A COHERENT DIFFERENTIAL PCF

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Abstract. The categorical models of the differential lambda-calculus are additive categories because of the Leibniz rule which requires the summation of two expressions. This means that, as far as the differential lambda-calculus and differential linear logic are concerned, these models feature finite non-determinism and indeed these languages are essentially non-deterministic. In a previous paper we introduced a categorical framework for differentiation which does not require additivity and is compatible with deterministic models such as coherence spaces and probabilistic models such as probabilistic coherence spaces. Based on this semantics we develop a syntax of a deterministic version of the differential lambda-calculus. One nice feature of this new approach to differentiation is that it is compatible with general fixpoints of terms, so our language is actually a differential extension of PCF for which we provide a fully deterministic operational semantics.

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1. Introduction

The differential lambda-calculus [ER03] extends the (typed) lambda-calculus with a syntactic notion of differentiation which is compatible with the basic intuition of differentiation in Calculus: given $f : E \to F$ a sufficiently regular function between two $\mathbb{R}$-vector spaces (finite dimensional, or Banach…), $f' : E \to (E \multimap F)$ where $E \multimap F$ is the space of linear (and continuous if we are in infinite dimension) maps $E \to F$ such that $f(x+u) = f(x)+f'(x)\cdot u+o(\|u\|)$. More generally $f'(x)$ is such that $y \mapsto f(x)+f'(x)\cdot(y-x)$ is the best affine approximation of f which coincides with f at x.

Syntactically this means that, given a term M such that $\Gamma \vdash M : A \Rightarrow B$ and a term N such that $\Gamma \vdash N : A$ we introduce a term $\mathsf{D}M\cdot N$ such that $\Gamma \vdash \mathsf{D}M \cdot N : A \Rightarrow B$. Intuitively M represents a function $f : A \to B$, N an element u of A and $\mathsf{D}M\cdot N$ represents the function $g : A \to B$ such that $g(x) = f'(x) \cdot u$. This syntactic presentation is a convenient way to express the fact that the derivative has the same regularity as the differentiated function (intuitively terms represent "smooth" maps, so their derivatives are themselves smooth) and allows easy iteration of differentiation (n-th derivatives).

Differentiation is inherently related to the algebraic operation of *addition* and the associated operation of subtraction, this is obvious in the definition of derivative we have all been thought at school: $f'(x) = \lim_{\varepsilon\to 0} \frac{f(x+\varepsilon)-f(x)}{\varepsilon}$. More algebraically this connection manifests itself for instance by the *Leibniz rule* $(fg)' = f'g + fg'$. In the differential lambda-calculus, beyond the ordinary β-reduction, there is a *differential β-reduction*: $\mathsf{D}(\lambda x^A\, M)\cdot N \to \frac{\partial M}{\partial x}\cdot N$ which uses a *linear substitution* $\frac{\partial M}{\partial x} \cdot N$ defined by induction on M. The definition of this operation involves the Leibniz rule[1] in the syntactic constructs where the variable x can be used at various places, the most typical example being:

$$\frac{\partial(P)Q}{\partial x} \cdot N = (\frac{\partial P}{\partial x} \cdot N)Q + (\mathsf{D}P \cdot \left(\frac{\partial Q}{\partial x} \cdot N\right))Q \tag{1.1}$$

which is a combination of the Leibniz rule and of the chain rule. Because of this feature of the definition of $\frac{\partial M}{\partial x} \cdot N$ we had to extend the syntax of the lambda-calculus with an addition operation typed as follows

$$\frac{\Gamma \vdash M_0 : A \quad \Gamma \vdash M_1 : A}{\Gamma \vdash M_0 + M_1 : A}$$

This rule is most natural if we have in mind the usual mathematical intuitions about differentiation. However the lambda-calculus is not only a nice and convenient syntax for denoting mathematical functions, it is also an expressive programming language featuring crucial properties of determinism. But the most natural operational interpretation of this $+$ operation is a kind of non-deterministic superposition. For instance if our calculus has a type ι of integers with $\Gamma \vdash \underline{n} : \iota$ for all $n \in \mathbb{N}$ (numerals) then we can write terms like $\underline{42} + \underline{57}$ which is a superposition of two values (nothing to do with $\underline{99}$ of course!).

A natural question is then whether differentiation requires such a general addition operation on terms and is therefore incompatible with determinism. In [Ehr21] we have provided a semantical negative answer to this question, based on a new categorical setting that we call *coherent differentiation*. The basic idea is to replace *additive categories*[2] with categories equipped with a weaker structure that we call a *summability structure*: such a category $\mathcal{L}$ is endowed with an endofunctor $\mathbf{S}$ which intuitively maps any object X to the object $\mathbf{S}\,X$ of pairs (x_0, x_1) such that $x_0 + x_1$ is well defined. This functor comes with natural transformations $\pi_0, \pi_1, \sigma \in \mathcal{L}(\mathbf{S}\,X, X)$ satisfying suitable axioms (intuitively they map (x_0, x_1) to x_0, x_1 and $x_0 + x_1$ respectively). Thanks to these axioms it is possible to equip $\mathbf{S}$ with a monad structure.

Then assuming that $\mathcal{L}$ is a resource category (that is, a cartesian symmetric monoidal category equipped with a resource modality comonad $!\_$), the differential structure is axiomatized as a natural morphism $\partial_X \in \mathcal{L}(!\mathbf{S}\,X, \mathbf{S}\,!X)$ which is a distributive law between the functor $\mathbf{S}$ and the comonad $!\_$, and also between the monad $\mathbf{S}$ and the functor $!\_$. This allows to extend the monad $\mathbf{S}$ to the Kleisli category $\mathcal{L}_!$ into a monad that we denote as $(\mathsf{D}, \zeta, \theta)$. The category $\mathcal{L}_!$ has the same objects as $\mathcal{L}$ but an element of $\mathcal{L}_!(X, Y)$ should not be considered as a linear morphism $X \to Y$ as in $\mathcal{L}(X, Y)$, but as a morphism which is only "smooth" (and actually analytic in several concrete models). The functor D acts exactly as $\mathbf{S}$ on objects but its action on morphisms implements differentiation: given $f \in \mathcal{L}_!(X, Y)$, considered as a smooth map $X \to Y$, the smooth map $\mathsf{D}\,f = (\mathbf{S}\,f)\,\partial_X \in \mathcal{L}_!(\mathsf{D}\,X\ \mathsf{D}\,Y)$ intuitively maps (x_0, x_1) to $(f(x_0), f'(x_0) \cdot x_1)$. The basic observation at the core of this work is indeed that, if $x_0 + x_1$ is defined, then so must be $f(x_0) + f'(x_0) \cdot x_1$, as the beginning of the Taylor expansion of $f(x_0 + x_1)$ at x_0. The monad structure of D accounts for addition: intuitively the natural transformation $\theta_X \in \mathcal{L}_!(\mathsf{D}^2\,X, \mathsf{D}\,X)$, which is linear[3], maps $((x_{00}, x_{01}), (x_{10}, x_{11}))$ to $(x_{00}, x_{01} + x_{10})$ and $\zeta_X \in \mathcal{L}_!(X, \mathsf{D}\,X)$ maps x to $(x, 0)$. The

[1] Actually the Leibniz rule is not really related to multiplication but more fundamentally to the fact that the parameter of a function can be used more than once that is, to the logical rule of contraction.

[2] Categories enriched over commutative monoids.

[3] In the sense that it is obtained from a morphism of $\mathcal{L}(\mathbf{S}^2\,X, \mathbf{S}\,X)$ by composition with the counit $\mathsf{der}_{\mathbf{S}^2\,X} \in \mathcal{L}(!\mathbf{S}^2\,X, \mathbf{S}^2\,X)$ of the comonad $!\_$.

naturality of these two transformations in $\mathcal{L}_!$ expresses exactly that the differential is linear with respect to this 0 and to this addition.

The major benefit of using the operation D for presenting the derivative of morphisms of $\mathcal{L}_!$ is that doing so we preserve the information of summability: we know that the two components of $\mathsf{D}\,f$ can be added without requiring that *all* pairs of morphisms can be added. The categorical axiomatization described in [Ehr21] involves other natural linear transformations: $\pi_i \in \mathcal{L}_!(\mathsf{D}\,X, X)$ for $i = 0, 1$ (the two obvious projections), $\iota_i \in \mathcal{L}_!(X, \mathsf{D}\,X)$ for $i = 0, 1$ (the two injections, $\iota_0 = \zeta$ and ι_1 maps x to $(0, x)$) and the "canonical flip" $\mathsf{c} \in \mathcal{L}_!(\mathsf{D}^2\,X, \mathsf{D}^2\,X)$ which maps $((x_{00}, x_{01}), (x_{10}, x_{11}))$ to $((x_{00}, x_{10}), (x_{01}, x_{11}))$.

As explained in the introduction of [Ehr21] this categorical axiomatization is very close to the *tangent categories* axiomatization of the differential calculus on smooth manifolds [Ros84], one major difference being that, in tangent categories there is an unrestricted addition operation available on "tangent spaces" which is not available here.

1.1. **Contents.** In the present paper we propose a differential lambda-calculus which uses the same idea, but now at the syntactical level. This new simply typed lambda-calculus (*à la* Church) is very much inspired by our categorical description of coherent differentiation in [Ehr21]. A major feature of this new presentation of differentiation is that it is fully compatible with *general recursive definitions* in PCF style, see [Plo77]. This is deeply related to the limitation we put on the + operation: contrarily to the ordinary differential lambda-calculus, our typing rules do not allow to write a term such that $\lambda x^\iota\,(x + \underline{42})$ which obviously has no fixpoint (ι is the type of integers).

1.1.1. *Syntax.* So our calculus is a differential extension of PCF[4] where the main novelty is a differentiation operation on terms: if $\Gamma \vdash M : A \Rightarrow B$ then $\Gamma \vdash \mathsf{D}M : \mathsf{D}A \Rightarrow \mathsf{D}B$ and this requires a new type construct $\mathsf{D}A$. The semantics tells us that we should have $\mathsf{D}(A \Rightarrow B) = (A \Rightarrow \mathsf{D}B)$ and we can consider this equation as a *definition* of $\mathsf{D}(A \Rightarrow B)$. So the only basic differential construct on types that we need is $\mathsf{D}^d\iota$ for all $d \in \mathbb{N}$. This differential term construction induces a new redex, namely the term $\mathsf{D}(\lambda x^A\,M)$ (for a term M such that $\Gamma, x : A \vdash M : B$) and we stipulate that it reduces to $\lambda x^{\mathsf{D}A}\,\partial(x, M)$ where the term $\partial(x, M)$ is defined by induction on M and satisfies $\Gamma, x : \mathsf{D}A \vdash \partial(x, M) : \mathsf{D}B$.

This inductive definition of $\partial(x, M)$ involves the use of constructs $\theta^d(N)$, $\iota_0^d(N)$ etc which syntactically account for the natural transformations alluded to above. The additional superscript d is required to express the fact that the corresponding operations are applied under d applications of the D functor. For instance in the rule $\partial(x, (N)P) = (\theta^0(\mathsf{D}\partial(x, N)))\partial(x, P)$, the $\theta^0(\_)$ construction implements the addition involved in the Leibniz rule required by the fact that the variable x may occur in both N and P, as in the Equation (1.1) of the differential lambda-calculus. We also stipulate that $\partial(x, \theta^d(N)) = \theta^{d+1}(\partial(x, N))$ which shows why the d superscripts are required. The same superscripts appear in the basic "arithmetic" constructs $\mathsf{succ}^d(N)$, $\mathsf{pred}^d(N)$, $\mathsf{if}^d(N, P_0, P_1)$ and also in the aforementioned let construct $\mathsf{let}^d(y, N, M)$. When applying the $\partial(x, \_)$ to these constructs, the d superscript is similarly incremented, for instance $\partial(x, \mathsf{pred}^d(N)) = \mathsf{pred}^{d+1}(\partial(x, N))$. The $\mathsf{c}^d(\_)$ construct is required because, guided by the semantics, we set $\partial(x, \mathsf{D}N) =$

[4]More precisely, of a version of PCF extended with a let operation restricted to the unique ground type ι of integers; this is particularly relevant for probabilistic extensions of this calculus in the spirit of [EPT18a], which is perfectly possible since it admits **Pcoh** as a natural denotational model.

$\mathsf{c}^0(\mathsf{D}\partial(x,N))$. Notice that the simple typing rules are not sufficient to "guess" this rule since in this situation we have $\Gamma, x : A \vdash N : B \Rightarrow C$ and hence $\Gamma, x : \mathsf{D}A \vdash \mathsf{D}\partial(x,N) : \mathsf{D}A \Rightarrow \mathsf{D}^2 B$ and also $\Gamma, x : \mathsf{D}A \vdash \mathsf{c}^0(\mathsf{D}\partial(x,N)) : \mathsf{D}A \Rightarrow \mathsf{D}^2 B$, the semantical analysis that we develop in Section 4 is really mandatory. The syntax also contains a *projection* construct $\pi^r_d(M)$ where $r \in \{0,1\}$ and $d \in \mathbb{N}$ is the depth we are now acquainted with. This construct allows to "extract" the first (when $r = 0$) or second (when $r = 1$) component of a term of type $\mathsf{D}A$ which represents a kind of summable pair[5]. The language also contains a $+$ operation[6] on terms which is used in a single reduction rule, namely $\pi^d_1(\theta^d(M)) \to_{\Lambda_{\mathsf{cd}}} \pi^d_0(\pi^d_1(M)) + \pi^d_1(\pi^d_0(M))$. So we can understand the construction $\theta^d(\_)$ as a tag identifying a place where a sum will have to be introduced when we will need to extract a component from a "summable 4-tuple" of type $\mathsf{D}^2 A$ for some type A and the reduction rules show how these tags are produced and modified during computations. The $+$ term construct is still necessary, just as in the original differential lambda-calculus of [ER03], but thanks to the $\theta^d(\_)$ construct we can prove subject reduction *without* the very strong typing rule according to which $M + N$ has type A as soon M and N have type A, which was the source of the non-determinism of the differential lambda-calculus[7].

1.1.2. *Categorical semantics and soundness.* After having presented the syntax of our calculus Λ_{cd} in Section 3, we recall in section 4 the general categorical setting introduced in [Ehr21] for coherent differentiation. Our aim in this section is to present a general categorical semantics of our calculus Λ_{cd} in the cartesian closed category associated with such a model by the familiar Kleisli construction associated with the $!\_$ functor and to prove that this interpretation is invariant under the $\to_{\Lambda_{\mathsf{cd}}}$ reduction relation (soundness). This requires to prove a number of categorical equations involving in particular *additive strengths*[8] $\psi^0_{X_0,X_1} \in \mathcal{L}_!(\mathsf{D}\,X_0 \;\&\; X_1, \mathsf{D}\,(X_0 \;\&\; X_1))$ and $\psi^1_{X_0,X_1} \in \mathcal{L}_!(X_0 \;\&\; \mathsf{D}\,X_1, \mathsf{D}\,(X_0 \;\&\; X_1))$ of the monad D on $\mathcal{L}_!$. These strengths are linear and extremely easy to define because D commutes with &, but their properties are not so straightforward due to the definition of the functor D which involves the distributive law ∂. Their main purpose is to define *partial derivatives*: given $f \in \mathcal{L}_!(X_0 \;\&\; X_1, Y)$ the first partial derivative of f is $(\mathsf{D}\,f) \circ \psi^0_{X_0,X_1} \in \mathcal{L}_!(\mathsf{D}\,X_0 \;\&\; X_1, Y)$ and the second partial derivative is $(\mathsf{D}\,f) \circ \psi^1_{X_0,X_1} \in \mathcal{L}_!(X_0 \;\&\; \mathsf{D}\,X_1, Y)$. These constructions are of course crucial in the interpretation of Λ_{cd} since, for instance, $\partial(x,M)$ must be interpreted as a derivative with respect to the variable x but not to the other variables occurring in M. Notice that we use $\circ$ for the composition operation in $\mathcal{L}_!$ whereas composition in $\mathcal{L}$ is denoted as simple juxtaposition.

[5]From our LL point of view it is important to understand that it is not a *multiplicative pair* in which both components are actually available, but an *additive pair* which offers two possible options among which one must be chosen: this is precisely the purpose of our projection construct.

[6]Of course there is also its associated "neutral" constant 0.

[7]To be more precise, as it becomes apparent in Section 5.1 where a specific reduction strategy is presented by means of a Krivine machine, we manage to reduce the use of the $+$ to the ground type ι of integers. This is quite similar to what happens in automated differentiation, see [MP21], the main differences being that our addition is not at all the arithmetic addition of the ground type and that we alloww differentiation wrt. parameters of all types, not only ground types.

[8]The adjective "additive" refers to the fact that this strength deals with the additive operation of cartesian product and not to the multiplicative operation of tensor product. It is shown in [Ehr21] that the monad $\mathbf{S}$ has multiplicative strengths as well which are deeply related to these additive strengths. Notice that both strong monads are commutative.

Thanks to this series of categorical lemmas we can prove soundness for the $\rightarrow_{\Lambda_{\mathsf{cd}}}$, the proof is lengthy due mainly to the number of rewriting rules.

1.1.3. *Intersection types, Krivine machine and completeness.* We address in Section 5 the major issue of *completeness*: is our rewriting system $\rightarrow_{\Lambda_{\mathsf{cd}}}$ sufficiently rich for performing all required reductions? To give a precise mathematical content to this question, we use the simplest categorical model (in the sense of Section 4) of Λ_{cd} which is **Rel** and we present the associated interpretation of terms by means of an *non-idempotent intersection typing system* for Λ_{cd}. The main result of this section is that if the interpretation in **Rel** of a closed term M contains an integer ν then M reduces to $\underline{\nu}$ using the $\rightarrow_{\Lambda_{\mathsf{cd}}}$ reduction relation. As usual in this kind of situation we prove this property for a particular reduction strategy that we prefer to present as an abstract machine in Krivine style. A state, or *command*, of this machine is a triple $c = (\delta, M, s)$ where

- M is a closed term of type $\mathsf{D}^d F$ where F is a type which is *sharp* in the sense that it is not of shape $\mathsf{D}A$; in other words $F = (A_1 \Rightarrow \cdots \Rightarrow A_k \Rightarrow \iota)$,
- $\delta = \langle r_1, \dots, r_d \rangle$ is a sequence of length d of elements of $\{0, 1\}$ called access word
- and s is a stack of type $F \vdash \iota$, meaning that it represents an evaluation context $\langle s \rangle$ of type ι whose "hole" has type F.

