ and $\mathcal{S}$ be localized relations. We define a localized relation $\mathcal{S} \circ \mathcal{R}$ as follows: $(P, H, R) \in \mathcal{S} \circ \mathcal{R}$ if $H \subseteq |P| \times |R|$ and there exist $Q$, $E$ and $F$ such that $(P, E, Q) \in \mathcal{R}$, $(Q, F, R) \in \mathcal{S}$ and $F \circ E \subseteq H$.

**Lemma 8.** *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are weak bisimulations, then so is* $\mathcal{S} \circ \mathcal{R}$*.*

*Proof.* First, observe that $\mathcal{S} \circ \mathcal{R}$ is symmetric.

We use the characterization of weak bisimulations given by Lemma 6. Let $(P, H, R) \in \mathcal{S} \circ \mathcal{R}$. Let $Q$, $E$ and $F$ be such that $(P, E, Q) \in \mathcal{R}$, $(Q, F, R) \in \mathcal{S}$ and $F \circ E \subseteq H$.

► Assume first that $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda,\lambda_1,\lambda'}{\Longrightarrow}} P'$. Then we have $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ with $(\lambda(p), \rho(q)) \in E$ and $(P', E', Q') \in \mathcal{R}$ with $E'$ such that if $(p', q') \in E'$ then $(\lambda\lambda_1\lambda'(p'), \rho\rho_1\rho'(q')) \in E$ and, if $n \geq 2$ then $(\lambda'(p'), \rho'(q')) \in \bigcup_i (L_i \times M_i)$ or $\lambda'(p') \notin \bigcup_i L_i$ and $\rho'(q') \notin \bigcup_i M_i$. Therefore we have $R \overset{r:f\cdot(\boldsymbol{N})}{\underset{\sigma,\sigma_1,\sigma'}{\Longrightarrow}} R'$ with $(\rho(q), \sigma(r)) \in F$ and $(Q', F', R') \in \mathcal{S}$ with $F'$ such that if $(q', r') \in F'$ then $(\rho\rho_1\rho'(q'), \sigma\sigma_1\sigma'(r')) \in F)$ and, if $n \geq 2$ then $(\rho'(q'), \sigma'(r')) \in \bigcup_i (M_i \times N_i)$ or $\rho'(q') \notin \bigcup_i M_i$ and $\sigma'(r') \notin \bigcup_i N_i$. So we have $(\lambda(p), \sigma(r)) \in F \circ E \subseteq H$. Let

$$H' = \{(p', r') \in |P'| \times |R'| \mid (\lambda\lambda_1\lambda'(p'), \sigma\sigma_1\sigma'(r')) \in H \text{ and if } n \geq 2 \text{ then}$$
$$(\lambda'(p'), \sigma'(r')) \in \bigcup_{i=1}^{n} (L_i \times N_i) \text{ or } \lambda'(p') \notin \bigcup_{i=1}^{n} L_i \text{ and } \sigma'(r') \notin \bigcup_{i=1}^{n} N_i\}$$

By definition of $H'$, the triple $(P', H', R')$ satisfies the conditions on residuals, and we are left with proving that $F' \circ E' \subseteq H'$ which will show that $(P', H', R') \in \mathcal{S} \circ \mathcal{R}$. Let $(p', r') \in F' \circ E'$, there exists $q'$ such that $(p', q') \in E'$ and $(q', r') \in F'$.

We know that $(\lambda\lambda_1\lambda'(p'), \rho\rho_1\rho'(q')) \in E$ and $(\rho\rho_1\rho'(q'), \sigma\sigma_1\sigma'(r)) \in F$ and therefore $(\lambda\lambda_1\lambda'(p'), \sigma\sigma_1\sigma'(r)) \in F \circ E \subseteq H$. So assume now that $n \geq 2$. We must prove that if $\lambda'(p') \in \bigcup_{i=1n} L_i$ or $\sigma'(r') \in \bigcup_{i=1}^{n} N_i$ then $(\lambda'(p'), \sigma'(r')) \in L_i \times N_i$ for some $i$. Without loss of generality, we can assume that $\lambda'(p') \in \bigcup_{i=1n} L_i$ (because the situation is symmetric). Then by the condition on residuals for $E'$ we know that $(\lambda'(p'), \rho'(q')) \in L_j \times M_j$ for some $j \in \{1, \ldots, n\}$, because $n \geq 2$. Therefore $(\rho'(q'), \sigma'(r')) \in M_i \times N_i$ by the conditions on residuals satisfied by $F'$. It follows that $(\lambda'(p'), \sigma'(r')) \in L_i \times N_i$ as required.

► Assume now that $P \xrightarrow[\lambda]{\tau*} P'$. Since $(P, E, Q) \in \mathcal{R}$ we have $Q \xrightarrow[\rho]{\tau*} Q'$ and there exists $E'$ such that $(P', E', Q') \in \mathcal{R}$ and, if $(p', q') \in E'$, then $(\lambda(p'), \rho(q')) \in E$. Since $(Q, F, R) \in \mathcal{S}$, we have $R \xrightarrow[\sigma]{\tau*} R'$ and there exists $F'$ such that $(Q', F', R') \in \mathcal{S}$ and for any $(q', r') \in F'$, one has $(\rho(q'), \sigma(r')) \in F$. We have $(P', F' \circ E', Q') \in \mathcal{S} \circ \mathcal{R}$ and it is obvious that $F' \circ E'$ satisfies the condition on residuals. □

We say that two processes $P$ and $Q$ are weakly bisimilar, and write $P \approx Q$, if there exists a weak bisimulation $\mathcal{R}$ and a relation $E \subseteq |P| \times |Q|$ such that $(P, E, Q) \in \mathcal{R}$.

**Proposition 2.** *The relation $\approx$ is an equivalence relation on processes.*

*Proof.* Reflexivity results from Lemma 7, and symmetry from the symmetry hypothesis on weak bisimulations. Transitivity is a straightforward consequence of Lemma 8. □

**Proposition 3.** *If $P \approx Q$ then $P \stackrel{\bullet}{\approx} Q$.*

*Proof.* Let $\mathcal{R}$ be a weak bisimulation. Let $\mathcal{B}$ be the binary relation on processes defined by: $(P,Q) \in \mathcal{B}$ if there exists $E \subseteq |P| \times |Q|$ such that $(P,E,Q) \in \mathcal{R}$. We contend that $\mathcal{B}$ is a weak barbed bisimulation, and this will prove the proposition. First observe that $\mathcal{B}$ is symmetric because $\mathcal{R}$ is a symmetric localized relation.

► Let $(P,Q) \in \mathcal{B}$ and assume first that $P \to^* P'$, that is $P \xrightarrow[\lambda]{\tau *} P'$ for some residual function $\lambda$. Let $E \subseteq |P| \times |Q|$ be such that $(P,E,Q) \in \mathcal{R}$. By Lemma 6, one has $Q \xrightarrow[\rho]{\tau *} Q'$ for some residual function $\rho$, and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P',E',Q') \in \mathcal{R}$ and therefore $(P',Q') \in \mathcal{B}$ as required; this shows that $\mathcal{B}$ is a weak reduction bisimulation.