Then the command c represents the term $\langle c \rangle = \langle s \rangle [\pi^0_{r_1}(\cdots \pi^0_{r_d}(M) \cdots)]$ which is closed of type ι.

We introduce a deterministic set of reduction rules $\rightarrow_{\Theta_{\mathsf{cd}}}$ for this machine and prove that these reduction rules are simulated in the $\rightarrow_{\Lambda_{\mathsf{cd}}}$ rewriting system through the $c \mapsto \langle c \rangle$ translation. More precisely this rewriting system acts on *finite multisets of commands* which are summable in the sense that the interpretations of their elements are summable in any model. We extend the semantics (and, accordingly, the intersection typing system) to stacks and commands and prove that, if a command is typeable in the intersection typing system (and then its type is a natural number ν) then its $\rightarrow_{\Theta_{\mathsf{cd}}}$ reduction leads to a summable multiset C of commands which contains the constant $\underline{\nu}$ (or more precisely the command $(\langle\rangle, \underline{\nu}, ()))$, thus proving our completeness result. This proof follows essentially the standard pattern of a reducibility argument, complicated by the fact that we have to take into account arbitrary iterations of the $\partial(x, M)$ construct. This method is developed in Section 5.3.2.

1.1.4. *Probabilistic semantics and determinism.* In Section 6 we prove that the integer ν above is unique, thus showing that the reduction on commands is essentially deterministic. To this end we use the fact that the LL model of probabilistic coherence spaces introduced in [DE11] is a model of coherent differentiation. In that model the type ι is interpreted as the set of all sub-probability distributions on $\mathbb{N}$ and we observe[9] that a summable multiset of commands must be interpreted as such a sub-probability distribution where *all probabilities belong to* $\mathbb{N}$ and hence is either equal to 0 or concentrated on a single element ν of $\mathbb{N}$. So all the elements c' of C distinct from the command $(\langle\rangle, \underline{\nu}, ())$ must have an $\emptyset$ interpretation in **Rel** and hence cannot reduce to a value. In spite of this strong result, the fact that the rewriting system for this machine has still to deal with multisets of states means that it still contains a little bit of non-determinism.

[9]This is due to the fact that the formalism Λ_{cd} considered here has no construct for generating random integers as in [EPT18a].

Finally using an idea suggested by Guillaume Geoffroy, we get rid of this non-determinism by slightly modifying our Krivine machine. The main change consists in making the access word writable. We can prove simulation results relating this new machine with the original one which allow to prove that the new machine, whilst being fully deterministic, computes the same thing as the original one, in the same number of steps.

Contents

2. Preliminaries

2.1. **Notations.** Given a sequence $\alpha = \langle i_1, \dots, i_k \rangle$, we set $\mathsf{len}(\alpha) = k$. We use simple juxtaposition to denote the concatenation of sequences and also $i\langle i_1, \dots, i_k \rangle = \langle i, i_1, \dots, i_k \rangle$. We use the following notation for circular permutations: $\underset{\rightarrow}{\alpha} = \langle i_k, i_1, \dots, i_{k-1} \rangle$ and $\underset{\leftarrow}{\alpha} = \langle i_2, \dots, i_k, i_1 \rangle$. Of course if $\mathsf{len}(\alpha) = 2$ we have $\underset{\rightarrow}{\alpha} = \underset{\leftarrow}{\alpha}$. We also use $\overline{\alpha}$ for the reversed word (that is, if $\alpha = a_1 \cdots a_l$ then $\overline{\alpha} = a_l \cdots a_1$).

We use $\mathcal{M}_{\mathrm{fin}}(I)$ for the set of finite multisets of elements of a set I. A multiset is a function $m : I \to \mathbb{N}$ such that $\mathsf{supp}(m) = \{i \in I \mid m(i) \neq 0\}$ is finite. We use additive notations for operations on multisets (0 for the empty multiset, $m + p$ for their pointwise sum). We use $[i_1, \dots, i_k]$ for the multiset m such that $m(i) = \#\{j \in \mathbb{N} \mid i_j = i\}$. If $m = [j_1, \dots, j_n] \in \mathcal{M}_{\mathrm{fin}}(J)$ and $i \in I$ we set $i * m = [(i, j_1), \dots, (i, j_n)] \in \mathcal{M}_{\mathrm{fin}}(I \times J)$.

We use $I \uplus J$ to denote $I \cup J$ when $I \cap J = \emptyset$.

2.2. **Rewriting.** Let $\mathcal{T} = (\underline{\mathcal{T}}, \to_{\mathcal{T}})$ be a rewriting system, that is $\underline{\mathcal{T}}$ is a set and $\to_{\mathcal{T}} \subseteq \underline{\mathcal{T}}^2$. We assume that $\underline{\mathcal{T}}$ contains a distinguished element 0 and that there is a binary operation + on $\underline{\mathcal{T}}$: given $t_1, t_2 \in \underline{\mathcal{T}}$ there is an element $t_1 + t_2 \in \underline{\mathcal{T}}$. We make no further assumptions, in particular we do not assume that equipped with 0 and +, the set $\underline{\mathcal{T}}$ is a monoid. We define a rewriting system $\mathcal{M}_{\mathrm{fin}}(\mathcal{T})$ by $\underline{\mathcal{M}_{\mathrm{fin}}(\mathcal{T})} = \mathcal{M}_{\mathrm{fin}}(\underline{\mathcal{T}})$ and rewriting relation defined by the following rules

$$\frac{}{[0] \to_{\mathcal{M}_{\mathrm{fin}}(\mathcal{T})} [\,]} \qquad \frac{}{[t_1 + t_2] \to_{\mathcal{M}_{\mathrm{fin}}(\mathcal{T})} [t_1, t_2]} \qquad \frac{t \to_{\mathcal{T}} t'}{[t] \to_{\mathcal{M}_{\mathrm{fin}}(\mathcal{T})} [t']}$$

$$\frac{S \to_{\mathcal{M}_{\mathrm{fin}}(\mathcal{T})} S'}{S + T \to_{\mathcal{M}_{\mathrm{fin}}(\mathcal{T})} S' + T}$$

In other words, we have $S \to_{\mathcal{M}_{\mathrm{fin}}(\mathcal{T})} S'$ iff one of the following conditions hold.

- $S = S_0 + [0]$ and $S' = S_0$.
- $S = S_0 + [t]$, $t \to_{\mathcal{T}} t'$ and $S' = S_0 + [t']$.
- $S = S_0 + [t_0 + t_1]$ and $S' = S_0 + [t_0, t_1]$.

3. Syntax of Λ_{cd}

Our choice of notations for Λ_{cd} is fully coherent with the notations chosen to describe the model, suggesting a straightforward denotational interpretation. The types are $A, B, \dots := \mathsf{D}^d\iota \mid A \Rightarrow B$ (with $d \in \mathbb{N}$) and then for any type A we define $\mathsf{D}A$ as follows: $\mathsf{D}(\mathsf{D}^d\iota) = \mathsf{D}^{d+1}\iota$ and $\mathsf{D}(A \Rightarrow B) = (A \Rightarrow \mathsf{D}B)$. Terms are given by

$$\begin{aligned} M, N, \dots := x \mid \lambda x^A\, M \mid (M)N \mid \mathsf{Y}M \mid \underline{n} \mid \mathsf{succ}^d(M) \mid \mathsf{pred}^d(M) \mid \mathsf{if}^d_A(M, P, Q) \\ \mid \mathsf{let}^d_A(x, M, P) \mid \mathsf{D}M \mid \pi^d_i(M) \mid \iota^d_i(M) \mid \theta^d(M) \mid \mathsf{c}^d_l(M) \mid 0^A \mid M + N \end{aligned}$$

where $n, d, l \in \mathbb{N}$ and $i \in \{0, 1\}$, so that our syntax has countably many constructs.

3.1. **The typing system.** The typing system uses a reduction relation $\to_{\mathsf{lin}}$ expressing that most constructs are linear wrt. 0 and addition of terms; it is specified in Figure 1 and is based on the following notion of *linear context*

$$\begin{aligned} L := [\,] \mid \lambda x^A\, L \mid (L)N \mid \mathsf{succ}^d(L) \mid \mathsf{pred}^d(L) \mid \mathsf{if}^d(L, P, Q) \mid \mathsf{let}^d(x, L, P) \\ \mid \mathsf{D}L \mid \pi^d_i(L) \mid \iota^d_i(L) \mid \theta^d(L) \mid \mathsf{c}^d_l(L)\,. \end{aligned} \tag{3.1}$$

The *height* $\mathsf{lh}(L)$ of a linear context L is the distance between its hole and its root, in other words $\mathsf{lh}([\,]) = 0$, $\mathsf{lh}(\lambda x^A\, L) = 1 + \mathsf{lh}(L)$, $\mathsf{lh}(\mathsf{if}^d(L, P, Q)) = 1 + \mathsf{lh}(L)$ etc.

$$\frac{}{L[0] \to_{\mathsf{lin}} 0} \qquad \frac{}{L[M_0 + M_1] \to_{\mathsf{lin}} L[M_0] + L[M_1]} \qquad \frac{M \to_{\mathsf{lin}} M'}{L[M] \to_{\mathsf{lin}} L[M']}$$

FIGURE 1. Linear reduction, L must be a linear context of height 1.

$$\frac{i \in \{1,\dots,k\}}{(x_1 : A_1,\dots,x_k : A_k) \vdash x_i : A_i}\ (\mathbf{var}) \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A\, M : A \Rightarrow B}\ (\mathbf{abs})$$

$$\frac{\Gamma \vdash M : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B}\ (\mathbf{app}) \qquad \frac{\Gamma \vdash M : A \Rightarrow A}{\Gamma \vdash \mathsf{Y}M : A}\ (\mathbf{fix}) \qquad \frac{n \in \mathbb{N}}{\Gamma \vdash \underline{n} : \iota}\ (\mathbf{num})$$

$$\frac{\Gamma \vdash M : \mathsf{D}^d\iota}{\Gamma \vdash \mathsf{succ}^d(M) : \mathsf{D}^d\iota}\ (\mathbf{suc}) \qquad \frac{\Gamma \vdash M : \mathsf{D}^d\iota}{\Gamma \vdash \mathsf{pred}^d(M) : \mathsf{D}^d\iota}\ (\mathbf{prd})$$

$$\frac{\Gamma \vdash M : \mathsf{D}^d\iota \quad \Gamma \vdash P : A \quad \Gamma \vdash Q : A}{\Gamma \vdash \mathsf{if}^d_A(M,P,Q) : \mathsf{D}^d A}\ (\mathbf{if}) \qquad \frac{}{\Gamma \vdash 0^A : A}\ (\mathbf{0})$$

$$\frac{\Gamma \vdash M : \mathsf{D}^{d+1}A \quad i \in \{0,1\}}{\Gamma \vdash \pi^d_i(M) : \mathsf{D}^d A}\ (\mathbf{proj1}) \qquad \frac{\Gamma \vdash M : \mathsf{D}^{d+1}A}{\Gamma \vdash \pi^d_0(M) + \pi^d_1(M) : \mathsf{D}^d A}\ (\mathbf{proj2})$$

$$\frac{\Gamma \vdash M_0 + M_1 : \mathsf{D}^{d+1}A}{\Gamma \vdash \pi^d_1(M_0) + \pi^d_0(M_1) : \mathsf{D}^d A}\ (\mathbf{projd}) \qquad \frac{\Gamma \vdash M : A \quad M \to_{\mathsf{lin}} M'}{\Gamma \vdash M' : A}\ (\mathbf{lin})$$

$$\frac{\Gamma \vdash M : \mathsf{D}^d A \quad i \in \{0,1\}}{\Gamma \vdash \iota^d_i(M) : \mathsf{D}^{d+1}A}\ (\mathbf{inj}) \qquad \frac{\Gamma \vdash M : \mathsf{D}^{d+2}A}{\Gamma \vdash \theta^d(M) : \mathsf{D}^{d+1}A}\ (\mathbf{sum})$$

$$\frac{\Gamma \vdash M : \mathsf{D}^{d+l+2}A}{\Gamma \vdash \mathsf{c}^d_l(M) : \mathsf{D}^{d+l+2}A}\ (\mathbf{circ})$$

$$\frac{\Gamma \vdash M : A \Rightarrow B}{\Gamma \vdash \mathsf{D}M : \mathsf{D}A \Rightarrow \mathsf{D}B}\ (\mathbf{diff}) \qquad \frac{\Gamma \vdash M : \mathsf{D}^d\iota \quad \Gamma, x : \iota \vdash N : B}{\Gamma \vdash \mathsf{let}^d_A(x, M, N) : \mathsf{D}^d B}\ (\mathbf{let})$$

FIGURE 2. Typing rules

Remark 3.1. We have decorated the conditional and the let constructs with a type, which is intended to be the type of its last parameter(s). The only purpose of this decoration is to provide a type for the resulting 0 in the linear reduction of $\mathsf{if}^d(0, P, Q)$ and $\mathsf{let}^d(x, 0, P)$ in Figure 1. Most often, we will drop this type decoration which can easily be retrieved from the context.

Lemma 3.2. *For any linear context L we have $L[0] \to^*_{\mathsf{lin}} 0$ and $L[M_0 + M_1] \to^*_{\mathsf{lin}} L[M_0] + L[L_1]$.*

Proof. By induction on $\mathsf{lh}(L)$. If $\mathsf{lh}(L) = 0$ we use the fact that $R \to^*_{\mathsf{lin}} R$. Otherwise we have $L = K[L']$ where $\mathsf{lh}(K) = 1$ and $\mathsf{lh}(L') = \mathsf{lh}(L) - 1$. By inductive hypothesis $L'[M_0 + M_1] \to^*_{\mathsf{lin}} L'[M_0] + L'[M_1]$ and hence by definition of $\to_{\mathsf{lin}}$ we have $L[M_0 + M_1] \to^*_{\mathsf{lin}} K[L'[M_0] + L'[M_1]]$ and by definition of $\to_{\mathsf{lin}}$ again we have $K[L'[M_0] + L'[M_1]] \to_{\mathsf{lin}} L[M_0] + L[M_1]$. The case of $L[0]$ is similar. □

One should think of a term of type $\mathsf{D}^k A$ as a complete binary tree of height k whose leaves have type A. In constructs such as $\mathsf{succ}^d(M)$, the integer d represents the "depth" at which the corresponding operation is performed in a tree of type $\mathsf{D}^d A$ with $k \geq d$. The

$$\partial(x,y) = \begin{cases} x & \text{if } y = x \\ \iota_0(y) & \text{otherwise} \end{cases} \qquad \partial(x,\lambda y^B\, P) = \lambda y^B\, \partial(x,P)$$
$$\partial(x,\mathsf{D}M) = \mathsf{c}(\mathsf{D}\partial(x,M)) \qquad \partial(x,(P)Q) = (\theta(\mathsf{D}\partial(x,P)))\partial(x,Q)$$
$$\partial(x,\mathsf{Y}M) = \mathsf{Y}(\theta(\mathsf{D}\partial(x,M))) \qquad \partial(x,\underline{n}) = \iota_0(\underline{n})$$
$$\partial(x,\mathsf{succ}^d(M)) = \mathsf{succ}^{d+1}(\partial(x,M)) \qquad \partial(x,\mathsf{pred}^d(M)) = \mathsf{pred}^{d+1}(\partial(x,M))$$
$$\partial(x,\mathsf{if}^d(M,P,Q)) = \theta(\mathsf{c}_d(\mathsf{if}^{d+1}(\partial(x,M),\partial(x,P),\partial(x,Q))))$$
$$\partial(x,\mathsf{let}^d(y,P,Q)) = \theta(\mathsf{c}_d(\mathsf{let}^{d+1}(y,\partial(x,P),\partial(x,Q))))$$
$$\partial(x,0) = 0 \qquad \partial(x,M_0+M_1) = \partial(x,M_0)+\partial(x,M_1)$$
$$\partial(x,\pi_i^d(M)) = \pi_i^{d+1}(\partial(x,M)) \qquad \partial(x,\theta^d(M)) = \theta^{d+1}(\partial(x,M))$$
$$\partial(x,\iota_i^d(M)) = \iota_i^{d+1}(\partial(x,M)) \qquad \partial(x,\mathsf{c}_l^d(M)) = \mathsf{c}_l^{d+1}(\partial(x,M))$$

FIGURE 3. Inductive definition of the differential of a term

main intuitive feature of such a tree is that its leaves are summable. When $d = 0$ we often drop the superscript.

We provide a typing system in Figure 2 allowing to prove typing judgments $\Gamma \vdash M : A$. Notice that in general, when $\Gamma \vdash N_0 : A$ and $\Gamma \vdash N_1 : A$, it is not necessarily true that $\Gamma \vdash N_0 + N_1 : A$.

3.2. **Differential.** Given a variable x and a term N, we define a term $\partial(x, M)$ in Figure 3 which is called the *differential* of M with respect to x.

Lemma 3.3. *Let L be a linear context. There is a linear context $\partial(x, L)$ such that, for any term M, we have $\partial(x, L[M]) = \partial(x, L)[\partial(x, M)]$.*

Proof. It suffices to deal with linear contexts L such that $\mathsf{lh}(L) = 1$, the general result is then obtained by a straightforward induction. The announced property results from a simple analysis of the definition of $\partial(x, M)$ in Figure 3. We give a few examples.

- ▶ $\partial(x, \lambda y^B\,[\,]) = \lambda y^B\,[\,]$
- ▶ $\partial(x, ([\,])P) = \theta((\mathsf{D}[\,])\partial(x, P))$
- ▶ $\partial(x, \mathsf{D}[\,]) = \mathsf{c}_0(\mathsf{D}[\,])$
- ▶ $\partial(x, \mathsf{if}^d([\,], P, Q)) = \theta(\mathsf{c}_d(\mathsf{if}^{d+1}([\,], \partial(x, P), \partial(x, Q))))$
- ▶ $\partial(X, \mathsf{let}^d(y, [\,], Q)) = \theta(\mathsf{c}_d(\mathsf{let}^{d+1}(y, [\,], \partial(x, Q))))$
- ▶ $\partial(x, \pi_i^d([\,])) = \pi_i^{d+1}([\,])$. □

Lemma 3.4. *If $R \to_{\mathsf{lin}} R'$ and L is a linear context then $L[R] \to_{\mathsf{lin}} L[R']$. We also have $L[0] \to_{\mathsf{lin}}^* 0$ and $L[R_0 + R_1] \to_{\mathsf{lin}}^* L[R_0] + L[R_1]$.*

Proof. Straightforward inductions on $\mathsf{lh}(L)$. □

Lemma 3.5. *If $R \to_{\mathsf{lin}} R'$ then $\partial(x, R) \to_{\mathsf{lin}}^* \partial(x, R')$.*

Proof. By induction on the derivation of $R \to_{\mathsf{lin}} R'$. Assume that $R = L[R_0 + R_1]$ and $R' = L[R_0] + L[R_1]$ with $\mathsf{lh}(L) = 1$. Using Lemma 3.3, we have

$$\begin{aligned}\partial(x, R) &= \partial(x, L)[\partial(x, R_0 + R_1)] \\ &= \partial(x, L)[\partial(x, R_0) + \partial(x, R_1)] \\ &\to_{\mathsf{lin}}^* \partial(x, L)[\partial(x, R_0)] + \partial(x, L)[\partial(x, R_1)] \\ &= \partial(x, L[R_0]) + \partial(x, L[R_1]) \\ &= \partial(x, L[R_0] + L[R_1])\end{aligned}$$

by Lemma 3.4.

Assume now that $R = L[M]$, $R' = L[M']$ and $M \to_{\mathsf{lin}} M'$. By inductive hypothesis we know that $\partial(x, M) \to_{\mathsf{lin}}^* \partial(x, M')$. We have $\partial(x, R) = \partial(x, L)[\partial(x, M)]$ and $\partial(x, R') = \partial(x, L)[\partial(x, M')]$ by Lemma 3.3 and hence $\partial(x, R) \to_{\mathsf{lin}}^* \partial(x, R')$ by Lemma 3.4. □

Lemma 3.6. *If* $\Gamma, x : A \vdash M : B$ *then* $\Gamma, x : \mathsf{D}A \vdash \partial(x, M) : \mathsf{D}B$.

Proof. We consider the following cases, the others are left to the reader.

▶ Assume first that $M = \mathsf{if}^d(P, Q_0, Q_1)$ and that the last typing rule is (**if**) so that $\Gamma, x : A \vdash P : \mathsf{D}^d\iota$ and $\Gamma, x : A \vdash Q_i : B$ for $i = 0, 1$. By inductive hypothesis we have $\Gamma, x : \mathsf{D}A \vdash \partial(x, P) : \mathsf{D}^{d+1}\iota$ and $\Gamma, x : \mathsf{D}A \vdash \partial(x, Q_i) : \mathsf{D}B$ for $i = 0, 1$. Applying the rule (**if**) we get $\Gamma, x : \mathsf{D}A \vdash \mathsf{if}^{d+1}(\partial(x, P), \partial(x, Q_0), \partial(x, Q_1)) : \mathsf{D}^{d+2}B$ and hence we have $\Gamma, x : \mathsf{D}A \vdash \mathsf{c}_d(\mathsf{if}^{d+1}(\partial(x, P), \partial(x, Q_0), \partial(x, Q_1))) : \mathsf{D}^{d+2}B$. Therefore

$$\Gamma, x : \mathsf{D}A \vdash \theta(\mathsf{c}_d(\mathsf{if}^{d+1}(\partial(x, P), \partial(x, Q_0), \partial(x, Q_1)))) : \mathsf{D}^{d+1}B\,.$$

Notice finally that $\mathsf{D}^{d+1}B = \mathsf{D}\mathsf{D}^dB$ is exactly the type expected for $\partial(x, M)$ in that case.

▶ Assume that $M = \mathsf{D}P$ and that the last typing rule is (**diff**) so that $\Gamma, x : A \vdash P : C \Rightarrow D$, $\Gamma, x : A \vdash M : \mathsf{D}C \Rightarrow \mathsf{D}D$ and $B = (\mathsf{D}C \Rightarrow \mathsf{D}D)$. By inductive hypothesis we have $\Gamma, x : \mathsf{D}A \vdash \partial(x, P) : C \Rightarrow \mathsf{D}D$ and hence $\Gamma, x : \mathsf{D}A \vdash \mathsf{D}\partial(x, P) : \mathsf{D}C \Rightarrow \mathsf{D}^2D = \mathsf{D}^2(\mathsf{D}C \Rightarrow D) = \mathsf{D}B$. It follows that $\Gamma, x : \mathsf{D}A \vdash \mathsf{c}(\mathsf{D}\partial(x, P)) : \mathsf{D}B$ as required.

▶ Assume next that $M = \lambda y^C\, P$ and that the last typing rule is (**abs**) so that $\Gamma, x : A, y : C \vdash P : D$ (and hence $B = (C \Rightarrow D)$). By inductive hypothesis $\Gamma, x : \mathsf{D}A, y : C \vdash \partial(x, P) : \mathsf{D}D$ and hence $\Gamma, x : \mathsf{D}A \vdash \lambda y^B\, \partial(x, P) : (C \Rightarrow \mathsf{D}D) = \mathsf{D}B$ as required.

▶ Assume now that $M = (P)Q$ and that the last typing rule is (**app**) with $\Gamma, x : A \vdash P : C \Rightarrow B$ and $\Gamma, x : A \vdash Q : C$. Then by inductive hypothesis we have $\Gamma, x : \mathsf{D}A \vdash \partial(x, P) : \mathsf{D}(C \Rightarrow B) = (B \Rightarrow \mathsf{D}C)$ and $\Gamma, x : \mathsf{D}A \vdash \partial(x, Q) : \mathsf{D}C$. Therefore

$$\Gamma, x : \mathsf{D}A \vdash \mathsf{D}\partial(x, P) : \mathsf{D}C \Rightarrow \mathsf{D}^2B = \mathsf{D}^2(\mathsf{D}C \Rightarrow B)$$

and hence $\Gamma, x : \mathsf{D}A \vdash \theta(\mathsf{D}\partial(x, P)) : \mathsf{D}C \Rightarrow \mathsf{D}B$ so that $\Gamma, x : \mathsf{D}A \vdash (\theta(\mathsf{D}\partial(x, P)))\partial(x, Q) : \mathsf{D}B$ by the rules (**sum**) and (**app**).

▶ Assume that $M = \mathsf{c}_l^d(P)$ and that the last typing rule is (**circ**) with $\Gamma, x : A \vdash P : \mathsf{D}^{l+d+2}C = B$. By inductive hypothesis we have $\Gamma, x : \mathsf{D}A \vdash \partial(x, P) : \mathsf{D}^{l+d+3}C$ and hence $\Gamma, x : \mathsf{D}A \vdash \mathsf{c}_l^{d+1}(\partial(x, P)) : \mathsf{D}^{l+d+3}C = \mathsf{D}B$ by applying the rule (**circ**).

▶ Assume that $M = \mathsf{Y}P$ and that the last typing rule is (**fix**) with $\Gamma, x : A \vdash P : B \Rightarrow B$ so that $\Gamma, x : \mathsf{D}A \vdash \partial(x, P) : B \Rightarrow \mathsf{D}B$ and hence $\Gamma, x : \mathsf{D}A \vdash \mathsf{D}\partial(x, P) : \mathsf{D}B \Rightarrow \mathsf{D}^2B$ by (**diff**) and therefore $\Gamma, x : \mathsf{D}A \vdash \theta(\mathsf{D}\partial(x, P)) : \mathsf{D}B \Rightarrow \mathsf{D}B$ by (**sum**) and finally $\Gamma, x : \mathsf{D}A \vdash \mathsf{Y}(\theta(\mathsf{D}\partial(x, P))) : \mathsf{D}B$ by (**fix**).

$$
\begin{array}{rlrl}
(\lambda x^A\, M)N & \to_{\Lambda_{\mathsf{cd}}} M\,[N/x] & \mathsf{D}(\lambda x^A\, M) & \to_{\Lambda_{\mathsf{cd}}} \lambda x^{\mathsf{D}A}\, \partial(x, M) \\
\mathsf{succ}^0(\underline{n}) & \to_{\Lambda_{\mathsf{cd}}} \underline{n+1} & \mathsf{pred}^0(\underline{0}) & \to_{\Lambda_{\mathsf{cd}}} \underline{0} \\
\mathsf{pred}^0(\underline{n+1}) & \to_{\Lambda_{\mathsf{cd}}} \underline{n} & \mathsf{if}^0(\underline{0}, P, Q) & \to_{\Lambda_{\mathsf{cd}}} P \\
\mathsf{if}^0(\underline{n+1}, P, Q) & \to_{\Lambda_{\mathsf{cd}}} Q & \mathsf{let}^0(x, \underline{n}, P) & \to_{\Lambda_{\mathsf{cd}}} P\,[x/\underline{n}] \\
\mathsf{Y}P & \to_{\Lambda_{\mathsf{cd}}} (P)\mathsf{Y}P & &
\end{array}
$$

FIGURE 4. Main reduction rules

▶ Assume that the last typing rule is (**proj1**) meaning that we have $M = \pi_i^d(P)$ and $B = \mathsf{D}^d C$ with $\Gamma, x : A \vdash P : \mathsf{D}^{d+1}C$. Then by inductive hypothesis we have $\Gamma, x : \mathsf{D}A \vdash \partial(x, P) : \mathsf{D}^{d+2}C$ and hence $\Gamma, x : \mathsf{D}A \vdash \pi_i^{d+1}(\partial(x, P)) : \mathsf{D}^{d+1}C$ by the rule (**proj1**).

▶ Assume that the last typing rule is (**proj2**) so that $M = \pi_0^d(P) + \pi_1^d(P)$ and $B = \mathsf{D}^d C$ with $\Gamma, x : A \vdash P : \mathsf{D}^{d+1}C$. By inductive hypothesis we have $\Gamma, x : \mathsf{D}A \vdash \partial(x, P) : \mathsf{D}^{d+2}B$ and hence $\Gamma, x : \mathsf{D}A \vdash \pi_0^{d+1}(\partial(x, P)) + \pi_1^{d+1}(\partial(x, P)) : \mathsf{D}^{d+1}C$ by the rule (**proj2**). That is $\Gamma, x : \mathsf{D}A \vdash \partial(x, M) : \mathsf{D}B$ as expected.

▶ Assume that the last typing rule is (**projd**) so that $M = \pi_0^d(P_0) + \pi_1^d(P_1)$ with $\Gamma, x : A \vdash P_0 + P_1 : \mathsf{D}^{d+1}C$ and $B = \mathsf{D}^d C$. By inductive hypothesis we have

$$\Gamma, x : \mathsf{D}A \vdash \partial(x, P_0) + \partial(x, P_1) : \mathsf{D}^{d+2}C$$

and hence $\Gamma, x : \mathsf{D}A \vdash \pi_0^{d+1}(\partial(x, P_0)) + \pi_1^{d+1}(\partial(x, P_1)) : \mathsf{D}^{d+1}C$ that is $\Gamma, x : \mathsf{D}A \vdash \partial(x, M) : \mathsf{D}B$ as expected.

▶ Assume that the last typing rule is (**lin**) so that $\Gamma, x : A \vdash P : B$ and $P \to_{\mathsf{lin}} M$. By inductive hypothesis $\Gamma, x : \mathsf{D}A \vdash \partial(x, P) : \mathsf{D}B$ and we have $\partial(x, P) \to_{\mathsf{lin}}^* \partial(x, M)$ by Lemma 3.5 and hence $\Gamma, x : \mathsf{D}A \vdash \partial(x, M) : \mathsf{D}B$ by the rule (**lin**). □

3.3. **Reduction rules.** We define a rewriting system Λ_{cd}. The elements of $\underline{\Lambda_{\mathsf{cd}}}$ are the terms of the syntax introduced above. The main reduction rules are given in Figure 4. A second series of reduction rules given in Figure 5 specifies how the projections $\pi_j^d(M)$ interact with the other constructs. They are crucially used for "reading" the result of a computations by accessing leaves of a "tree" of type $\mathsf{D}^d A$ (complete binary tree of height d; the leaves can themselves be trees if $A = \mathsf{D}^e B$ with $e > 0$).

Remark 3.7. The two rules $\pi_i^d(\pi_j^e(M)) \to_{\Lambda_{\mathsf{cd}}} \pi_j^{e-1}(\pi_i^d(M))$ if $d < e$ and $\pi_i^d(\pi_j^e(M)) \to_{\Lambda_{\mathsf{cd}}} \pi_j^e(\pi_i^{d+1}(M))$ if $e \leq d$ lead clearly to infinite sequences of computations so it would be tempting to remove one of them from the rewriting system. However both seem necessary in order to prove the soundness of the stack machine that we introduce in Section 5.1.

We also need the reduction rule

$$\frac{M \to_{\mathsf{lin}} M'}{M \to_{\Lambda_{\mathsf{cd}}} M'}$$

$$\pi_i^d(\lambda x^A\, M) \to_{\Lambda_{\mathsf{cd}}} \lambda x^A\, \pi_i^d(M) \qquad \pi_i^d((M)N) \to_{\Lambda_{\mathsf{cd}}} (\pi_i^d(M))N$$

$$\pi_i^d(\mathsf{succ}^e(M)) \to_{\Lambda_{\mathsf{cd}}} \mathsf{succ}^{e-1}(\pi_i^d(M)) \quad \text{if } d < e \qquad \pi_i^d(\mathsf{pred}^e(M)) \to_{\Lambda_{\mathsf{cd}}} \mathsf{pred}^{e-1}(\pi_i^d(M)) \quad \text{if } d < e$$

$$\pi_i^d(\mathsf{if}^e(M,P,Q)) \to_{\Lambda_{\mathsf{cd}}} \mathsf{if}^{e-1}(\pi_i^d(M),P,Q) \quad \text{if } d < e \qquad \pi_i^d(\mathsf{if}^e(M,P,Q)) \to_{\Lambda_{\mathsf{cd}}} \mathsf{if}^e(M,\pi_i^{d-e}(P),\pi_i^{d-e}(Q)) \quad \text{if } e \leq d$$

$$\pi_i^d(\mathsf{let}^e(x,M,P)) \to_{\Lambda_{\mathsf{cd}}} \mathsf{let}^{e-1}(x,\pi_i^d(M),P) \quad \text{if } d < e \qquad \pi_i^d(\mathsf{let}^e(x,M,P)) \to_{\Lambda_{\mathsf{cd}}} \mathsf{let}^e(x,M,\pi_i^{d-e}(P)) \quad \text{if } e \leq d$$

$$\pi_0^d(\theta^d(M)) \to_{\Lambda_{\mathsf{cd}}} \pi_0^d(\pi_0^d(M)) \qquad \pi_1^d(\theta^d(M)) \to_{\Lambda_{\mathsf{cd}}} \pi_1^d(\pi_0^d(M)) + \pi_0^d(\pi_1^d(M))$$

$$\pi_i^d(\theta^e(M)) \to_{\Lambda_{\mathsf{cd}}} \theta^{e-1}(\pi_i^d(M)) \quad \text{if } d < e \qquad \pi_i^d(\theta^e(M)) \to_{\Lambda_{\mathsf{cd}}} \theta^e(\pi_i^{d+1}(M)) \quad \text{if } e < d$$

$$\pi_{i_{l+1}}^d(\cdots \pi_{i_0}^d(\mathsf{c}_l^d(M))) \to_{\Lambda_{\mathsf{cd}}} \pi_{i_0}^d(\pi_{i_{l+1}}^d(\cdots \pi_{i_1}^d(M)))$$

$$\pi_i^d(\mathsf{c}_l^e(M)) \to_{\Lambda_{\mathsf{cd}}} \mathsf{c}_l^{e-1}(\pi_i^d(M)) \quad \text{if } d < e \qquad \pi_i^d(\mathsf{c}_l^e(M)) \to_{\Lambda_{\mathsf{cd}}} \mathsf{c}_l^e(\pi_i^d(M)) \quad \text{if } e+l+2 \leq d$$

$$\pi_i^d(\iota_j^d(M)) \to_{\Lambda_{\mathsf{cd}}} M \quad \text{if } i = j \qquad \pi_i^d(\iota_j^d(M)) \to_{\Lambda_{\mathsf{cd}}} 0 \quad \text{if } i \neq j$$

$$\pi_i^d(\iota_j^e(M)) \to_{\Lambda_{\mathsf{cd}}} \iota_j^{e-1}(\pi_i^d(M)) \quad \text{if } d < e \qquad \pi_i^d(\iota_j^e(M)) \to_{\Lambda_{\mathsf{cd}}} \iota_j^e(\pi_i^{d-1}(M)) \quad \text{if } e < d$$

$$\pi_i^d(\pi_j^e(M)) \to_{\Lambda_{\mathsf{cd}}} \pi_j^{e-1}(\pi_i^d(M)) \quad \text{if } d < e \qquad \pi_i^d(\pi_j^e(M)) \to_{\Lambda_{\mathsf{cd}}} \pi_j^e(\pi_i^{d+1}(M)) \quad \text{if } e \leq d$$

$$\pi_i^{d+1}(\mathsf{D}M) \to_{\Lambda_{\mathsf{cd}}} \mathsf{D}\pi_i^d(M) \qquad \mathsf{D}\pi_i^d(M) \to_{\Lambda_{\mathsf{cd}}} \pi_i^{d+1}(\mathsf{D}M)$$

FIGURE 5. Projection reduction rules

3.3.1. *Reducing sums, and the evaluation contexts.* These reduction rules can be applied almost anywhere in a term (taking care as usual of not binding free variables of N in the ordinary substitution $M\,[N/x]$ and in the differential substitution $\partial(x,M)$).