► Assume now that $(P,Q) \in \mathcal{B}$ and that $P \to^* P'$ with $P' \downarrow_f$ (with $f \in \bar{\Sigma}$ of arity $n$), meaning that $P' \xrightarrow[\lambda'_1]{p' : f \cdot (\boldsymbol{L})} P''$ for some $p' \in |P'|$, some sequence of sets of locations $\boldsymbol{L}$ and some residual function $\lambda'_1$.

Let $E \subseteq |P| \times |Q|$ be such that $(P,E,Q) \in \mathcal{R}$. By Lemma 6, one has $Q \xrightarrow[\rho]{\tau *} Q'$ for some residual function $\rho$, and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P',E',Q') \in \mathcal{R}$. Since $\mathcal{R}$ is a weak bisimulation we have therefore $Q' \overset{q' : f \cdot (\boldsymbol{M})}{\underset{\rho', \rho_1, \rho''}{\Longrightarrow}} Q''$ and hence $Q' \to^* Q'_1$ with $Q'_1 \downarrow_f$. This shows that $\mathcal{B}$ is weak barb preserving since $Q \to^* Q'_1$. □

We want now to prove a much stronger result, namely that weak bisimilarity implies weak barbed congruence (and not just weak barbed bisimilarity). This boils down to proving that weak bisimilarity is a congruence. Let us first give an example which illustrates this implication.

*Example 2.* Let first $\Sigma$ be such that $\Sigma_1 = \{a,b\}$ and $\Sigma_i = \emptyset$ if $i \neq 1$. Then it is easy to see that $a \cdot \varepsilon \mid b \cdot \varepsilon$ and $a \cdot b \cdot \varepsilon + b \cdot a \cdot \varepsilon$ are weakly bisimilar just as in usual CCS.

Let now $\Sigma$ be such that $\Sigma_1 = \{a\}$, $\Sigma_2 = \{f,g\}$ and $\Sigma_i = \emptyset$ for $i > 2$. Let $P = f \cdot (g \cdot (\varepsilon,\varepsilon),\varepsilon) + g \cdot (f \cdot (\varepsilon,\varepsilon),\varepsilon)$ and $Q = f \cdot (\varepsilon,\varepsilon) \mid g \cdot (\varepsilon,\varepsilon)$. Then we cannot prove that $P$ and $Q$ are weakly bisimilar (because, in the definition of a localized bisimulation, we are in the case $n > 1$). And indeed, surprisingly, $P$ and $Q$ are not weak barbed bisimilar. Actually, let $R = \overline{f} \cdot (\varepsilon, \overline{g} \cdot (a \cdot \varepsilon, \varepsilon)))$. Then $Q \mid R \to^* a \cdot \varepsilon$ and $a \cdot \varepsilon \downarrow_a$ whereas there is no process $M$ such that $P \mid R \to^* M$ with $M \downarrow_a$. The best we can do is reduce $P \mid R$ to $g \cdot (\varepsilon,\varepsilon) \oplus \overline{g} \cdot (a \cdot \varepsilon, \varepsilon)$.

## 4 Weak bisimilarity is a congruence

As in the standard method used in ordinary CCS, the main step for proving that weak bisimilarity is a congruence consists in extending a localized relation $\mathcal{R}$ on processes into another localized relation $\mathcal{R}'$ which is, intuitively, a congruence wrt. "parallel composition". Since parallel composition here is parametrized by a relation, the definition is more involved than in ordinary CCS and strongly involves locations.

*Adapted triples of relations.* We say that a triple of relations $(D, D', E)$ with $D \subseteq A \times B$, $D' \subseteq A \times B'$ and $E \subseteq B \times B'$ is *adapted*, if, for any $(a, b, b') \in A \times B \times B'$, with $(b, b') \in E$, one has $(a, b) \in D$ iff $(a, b') \in D'$.

*Parallel extension of a localized relation.* Let $\mathcal{R}$ be a localized relation on processes. One defines a new localized relation $\mathcal{R}'$ by stipulating that $(U, F, V) \in \mathcal{R}'$ if there is a process $S$, and a triple $(P, E, Q) \in \mathcal{R}$ as well are two relations $C \subseteq |S| \times |P|$ and $D \subseteq |S| \times |Q|$ such that $U = S \oplus_C P$, $V = S \oplus_D Q$ (these notations are introduced in Section 1.6), the triple of relations $(C, D, E)$ is adapted and $F$ is the relation $\mathrm{Id}_{|S|} \cup E \subseteq |U| \times |V|$. This localized relation will be called *the parallel extension* of $\mathcal{R}$.

Intuitively, we express here that $U$ is the parallel composition of $S$ and $P$, with connections between the processes of $S$ and those of $P$ specified by $C$. And similarly for $V$, defined as the parallel composition of $S$ and $Q$ through the relation $D$. The hypothesis that $(C, D, E)$ should be adapted means that $C$ and $D$ specify the same connections between processes up to $E$.

**Lemma 9.** *If $\mathcal{R}$ is symmetric, then so is its parallel extension $\mathcal{R}'$.*

*Proof.* Observe that $(C, D, E)$ is adapted iff $(D, C, {}^{\mathrm{t}}E)$ is adapted. □

The next proposition is an essential tool for proving that weak bisimulation is a congruence.

**Proposition 4.** *If $\mathcal{R}$ is a weak bisimulation, so is its parallel extension $\mathcal{R}'$.*

*Proof.* Symmetry of $\mathcal{R}'$ results from the symmetry of $\mathcal{R}$ and from Lemma 9.

Let $(U, F, V) \in \mathcal{R}'$ with $U = S \oplus_C P$, $V = S \oplus_D Q$, $(P, E, Q) \in \mathcal{R}$, $(C, D, E)$ adapted and $F = \mathrm{Id}_{|S|} \cup E$.

*Case of a $\tau$-transition.* Assume that $U \xrightarrow[\lambda]{\tau} U'$. We must show that $V \xrightarrow[\rho]{\tau *} V'$ with $(U', F', V') \in \mathcal{R}'$ and $(\lambda(u'), \rho(v')) \in F$ for each $(u', v') \in F'$ (condition on residuals). There are three cases as to the locations of the two guarded sums involved in that reduction.

► Assume first that they are located in $S$, in other words there are $s, t \in |S|$ with $s \frown_S t$, $\mathsf{cs}(S(s)) = f \cdot \boldsymbol{S} + \tilde{S}$ ($\tilde{S}$ is a guarded sum) and $\mathsf{cs}(S(t)) = \overline{f} \cdot \boldsymbol{T} + \tilde{T}$ ($\tilde{T}$ is a guarded sum), and we have $S \xrightarrow[\mu]{\tau} S'$ with

- $|S'| = (|S| \setminus \{s, t\}) \cup \bigcup_i |S_i| \cup \bigcup_i |T_i|$
- and $\frown_{S'}$ is the least symmetric relation on $|S'|$ such that $s' \frown_{S'} t'$ if $s' \frown_{S_i} t'$, or $s' \frown_{T_i} t'$, or $(s', t') \in |S_i| \times |T_i|$ for some $i = \{1, \ldots, n\}$, or $\{s', t'\} \not\subseteq \bigcup_{i=1}^n |S_i| \cup \bigcup_{i=1}^n |T_i|$ and $\mu(s') \frown_S \mu(t')$.

Remember that the residual function $\mu$ is given by $\mu(s') = s$ if $s' \in \bigcup_i |S_i|$, $\mu(s') = t$ if $s' \in \bigcup_i |T_i|$ and $\mu(s') = s'$ otherwise. We have $U' = S' \oplus_{C'} P$ where $C' = \{(s', p) \in |S'| \times |P| \mid (\mu(s'), p) \in C\}$ and $\lambda = \mu \cup \mathrm{Id}_{|P|}$.

Then we have similarly $V = S \oplus_D Q \xrightarrow[\rho]{\tau} V' = S' \oplus_{D'} Q$ with $\rho = \mu \cup \mathrm{Id}_{|Q|}$, and $D' = \{(s', q) \in |S'| \times |Q| \mid (\mu(s'), q) \in D\}$.

The triple $(C', D', E)$ is adapted: let $s' \in |S'|$, $p \in |P|$ and $q \in |Q|$ be such that $(p, q) \in E$. If $(s', p) \in C'$ then $(\mu(s'), p) \in C$ and hence $(\mu(s'), q) \in D$ since $(C, D, E)$ is adapted, that is $(s', q) \in D'$, and similarly for the converse implication.

Coming back to the definition of $\mathcal{R}'$, we see that $(U', F', V') \in \mathcal{R}'$ where $F' = \mathrm{Id}_{|S'|} \cup E$. Moreover, the condition on residuals is satisfied, since, given $(u', v') \in F'$, we have either $u' = v' \in |S'|$ and then $\lambda(u') = \rho(v') \in |S|$ or $(u', v') \in E$ and $(\lambda(u'), \rho(v')) = (u', v') \in E$. In both cases $(\lambda(u'), \rho(v')) \in F$.

► Assume next that they are located in $P$, in other words there are $p, r \in |P|$ with $\mathsf{cs}(P(p)) = f \cdot \boldsymbol{P} + \tilde{P}$ (where $\tilde{P}$ is a guarded sum) and $\mathsf{cs}(P(r)) = \overline{f} \cdot \boldsymbol{R} + \tilde{R}$ (where $\tilde{R}$ is a guarded sum), and we have $P \xrightarrow[\mu]{\tau} P'$ with

- $|P'| = (|P| \setminus \{p, r\}) \cup \bigcup_i |P_i| \cup \bigcup_i |R_i|$
- and $\frown_{P'}$ is the least symmetric relation on $|P'|$ such that $p' \frown_{P_i} r'$ or $p' \frown_{R_i} r'$ or $(p', r') \in |P_i| \times |R_i|$ for some $i \in \{1, \ldots, n\}$, or $\{p', r'\} \not\subseteq \bigcup_i |P_i| \cup \bigcup_i |R_i|$ and $\mu(p') \frown_P \mu(r')$.

We recall that the residual function $\mu$ is given by $\mu(p') = p$ if $p' \in \bigcup_i |P_i|$, $\mu(p') = r$ if $p' \in \bigcup_i |R_i|$ and $\mu(p') = p'$ otherwise. With these notations, the process $U'$ is $U' = S \oplus_{C'} P'$ where $C' = \{(s, p') \in |S| \times |P'| \mid (s, \mu(p')) \in C\}$ and the residual function $\lambda$ is defined as $\lambda = \mathrm{Id}_{|S|} \cup \mu$. Since $(P, E, Q) \in \mathcal{R}$ and $P \xrightarrow[\mu]{\tau *} P'$, one has $Q \xrightarrow[\nu]{\tau *} Q'$ with $(P', E', Q') \in \mathcal{R}$ where $E' \subseteq |P'| \times |Q'|$ satisfies the condition on residuals $(p', q') \in E' \Rightarrow (\mu(p'), \nu(q')) \in E$. Let $D' = \{(s, q') \in |S| \times |Q'| \mid (s, \nu(q')) \in D\}$. Setting $V' = S \oplus_{D'} Q'$, we have $V \xrightarrow[\rho]{\tau *} V'$ where $\rho = \mathrm{Id}_{|S|} \cup \nu$.

The triple $(C', D', E')$ is adapted: let $(p', q') \in E'$ and let $s \in |S|$. If $(s, p') \in C'$, we have $(s, \mu(p')) \in C$. Since $(\mu(p'), \nu(q')) \in E$ (by definition of $E'$), we have $(s, \nu(q')) \in D$ because $(C, D, E)$ is adapted. That is $(s, q') \in D'$. The converse implication is proved similarly.

Let $F' = \mathrm{Id}_{|S|} \cup E' \subseteq |U'| \times |V'|$, we have therefore $(U', F', V') \in \mathcal{R}'$ (by definition of $\mathcal{R}'$). Last we check the condition on residuals. Let $(u', v') \in F'$, then either $u' = v' \in |S|$ and then $\lambda(u') = u' = v' = \rho(v')$ or $u' \in |P'|$, $v' \in |Q'|$ and $(u', v') \in E'$ and then $(\lambda(u'), \rho(v')) = (\mu(u'), \nu(v')) \in E$ by the condition on residuals satisfied by $E$.

► Assume last that one of the involved guarded sums is located in $S$ and that the other one is located in $P$, this is of course the most interesting case in this first part of the proof.

By definition of internal reduction (see Section 2.1) we have $s \in |S|$ and $p \in |P|$ with $(s, p) \in C$ and with $\mathsf{cs}(S(s)) = \overline{f} \cdot \boldsymbol{S} + \tilde{S}$ and $\mathsf{cs}(P(p)) = f \cdot \boldsymbol{P} + \tilde{P}$ with the usual notational conventions, and $U' = S' \oplus_{C'} P'$ where $S' = S\,[\oplus \boldsymbol{S}/s]$, $P' = P\,[\oplus \boldsymbol{P}/p]$, and $C' \subseteq |S'| \times |P'|$ is defined as follows: $(s', p') \in C'$ if

- $(s', p') \in |S_i| \times |P_i|$ for some $i$,

– or $(s',p') \notin (\bigcup_i |S_i|) \times (\bigcup_i |P_i|)$ and $(\lambda(s'), \lambda(p')) \in C$,

where the residual map $\lambda : |U'| = |S'| \cup |P'| \to |U| = |S| \cup |P|$ is defined by $\lambda(u') = u'$ if $u' \in (|S'| \setminus \bigcup_i |S_i|) \cup (|P'| \setminus \bigcup_i |P_i|)$, $\lambda(s') = s$ if $s' \in \bigcup_i |S_i|$ and $\lambda(p') = p$ if $p' \in \bigcup_i |P_i|$.