However, in order to make the proof of subject reduction possible, we forbid reductions within subterms of the shape $M_0 + M_1$. Indeed by the very nature of the coherence we want to implement in this programming language, we have provided very restricted ways to type sums. For that reason allowing for instance to reduce M_0 to some M_0' by performing, say, a β-reduction would lead to a term $M_0' + M_1$ whose typability is not at all obvious (imagine for instance that $M_i = \pi_i(M)$ for some M such that $\Gamma \vdash M : \mathsf{D}A$). One option would be to develop a theory of "parallel" reductions generalizing the observation that in the example at hand the β-reduction performed in M_0 is also available in M_1 because both come from the same term M. This kind of approach will certainly be developed in further work. For the time being we adopt a much simpler and conservative approach. So here is the syntax of our evaluation contexts:

$$\begin{aligned} E := [\,] &\mid \lambda x^A\, E \mid (E)N \mid (M)E \mid \mathsf{Y}E \mid \mathsf{succ}^d(E) \mid \mathsf{pred}^d(E) \\ &\mid \mathsf{if}^d(E,P,Q) \mid \mathsf{if}^d(M,E,Q) \mid \mathsf{if}^d(M,P,E) \mid \mathsf{let}^d(x,E,P) \mid \mathsf{let}^d(x,M,E) \\ &\mid \mathsf{D}E \mid \pi_i^d(E) \mid \iota_i^d(E) \mid \theta^d(E) \mid \mathsf{c}_l^d(E) \end{aligned}$$

and the associated inference rule is as usual

$$\frac{M \to_{\Lambda_{\mathsf{cd}}} M'}{E[M] \to_{\Lambda_{\mathsf{cd}}} E[M']}$$

We will need however to perform reduction within sums at some point otherwise our computations will remain stuck for artificial reasons. So we do allow such reductions but only at "toplevel": this is precisely the purpose of the associated rewriting system $\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})$ defined in Section 2.2.

3.3.2. *Term multiset typing.* Let $S = [M_1, \dots, M_k] \in \mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})$, Γ be a context and A be a type. We write $\Gamma \vdash S : A$ if $\Gamma \vdash M_i : A$ for $i = 1, \dots, k$. This notion of typing for multisets (which represent sums of terms) is quite weak: $\Gamma \vdash [M_0, M_1] : A$ does not imply $\Gamma \vdash M_0 + M_1 : A$. It is only for that reason that we will be able to prove subject reduction for $\rightarrow_{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})}$. This is not really an issue because we will prove that the semantics is invariant by reduction (including the $\rightarrow_{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})}$ reduction) so we know that actually the terms that we obtain by performing the $\rightarrow_{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})}$ reduction belong to the expected type even if we are not necessarily able to prove it syntactically.

3.3.3. *Subject reduction.*

Lemma 3.8. *If $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : A$ then $\Gamma \vdash M\,[N/x] : B$.*

Proof. Straightforward induction on the typing derivation of M. □

Theorem 3.9 (Subject reduction)**.** *If $\Gamma \vdash M : A$ and $M \rightarrow_{\Lambda_{\mathsf{cd}}} M'$ then $\Gamma \vdash M' : A$.*

Proof. The last possible typing rules for the derivation of $\Gamma \vdash M : A$ cannot be any of the rules (**proj2**), (**projd**) or (**lin**) since the reduction $\rightarrow_{\Lambda_{\mathsf{cd}}}$ does not apply to sums and to 0. For the remaining typing rules, observe that the typing system is syntax directed, we consider a few reductions. The proof is by induction on the derivation of $M \rightarrow_{\Lambda_{\mathsf{cd}}} M'$.

▶ Assume that $M = \mathsf{D}(\lambda x^B\, N)$ and $M' = \lambda x^{\mathsf{D}B}\, \partial(x, N)$ so that the last typing rule is (**diff**), with $\Gamma, x : B \vdash N : C$ and hence $\Gamma \vdash \lambda x^B\, N : B \Rightarrow C$ and $A = (\mathsf{D}B \Rightarrow \mathsf{D}C)$. Then we have $\Gamma, x : \mathsf{D}B \vdash \partial(x, N) : \mathsf{D}C$ by Lemma 3.6, and therefore $\Gamma \vdash \lambda x^{\mathsf{D}B}\, \partial(x, N) : \mathsf{D}B \Rightarrow \mathsf{D}C$ by (**abs**). All the other reduction rules of Figure 4 are dealt with as usual in the typed λ-calculus, using Lemma 3.8.

The fact that if $\Gamma \vdash M : A$ and $M \rightarrow_{\mathsf{lin}} M'$ then $\Gamma \vdash M' : A$ is by a straightforward application of rule (**lin**).

So we consider now some of the rules of Figure 5.

▶ Assume that $M = \pi_i^d(\lambda x^B\, N)$ with $\Gamma, x : B \vdash N : \mathsf{D}^{d+1}C$ so that $A = (B \Rightarrow \mathsf{D}^d C) = \mathsf{D}^d(B \Rightarrow C)$. Then we have $\Gamma, x : B \vdash \pi_i^d(N) : \mathsf{D}^d C$ and hence $\Gamma \vdash \lambda x^B\, \pi_i^d(N) : A$. And on the other hand $\Gamma \vdash \pi_i^d(\lambda x^B\, N) : A$.

▶ Assume that $M = \pi_i^d((N)P)$ with $\Gamma \vdash N : B \Rightarrow \mathsf{D}^{d+1}C = \mathsf{D}(B \Rightarrow \mathsf{D}^d C)$ and $\Gamma \vdash P : B$ so that $\Gamma \vdash (N)P : \mathsf{D}^{d+1}C$ and hence $\Gamma \vdash M : A$ where $A = \mathsf{D}^d C$, and on the other hand $\Gamma \vdash \pi_i^d(N) : B \Rightarrow \mathsf{D}^d C$ and hence $\Gamma \vdash (\pi_i^d(N))P : A$.

▶ Assume that $M = \pi_i^d(\mathsf{succ}^e(N))$ with $\Gamma \vdash N : \mathsf{D}^e\iota$. For M to be typeable we need to have $d < e$ and then $\Gamma \vdash M : \mathsf{D}^{e-1}\iota = A$. Then we have $\Gamma \vdash \pi_i^d(N) : \mathsf{D}^{e-1}\iota$ and hence $\Gamma \vdash \mathsf{succ}^{e-1}(\pi_i^d(N)) : A$.

▶ Assume that $M = \pi_i^d(\mathsf{if}^e(N, P, Q))$ and $d < e$, with $\Gamma \vdash N : \mathsf{D}^e\iota$ and $\Gamma \vdash P : B$ and $\Gamma \vdash Q : B$ so that $\Gamma \vdash \mathsf{if}^e(N, P, Q) : \mathsf{D}^e B$ and hence $\Gamma \vdash M : \mathsf{D}^{e-1}B$, which means that $A = \mathsf{D}^{e-1}B$. On the other hand we have $\Gamma \vdash \pi_i^d(N) : \mathsf{D}^{e-1}\iota$ and hence $\Gamma \vdash \mathsf{if}^{e-1}(\pi_i^d(N), P, Q) : \mathsf{D}^{e-1}B$ as expected.

▶ Assume that $M = \pi_i^d(\mathsf{if}^e(N, P, Q))$ and $e \leq d$, with $\Gamma \vdash N : \mathsf{D}^e\iota$ and assume that $\Gamma \vdash P : B$ and $\Gamma \vdash Q : B$ so that $\Gamma \vdash \mathsf{if}^e(N, P, Q) : \mathsf{D}^e B = A$. For the term M to be typeable, we need B to be of shape $\mathsf{D}^{d-e+1}C$ for some (uniquely defined) type C so that $A = \mathsf{D}^{d+1}C$ and hence $\Gamma \vdash \pi_i^d(M) : \mathsf{D}^d C$. On the other hand we have $\Gamma \vdash \pi_i^{d-e}(P) : \mathsf{D}^{d-e}C$ and hence $\Gamma \vdash \mathsf{if}^e(N, \pi_i^{d-e}(P), \pi_i^{d-e}(Q)) : \mathsf{D}^d C$ as expected.

▶ Assume that $M = \pi_0^d(\theta^d(N))$ with $\Gamma \vdash N : \mathsf{D}^{d+2}C$, and hence $\Gamma \vdash \theta^d(N) : \mathsf{D}^{d+1}C$ and therefore $\Gamma \vdash \pi_0^d(\theta^d(N)) : C$, so that $A = \mathsf{D}^d C$. Then we have $\Gamma \vdash \pi_0^d(N) : \mathsf{D}^{d+1}C$ by (**proj1**) and hence $\Gamma \vdash \pi_0^d(\pi_0^d(N)) : A$ by (**proj1**) again.

▶ Assume that $M = \pi_1^d(\theta^d(N))$ with $\Gamma \vdash N : \mathsf{D}^{d+2}C$, and hence $\Gamma \vdash \theta^d(N) : \mathsf{D}^{d+1}C$ and therefore $\Gamma \vdash \pi_1^d(\theta^d(N)) : C$, so that $A = \mathsf{D}^d C$. Then we have $\Gamma \vdash \pi_0^d(N) + \pi_1^d(N) : \mathsf{D}^{d+1}C$ by (**proj2**) and hence $\Gamma \vdash \pi_1^d(\pi_0^d(N)) + \pi_0^d(\pi_1^d(N)) : A$ by (**projd**).

▶ Assume that $M = \pi_i^d(\theta^e(N))$ with $d < e$. So we must have $\Gamma \vdash N : \mathsf{D}^{e+2}A$ so that $\Gamma \vdash \theta^e(N) : \mathsf{D}^{e+1}A = \mathsf{D}^{d+1}\mathsf{D}^{e-d}A$ and hence $\Gamma \vdash \pi_i^d(\theta^e(N)) : \mathsf{D}^e A$. We have $\Gamma \vdash \pi_i^d(N) : \mathsf{D}^{e+1}A$ and hence $\Gamma \vdash \theta^{e-1}(\pi_i^d(N)) : \mathsf{D}^e A$ as required.

▶ Assume that $M = \pi_i^d(\theta^e(N))$ with $e < d$. We must have $\Gamma \vdash N : \mathsf{D}^{e+2}A$ so that $\Gamma \vdash \theta^e(N) : \mathsf{D}^{e+1}A$ and for $\pi_i^d(\theta^e(N))$ to be typeable we need A to be of shape $\mathsf{D}^{d-e}B$ (for a uniquely defined type B) so that $\Gamma \vdash \theta^e(N) : \mathsf{D}^{d+1}B$ and hence $\Gamma \vdash M = \pi_i^d(\theta^e(N)) : \mathsf{D}^d B = \mathsf{D}^e A$. We have $\Gamma \vdash N : \mathsf{D}^{d+2}B$ and hence $\Gamma \vdash \pi_i^{d+1}(N) : \mathsf{D}^{d+1}B$ and therefore (using the fact that $d > 0$) $\Gamma \vdash \theta^e(\pi_i^{d+1}(N)) : \mathsf{D}^d B = \mathsf{D}^e A$.

▶ Assume that $M = \pi_{i_{l+1}}^d(\cdots \pi_{i_0}^d(\mathsf{c}_l^d(N)))$ with $\Gamma \vdash N : \mathsf{D}^{l+d+2}C$ and hence $\Gamma \vdash M : \mathsf{D}^{d+2}C = A$ by (**circ**) and l applications of (**proj1**). Then we have $\Gamma \vdash \pi_{i_0}^d(\pi_{i_{l+1}}^d(\cdots \pi_{i_1}^d(N))) : A$ by l applications of (**proj1**).

▶ Assume that $M = \pi_i^d(\mathsf{c}_l^e(N))$ with $d < e$ so that we have $\Gamma \vdash N : \mathsf{D}^{e+l+2}C$ for a type C such that $A = \mathsf{D}^{e+l+1}C$. Then we have $\Gamma \vdash \pi_i^d(N) : \mathsf{D}^{e+l+1}C$ (since $d < e + l + 1$) and hence $\Gamma \vdash \mathsf{c}_l^{e-1}(\pi_i^d(N)) : A$ since $e > 0$ and hence $e + l + 1 = (e-1) + l + 2$.

▶ Assume that $M = \pi_i^d(\mathsf{c}_l^e(N))$ with $e + l + 2 \leq d$ so that we have $\Gamma \vdash N : \mathsf{D}^{e+l+2}C$ for a type C such that $A = \mathsf{D}^{e+l+1}C$, and moreover $A = \mathsf{D}^d D$ for some type D, meaning that $C = \mathsf{D}^{d-e-l-1}D$ (remember that $d - e - l - 1 > 0$). Then we have $\Gamma \vdash \pi_i^d(N) : \mathsf{D}^{e+l+1}C = A$ by (**proj1**) and hence $\Gamma \vdash \mathsf{c}_l^e(\pi_i^d(N)) : A$ by (**circ**) which can be applied since $A = \mathsf{D}^{e+l+2}\mathsf{D}^{d-e-l-2}D$.

The remaining cases are similar. □

Given a derivation δ in the typing system we use $\mathsf{sz}(\delta)$ for the number of inference rule occurrences δ contains.

Lemma 3.10. *Let δ be a typing derivation of $\Gamma \vdash M_0 + M_1 : A$. For $j = 0, 1$, there is a derivation δ_j of $\Gamma \vdash M_j : A$ such that $\mathsf{sz}(\delta_j) \leq \mathsf{sz}(\delta)$.*

Proof. By induction on δ. The following cases can arise.

▶ The last rule of δ is (**proj2**) so that $M_j = \pi_j^d(M)$, $A = \mathsf{D}^d B$ and $\Gamma \vdash M : \mathsf{D}^{d+1}B$ by a derivation δ' such that $\mathsf{sz}(\delta') = \mathsf{sz}(\delta) - 1$ that we can extend with a rule (**proj1**) to get the required derivation δ_j of $\Gamma \vdash M_j : A$ which satisfies $\mathsf{sz}(\delta_j) = \mathsf{sz}(\delta') + 1 = \mathsf{sz}(\delta)$.

▶ The last rule of δ is (**projd**) so that $M_0 = \pi_1^d(N_0)$, $M_1 = \pi_0^d(N_1)$, $A = \mathsf{D}^d B$ and $\Gamma \vdash N_0 + N_1 : \mathsf{D}^{d+1}B$ by a derivation δ' such that $\mathsf{sz}(\delta') = \mathsf{sz}(\delta) - 1$ and hence by inductive hypothesis, for $j = 0, 1$, we have a derivation δ'_j of $\Gamma \vdash N_j : \mathsf{D}^{d+1}B$ such that $\mathsf{sz}(\delta'_j) \leq \mathsf{sz}(\delta')$ that we can extend with a (**proj1**) rule to get a derivation δ_j of $\Gamma \vdash M_j : A$. We have $\mathsf{sz}(\delta_j) = \mathsf{sz}(\delta'_j) + 1 \leq \mathsf{sz}(\delta)$.

▶ The last rule of δ is (**lin**) so that there is a linear context L of height 1 and terms N_0, N_1 such that $M_j = L[N_j]$ and $\Gamma \vdash L[N_0 + N_1] : A$ by a derivation δ' such that $\mathsf{sz}(\delta') = \mathsf{sz}(\delta) - 1$. This implies (by a simple inspection of the various possibilities for L which has height 1) that for some context Δ and some type B one has $\Delta \vdash N_0 + N_1 : B$ by a derivation δ'' such that $\mathsf{sz}(\delta'') = \mathsf{sz}(\delta') - k_L$ where $k_L \in \mathbb{N}^+$ *depends only on* L (if for instance $L = \mathsf{if}^d([\,], P_0, P_1)$ then $k_L = 1 + k_0 + k_1$ where k_i is the size of the typing derivation of P_i). So that by inductive hypothesis we have derivations δ''_j of $\Delta \vdash N_j : B$ for $j = 0, 1$ such that $\mathsf{sz}(\delta''_j) \leq \mathsf{sz}(\delta'')$. We can extend δ''_j with exactly the typing rule associated with L to get a derivation δ_j of $\Gamma \vdash L[N_j] : A$ such that $\mathsf{sz}(\delta_j) = \mathsf{sz}(\delta''_j) + k_L \leq \mathsf{sz}(\delta'') + k_L = \mathsf{sz}(\delta') = \mathsf{sz}(\delta) - 1$ and hence $\mathsf{sz}(\delta_j) < \mathsf{sz}(\delta)$. □

Theorem 3.11 (Subject reduction for multisets)**.** *Assume that* $\Gamma \vdash S : A$ *where* $S \in \underline{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})}$ *and that* $S \to_{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})} S'$. *Then* $\Gamma \vdash S' : A$.

Proof. The following cases are possible.

► $S = S_0 + [M]$, $M \to_{\Lambda_{\mathsf{cd}}} 0$ and $S' = S_0$. We have $\Gamma \vdash S' : A$ since all elements of S' belong to S.

► $S = S_0 + [M]$, $M \to_{\Lambda_{\mathsf{cd}}} M'$ and $S' = S_0 + [M']$. Then we have $\Gamma \vdash M' : A$ by Theorem 3.9 and hence $\Gamma \vdash S' : A$.

► $S = S_0 + [M]$, $M \to_{\Lambda_{\mathsf{cd}}} M_0 + M_1$ and $S' = S_0 + [M_0, M_1]$. Since $\Gamma \vdash M : A$, we have $\Gamma \vdash M_i : A$ for $i = 0, 1$ by Lemma 3.10 and hence $\Gamma \vdash S' : A$. □

4. Semantics

We provide first a bird's eye view on the categorical setting introduced in [Ehr21] for coherent differentiation. For more detailed definitions, we refer to that paper. Then we introduce the specific operations and properties which are used for interpreting Λ_{cd}.

4.1. A summary of summable differential categories.

A *summable category* is a tuple $(\mathcal{L}, \mathbf{S}, \pi_0, \pi_1, \sigma)$ where $\mathcal{L}$ is a category with 0-morphisms, $\mathbf{S} : \mathcal{L} \to \mathcal{L}$ is a functor and $\pi_0, \pi_1, \sigma : \mathbf{S} \Rightarrow \mathsf{Id}$ are natural transformations such that $\pi_0, \pi_1 \in \mathcal{L}(\mathbf{S}\, X, X)$ are jointly monic: this means that a morphism $f \in \mathcal{L}(X, \mathbf{S}\, Y)$ is fully characterized by $\pi_0\ f$ and $\pi_1\ f$. Then we say that $f_0, f_1 \in \mathcal{L}(X, Y)$ are *summable* if there is $h \in \mathcal{L}(X, \mathbf{S}\, Y)$ such that $\pi_i\ h = f_i$ for $i = 0, 1$. This h is unique and is denoted $\langle f_0, f_1 \rangle_{\mathbf{S}}$. In that situation the sum $f_0 + f_1$ of f_0, f_1 is defined as $f_0 + f_1 = \sigma \, \langle f_0, f_1 \rangle_{\mathbf{S}}$. There are further axioms (**S-com**), (**S-zero**), (**S-witness**) and (**S-assoc**) which imply in particular that, equipped with this partially defined addition, any homset $\mathcal{L}(X, Y)$ is a partial commutative monoid with 0 as neutral element, and the naturality of π_0, π_1 and σ implies that composition commutes with this partially defined addition, that is, $\mathcal{L}$ is a partially additive category. These axioms also imply that there is a natural transformation $\mathsf{c} : \mathbf{S}^2 \Rightarrow \mathbf{S}^2$ uniquely characterized by $\pi_i\ \pi_j\ \mathsf{c} = \pi_j\ \pi_i$ for all $i, j \in \{0, 1\}$, it is called the *standard flip*. One also defines uniquely two natural injections $\iota_0 = \langle X, 0 \rangle_{\mathbf{S}}, \iota_1 = \langle 0, X \rangle_{\mathbf{S}} : X \to \mathbf{S}\, X$.