We have $P \overset{p:f\cdot(\boldsymbol{L})}{\underset{\lambda}{\longrightarrow}} P'$ (where $L_i = |P_i|$ for each $i = 1, \ldots, n$) and hence, since we have assumed that $(P, E, Q) \in \mathcal{R}$, we have $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ with $(p, \rho(q)) \in E$ and $(P', E', Q') \in \mathcal{R}$ where $E'$ is such that if $(p', q') \in E'$ then $(\lambda(p'), \rho\rho_1\rho'(q')) \in E$ and, if $n \geq 2$, then $(p', \rho'(q')) \in L_i \times M_i$ for some $i$, or $p' \notin \bigcup_i L_i$ and $\rho'(q') \notin \bigcup M_i$.

We can decompose this transition as follows

$$Q \overset{\tau*}{\underset{\rho}{\longrightarrow}} Q_1 \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho_1}{\longrightarrow}} Q'_1 \overset{\tau*}{\underset{\rho'}{\longrightarrow}} Q' \, .$$

With these notations we have $V \overset{\tau*}{\underset{\mu}{\longrightarrow}} V_1$ with $V_1 = S \oplus_{D_1} Q_1$ where $D_1 = \{(s, q_1) \in |S| \times |Q_1| \mid (s, \rho(q_1)) \in D\}$, and $\mu = \mathrm{Id}_{|S|} \cup \rho$.

We have $q \in |Q_1|$ with $\mathsf{cs}(Q_1(q)) = f \cdot \boldsymbol{R} + \tilde{R}$ and $|R_i| = M_i$ for $i = 1, \ldots, n$. Moreover, since $(p, \rho(q)) \in E$ and $(s, p) \in C$, and since $(C, D, E)$ is adapted, we have $(s, \rho(q)) \in D$, that is $(s, q) \in D_1$. Therefore, since $\mathsf{cs}(S(s)) = \overline{f} \cdot \boldsymbol{S} + \tilde{S}$, we have $V_1 \overset{\tau}{\underset{\theta}{\longrightarrow}} V'_1 = S' \oplus_{D'_1} Q'_1$ where $D'_1 \subseteq |S'| \times |Q'_1|$ is defined as follows: given $(s', q'_1) \in |S'| \times |Q'_1|$, we have $(s', q'_1) \in D'_1$

– if $s' \in |S_i|$ and $q'_1 \in |R_i|$ for some $i = 1, \ldots, n$
– or $s' \notin \bigcup_i |S_i|$ or $q'_1 \notin \bigcup_i |R_i|$ and $(\theta(s'), \theta(q'_1)) \in D_1$ (that is $(\theta(s'), \rho\theta(q'_1)) \in D$),

and the residual function $\theta$ is defined by $\theta(v'_1) = v'_1$ if $v'_1 \in (|S| \setminus \bigcup_i |S_i|) \cup (|Q_1| \setminus \bigcup_i |R_i|)$, $\theta(s') = s$ if $s' \in \bigcup_i |S_i|$ and $\theta(q'_1) = q$ if $q'_1 \in \bigcup_i |R_i|$.

Observe that $\theta(q'_1) = \rho_1(q'_1)$ for all $q'_1 \in |Q'_1|$.

Since $Q'_1 \overset{\tau*}{\underset{\rho'}{\longrightarrow}} Q'$, we have $V'_1 = S' \oplus_{D'_1} Q'_1 \overset{\tau*}{\underset{\mu'}{\longrightarrow}} V' = S' \oplus_{D'} Q'$ where $\mu' = \mathrm{Id}_{|S'|} \cup \rho'$ and $D' = \{(s', q') \in |S'| \times |Q'| \mid (s', \rho'(q')) \in D'_1\}$. So we have $V \overset{\tau*}{\underset{\mu\theta\mu'}{\longrightarrow}} V'$. Let $F' \subseteq |U'| \times |V'|$ be defined by $F' = \mathrm{Id}_{|S'|} \cup E'$. It is clear then that $(u', v') \in F' \Rightarrow (\lambda(u'), \mu\theta\mu'(v')) \in F$ because $(p', q') \in E' \Rightarrow (\lambda(p'), \rho\rho_1\rho'(q')) \in E$ and $\theta$ and $\rho_1$ coincide on $|Q'_1|$.

To finish, we must prove that $(U', F', V') \in \mathcal{R}'$ and to this end it suffices to show that the triple of relations $(C', D', E')$ is adapted. So let $s' \in |S'|$, $p' \in |P'|$ and $q' \in |Q'|$ with $(p', q') \in E'$ (so that in particular $(\lambda(p'), \rho\theta\rho'(q')) \in E$).

Assume first that $(s', p') \in C'$ and let us show that $(s', q') \in D'$, that is $(s', \rho'(q')) \in D'_1$. Coming back to the definition of $C'$, we can reduce our analysis to three cases.

– First case: $(s', p') \in |S_i| \times |P_i|$ for some $i$. We distinguish two cases as to the value of $n$ (the arity of $f$). Assume first that $n \geq 2$. Since $p' \in |P_i| = L_i$, we must have $\rho'(q') \in M_i = |Q_i|$ because $(p', q') \in E'$ and then $(s', \rho'(q')) \in D'_1$ as required. Assume now $n = 1$. If $\rho'(q') \in M_1$ we reason as above, so assume that $\rho'(q') \notin M_1 = \bigcup_{i=1}^{n} |R_i|$. Coming back to the definition of $D'_1$, it suffices to prove that $(\theta(s'), \rho\theta\rho'(q')) = (s, \rho\rho'(q')) \in D$. Since $(p', q') \in E'$ we have $(\lambda(p'), \rho\theta\rho'(q')) = (p, \rho\rho'(q')) \in E$. We also have $(s, p) \in C$, and hence $(s, \rho\rho'(q')) \in D$ as required, since $(C, D, E)$ is adapted.

- Second case: $s' \notin \bigcup_i |S_i|$. In order to prove $(s', q') \in D'$, it suffices to prove that $(\theta(s'), \rho\theta\rho'(q')) = (s', \rho\theta\rho'(q')) \in D$. But we have $(s', p') \in C'$ and $s' \notin \bigcup_i |S_i|$, hence $(\lambda(s'), \lambda(p')) = (s', \lambda(p')) \in C$. Since $(p', q') \in E'$, we have $(\lambda(p'), \rho\theta\rho'(q')) \in E$ and hence $(s', \rho\theta\rho'(q')) \in D$ since $(C, D, E)$ is adapted.
- Third case: $s' \in \bigcup_i |S_i|$ and $p' \notin \bigcup_i |P_i|$ so that we have $(s, p') \in C$ (by definition of $C'$ and because $(s', p') \in C'$). Assume first that $n \geq 2$. Since $(p', q') \in E'$, we must have $\rho'(q') \notin \bigcup_{i=1}^{n} M_i$. To prove that $(s', \rho'(q')) \in D'_1$, it suffices therefore to check that $(\theta(s'), \rho\theta\rho'(q')) = (s, \rho\rho'(q')) \in D$. This property holds because $(C, D, E)$ is adapted, $(s, p') \in C$ and $(p', \rho\rho'(q')) \in E$ because $(p', q') \in E'$. Assume now that $n = 1$. If $\rho'(q') \notin \bigcup_{i=1}^{n} M_i = M_1$, we can reason as above, so assume that $\rho'(q') \in M_1$. Then we have $(s', \rho'(q')) \in |S_1| \times M_1$ and hence $(s', \rho'(q')) \in D'_1$.