4.1.1. *The associated monad, its monoidal strength and its commutativity.* Under these assumptions $\mathbf{S}$ has a monad structure with unit $\iota_0 : \mathsf{Id} \Rightarrow \mathbf{S}$ and multiplication $\tau : \mathbf{S}^2 \Rightarrow \mathbf{S}$ characterized by $\pi_0\ \tau = \pi_0\ \pi_0$ and $\pi_1\ \tau = \pi_1\ \pi_0 + \pi_0\ \pi_1$ from which it follows easily that $\tau\ \mathsf{c} = \tau$. When $\mathcal{L}$ is symmetric monoidal (with monoidal unit 1 and monoidal product $\otimes$) a further axiom ($\mathbf{S}\otimes$**-dist**) expresses that $\otimes$ distributes over the sum of morphisms, when defined. It is then possible to define a tensorial strength $\varphi^0_{X_0,X_1} \in \mathcal{L}((\mathbf{S}\,X_0)\otimes X_1, \mathbf{S}(X_0 \otimes X_1))$ which is a natural transformation satisfying further commutations expressing its compatibility with the $\otimes$ monoidal structure of $\mathcal{L}$. Using the symmetry iso of the monoidal structure of $\mathcal{L}$ one can define then $\varphi^1_{X_0,X_1} \in \mathcal{L}(X_0 \otimes (\mathbf{S}\,X_1), \mathbf{S}(X_0 \otimes X_1))$ from φ^0. This strength is fully characterized by $\pi_i\ \varphi^0_{X_0,X_1} = \pi_i \otimes X_1$ for $i = 0, 1$. Equipped with this strength the monad $(\mathbf{S}, \iota_0, \tau)$ is a commutative monad. More precisely the following equation holds

$$\mathsf{c}_{X_0 \otimes X_1}\,(\mathbf{S}\,\varphi^1_{X_0,X_1})\ \varphi^0_{X_0,\mathbf{S}\,X_1} = (\mathbf{S}\,\varphi^0_{X_0,X_1})\ \varphi^1_{\mathbf{S}\,X_0,X_1}$$

as indeed $\pi_i\ \pi_j\,(\mathbf{S}\,\varphi^1_{X_0,X_1})\ \varphi^0_{X_0,\mathbf{S}\,X_1} = \pi_j \otimes \pi_i$ and $\pi_i\ \pi_j\,(\mathbf{S}\,\varphi^0_{X_0,X_1})\ \varphi^1_{\mathbf{S}\,X_0,X_1} = \pi_i \otimes \pi_j$. The induced natural symmetric monoidality morphism $\mathsf{L}_{X_0,X_1} = \tau\,(\mathbf{S}\,\varphi^0_{X_0,X_1})\ \varphi^1_{\mathbf{S}\,X_0,X_1} = \tau(\mathbf{S}\,\varphi^1_{X_0,X_1})\ \varphi^0_{X_0,\mathbf{S}\,X_1} \in \mathcal{L}(\mathbf{S}\,X_0 \otimes \mathbf{S}\,X_1, \mathbf{S}(X_0 \otimes X_1))$ is fully characterized by

$$\pi_0\ \mathsf{L}_{X_0,X_1} = \pi_0 \otimes \pi_0 \quad \text{and} \quad \pi_1\ \mathsf{L}_{X_0,X_1} = \pi_1 \otimes \pi_0 + \pi_0 \otimes \pi_1$$

and its 0-ary version is simply $\mathsf{L}^0 = \iota_0 \in \mathcal{L}(1, \mathbf{S}\,1)$.

If the SMC $\mathcal{L}$ is also closed then we require the summability structure to satisfy a further condition ($\mathbf{S}\otimes$**-fun**). We use $(X \multimap Y, \mathsf{ev} \in \mathcal{L}((X \multimap Y)\otimes X, Y))$ for the internal hom object of X and Y in $\mathcal{L}$ and $\mathsf{cur}\,f \in \mathcal{L}(Z, X \multimap Y)$ for the curryfied version of $f \in \mathcal{L}(Z \otimes X, Y)$. With these notations, ($\mathbf{S}\otimes$**-fun**) says that $\mathbf{S}\,(X \multimap Y)$ and $X \multimap \mathbf{S}\,Y$ are isomorphic (more precisely the morphism $\mathbf{S}\,(X \multimap Y) \to (X \multimap \mathbf{S}\,Y)$ that one can define using $\varphi^0_{X \multimap Y,X}$ is an iso).

4.1.2. *Differentiation as a double distributive law on a resource category.* We assume now moreover that $\mathcal{L}$ is a resource category which means that $\mathcal{L}$ is cartesian (with cartesian product $(\&_{i\in I}\,X_i, (\mathsf{pr}_i)_{i\in I})$ for any finite family of objects $(X_i)_{i\in I}$ of $\mathcal{L}$, the case $I = \emptyset$ yielding the terminal object $\top$ of $\mathcal{L}$. It is then assumed that the summability functor $\mathbf{S}$ preserves cartesian products and to simplify notations we assume that it preserves them strictly, that is $\mathbf{S}(\&_{i\in I}\,X_i) = \&_{i\in I}\,\mathbf{S}\,X_i$ and $\mathbf{S}\,\mathsf{pr}_i = \mathsf{pr}_i$ for each $i \in I$. Being a resource category means also that $\mathcal{L}$ is equipped with a *resource comonad*, that is a tuple $(!\_, \mathsf{der}, \mathsf{dig}, \mathsf{m}^0, \mathsf{m}^2)$ where $!\_$ is a functor $\mathcal{L} \to \mathcal{L}$ which is a comonad with counit der and comultiplication dig, and $\mathsf{m}^0 \in \mathcal{L}(1, !\top)$ and $\mathsf{m}^2 \in \mathcal{L}(!X \otimes !Y, !(X \,\&\, Y))$ are the Seely isomorphisms subject to conditions that we do not recall here, see for instance [Mel09] apart for the following which explains how dig interacts with m^2.

$$\begin{array}{ccccc} !X_0 \otimes !X_0 & & \xrightarrow{\mathsf{dig}_{X_0} \otimes \mathsf{dig}_{X_1}} & & !!X_0 \otimes !!X_1 \\ {\scriptstyle \mathsf{m}^2_{X_0,X_1}} \downarrow & & & & \downarrow {\scriptstyle \mathsf{m}^2_{!X_0,!X_1}} \\ !(X_0 \,\&\, X_1) & \xrightarrow{\mathsf{dig}_{X_0 \& X_1}} & !!(X_0 \,\&\, X_1) & \xrightarrow{!\langle !\mathsf{pr}_0, !\mathsf{pr}_1 \rangle} & !(!X_0 \,\&\, !X_1) \end{array} \tag{4.1}$$

Then $!\_$ inherits a *lax* symmetric monoidality on the SMC $(\mathcal{L}, \otimes)$. This means that one can define $\mu^0 \in \mathcal{L}(1, !1)$ and $\mu^2_{X_0,X_1} \in \mathcal{L}(!X_0 \otimes !X_1, !(X_0 \otimes X_1))$ satisfying suitable coherence commutations. Explicitly these morphisms are given by

$$1 \xrightarrow{\mathsf{m}^0} !\top \xrightarrow{\mathsf{dig}_\top} !!\top \xrightarrow{!(\mathsf{m}^0)^{-1}} !1$$

$$!X_0 \otimes !X_1 \xrightarrow{\mathsf{m}^2_{X_0,X_1}} !(X_0 \mathbin{\&} X_1) \xrightarrow{\mathsf{dig}_{X_0 \& X_1}} !!(X_0 \mathbin{\&} X_1) \xrightarrow{!(\mathsf{m}^2_{X_0,X_1})^{-1}} !(!X_0 \otimes !X_1) \xrightarrow{!(\mathsf{der}_{X_0} \otimes \mathsf{der}_{X_1})} !(X_0 \otimes X_1)$$

And by combining these morphisms in an arbitrary way one can define uniquely $\mu^n_{X_0,\dots,X_{n-1}} \in \mathcal{L}(!X_0 \otimes \cdots \otimes !X_{n-1}, !(X_0 \otimes \cdots \otimes X_{n-1}))$.

The Kleisli category $\mathcal{L}_!$ is cartesian with cartesian product of a family $(X_i)_{i\in I}$ of objects of $\mathcal{L}$ given by $(\&_{i\in I} X_i, (\mathsf{pr}^{\mathsf{K}}_i = \mathsf{Lin}_!(\mathsf{pr}_i))_{i\in I})$. Given a family of morphisms $f_i \in \mathcal{L}_!(Y, X_i)$ for $i \in I$, the morphism $\langle f_i\rangle_{i\in I} \in \mathcal{L}(!Y, \&_{i\in I} X_i) = \mathcal{L}_!(Y, \&_{i\in I} X_i)$ is uniquely characterized by the fact that $\mathsf{pr}^{\mathsf{K}}_j \circ \langle f_i\rangle_{i\in I} = f_j$ for each $j \in I$. We use the notation $g \circ f$ to denote composition in $\mathcal{L}_!$.

Notice that if now $f_i \in \mathcal{L}_!(X_i, Y_i)$ for each $i \in I$ we can define functorially $\&^{\mathsf{K}}{}_{i\in I} f_i \in \mathcal{L}_!(\&_{i\in I} X_i, \&_{i\in I} Y_i)$ by

$$\mathop{\&^{\mathsf{K}}}_{i\in I} f_i = \langle f_i \circ \mathsf{pr}^{\mathsf{K}}_i\rangle_{i\in I} = \langle f_i \, !\mathsf{pr}_i\rangle_{i\in I} = (\mathop{\&}_{i\in I} f_i)\, \langle !\mathsf{pr}_i\rangle_{i\in I}$$

Remember that we have a faithful functor $\mathsf{Lin}_! : \mathcal{L} \to \mathcal{L}_!$ defined by $\mathsf{Lin}_!(X) = X$ for objects and, given $f \in \mathcal{L}(X, Y)$, we set $\mathsf{Lin}_! f = f \, \mathsf{der}_X \in \mathcal{L}_!(X, Y)$. Functoriality results from the fact that $(!\_, \mathsf{der}, \mathsf{dig})$ is a comonad. The intuition is that this functor allows to see morphisms of $\mathcal{L}$ (considered as linear) can also be seen as morphisms in $\mathcal{L}_!$ where morphisms are not linear in general. This is why this functor is faithful but of course not full in general.

Lemma 4.1. *Let $f \in \mathcal{L}(X, Y)$ and $g \in \mathcal{L}_!(Y, Z)$, we have $g \circ \mathsf{Lin}_!(f) = g \, !f$.*

Proof. We have $!\mathsf{der}_X \circ \mathsf{dig}_X = \mathsf{Id}_{!X}$. □

Given $f \in \mathcal{L}(!X, Y)$ we set $f^! = !f \, \mathsf{dig}_X \in \mathcal{L}(!X, !Y)$ which is sometimes called !-lifting or *promotion* of f. Given $f \in \mathcal{L}(!X_0 \otimes \cdots \otimes !X_{n-1}, Y)$ one can define more generally $f^! \in \mathcal{L}(!X_0 \otimes \cdots \otimes !X_{n-1}, !Y)$ using $\mu^n_{X_0,\dots,X_{n-1}}$. Notice also that if $f \in \mathcal{L}(X, Y)$ one has $\mathsf{Lin}_!(f)^! = !f$.

A *coherent differential structure* on such a summable resource category consists of a natural transformation $\partial_X \in \mathcal{L}(!\mathbf{S}\, X, \mathbf{S}\, !X)$ satisfying a few axioms that we recall here.

(∂**-local**) $$\pi_0 \circ \partial_X = !\pi_0 : !\mathbf{S}\,X \to !X \qquad (!\mathbf{S}\,X \xrightarrow{\partial_X} \mathbf{S}\,!X \xrightarrow{\pi_0} !X)$$

(∂**-lin**)

$$
\begin{array}{ccc}
!X & & \\
{\scriptstyle !\iota_0}\downarrow & \searrow^{\iota_0} & \\
!\mathbf{S}\,X & \xrightarrow{\partial_X} & \mathbf{S}\,!X
\end{array}
\qquad
\begin{array}{ccccc}
!\mathbf{S}^2\,X & \xrightarrow{\partial_{\mathbf{S}\,X}} & \mathbf{S}\,!\mathbf{S}\,X & \xrightarrow{\mathbf{S}\,\partial_X} & \mathbf{S}^2\,!X \\
{\scriptstyle !\tau}\downarrow & & & & \downarrow{\scriptstyle \tau} \\
!\mathbf{S}\,X & & \xrightarrow{\partial_X} & & \mathbf{S}\,!X
\end{array}
$$

That is, ∂ is a distributive law between the monad $\mathbf{S}$ and the functor $!\_$. This means essentially that derivatives commute with sums and with 0, that is, are linear. This allows to extend the comonad $!\_$ to the Kleisli category of the monad $\mathbf{S}$ which is again a resource category. It can be understood as an *infinitesimal* extension of $\mathcal{L}$; this construction, as well as its syntactical outcomes, will be studied in further work.

(∂**-chain**)

$$
\begin{array}{ccc}
!\mathbf{S}\,X & \xrightarrow{\partial_X} & \mathbf{S}\,!X \\
& {\scriptstyle \mathsf{der}_{\mathbf{S}\,X}}\searrow & \downarrow{\scriptstyle \mathbf{S}\,\mathsf{der}_X} \\
& & \mathbf{S}\,X
\end{array}
\qquad
\begin{array}{ccccc}
!\mathbf{S}\,X & & \xrightarrow{\partial_X} & & \mathbf{S}\,!X \\
{\scriptstyle \mathsf{dig}_{\mathbf{S}\,X}}\downarrow & & & & \downarrow{\scriptstyle \mathbf{S}\,\mathsf{dig}_X} \\
!!\mathbf{S}\,X & \xrightarrow{!\partial_X} & !\mathbf{S}\,!X & \xrightarrow{\partial_{!X}} & \mathbf{S}\,!!X
\end{array}
$$

That is, ∂ is a distributive law between the comonad $!\_$ and the functor $\mathbf{S}$. This allows to extend $\mathbf{S}$ to an endofunctor on $\mathcal{L}_!$.

(∂-&)

$$
\begin{array}{ccccc}
!\mathbf{S}\,\top & & \xrightarrow{\partial_\top} & & \mathbf{S}\,!\top \\
{\scriptstyle !0}\downarrow & & & & \downarrow{\scriptstyle \mathbf{S}\,(\mathsf{m}^0)^{-1}} \\
!\top & \xrightarrow{(\mathsf{m}^0)^{-1}} & 1 & \xrightarrow{\iota_0} & \mathbf{S}\,1
\end{array}
$$

$$
\begin{array}{ccccc}
!\mathbf{S}\,(X_0 \mathbin{\&} X_1) & \xrightarrow{\partial_{X_0 \& X_1}} & \mathbf{S}\,!(X_0 \mathbin{\&} X_1) & \xrightarrow{\mathbf{S}\,(\mathsf{m}^2)^{-1}} & \mathbf{S}(!X_0 \otimes !X_1) \\
{\scriptstyle !\langle \mathbf{S}\,\mathsf{pr}_0, \mathbf{S}\,\mathsf{pr}_1\rangle}\downarrow & & & & \uparrow{\scriptstyle \mathsf{L}_{!X_0,!X_1}} \\
!(\mathbf{S}\,X_0 \mathbin{\&} \mathbf{S}\,X_1) & \xrightarrow{(\mathsf{m}^2)^{-1}} & !\mathbf{S}\,X_0 \otimes !\mathbf{S}\,X_1 & \xrightarrow{\partial_{X_0} \otimes \partial_{X_1}} & \mathbf{S}\,!X_0 \otimes \mathbf{S}\,!X_1
\end{array}
$$

Notice that our assumption that $\mathbf{S}$ preserves & on the nose implies that the morphism $\langle \mathbf{S}\,\mathsf{pr}_0, \mathbf{S}\,\mathsf{pr}_1\rangle$ is the identity.

Intuitively, this diagram expresses that a derivative wrt. a pair of variables ($\partial_{X_0 \& X_1}$) can be expressed as a sum of partial derivatives (∂_{X_0} and ∂_{X_1}): this is deeply related to the Leibniz Law.

(∂**-Schwarz**)

$$
\begin{array}{ccccc}
!\mathbf{S}^2\,X & \xrightarrow{\partial_{\mathbf{S}\,X}} & \mathbf{S}\,!\mathbf{S}\,X & \xrightarrow{\mathbf{S}\,\partial_X} & \mathbf{S}^2\,!X \\
{\scriptstyle !\mathsf{c}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{c}} \\
!\mathbf{S}^2\,X & \xrightarrow{\partial_{\mathbf{S}\,X}} & \mathbf{S}\,!\mathbf{S}\,X & \xrightarrow{\mathbf{S}\,\partial_X} & \mathbf{S}^2\,!X
\end{array}
$$

This expresses that the second derivative is a symmetric bilinear function.

4.1.3. *The induced coherent differentiation monad.* Thanks to (∂**-chain**) we extend the functor $\mathbf{S}$ to a functor $\mathsf{D} : \mathcal{L}_! \to \mathcal{L}_!$ on the Kleisli category of the comonad $!\_$ as follows: $\mathsf{D}\,X = \mathbf{S}\,X$ and, if $f \in \mathcal{L}_!(X, Y)$ then $\mathsf{D}\,f = (\mathbf{S}\,f)\,\partial_X \in \mathcal{L}_!(\mathsf{D}\,X, \mathsf{D}\,Y)$. Then we can define $\zeta_X = \mathsf{Lin}_!\iota_0 \in \mathcal{L}_!(X, \mathsf{D}\,X)$ and $\theta_X = \mathsf{Lin}_!\tau \in \mathcal{L}_!(\mathsf{D}^2\,X, \mathsf{D}\,X)$ and the condition (∂**-lin**) entails that these morphisms are natural; the intuitive meaning of that condition is that the differential of a map of the Kleisli category is linear in the sense that it commutes with the algebraic operation represented by ζ and θ. These natural transformations are easily seen to equip D with a monad structure.