Let us prove now the converse implication, assuming that $(s', \rho'(q')) \in D'_1$; we contend that $(s', p') \in C'$. Again, we consider three cases.

- First case: $s' \in |S_i|$ and $\rho'(q') \in M_i = |R_i|$ for some $i \in \{1, \ldots, n\}$. If $n \geq 2$ the fact that $(p', q') \in E'$ implies that $p' \in L_i = |P_i|$ and hence $(s', p') \in C'$ as required. Assume that $n = 1$ and $p' \notin L_1 = \bigcup_{i=1}^{n} |P_i|$, we have $(\lambda(s'), \lambda(p')) = (s, p') \in C$ because $(s, \rho\rho'(q')) \in D$ — since $(s', \rho'(q')) \in D'_1$, $\rho'(q') \notin M_1$ and $(\theta(s'), \rho\theta\rho'(q')) = (s, \rho\rho'(q'))$ —, $(p, \rho\rho'(q')) \in E$ and $(C, D, E)$ is adapted. Hence $(s', p') \in C'$.
- Second case: $s' \notin \bigcup_i |S_i|$. In view of the definition of $C'$, it suffices to prove that $(\lambda(s'), \lambda(p')) = (s', \lambda(p')) \in C$. Since $(s', \rho'(q')) \in D'_1$ and $s' \notin \bigcup_i |S_i|$, we have $(\theta(s'), \rho\theta\rho'(q')) = (s', \rho\theta\rho'(q')) \in D$. And since $(p', q') \in E'$ we have $(\lambda(p'), \rho\theta\rho'(q')) \in E$, and hence $(s', \lambda(p')) \in C$ because $(C, D, E)$ is adapted.
- Third case: $s' \in |S_i|$ for some $i \in \{1, \ldots, n\}$ and $\rho'(q') \notin \bigcup_i M_i$. If $n \geq 2$, we must have $p' \notin \bigcup_i L_i$ because $(p', q') \in E'$. Therefore, to check that $(s', p') \in C'$, it suffices to prove that $(\lambda(s'), \lambda(p')) = (s, p') \in C$. We have $(s', \rho'(q')) \in D'_1$ and hence $(\theta(s'), \rho\theta\rho'(q')) = (s, \rho\rho'(q')) \in D$. Since $(p', q') \in E'$ we have $(\lambda(p'), \rho\theta\rho'(q')) = (p', \rho\rho'(q')) \in E$ and hence $(s, p') \in C$ because $(C, D, E)$ is adapted. Assume now that $n = 1$. If $p' \in L_1$ we have $(s', p') \in C'$ since $(s', p') \in |S_1| \times |P_1|$. So assume that $p' \notin L_1$. Since then $p' \notin \bigcup_{i=1}^{n} |P_i|$, it suffices to prove that $(\lambda(s'), \lambda(p')) = (s, p') \in C$ (by definition of $C'$). We have $(p', \rho\theta\rho'(q')) = (p', \rho\rho'(q')) \in E$ because $(p', q') \in E'$ and $(s, \rho\theta\rho'(q')) = (s, \rho\rho'(q')) \in D$ because $(s', \rho'(q')) \in D'_1$ and $\rho'(q') \notin \bigcup_i M_i$. It follows that $(s, p') \in C$ as required.

This ends the first part of the proof.

*Case of a labeled transition.* We assume now that $U \xrightarrow[\mu_1]{r:f\cdot(\boldsymbol{L})} U'$. Since $U = S \oplus_C P$, we consider two cases as to the location of $r$.

► If $r \in |S|$ then we have $\mathsf{cs}(S(r)) = f \cdot \boldsymbol{S} + \tilde{S}$ and $S \xrightarrow[\sigma_1]{r:f\cdot(\boldsymbol{L})} S'$ where $S' = S[\oplus \boldsymbol{S}/r]$ (so that $L_i = |S_i|$ for each $i$), and $U' = S' \oplus_{C'} P$ where $C' = \{(s', p) \in |S'| \times |P| \mid (\sigma_1(s'), p) \in C\}$. Let $D' = \{(s', q) \in |S'| \times |Q| \mid (\sigma_1(s'), q) \in D\}$. We have $\mu_1 = \sigma_1 \cup \mathrm{Id}_{|P|}$. It is clear that $(C', D', E)$ is adapted, since $(C, D, E)$ is adapted.

Let $V' = S' \oplus_{D'} Q$, we have just seen that $(U', F', V') \in \mathcal{R}'$ where $F' = \mathrm{Id}_{|S'|} \cup E$. We have $(r, r) \in F$, $V \xrightarrow[\nu_1]{r:f\cdot(\boldsymbol{L})} V'$ (with $\nu_1 = \sigma_1 \cup \mathrm{Id}_{|Q|}$) and, given $(u', v') \in F'$, we have either $(u', v') \in \bigcup_i (L_i \times L_i)$ (and actually $u' = v'$) or $u' \notin \bigcup_i L_i$, $v' \notin \bigcup_i L_i$ and $(u', v') \in F$ as easily checked. Therefore the condition on residuals is satisfied.

► The last case to consider is when $r = p \in |P|$ and then we have $P(p) = f \cdot \boldsymbol{P} + \tilde{P}$ and $P \xrightarrow[\lambda_1]{p:f\cdot(\boldsymbol{L})} P'$. Then we have $U' = S \oplus_{C'} P'$ where $C' = \{(s, p') \in |S| \times |P'| \mid (s, \lambda_1(p')) \in C\}$.

Since $(P, E, Q) \in \mathcal{R}$ we have $Q \overset{q:f\cdot(\boldsymbol{M})}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ with $(p, \rho(q)) \in E$ and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P', E', Q') \in \mathcal{R}$ and, for any $(p', q') \in E'$, $(\lambda_1(p'), \rho\rho_1\rho'(q')) \in E$ and, if $n \geq 2$, either $(p', \rho'(q')) \in \bigcup_{i=1}^n (L_i \times M_i)$, or $p' \notin \bigcup_{i=1}^n L_i$ and $\rho'(q') \notin \bigcup_{i=1}^n M_i$.

Therefore we have $V \overset{q:f\cdot(\boldsymbol{M})}{\underset{\nu,\nu_1,\nu'}{\Longrightarrow}} V'$ where $V' = S \oplus_{D'} Q'$ with $D' = \{(s, q') \in |S| \times |Q'| \mid (s, \rho\rho_1\rho'(q')) \in D\}$. Moreover $\nu = \mathrm{Id}_{|S|} \cup \rho$, $\nu_1 = \mathrm{Id}_{|S|} \cup \rho_1$ and $\nu' = \mathrm{Id}_{|S|} \cup \rho'$.

Let $F' \subseteq |U'| \times |V'|$ be defined by $F' = \mathrm{Id}_{|S|} \cup E'$. Let $(u', v') \in F'$. If $u' \in |S|$ or $v' \in |S|$, we must have $u' = v'$. If $u' \notin |S|$ and $v' \notin |S|$ then we have $(u', v') \in E'$ and hence $(\mu_1(u'), \nu\nu_1\nu'(v')) = (\lambda_1(u'), \rho\rho_1\rho'(q')) \in E$ and, if $n \geq 2$, either there exists $i$ such that $u' \in L_i$ and $\nu'(v') = \rho'(v') \in M_i$, or $u' \notin \bigcup_i L_i$ and $\nu'(v') = \rho'(v') \notin \bigcup_i M_i$.

Moreover, the triple $(C', D', E')$ is adapted: let $(p', q') \in E'$ and $s \in |S|$. We have $(\lambda_1(p'), \rho\rho_1\rho'(q')) \in E$. We have $(s, p') \in C'$ iff $(s, \lambda_1(p')) \in C$ iff $(s, \rho\rho_1\rho'(q')) \in D$ iff $(s, q') \in D'$. □

Now we are in position of proving that weak bisimilarity is a congruence, a result which is interesting *per se* and will be essential for proving Theorem 3.

**Theorem 2.** *The weak bisimilarity relation $\approx$ is a congruence.*

*Proof.* Let $\mathcal{R}$ be a weak bisimulation. Let $R$ be a $Y$-context. We define a new localized relation denoted as $R[\mathcal{R}/Y]$:

- if $R = Y$ then $R[\mathcal{R}/Y] = \mathcal{R}$;
- if $R \neq Y$ then we stipulate that $(P', E', Q') \in R[\mathcal{R}/Y]$ if there exists $(P, E, Q) \in \mathcal{R}$ and if $E' = \mathrm{Id}_{|R|}$, $P' = R[P/Y]$ and $Q' = R[Q/Y]$ (observe that $|P'| = |Q'| = |R|$ because $R \neq Y$).

We define a localized relation $\mathcal{R}^+$ as the union of $\mathcal{I}$ (the set of all triples $(U, E, U)$ where $U$ is any process and $E = \mathrm{Id}_{|U|}$), of the parallel extension $\mathcal{R}'$ of $\mathcal{R}$ (see Proposition 4) and of all the relations of the shape $R\,[\mathcal{R}/Y]$ for all $Y$-contexts $R$.

We prove that $\mathcal{R}^+$ is a weak bisimulation and the theorem will follow easily.

Let $(U, F, V) \in \mathcal{R}^+$ and assume that we are in one of the two following situations

- $U \xrightarrow[\mu]{\tau} U'$ (called case (**1**) in the sequel)
- or $U \xrightarrow[\mu_1]{p:f\cdot(\boldsymbol{L})} U'$ (called case (**2**) in the sequel).

We describe explicitely our objectives.

- In case (**1**) we must show that $V \xrightarrow[\nu]{\tau*} V'$ with $(U', F', V') \in \mathcal{R}^+$ for some $F' \subseteq |U'| \times |V'|$ such that for any $(u', v') \in F'$, one has $(\mu(u'), \nu(v')) \in F$.
- In case (**2**) we must show that $V \overset{q:f\cdot(\boldsymbol{M})}{\underset{\nu,\nu_1,\nu'}{\Longrightarrow}} V'$ with $(p, \nu(q))) \in F$ and $(U', F', V') \in \mathcal{R}^+$, for some $F' \subseteq |U'| \times |V'|$ such that, for any $(u', v') \in F'$, one has $(\mu_1(u'), \nu\nu_1\nu'(v')) \in F$ and, if $n \geq 2$, then one has either $(u', \nu'(v')) \in \bigcup_{i=1}^n (L_i \times M_i)$ or $u' \notin \bigcup_i L_i$ and $\nu'(v') \notin \bigcup_i M_i$.

The case where $(U, F, V) \in \mathcal{I}$ is trivial.

If $(U, F, V) \in \mathcal{R}'$ we apply directly Proposition 4 in both cases.

Assume now that $(U, F, V) \in R\,[\mathcal{R}/Y]$ for some $Y$-context $R$, so that $U = R\,[P/Y]$, $V = R\,[Q/Y]$ with $(P, E, Q) \in \mathcal{R}$ and $F = E$ if $R = Y$ and $F = \mathrm{Id}_{|R|}$ otherwise. If $R = Y$ we use directly the fact that $\mathcal{R}$ is a weak bisimulation to exhibit $V'$ and $F'$ satisfying the required conditions.

So we assume from now on that $R \neq Y$ and therefore $F = \mathrm{Id}_{|R|}$.

By definition of a $Y$-context, there is exactly one $r \in |R|$ such that $Y$ occurs free in $R(r)$. Then $R(r)$ can be written uniquely as $R(r) = g \cdot \boldsymbol{R} + \tilde{R}$ where $Y$ does not occur in $\tilde{R}$ and occurs in exactly one of the processes $R_1, \ldots, R_n$; without loss of generality we can assume that $R_1$ is a $Y$-context and that $Y$ does not occur free in $R_2, \ldots, R_n$.

Assume first that $R_1 \neq Y$. In both cases (**1**) and (**2**), we have $U' = R'\,[P/Y]$ with $R \xrightarrow[\mu]{\tau} R'$ (case (**1**)) or $R \xrightarrow[\mu_1]{p:f\cdot(\boldsymbol{L})} R'$ (case (**2**)). Let $V' = R'\,[Q/Y]$. In case (**1**), we have $V \xrightarrow[\mu]{\tau} V'$ and in case (**2**) we have $V \xrightarrow[\mu_1]{q:f\cdot(\boldsymbol{L})} V'$, and since $R' \neq Y$ (by our hypothesis on $R_1$), we have $(U', \mathrm{Id}_{|R'|}, V') \in \mathcal{R}^+$ because $(P, E, Q) \in \mathcal{R}$. The condition on residuals is obviously satisfied in both cases.

Assume now that $R_1 = Y$.

► Suppose first that we are in case (**1**). There are two cases to consider as to the locations $s, t \in |U|$ of the sub-processes involved in the transition $U \xrightarrow[\mu]{\tau} U'$. The case where $s \neq r$ and $t \neq r$ is similar to the case above where $R_1 \neq Y$. By symmetry we are left with the case where $s = r$ (and hence $t \neq r$).

So $U(t) = R(t) = \overline{f} \cdot \boldsymbol{T} + \tilde{T}$ and the guarded sum $R(r)$ has an unique summand which is involved in the transition $U \xrightarrow[\mu]{\tau} U'$ (called *active summand* in the sequel), and this summand is of the shape $f \cdot \boldsymbol{S}$.