4.2. **Partial derivatives.** We assume to be given a summable resource category $\mathcal{L}$ which is closed (wrt. its symmetric monoidal structure) and is equipped with a differentiation ∂. We generalize the lax monoidality of the D functor to a natural transformation $\mathsf{L}_n \in \mathcal{L}(\mathsf{D}\,X_0 \otimes \cdots \otimes \mathsf{D}\,X_n, \mathsf{D}(X_0 \otimes \cdots \otimes X_n))$ by induction on n (there are various possible definitions, all leading to the same morphisms), for instance $\mathsf{L}_0 = \iota_0$ and $\mathsf{L}_{n+1} = \mathsf{L}_{X_0 \otimes \cdots \otimes X_n, X_{n+1}}\,(\mathsf{L}_n \otimes \mathsf{D}\,X_{n+1})$. The resulting morphism is fully characterized by the following property.

Lemma 4.2. *$\pi_0\ \mathsf{L}_n = \pi_0 \otimes \cdots \otimes \pi_0$ and $\pi_1\ \mathsf{L}_n = \pi_1 \otimes \pi_0 \otimes \cdots \otimes \pi_0 + \cdots + \pi_0 \otimes \cdots \otimes \pi_0 \otimes \pi_1$.*

4.2.1. *Additive strength.* We define morphisms $\psi^0_{X_0,X_1} \in \mathcal{L}_!(\mathsf{D}\,X_0 \mathbin{\&} X_1, \mathsf{D}\,(X_0 \mathbin{\&} X_1))$ and $\psi^1_{X_0,X_1} \in \mathcal{L}_!(X_0 \mathbin{\&} \mathsf{D}\,X_1, \mathsf{D}\,(X_0 \mathbin{\&} X_1))$ of D as

$$\psi^0_{X_0,X_1} = \mathsf{Lin}_!\,(\mathbf{S}\,X_0 \mathbin{\&} \iota_0) \quad \text{and} \quad \psi^1_{X_0,X_1} = \mathsf{Lin}_!\,(\iota_0 \mathbin{\&} \mathbf{S}\,X_1)\ .$$

Lemma 4.3. *The morphism $\psi^0_{X_0,X_1} \in \mathcal{L}_!(\mathsf{D}\,X_0 \mathbin{\&} X_1, \mathsf{D}\,(X_0 \mathbin{\&} X_1))$ is natural in X_0, X_1 and similarly for $\psi^1_{X_0,X_1}$.*

Proof. Let $f_i \in \mathcal{L}_!(X_i, Y_i)$ for $i = 0, 1$, we must show that the two following morphisms are equal:

$$\begin{aligned}
g &= \mathsf{D}\left(f_0 \mathbin{\&}^{\mathsf{K}} f_1\right) \circ \psi^0_{X_0,X_1} \\
&= (\mathbf{S}\,f_0 \mathbin{\&} \mathbf{S}\,f_1)\ \langle \mathbf{S}\,!\mathsf{pr}_0, \mathbf{S}\,!\mathsf{pr}_1\rangle\ \partial_{X_0 \& X_1}\ !(\mathbf{S}\,X_0 \mathbin{\&} \iota_0) \\
h &= \psi^0_{Y_0 \& Y_1} \circ \left(\mathsf{D}\,f_0 \mathbin{\&}^{\mathsf{K}} f_1\right) \\
&= (\mathbf{S}\,Y_0 \mathbin{\&} \iota_0)\ ((\mathbf{S}\,f_0)\ \partial_{X_0} \mathbin{\&} f_1)\ \langle !\mathsf{pr}_0, !\mathsf{pr}_1\rangle \\
&= (\mathbf{S}\,Y_0 \mathbin{\&} \iota_0)\ (\mathbf{S}\,f_0 \mathbin{\&} f_1)\ (\partial_{X_0} \mathbin{\&} !X_1)\ \langle !\mathsf{pr}_0, !\mathsf{pr}_1\rangle
\end{aligned}$$

and for this it suffices to prove that $\mathsf{pr}_i\,g = \mathsf{pr}_i\,h$ for $i = 0, 1$. We have

$$\begin{aligned}
\mathsf{pr}_0\,g &= (\mathbf{S}\,f_0)\,(\mathbf{S}\,!\mathsf{pr}_0)\ \partial_{X_0 \& X_1}\ !(\mathbf{S}\,X_0 \mathbin{\&} \iota_0) \\
&= (\mathbf{S}\,f_0)\ \partial_{X_0}\ !\mathsf{pr}_0\,!(\mathbf{S}\,X_0 \mathbin{\&} \iota_0) \quad \text{by naturality of } \partial \\
&= (\mathbf{S}\,f_0)\ \partial_{X_0}\ !\mathsf{pr}_0 = \mathsf{pr}_0\,h
\end{aligned}$$

and, by a similar computation

$$\mathsf{pr}_1\,g = (\mathbf{S}\,f_1)\ \partial_{X_1}\ !\iota_0\,!\mathsf{pr}_1 = (\mathbf{S}\,f_1)\ \iota_0\ !\mathsf{pr}_1 = \iota_0\ f_1\,!\mathsf{pr}_1$$

by (∂**-lin**) and by naturality of ι_0 and hence $\mathsf{pr}_1\,g = \mathsf{pr}_1\,h$. □

There is a simple connection between this additive strength and the tensorial strength of the monad $\mathbf{S}$ introduced in [Ehr21], Section 3, see also Section 4.1.1 in the present paper, through the strong symmetric monoidal structure of $!\_$ (Seely isomorphisms).

Theorem 4.4. *The following diagram commutes.*

$$\begin{array}{ccccc}
!\mathbf{S}\,X_0 \otimes !X_1 & \xrightarrow{\mathsf{m}^2_{\mathbf{S}\,X_0,X_1}} & !(\mathbf{S}\,X_0 \mathbin{\&} X_1) & \xrightarrow{(\psi^0_{X_0,X_1})^!} & !\mathbf{S}\,(X_0 \mathbin{\&} X_1) \\
\Big\downarrow{\scriptstyle \partial_{X_0} \otimes !X_1} & & & & \Big\downarrow{\scriptstyle \partial_{X_0 \& X_1}} \\
\mathbf{S}\,!X_0 \otimes !X_1 & \xrightarrow{\varphi^0_{!X_0,!X_1}} & \mathbf{S}(!X_0 \otimes !X_1) & \xrightarrow{\mathbf{S}\,\mathsf{m}^2_{X_0,X_1}} & \mathbf{S}\,!(X_0 \mathbin{\&} X_1)
\end{array}$$

and similarly for $\psi^1_{X_0,X_1}$ and $\varphi^1_{X_0,X_1}$.

Remember that $(\psi^0_{X_0,X_1})^!$ is the promotion of $\psi^0_{X_0,X_1} \in \mathcal{L}(!(\mathbf{S}\,X_0 \mathbin{\&} X_1), \mathbf{S}\,(X_0 \mathbin{\&} X_1))$, so that actually $(\psi^0_{X_0,X_1})^! = !(\mathbf{S}\,X_0 \mathbin{\&} \iota_0)$.

Proof. By (∂-&) we have

$$\partial_{X_0 \& X_1} = (\mathbf{S}\,\mathsf{m}^2_{X_0,X_1})\ \mathsf{L}_{!X_0,!X_1}\,(\partial_{X_0} \otimes \partial_{X_1})\,(\mathsf{m}^2_{\mathbf{S}\,X_0,\mathbf{S}\,X_1})^{-1}$$

so that

$$\begin{aligned}
&\partial_{X_0 \& X_1}\,(\psi^0_{X_0,X_1})^!\ \mathsf{m}^2_{\mathbf{S}\,X_0,X_1} \\
&\quad = (\mathbf{S}\,\mathsf{m}^2_{X_0,X_1})\ \mathsf{L}_{!X_0,!X_1}\,(\partial_{X_0} \otimes \partial_{X_1})\,(\mathsf{m}^2_{\mathbf{S}\,X_0,\mathbf{S}\,X_1})^{-1}\,!(\mathbf{S}\,X_0 \mathbin{\&} \iota_0)\ \mathsf{m}^2_{\mathbf{S}\,X_0,X_1} \\
&\quad = (\mathbf{S}\,\mathsf{m}^2_{X_0,X_1})\ \mathsf{L}_{!X_0,!X_1}\,(\partial_{X_0} \otimes \partial_{X_1})(!(\mathbf{S}\,X_0) \otimes !\iota_0) \\
&\quad = (\mathbf{S}\,\mathsf{m}^2_{X_0,X_1})\ \mathsf{L}_{!X_0,!X_1}\,(\partial_{X_0} \otimes \iota_0) \quad \text{by } (\partial\textbf{-lin}) \\
&\quad = (\mathbf{S}\,\mathsf{m}^2_{X_0,X_1})\,\tau\,(\mathbf{S}\,\varphi^1_{!X_0,!X_1})\ \varphi^0_{!X_0,\mathbf{S}\,!X_1}\,(\partial_{X_0} \otimes \iota_0) \\
&\qquad\quad \text{by definition of } \mathsf{L} \text{ in [Ehr21], Section 3} \\
&\quad = (\mathbf{S}\,\mathsf{m}^2_{X_0,X_1})\,\tau\,(\mathbf{S}\,\varphi^1_{!X_0,!X_1})\ \mathbf{S}(!X_0 \otimes \iota_0)\ \varphi^0_{!X_0,!X_1}\,(\partial_{X_0} \otimes !X_1) \\
&\quad = (\mathbf{S}\,\mathsf{m}^2_{X_0,X_1})\,\tau\,(\mathbf{S}\,\iota_0)\ \varphi^0_{!X_0,!X_1}\,(\partial_{X_0} \otimes !X_1) \\
&\qquad\quad \text{since } \varphi^0 \text{ is a strength of the monad } \mathbf{S} \text{ whose unit is } \iota_0 \\
&\quad = (\mathbf{S}\,\mathsf{m}^2_{X_0,X_1})\ \varphi^0_{!X_0,!X_1}\,(\partial_{X_0} \otimes !X_1)
\end{aligned}$$

by one of the monad commutations. □

Lemma 4.5. *The morphism $\psi^0_{X_0,X_1}$ is equal to the following composition of morphisms in $\mathcal{L}$:*

$$\begin{array}{ccccc}
!(\mathbf{S}\,X_0 \mathbin{\&} X_1) & \xrightarrow{(\mathsf{m}^2)^{-1}} & !\mathbf{S}\,X_0 \otimes !X_1 & \xrightarrow{\partial_{X_0} \otimes !X_1} & \mathbf{S}\,!X_0 \otimes !X_1 \\
 & & & & \downarrow{\scriptstyle \varphi^0_{!X_0,!X_1}} \\
\mathbf{S}\,(X_0 \mathbin{\&} X_1) & \xleftarrow[\mathbf{S}\,\mathsf{der}_{X_0 \& X_1}]{} & \mathbf{S}\,!(X_0 \mathbin{\&} X_1) & \xleftarrow[\mathbf{S}\,\mathsf{m}^2]{} & \mathbf{S}(!X_0 \otimes !X_1)
\end{array} \tag{4.2}$$

Proof. By Theorem 4.4, the morphism (4.2) is equal to

$$\begin{aligned}
(\mathbf{S}\,\mathsf{der}_{X_0 \& X_1})\ \partial_{X_0 \& X_1}\,(\psi^0_{X_0,X_1})^! &= \mathsf{der}_{\mathbf{S}(X_0 \& X_1)}\,(\psi^0_{X_0,X_1})^! \quad \text{by } (\partial\textbf{-chain}) \\
&= \psi^0_{X_0,X_1}\,.
\end{aligned}$$

□

Lemma 4.6.

$$\pi_0\ \psi^0_{X_0,X_1} = \pi_0 \mathbin{\&} X_1 \quad \textit{and} \quad \pi_1\ \psi^0_{X_0,X_1} = \pi_1 \mathbin{\&} 0 \tag{4.3}$$
$$\pi_0\ \psi^1_{X_0,X_1} = X_0 \mathbin{\&} \pi_0 \quad \textit{and} \quad \pi_1\ \psi^1_{X_0,X_1} = 0 \mathbin{\&} \pi_1\,. \tag{4.4}$$

Proof. Immediate consequence of the definitions. □

Theorem 4.7. *The natural morphisms ψ^0, ψ^1 are strengths for the monad $(\mathsf{D}, \zeta, \theta)$ on the category $\mathcal{L}_!$.*

This means that the following diagrams commute in $\mathcal{L}_!$.

$$\begin{array}{ccc} X_0 \mathbin{\&} X_1 & & \\ \downarrow{\scriptstyle \zeta_{X_0} \& X_1} & \searrow{\scriptstyle \zeta_{X_0 \& X_1}} & \\ \mathsf{D}\, X_0 \mathbin{\&} X_1 & \xrightarrow{\psi^0_{X_0,X_1}} & \mathsf{D}\,(X_0 \mathbin{\&} X_1) \end{array}$$

$$\begin{array}{ccccc} \mathsf{D}^2\, X_0 \mathbin{\&} X_1 & \xrightarrow{\psi^0_{\mathsf{D}\, X_0,X_1}} & \mathsf{D}\,(\mathsf{D}\, X_0 \mathbin{\&} X_1) & \xrightarrow{\mathsf{D}\, \psi_{X_0,X_1}} & \mathsf{D}^2\,(X_0 \mathbin{\&} X_1) \\ \downarrow{\scriptstyle \theta_{X_0} \& X_1} & & & & \downarrow{\scriptstyle \theta_{X_0 \& X_1}} \\ \mathsf{D}\, X_0 \mathbin{\&} X_1 & & \xrightarrow{\qquad \psi^0_{X_0,X_1} \qquad} & & \mathsf{D}\,(X_0 \mathbin{\&} X_1) \end{array}$$

$$\begin{array}{ccc} \mathsf{D}\, X \mathbin{\&} \top & \xrightarrow{\psi_{X,\top}} & \mathsf{D}\,(X \mathbin{\&} \top) \\ & \searrow{\scriptstyle \mathsf{pr}_0} & \downarrow{\scriptstyle \mathsf{D}\, \mathsf{pr}_0} \\ & & \mathsf{D}\, X \end{array} \qquad \begin{array}{ccc} \mathsf{D}\, X_0 \mathbin{\&} X_1 \mathbin{\&} X_2 & \xrightarrow{\psi^0_{X_0,X_1} \& X_2} & \mathsf{D}\,(X_0 \mathbin{\&} X_1) \mathbin{\&} X_2 \\ & \searrow{\scriptstyle \psi^0_{X_0,X_1 \& X_2}} & \downarrow{\scriptstyle \psi^0_{X_0 \& X_1, X_2}} \\ & & \mathsf{D}(X_0 \mathbin{\&} X_1 \mathbin{\&} X_2) \end{array}$$

where we keep the associativity isomorphisms of & implicit.

Proof. It suffices to prove the corresponding commutations in $\mathcal{L}$ rather than $\mathcal{L}_!$ since all the involved morphisms are images of morphisms in $\mathcal{L}$ through $\mathsf{Lin}_!$, and this is quite easy. □

The commutativity of this strength takes a particularly strong form in this setting.

Lemma 4.8. *The following diagram commutes in* $\mathcal{L}$

$$\begin{array}{ccccc} \mathbf{S}\, X_0 \mathbin{\&} \mathbf{S}\, X_1 & \xrightarrow{\psi^1_{\mathbf{S}\, X_0,X_1}} & \mathbf{S}\,(\mathbf{S}\, X_0 \mathbin{\&} X_1) & \xrightarrow{\mathbf{S}\, \psi^0_{X_0,X_1}} & \mathbf{S}^2\,(X_0 \mathbin{\&} X_1) \\ \downarrow{\scriptstyle \psi^0_{X_0,\mathbf{S}\, X_1}} & & & & \downarrow{\scriptstyle \mathsf{c}} \\ \mathbf{S}\,(X_0 \mathbin{\&} \mathbf{S}\, X_1) & & \xrightarrow{\qquad \mathbf{S}\, \psi^1_{X_0,X_1} \qquad} & & \mathbf{S}^2\,(X_0 \mathbin{\&} X_1) \end{array}$$

Proof. We prove that for each $j, k \in \{0,1\}$ one has

$$\pi_k\ \pi_j\,(\mathbf{S}\, \psi^1_{X_0,X_1})\, \psi^0_{X_0,\mathbf{S}\, X_1} = \pi_j\ \pi_k\,(\mathbf{S}\, \psi^0_{X_0,X_1})\, \psi^1_{X_0,\mathbf{S}\, X_1}$$

which will prove our contention since $\pi_k\ \pi_j\ \mathsf{c} = \pi_j\ \pi_k$. This amounts to proving that

$$\pi_k\ \psi^1_{X_0,X_1}\ \pi_j\ \psi^0_{X_0,\mathbf{S}\, X_1} = \pi_j\ \psi^0_{X_0,X_1}\ \pi_k\ \psi^1_{\mathbf{S}\, X_0,X_1}$$

for which we apply Equations (4.3) and (4.4). We have

$$\begin{aligned} \pi_0\ \psi^1_{X_0,X_1}\ \pi_0\ \psi^0_{X_0,\mathbf{S}\, X_1} &= (X_0 \mathbin{\&} \pi_0)\ (\pi_0 \mathbin{\&} \mathbf{S}\, X_1) = \pi_0 \mathbin{\&} \pi_0 = \pi_0\ \psi^0_{X_0,X_1}\ \pi_0\ \psi^1_{\mathbf{S}\, X_0,X_1} \\ \pi_1\ \psi^1_{X_0,X_1}\ \pi_1\ \psi^0_{X_0,\mathbf{S}\, X_1} &= (0 \mathbin{\&} \pi_1)\ (\pi_1 \mathbin{\&} 0) = 0 = \pi_1\ \psi^0_{X_0,X_1}\ \pi_1\ \psi^1_{\mathbf{S}\, X_0,X_1} \\ \pi_0\ \psi^1_{X_0,X_1}\ \pi_1\ \psi^0_{X_0,\mathbf{S}\, X_1} &= (X_0 \mathbin{\&} \pi_0)\ (\pi_1 \mathbin{\&} 0) = \pi_1 \mathbin{\&} 0 \\ &= (\pi_1 \mathbin{\&} 0)\ (\mathbf{S}\, X_0 \mathbin{\&} \pi_0) = \pi_1\ \psi^0_{X_0,X_1}\ \pi_0\ \psi^1_{\mathbf{S}\, X_0,X_1} \end{aligned}$$

and the last case is symmetrical. □

As a consequence

$$\tau\,(\mathbf{S}\, \varphi_1^{X_0,X_1})\ \varphi_0^{X_0,\mathbf{S}\, X_1} = \tau\,(\mathbf{S}\, \varphi_0^{X_0,X_1})\ \varphi_1^{\mathbf{S}\, X_0,X_1} \in \mathcal{L}(\mathbf{S}\, X_0 \mathbin{\&} \mathbf{S}\, X_1, \mathbf{S}\,(X_0 \mathbin{\&} X_1))\,.$$

Thanks to our assumption that $\mathbf{S}$ preserves & on the nose this morphism is actually the identity.