If the active summand is $g \cdot \boldsymbol{R}$[6] (so that $g = f$) then $U(r) = f \cdot (P, R_2, \ldots, R_n) + \tilde{S}$ and $U'$ can be written $U' = R' \oplus_C P$ for some process $R'$ which can be defined using only $R$, and $C \subseteq |R'| \times |P|$. Explicitly, $R'$ is defined as follows:

- $|R'| = (|R| \setminus \{r, t\}) \cup \bigcup_{i=2}^{n} |R_i| \cup \bigcup_{i=1}^{n} |T_i|$
- and $\frown_{R'}$ is the least symmetric relation on $|R'|$ such that $r' \frown_{R'} t'$ if $r' \frown_{R_i} t'$ for some $i = 2, \ldots, n$ or $r' \frown_{T_i} t'$ for some $i = 1, \ldots, n$, or $(r', t') \in |R_i| \times |T_i|$ for some $i \in \{2, \ldots, n\}$, or $r' \notin \bigcup_{i=2}^{n} |R_i|$ or $t' \notin \bigcup_{i=1}^{n} |T_i|$ and $r' \frown_R t$ and $\mu(r') \frown_R \mu(t')$

where the residual function $\mu : |U'| \to |U|$ is given by $\mu(r') = r$ if $r' \in |P| \cup \bigcup_{i=2}^{n} |R_i|$, $\mu(r') = t$ if $r' \in \bigcup_{i=1}^{n} |T_i|$ and $\mu(r') = r'$ when $r'$ belongs to none of these two sets.

The relation $C$ is defined as follows: given $(r', p) \in |R'| \times |P|$, one has $(r', p) \in C$ if $r' \in |T_1|$, or $r' \notin \bigcup_{i=2}^{n} |R_i| \cup \bigcup_{i=1}^{n} |T_i|$ and $r' \frown_R r$.

Let $V' = R' \oplus_D Q$, where $D \subseteq |R'| \times |Q|$ is defined exactly like $C$ (just replace $P$ by $Q$ in the definition). Then $(C, D, E)$ is adapted (because the property for $(r', p) \in |R'| \times |P|$ of belonging or not to $C$ depends only on $r'$, and does not depend on $p$, and similarly for $D$). We can mimic that reduction on $V$, so that $V \xrightarrow[\nu]{\tau} V'$ for the residual function $\nu$ which is defined like $\mu$ (replacing $P$ by $Q$). We have $(U', F', V') \in \mathcal{R}' \subseteq \mathcal{R}^+$ where $F' = \mathrm{Id}_{|R'|} \cup E$. Given $(u', v') \in F'$, we have $\mu(u') = \nu(v')$, that is $(\mu(u'), \nu(v')) \in F$ so that the condition on residuals holds[7].

Assume now that the active summand is not $g \cdot \boldsymbol{R}$. In that case we also have $V \xrightarrow[\mu]{\tau} U'$ (both $P$ and $Q$ vanish in the corresponding reductions), and we are done because $(U', \mathrm{Id}_{|U'|}, U') \in \mathcal{I} \subseteq \mathcal{R}^+$.

► We suppose now that we are in case (**2**). Assume first that $p \neq r$. In that case we have $R \xrightarrow[\theta_1]{p:f \cdot (\boldsymbol{L})} R'$ and $U' = R'\,[P/Y]$ and we also have $V \xrightarrow[\theta_1]{p:f \cdot (\boldsymbol{L})} V' = R'\,[Q/Y]$ so $(U', \mathrm{Id}_{|R'|}, V') \in R'\,[\mathcal{R}/Y] \subseteq \mathcal{R}^+$, and the condition on residuals is obvious.

Assume now that $p = r$. Then exactly one of the summands of the guarded sum $R(r)$ is the prefixed process performing the action $f$ in the considered transition on $U$ (again, this summand is called the active summand in the sequel).

The case where the active summand is not $g \cdot (P, R_2, \ldots, R_n)$ is completely similar to the previous one ($P$ vanishes in the transition).

Assume that the active summand is $g \cdot (P, R_2, \ldots, R_n)$ (so that $g = f$), then $U' = R' \oplus_C P$ where $R'$ is defined by

- $|R'| = (|R| \setminus \{r\}) \cup \bigcup_{i=2}^{n} |R_i|$ and $\frown_{R'}$ is the least symmetric relation on $|R'|$ such that $r' \frown_{R'} t'$ if $r' \frown_{R_i} t'$ for some $i = 2, \ldots, n$ or $\theta_1(r') \frown_R \theta_1(t')$.

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[6] Remember that $g \cdot \boldsymbol{R}$ is the unique summand of $R(r)$ which contains $Y$.

[7] It is in this part of the proof that one understand the importance of adapted triples of relations in the definition of the parallel extension of a weak bisimulation.

– The relation $C \subseteq |R'| \times |P|$ is defined by $(r', q) \in C$ if $r' \notin \bigcup_{i=2}^{n} |R_i|$ and $r' \frown_R r$ (this does not depend on $q$).

Then we have $V \xrightarrow[\varphi_1]{p:f\cdot(\boldsymbol{M})} V'$ (with $M_1 = |Q|$ and $M_i = L_i = |R_i|$ for $i = 2, \ldots, n$) with $V' = R' \oplus_D Q$ where $D$ is defined like $C$ (replacing $P$ by $Q$ in the definition). Then we have $(U', F', V') \in \mathcal{R}' \subseteq \mathcal{R}^+$ where $F' = \mathrm{Id}_{|R'|} \cup E$ since $(C, D, E)$ is obviously adapted (as above). Moreover the condition on residuals is obviously satisfied. This ends the proof of the fact that $\mathcal{R}^+$ is a weak bisimulation.

We can now prove that $\approx$ is a congruence. Assume that $P \approx Q$ and let $R$ be a $Y$-context. Let $E \subseteq |P| \times |Q|$ and let $\mathcal{R}$ be a weak bisimulation such that $(P, E, Q) \in \mathcal{R}$. Then we have $(R[P/Y], \mathrm{Id}_{|R|}, R[Q/Y]) \in R[\mathcal{R}/Y] \subseteq \mathcal{R}^+$ and hence $R[P/Y] \approx R[Q/Y]$ since $\mathcal{R}^+$ is a weak bisimulation. □

We can prove now the main theorem of the paper.

**Theorem 3.** *Let $P$ and $Q$ be processes. If $P \approx Q$ ($P$ and $Q$ are weakly bisimilar) then $P \cong Q$ ($P$ and $Q$ are weakly barb congruent).*

*Proof.* Assume that $P \approx Q$ and let $R$ be a $Y$-context. We have $R[P/Y] \approx R[Q/Y]$ by Theorem 2 and hence $R[P/Y] \stackrel{\bullet}{\approx} R[Q/Y]$ by Proposition 3. □

## 5 Weak bisimilarity on CCS

We assume in this section that $\Sigma_n = \emptyset$ if $n \neq 1$ (see the end of Section 2.2). All processes $P$ considered in this section are $\mathsf{CCS}$ processes built on $\Sigma$, meaning that, in any subprocess of $P$ which is of shape $G\langle\Phi\rangle$, the graph $G$ is a complete graph (for all $p, q \in |G|$, $p \frown_G q$).

We answer here a very natural question: when restricted to ordinary $\mathsf{CCS}$, does our weak localized bisimilarity coincide with standard weak bisimilarity?

Let $\mathcal{R}$ be a localised weak bisimulation. Let $\mathcal{R}^0$ be the following relation on $\mathsf{CCS}$ processes: $P\, \mathcal{R}^0\, Q$ if $(P, E, Q) \in \mathcal{R}$ for some $E \subseteq |P| \times |Q|$. We prove that $\mathcal{R}^0$ is a weak bisimulation on $\mathsf{CCS}$ processes.

**Lemma 10.** *Let $\mathcal{R}$ be a localized weak bisimulation. Then $\mathcal{R}^0$ is weak bisimulation on $\mathsf{CCS}$ processes.*

*Proof.* Let $P$ and $Q$ be $\mathsf{CCS}$ processes such that $P\, \mathcal{R}^0\, Q$. Let $E \subseteq |P| \times |Q|$ be such that $(P, E, Q) \in \mathcal{R}$.

Assume first that $P \xrightarrow{\tau} P'$. Let $p_1, p_2 \in |P|$ with $\mathsf{cs}(P(p_1)) = a \cdot P_1 + S_1$ and $\mathsf{cs}(P(p_2)) = \overline{a} \cdot P_2 + S_2$ (the two sub-processes involved in this reduction). Then, by definition of the internal reduction in $\mathsf{CCTS}$, $P' = G\langle\Phi\rangle$ where $G$ is the complete graph on $|G| = |P| \setminus \{p_1, p_2\} \cup |P_1| \cup |P_2|$ and $\Phi(r) = P(r)$ if $r \in |P|$, $\Phi(r) = P_i(r)$ if $r \in |P_i|$ for $i = 1, 2$. In other words $P' = P[P_1/p_1, P_2/p_2]$

Let $\lambda_1 : |P'| \to |P|$ be the corresponding residual map ($\lambda_1(r) = r$ if $r \in |P|$ and $\lambda_1(r) = p_i$ if $r \in |P_i|$), we have $P \xrightarrow[\lambda_1]{\tau} P'$ and therefore there is a $\mathsf{CCTS}$

process $Q'$ such that $(P', E', Q') \in \mathcal{R}$ for some relation $E' \subseteq |P'| \times |Q'|$, and a function $\rho : |Q'| \to |Q|$ with $Q \xrightarrow[\rho]{\tau*} Q'$ and $(p', q') \in E' \Rightarrow (\lambda_1(p'), \rho(q')) \in E$. Therefore we have $P' \mathcal{R}^0 Q'$ as required.

Assume now that $P \xrightarrow{a} P'$. Let $p \in |P|$ with $\mathsf{cs}(P(p)) = a \cdot P_1 + S_1$ and $P' = P[P_1/p]$. Then we have $P \xrightarrow[\lambda_1]{p:a\cdot(L)} P'$ where $L = |P_1|$ and $\lambda_1 : |P'| \to |P|$ is given by $\lambda_1(r) = p$ if $r \in |P_1|$ and $\lambda_1(r) = r$ otherwise. Since $(P, E, Q) \in \mathcal{R}$, we have $Q \overset{q:a\cdot(M)}{\underset{\rho,\rho_1,\rho'}{\Longrightarrow}} Q'$ with $(p, \rho(q)) \in E$, and there exists $E' \subseteq |P'| \times |Q'|$ such that $(P', E', Q') \in \mathcal{R}$, and $(\lambda_1(p'), \rho\rho_1\rho'(q')) \in E$ for each $(p', q') \in E'$. In particular $P' \mathcal{R}^0 Q'$.

Since $\mathcal{R}$ is a localized bisimulation, the relation $\mathcal{R}^0$ is symmetric and is therefore a bisimulation on $\mathsf{CCS}$ processes. □

We need now to prove the converse. Let $\mathcal{U}$ be a binary relation on $\mathsf{CCS}$ processes. Let $\widehat{\mathcal{U}}$ be the set of all triples $(P, E, Q)$ where $P$ and $Q$ are $\mathsf{CCS}$ processes such that $P \mathcal{U} Q$ and $E = |P| \times |Q|$.

**Lemma 11.** *If $\mathcal{U}$ is a bisimulation, then $\widehat{\mathcal{U}}$ is a localized bisimulation.*

*Proof.* Let $P$ and $Q$ be $\mathsf{CCS}$ processes and let $E$ be such that $(P, E, Q) \in \widehat{\mathcal{U}}$, so that $E = |P| \times |Q|$ and $P \mathcal{U} Q$.

Assume first that $P \xrightarrow[\lambda_1]{\tau} P'$ so that $P \xrightarrow{\tau} P'$ (in $\mathsf{CCS}$) and hence there exists $Q'$ such that $Q \xrightarrow{\tau*} Q'$ and $P' \mathcal{U} Q'$. Then there is a function $\rho : |Q'| \to |Q|$ such that $Q \xrightarrow[\rho]{\tau*} Q'$ and we have $(P', E', Q') \in \widehat{\mathcal{U}}$. The condition on residuals holds obviously, by definition of $E$.

The case of a labeled transition is completely similar and the condition on residuals holds again by definition of $\widehat{\mathcal{U}}$ and because we are in the case where $n = 1$ (all function symbols are of arity 1). □

So we can conclude that, when restricted to $\mathsf{CCS}$ processes, our notion of weak bisimilarity coincides with the usual one.

**Proposition 5.** *Two CCS processes are weakly bisimilar (in the usual CCS sense) iff they are weakly bisimilar in the localized sense.*

## Conclusion

We have presented an extension of $\mathsf{CCS}$ which deals with trees instead of words, and various concepts and tools associated with this new process algebra. The notion of barbed bisimilarity, as it is defined here, is a straightforward generalization of the corresponding notion for $\mathsf{CCS}$ and therefore is hardly questionable, but we cannot say the same of weak bisimilarity. It will be crucial to understand if weak bisimilarity is equivalent to weak barbed congruence here and, if not, to look for a more liberal notion of weak bisimilarity in order to get such a full abstraction property. Another more conceptual task will be to extend this approach to more expressive settings such as for instance the $\pi$-calculus, and of course to understand if CCTS can be encoded in such settings.

This work also originated from the encodings of the $\pi$-calculus and of the solos calculus in differential interaction nets by the first author and Laurent [EL10]. In these nets, which are graphical objects, parallel compositions appear as complete graphs, and it is clear that more general graphs (actually, arbitrary graphs) could be encoded as well in the very same formalism. A graphical approach to CCTS, in the spirit of interaction nets, will be presented in a forthcoming paper.

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