Theorem 4.9. *The morphism* $\tau\,(\mathbf{S}\,\psi^1_{X_0,X_1})\,\psi^0_{X_0,\mathbf{S}\,X_1} = \tau\,(\mathbf{S}\,\psi^0_{X_0,X_1})\,\psi^1_{\mathbf{S}\,X_0,X_1}$ *is the identity morphism.*

Proof. The first equation results from Lemma 4.8. From the proof of that lemma we get

$$\begin{aligned}\pi_0\ \tau\,(\mathbf{S}\,\psi^1_{X_0,X_1})\,\psi^0_{X_0,\mathbf{S}\,X_1} &= \pi_0\ \pi_0\,(\mathbf{S}\,\psi^1_{X_0,X_1})\,\psi^0_{X_0,\mathbf{S}\,X_1}\\ &= \pi_0 \mathbin{\&} \pi_0\end{aligned}$$

and

$$\begin{aligned}\pi_1\ \tau\,(\mathbf{S}\,\psi^1_{X_0,X_1})\,\psi^0_{X_0,\mathbf{S}\,X_1} &= \pi_1\ \pi_0\,(\mathbf{S}\,\psi^1_{X_0,X_1})\,\psi^0_{X_0,\mathbf{S}\,X_1} + \pi_0\ \pi_1\,(\mathbf{S}\,\psi^1_{X_0,X_1})\,\psi^0_{X_0,\mathbf{S}\,X_1}\\ &= (0 \mathbin{\&} \pi_1) + (\pi_1 \mathbin{\&} 0) = \pi_1 \mathbin{\&} \pi_1\end{aligned}$$

by linearity of & on morphisms. □

More generally given objects $X_0,\dots,X_n$ we have an additive strength morphism

$$\psi^i \in \mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}\,X_i \mathbin{\&} \cdots \mathbin{\&} X_n, \mathsf{D}(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n))\,.$$

which is actually linear and comes from $\psi^i \in \mathcal{L}(X_0 \mathbin{\&} \cdots \mathbin{\&} \mathbf{S}\,X_i \mathbin{\&} \cdots \mathbin{\&} X_n, \mathbf{S}(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n))$. Up to the identification of $\mathbf{S}(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n)$ with $\mathbf{S}\,X_0 \mathbin{\&} \cdots \mathbin{\&} \mathbf{S}\,X_n$, this morphism of $\mathcal{L}$ can simply be written as

$$\psi^i = \iota_0^{X_0} \mathbin{\&} \cdots \mathbin{\&} \iota_0^{X_{i-1}} \mathbin{\&} X_i \mathbin{\&} \iota_0^{X_{i+1}} \mathbin{\&} \cdots \mathbin{\&} \iota_0^{X_n}\ . \tag{4.5}$$

When we will need to be explicit as to the list of objects $X_0,\dots,X_n$, we will write $\psi^i_{X_0,\dots,X_n}$ instead of ψ^i.

Lemma 4.10. *Let* $i,l \in \{0,\dots,n\}$. *If* $i \neq l$ *we have*

$$(\mathbf{S}\,\psi^i_{X_0,\dots,X_n})\,\psi^l_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n} = \mathsf{c}\,(\mathbf{S}\,\psi^l_{X_0,\dots,X_n})\,\psi^i_{X_0,\dots,\mathbf{S}\,X_l,\dots,X_n}\,.$$

And for any $i,l \in \{0,1\}$, *we have*

$$\begin{aligned}\tau\,(\mathbf{S}\,\psi^i_{X_0,\dots,X_n})\,\psi^l_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n} &= \tau\,(\mathbf{S}\,\psi^l_{X_0,\dots,X_n})\,\psi^i_{X_0,\dots,\mathbf{S}\,X_l,\dots,X_n}\\ \tau\,(\mathbf{S}\,\psi^i_{X_0,\dots,X_n})\,\psi^i_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n} &= \psi^i_{X_0,\dots,X_n}\,(X_0 \mathbin{\&} \cdots \mathbin{\&} \tau \mathbin{\&} \cdots \mathbin{\&} X_n)\,.\end{aligned}$$

Proof. The proof of the first equation is exactly as the one of Lemma 4.8. In the case $l \neq i$, the second equation follows from the first one and from $\tau\,\mathsf{c} = \tau$ and in the case $l = i$, it is trivial. We prove the last equation. We have

$$\begin{aligned}\pi_0\ \tau\,(\mathbf{S}\,\psi^i_{X_0,\dots,X_n})\,\psi^i_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n} &= \pi_0\ \pi_0\,(\mathbf{S}\,\psi^i_{X_0,\dots,X_n})\,\psi^i_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n}\\ &= \pi_0\ \psi^i_{X_0,\dots,X_n}\ \pi_0\ \psi^i_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n}\\ &= X_0 \mathbin{\&} \cdots \mathbin{\&} (\pi_0\ \pi_0) \mathbin{\&} \cdots \mathbin{\&} X_n\end{aligned}$$

and

$$\begin{aligned}\pi_0\ \psi^i_{X_0,\dots,X_n}\,(X_0 \mathbin{\&} \cdots \mathbin{\&} \tau \mathbin{\&} \cdots \mathbin{\&} X_n) &= X_0 \mathbin{\&} \cdots \mathbin{\&} (\pi_0\ \tau) \mathbin{\&} \cdots \mathbin{\&} X_n\\ &= X_0 \mathbin{\&} \cdots \mathbin{\&} (\pi_0\ \pi_0) \mathbin{\&} \cdots \mathbin{\&} X_n\,.\end{aligned}$$

Next we have

$$\begin{aligned}
\pi_1\ \tau\,(\mathbf{S}\,\psi^i_{X_0,\dots,X_n})\,\psi^i_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n} &= (\pi_0\ \pi_1 + \pi_1\ \pi_0)\,(\mathbf{S}\,\psi^i_{X_0,\dots,X_n})\,\psi^i_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n} \\
&= \pi_0\ \psi^i_{X_0,\dots,X_n}\ \pi_1\ \psi^i_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n} \\
&\quad + \pi_1\ \psi^i_{X_0,\dots,X_n}\ \pi_0\ \psi^i_{X_0,\dots,\mathbf{S}\,X_i,\dots,X_n} \\
&= (X_0 \mathbin{\&} \cdots \mathbin{\&} \pi_0 \mathbin{\&} \cdots \mathbin{\&} X_n)\,(0 \mathbin{\&} \cdots \mathbin{\&} \pi_1 \mathbin{\&} \cdots \mathbin{\&} 0) \\
&\quad + (0 \mathbin{\&} \cdots \mathbin{\&} \pi_1 \mathbin{\&} \cdots \mathbin{\&} 0)\,(X_0 \mathbin{\&} \cdots \mathbin{\&} \pi_0 \mathbin{\&} \cdots \mathbin{\&} X_n) \\
&= 0 \mathbin{\&} \cdots \mathbin{\&} (\pi_0\ \pi_1 + \pi_1\ \pi_0) \mathbin{\&} \cdots \mathbin{\&} 0
\end{aligned}$$

Notice that & is not a multilinear operation on morphisms, so in the last equality we are crucially using the fact that all factors but the ith are equal to 0 in both summands. On the other hand we have

$$\begin{aligned}
\pi_1\ \psi^i_{X_0,\dots,X_n}\,(X_0 \mathbin{\&} \cdots \mathbin{\&} \tau \mathbin{\&} \cdots \mathbin{\&} X_n) &= 0 \mathbin{\&} \cdots \mathbin{\&} (\pi_1\ \tau) \mathbin{\&} \cdots \mathbin{\&} 0 \\
&= 0 \mathbin{\&} \cdots \mathbin{\&} (\pi_0\ \pi_1 + \pi_1\ \pi_0) \mathbin{\&} \cdots \mathbin{\&} 0
\end{aligned}$$

proving our contention by the fact that π_0, π_1 are jointly monic. □

Given $f \in \mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n, Y)$, we define the *i-th partial derivative of f* as $\mathsf{D}_i\, f = \mathsf{D}\, f \circ \psi^i \in \mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}\, X_i \mathbin{\&} \cdots \mathbin{\&} X_n, \mathsf{D}\, Y)$.

Theorem 4.11. *Let $i, l \in \{0, 1\}$. If $i \neq l$ then*

$$\mathsf{D}_i\, \mathsf{D}_l\, f = \mathsf{c} \circ \mathsf{D}_l\, \mathsf{D}_i\, f$$

so that for any $i, l \in \{0, 1\}$ we have $\theta \circ \mathsf{D}_i\, \mathsf{D}_l\, f = \theta \circ \mathsf{D}_l\, \mathsf{D}_i\, f$. Moreover $\theta \circ \mathsf{D}_i\, \mathsf{D}_i\, f = \mathsf{D}_i\, f \circ (X_0 \mathbin{\&} \cdots \mathbin{\&} \theta \mathbin{\&} \cdots \mathbin{\&} X_n)$.

This is an immediate consequence of Lemma 4.10 and of the naturality of c and of θ in the category $\mathcal{L}_!$.

Notice that the morphism $\theta \circ \mathsf{D}_i\, \mathsf{D}_l\, f$ in the statement of this result belongs to $\mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}\, X_i \mathbin{\&} X_{i+1} \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}\, X_l \mathbin{\&} \cdots \mathbin{\&} X_n, \mathsf{D}\, Y)$ if $i < l$ and to $\mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}^2\, X_i \mathbin{\&} \cdots \mathbin{\&} X_n, \mathsf{D}\, Y)$ if $i = l$.

Theorem 4.12. *Let $f \in \mathcal{L}_!(X_0 \mathbin{\&} X_1, Y)$ so that $\mathsf{D}\, f \in \mathcal{L}_!(\mathsf{D}\, X_0 \mathbin{\&} \mathsf{D}\, X_1, \mathsf{D}\, Y)$. Then*

$$\mathsf{D}\, f = \theta \circ \mathsf{D}_1\, \mathsf{D}_0\, f = \theta \circ \mathsf{D}_0\, \mathsf{D}_1\, f\,.$$

Proof. The second equation holds by Theorem 4.11. Next we have

$$\begin{aligned}
\theta \circ \mathsf{D}_1\, \mathsf{D}_0\, f &= \theta \circ \mathsf{D}(\mathsf{D}_0\, f \circ \psi^0_{X_0,X_1}) \\
&= \theta \circ \mathsf{D}^2\, f \circ \mathsf{D}\, \psi^0_{X_0,X_1} \circ \psi^1_{\mathsf{D}\, X_0,X_1} \\
&= \mathsf{D}\, f \circ \theta \circ \mathsf{D}\, \psi^0_{X_0,X_1} \circ \psi^1_{\mathsf{D}\, X_0,X_1} \\
&= \mathsf{D}\, f
\end{aligned}$$

by Theorem 4.9. □

Remark 4.13. The intuitive meaning of this result is that the derivative of a function acting on pairs is obtained as the sum of its partial derivatives. This sum is computed by the θ natural transformation.

Given $e \in \mathbb{N}$ we can more generally define a linear

$$\psi^i_{X_0,\dots,X_n}(e) \in \mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}^e\, X_i \mathbin{\&} \cdots \mathbin{\&} X_n, \mathsf{D}^e\, X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}^e\, X_n)$$

by induction on e (we give only the definition for $n = 1$, the generalization is easy and not really required for our purpose): we set $\psi^0_{X_0,X_1}(0) = \mathsf{Id}$ and $\psi^0_{X_0,X_1}(e+1) = \mathsf{D}\,\psi^0_{X_0,X_1}(e) \circ \psi^0_{\mathsf{D}^e\, X_0,X_1}$ typed as follows:

$$\mathsf{D}^{e+1}\, X_0 \mathbin{\&} X_1 \xrightarrow{\psi^0_{\mathsf{D}^e\, X_0,X_1}} \mathsf{D}^{e+1}\, X_0 \mathbin{\&} \mathsf{D}\, X_1 \xrightarrow{\mathsf{D}\,\psi^0_{X_0,X_1}(e)} \mathsf{D}^{e+1}\, X_0 \mathbin{\&} \mathsf{D}^{e+1}\, X_1$$

and similarly for $\psi^1(e)$. We can easily give a direct description of this morphism.

Lemma 4.14. $\psi^i_{X_0,\dots,X_n}(e) = \iota_0(e) \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}^e\, X_i \mathbin{\&} \cdots \mathbin{\&} \iota_0(e)$ *where* $\iota_0(e) \in \mathcal{L}(X, \mathbf{S}^e\, X)$ *is defined inductively by* $\iota_0(e) = \mathsf{Id}$ *and* $\iota_0(e+1) = (\mathbf{S}\,\iota_0(e))\ \iota_0$.

That is, in $\mathcal{L}_!$, $\iota_0(e+1) = (\mathsf{D}\,\iota_0(e)) \circ \iota_0$. The proof is a straightforward induction.

Lemma 4.15. *If* $f \in \mathcal{L}_!(X_0 \mathbin{\&} X_1, Y)$, $d \in \mathbb{N}$ *and* $i \in \{0,1\}$, *we have* $\mathsf{D}^d_i\, f = \mathsf{D}^d\, f \circ \psi^i_{X_0,X_1}(d)$.

Proof. Immediate consequence of the functoriality of D in $\mathcal{L}_!$ and of the definition of $\psi^i_{X_0,X_1}(d)$. ☐

Lemma 4.16. *If* $d < e$ *we have* $\mathsf{D}^d\,\pi_0 \circ \iota_0(e) = \iota_0(e-1)$ *and* $\mathsf{D}^d\,\pi_1 \circ \iota_0(e) = 0$.

Proof. By induction on d. Notice first that we are actually dealing with linear morphisms so that we can do our computations in $\mathcal{L}$. For the base case we have, since $e > 0$: $\pi_i\ \iota_0(e) = \pi_i\ \mathbf{S}(\iota_0(e-1))\ \iota_0 = \iota_0(e-1)\ \pi_i\ \iota_0$ and we have $\pi_i\ \iota_0 = \boldsymbol{\delta}_{i,0}\,\mathsf{Id}$ (where $\boldsymbol{\delta}_{i,j}$ is the Kronecker symbol).

For the inductive case observe first that since $d + 1 < e$ we have $e \geq 2$. Then

$$\begin{aligned}(\mathbf{S}^{d+1}\,\pi_i)\ \iota_0(e) &= (\mathbf{S}^{d+1}\,\pi_i)\ \mathbf{S}(\iota_0(e-1)) \circ \iota_0\\ &= \mathbf{S}((\mathbf{S}^d\,\pi_i)\ \iota_0(e-1))\ \iota_0\\ &= \mathbf{S}(\boldsymbol{\delta}_{i,0}\,\iota_0(e-2))\ \iota_0 \quad \text{by ind. hypothesis}\\ &= \boldsymbol{\delta}_{i,0}\,\iota_0(e-1)\end{aligned}$$

as contended. ☐

We generalize the canonical flip $\mathsf{c} \in \mathcal{L}(\mathbf{S}^2\, X, \mathbf{S}^2\, X)$ to an iso $\mathsf{c}(l) \in \mathcal{L}(\mathbf{S}^{l+2}\, X, \mathbf{S}^{l+2}\, X)$ for each $l \in \mathbb{N}$ defined inductively by

$$\mathsf{c}(0) = \mathsf{c} \quad \text{and} \quad \mathsf{c}(l+1) = \mathsf{c}\,(\mathbf{S}\,\mathsf{c}(l))\,.$$

Lemma 4.17. *Given* $i_0, \dots, i_{l+1} \in \{0,1\}$, *one has*

$$\pi_{i_{l+1}} \cdots \pi_{i_0}\ \mathsf{c}(l) = \pi_{i_0}\ \pi_{i_{l+1}} \cdots \pi_{i_1}\ .$$

Proof. By induction on l. For $l = 0$ the property holds by the very definition of c. Assume that the property holds for l and let us prove it for $l+1$. So let $i_0, \dots, i_{l+2} \in \{0,1\}$. We have

$$\begin{aligned}\pi_{i_{l+2}} \cdots \pi_{i_0}\ \mathsf{c}(l+1) &= \pi_{i_{l+2}} \cdots \pi_{i_0}\ \mathsf{c}\,(\mathbf{S}\,\mathsf{c}(l))\\ &= \pi_{i_{l+2}} \cdots \pi_{i_2}\ \pi_{i_0}\ \pi_{i_1}\,(\mathbf{S}\,\mathsf{c}(l)\\ &= \pi_{i_{l+2}} \cdots \pi_{i_2}\ \pi_{i_0}\ \mathsf{c}(l)\ \pi_{i_1} \quad \text{by nat. of } \pi_{i_1}\\ &= \pi_{i_0}\ \pi_{i_{l+2}} \cdots \pi_{i_2}\ \pi_{i_1} \quad \text{by ind. hyp.}\end{aligned}$$

□

This means that $\mathsf{c}(l)$ implements a circular permutation of length $l+2$ on the indices.

Lemma 4.18. *Let* $f \in \mathcal{L}_!(X_0 \mathbin{\&} X_1, Y)$ *and let* $k \in \mathbb{N}$. *Then* $\mathsf{D}_1\,\mathsf{D}_0^{k+1}\,f, \mathsf{D}_0^{k+1}\,\mathsf{D}_1\,f \in \mathcal{L}_!(\mathsf{D}^{k+1}\,X \mathbin{\&} \mathsf{D}\,X, \mathsf{D}^{k+2}\,Y)$ *satisfy the relation* $\mathsf{D}_1\,\mathsf{D}_0^{k+1}\,f = \mathsf{c}(k) \circ \mathsf{D}_0^{k+1}\,\mathsf{D}_1\,f$.

Proof. For $k=0$, this is just Theorem 4.11. For the inductive step we have

$$\begin{aligned}
\mathsf{D}_1\,\mathsf{D}_0^{k+2}\,f &= \mathsf{c} \circ \mathsf{D}_0\,\mathsf{D}_1\,\mathsf{D}_0^{k+1}\,f \quad \text{by Theorem 4.11} \\
&= \mathsf{c} \circ \mathsf{D}_0(\mathsf{c}(k) \circ \mathsf{D}_0^{k+1}\,\mathsf{D}_1\,f) \quad \text{by ind. hyp.} \\
&= \mathsf{c} \circ \mathsf{D}\,\mathsf{c}(k) \circ \mathsf{D}_0^{k+2}\,\mathsf{D}_1\,f \quad \text{by def. of } \mathsf{D}_0 \\
&= \mathsf{c}(k+1) \circ \mathsf{D}_0^{k+2}\,\mathsf{D}_1\,f \quad \text{by def. of } \mathsf{c}(k+1)\,.
\end{aligned}$$

□

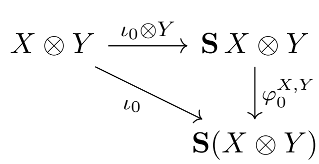
4.3. **Differentiation in the closed case.** Since our purpose is to provide the categorical foundations of Λ_{cd}, we require the category $\mathcal{L}$ to be closed wrt. its SMC structure.

Remember that we consider the isos $\mathsf{D}(Z \mathbin{\&} X) \simeq \mathsf{D}\,Z \mathbin{\&} \mathsf{D}\,X$ and $\mathsf{D}\,(X \Rightarrow Y) \simeq X \Rightarrow \mathsf{D}\,Y$ as identities: this is our (**S**⊗**-fun**) axiom and we assume that the corresponding iso, which is $\mathsf{Cur}(\mathsf{D}_0\,\mathsf{Ev}) = \mathsf{Cur}((\mathsf{D}\,\mathsf{Ev}) \circ \psi^0_{X \Rightarrow Y, X}) \in \mathcal{L}_!(\mathsf{D}(X \Rightarrow Y), X \Rightarrow \mathsf{D}\,Y)$, is the identity morphism. With these identifications we have the following equation.

Lemma 4.19. *Let* $f \in \mathcal{L}_!(Z \mathbin{\&} X, Y)$ *so that* $\mathsf{Cur}\,f \in \mathcal{L}_!(Z, X \Rightarrow Y)$, $\mathsf{D}_0\,f \in \mathcal{L}_!(\mathsf{D}\,Z \mathbin{\&} X, \mathsf{D}\,Y)$ *and* $\mathsf{D}(\mathsf{Cur}\,f) \in \mathcal{L}_!(\mathsf{D}\,Z, X \Rightarrow \mathsf{D}\,Y)$. *Then* $\mathsf{D}(\mathsf{Cur}\,f) = \mathsf{Cur}(\mathsf{D}_0\,f)$.

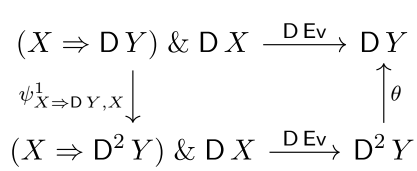
Proof. More precisely we must prove that $\mathsf{Cur}(\mathsf{D}_0\,\mathsf{Ev}) \circ \mathsf{D}(\mathsf{Cur}\,f) = \mathsf{Cur}(\mathsf{D}_0\,f)$ which boils down to the naturality of ψ^0 by simple computations in the CCC $\mathcal{L}_!$. □

Lemma 4.20. *The following diagram commutes in* $\mathcal{L}$



This is easily proven using as usual the fact that π_0, π_1 are jointly monic.

Lemma 4.21. *The following diagram commutes in* $\mathcal{L}_!$



Proof. We need to come back to the definition of the functor D. We expand the definition of $\mathsf{D}\,\mathsf{Ev} : (X \Rightarrow \mathsf{D}\,Y) \mathbin{\&} \mathsf{D}\,X \to \mathsf{D}\,Y$ in $\mathcal{L}$.

$$\begin{aligned}
\mathsf{D}\,\mathsf{Ev} &= (\mathbf{S}\,\mathsf{Ev})\ \partial_{(!X\multimap Y)\&X}\\
&= (\mathbf{S}\,\mathsf{ev})\ \mathbf{S}(\mathsf{der}_{!X\multimap Y}\otimes !X)\ \mathbf{S}(\mathsf{m}^2)^{-1}\ \partial_{(!X\multimap Y)\&X} \quad \text{by def. of } \mathsf{Ev}\\
&= (\mathbf{S}\,\mathsf{ev})\ \mathbf{S}(\mathsf{der}_{!X\multimap Y}\otimes !X)\ \mathsf{L}_{!(!X\multimap Y),!X}\,(\partial_{!X\multimap Y}\otimes\partial_X)\,(\mathsf{m}^2)^{-1} \quad \text{by } (\partial\text{-}\&)\\
&= (\mathbf{S}\,\mathsf{ev})\ \mathsf{L}_{(!X\multimap Y),!X}\,(\mathbf{S}\,\mathsf{der}_{!X\multimap Y}\otimes \mathbf{S}\,!X)\,(\partial_{!X\multimap Y}\otimes\partial_X)\,(\mathsf{m}^2)^{-1}\\
&= (\mathbf{S}\,\mathsf{ev})\ \mathsf{L}_{(!X\multimap Y),!X}\,(\mathsf{der}_{\mathbf{S}(!X\multimap Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1} \quad \text{by } (\partial\text{-}\mathbf{chain})\\
&= (\mathbf{S}\,\mathsf{ev})\,\tau\,(\mathbf{S}\,\varphi^0_{(!X\multimap Y),!X})\ \varphi^1_{!X\multimap\mathbf{S}\,Y,!X}\,(\mathsf{der}_{\mathbf{S}(!X\multimap Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1} \quad \text{by def. of } \mathsf{L}\\
&= \tau\,(\mathbf{S}^2\,\mathsf{ev})\,(\mathbf{S}\,\varphi^0_{(!X\multimap Y),!X})\ \varphi^1_{!X\multimap\mathbf{S}\,Y,!X}\,(\mathsf{der}_{\mathbf{S}(!X\multimap Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1}\\
&= \tau\,(\mathbf{S}\,\mathsf{ev})\ \varphi^1_{!X\multimap\mathbf{S}\,Y,!X}\,(\mathsf{der}_{\mathbf{S}(!X\multimap Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1}
\end{aligned}$$

by the identification $\mathbf{S}\,(!X \multimap Y) = (!X \multimap \mathbf{S}\,Y)$. On the other hand we have

$$\begin{aligned}
(\mathsf{D}\,\mathsf{Ev})\circ\psi^1_{X\Rightarrow\mathsf{D}\,Y,X} &= (\mathbf{S}\,\mathsf{ev})\ \mathsf{L}_{(!X\multimap\mathbf{S}\,Y),!X}\,(\mathsf{der}_{\mathbf{S}(!X\multimap\mathbf{S}\,Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1}\,!\psi^1_{X\Rightarrow\mathsf{D}\,Y,X} \quad \text{as above}\\
&= (\mathbf{S}\,\mathsf{ev})\ \mathsf{L}_{(!X\multimap\mathbf{S}\,Y),!X}\,(\mathsf{der}_{\mathbf{S}(!X\multimap\mathbf{S}\,Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1}\,!(\iota_0\mathbin{\&}\mathbf{S}\,X)\\
&\qquad\qquad \text{by def. of } \psi^1_{X\Rightarrow\mathsf{D}\,Y,X}\\
&= (\mathbf{S}\,\mathsf{ev})\ \mathsf{L}_{(!X\multimap\mathbf{S}\,Y),!X}\,(\mathsf{der}_{\mathbf{S}(!X\multimap\mathbf{S}\,Y)}\otimes\partial_X)\,(!\iota_0\otimes !\mathbf{S}\,X)\,(\mathsf{m}^2)^{-1}\\
&= (\mathbf{S}\,\mathsf{ev})\ \mathsf{L}_{(!X\multimap\mathbf{S}\,Y),!X}\ (\iota_0\otimes\partial_X)\,(\mathsf{der}_{(!X\multimap\mathbf{S}\,Y)}\otimes !\mathbf{S}\,X)\,(\mathsf{m}^2)^{-1}\\
&= (\mathbf{S}\,\mathsf{ev})\,\tau\,(\mathbf{S}\,\varphi^1_{!X\multimap\mathbf{S}\,Y,!X})\ \varphi^0_{!X\multimap\mathbf{S}\,Y,\mathbf{S}\,!X}\ (\iota_0\otimes\mathbf{S}\,!X)\\
&\qquad ((!X\multimap\mathbf{S}\,Y)\otimes\partial_X)\,(\mathsf{der}_{(!X\multimap\mathbf{S}\,Y)}\otimes !\mathbf{S}\,X)\,(\mathsf{m}^2)^{-1} \quad \text{by def. of } \mathsf{L}\\
&= (\mathbf{S}\,\mathsf{ev})\,\tau\,(\mathbf{S}\,\varphi^1_{!X\multimap\mathbf{S}\,Y,!X})\ \iota_0\,(\mathsf{der}_{(!X\multimap\mathbf{S}\,Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1} \quad \text{by Lemma 4.20}\\
&= (\mathbf{S}\,\mathsf{ev})\,\tau\ \iota_0\ \varphi^1_{!X\multimap\mathbf{S}\,Y,!X}\,(\mathsf{der}_{(!X\multimap\mathbf{S}\,Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1}\\
&= (\mathbf{S}\,\mathsf{ev})\ \varphi^1_{!X\multimap\mathbf{S}\,Y,!X}\,(\mathsf{der}_{(!X\multimap\mathbf{S}\,Y)}\otimes\partial_X)\,(\mathsf{m}^2)^{-1}
\end{aligned}$$

which proves our contention. □

4.4. **The case of multilinear morphisms.** For each $i \in \{0, \dots, n\}$ we can define a tensorial generalized strength

$$\varphi^i_{X_0,\dots,X_n} \in \mathcal{L}(X_0 \otimes \cdots \otimes \mathbf{S}\,X_i \otimes \cdots \otimes X_n, \mathbf{S}\,(X_0 \otimes \cdots \otimes X_n))\,.$$

Let $l \in \mathcal{L}(X_0 \otimes \cdots \otimes X_n, Y)$, then we define $\widetilde{l} \in \mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n, Y)$ as the following composition of morphisms

$$!(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n) \xrightarrow{(\mathsf{m}^n)^{-1}} !X_0 \otimes \cdots \otimes !X_n \xrightarrow{\mathsf{der}_{X_0}\otimes\cdots\otimes\mathsf{der}_{X_n}} X_0 \otimes \cdots \otimes X_n \xrightarrow{l} Y$$

A morphism in $\mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n, Y)$ definable in that way can be called an $n+1$-*linear* morphism (that is, a multilinear morphisms with $n+1$ arguments) for the following reason.

Lemma 4.22. *With these notations, we have*

$$\widetilde{l} \circ (X_0 \mathbin{\&} \cdots \mathbin{\&} X_{i-1} \mathbin{\&} 0 \mathbin{\&} X_{i+1} \cdots \mathbin{\&} X_n) = 0$$

and if $f_0, f_1 \in \mathcal{L}(Z, X_i)$ *are summable then so are*

$$\widetilde{l} \circ (X_0 \mathbin{\&} \cdots \mathbin{\&} X_{i-1} \mathbin{\&} f_j \mathbin{\&} X_{i+1} \cdots \mathbin{\&} X_n)$$

for $j = 0, 1$ *and*

$$\begin{aligned}\widetilde{l} \circ (X_0 \mathbin{\&} \cdots \mathbin{\&} X_{i-1} \mathbin{\&} (f_0 + f_1) \mathbin{\&} X_{i+1} \cdots \mathbin{\&} X_n)& \\ = \widetilde{l} \circ (X_0 \mathbin{\&} \cdots \mathbin{\&} X_{i-1} \mathbin{\&} f_0 \mathbin{\&} X_{i+1} \cdots \mathbin{\&} X_n)& \\ + \widetilde{l} \circ (X_0 \mathbin{\&} \cdots \mathbin{\&} X_{i-1} \mathbin{\&} f_1 \mathbin{\&} X_{i+1} \cdots \mathbin{\&} X_n)&\,.\end{aligned}$$

Proof. For the summability, we have $\widetilde{l} \circ (X_0 \mathbin{\&} \cdots \mathbin{\&} X_{i-1} \mathbin{\&} f_j \mathbin{\&} X_{i+1} \cdots \mathbin{\&} X_n) = \widetilde{l_j}$ where $l_j = l\,(X_0 \otimes \cdots \otimes f_j \otimes \cdots \otimes X_n)$ and since f_0, f_1 are summable so are $X_0 \otimes \cdots \otimes f_j \otimes \cdots \otimes X_n$ for $j = 0, 1$ by (**S**$\otimes$**-dist**) with $X_0 \otimes \cdots \otimes (f_0 + f_1) \otimes \cdots \otimes X_n$ as sum. The result follows by Lemma 2.3 of [Ehr21]. For the property relative to 0 we use similarly the fact that 0 is absorbing for $\otimes$ and for composition in $\mathcal{L}$. ☐

Theorem 4.23. *Let* $f \in \mathcal{L}(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n, Y)$ *be* $n+1$*-linear. Then* $\mathsf{D}\, f \in \mathcal{L}_!(\mathsf{D}\, X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}\, X_n, \mathsf{D}\, Y)$ *is also* $n+1$*-linear.*

Proof. We assume $n = 1$ for the sake of readability, the general case is not more difficult. Let $l \in \mathcal{L}(X_0 \otimes X_1, Y)$ be such that $f = \widetilde{l}$. Then we set $l' = (\mathbf{S}\, l)\ \mathsf{L}_{X_0,X_1} \in \mathcal{L}(\mathbf{S}\, X_0 \otimes \mathbf{S}\, X_1, \mathbf{S}\, Y)$ and using (∂**-**&) and (∂**-chain**) one shows that $\widetilde{l'} = \mathsf{D}\, f$. ☐

Lemma 4.24. *Given* $X_0, \dots, X_n \in \mathsf{Obj}(\mathcal{L})$ *and* $X = X_0 \mathbin{\&} \cdots \mathbin{\&} X_n$*, we have*

$$\begin{aligned}&\pi_0\ \partial_X\ \mathsf{m}^n = \mathsf{m}^n\,(!\pi_0 \otimes \cdots \otimes !\pi_0)\\ &\pi_1\ \partial_X\ \mathsf{m}^n = \mathsf{m}^n\,((\pi_1\ \partial_{X_0}) \otimes !\pi_0 \otimes \cdots \otimes !\pi_0 + \cdots + !\pi_0 \otimes \cdots \otimes !\pi_0 \otimes (\pi_1\ \partial_{X_0}))\,.\end{aligned}$$

which both belong to $\mathcal{L}(!\mathbf{S}\, X_0 \otimes \cdots \otimes !\mathbf{S}\, X_n, !X)$.

Proof. For the sake of readability we assume that $n = 1$, the general case is not more difficult. By Axiom (∂**-**&) we have $\partial_X\ \mathsf{m}^2 = \mathbf{S}\,\mathsf{m}^2\ \mathsf{L}_2\,(\partial_{X_0} \otimes \partial_{X_1})$ hence

$$\begin{aligned}\pi_0\ \partial_X\ \mathsf{m}^2 &= \pi_0\ \mathbf{S}\,\mathsf{m}^2\ \mathsf{L}_2\,(\partial_{X_0} \otimes \partial_{X_1})\\ &= \mathsf{m}^2\ \pi_0\ \mathsf{L}_2\,(\partial_{X_0} \otimes \partial_{X_1})\\ &= \mathsf{m}^2\,(\pi_0 \otimes \pi_0)\,(\partial_{X_0} \otimes \partial_{X_1})\\ &= \mathsf{m}^2(!\pi_0 \otimes !\pi_0)\end{aligned}$$

and

$$\begin{aligned}\pi_1\ \partial_X\ \mathsf{m}^2 &= \mathsf{m}^2\ \pi_1\ \mathsf{L}_2\,(\partial_{X_0} \otimes \partial_{X_1})\\ &= \mathsf{m}^2\,(\pi_1 \otimes \pi_0 + \pi_0 \otimes \pi_1)\,(\partial_{X_0} \otimes \partial_{X_1})\\ &= \mathsf{m}^2\,((\pi_1\ \partial_{X_0}) \otimes !\pi_0 + !\pi_0 \otimes (\pi_1\ \partial_{X_1}))\end{aligned}$$

☐

Theorem 4.25. *With the same notations,* $\mathsf{D}\,\widetilde{l} \in \mathcal{L}_!(\mathsf{D}\, X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}\, X_n, \mathsf{D}\, Y)$ *satisfies*

$$\begin{aligned}&\pi_0 \circ \mathsf{D}\,\widetilde{l} = \widetilde{l} \circ (\pi_0 \mathbin{\&} \cdots \mathbin{\&} \pi_0)\\ &\pi_1 \circ \mathsf{D}\,\widetilde{l} = \widetilde{l} \circ (\pi_1 \mathbin{\&} \cdots \mathbin{\&} \pi_0) + \cdots + \widetilde{l} \circ (\pi_0 \mathbin{\&} \cdots \mathbin{\&} \pi_1)\,.\end{aligned}$$

Proof. Immediate consequence of Lemma 4.24 and of the naturality of the π_j's wrt. $\mathbf{S}$ in $\mathcal{L}$. □

For $i \in \{1, \dots, n\}$, we have $(\mathbf{S}\, l)\ \varphi^i \in \mathcal{L}(X_0 \otimes \cdots \otimes \mathbf{S}\, X_i \otimes \cdots \otimes X_n, \mathbf{S}\, Y)$. Remember that $\pi_j \in \mathcal{L}(\mathbf{S}\, X, X)$. Given a "linear" morphism $h \in \mathcal{L}(X, Y)$, we use the same notation h for the corresponding morphism $\mathsf{Lin}_! h = h\ \mathsf{der}_X \in \mathcal{L}_!(X, Y)$.

Let $l \in \mathcal{L}(X_0 \otimes \cdots \otimes X_n, Y)$, since $\widetilde{l}$ is multilinear, its partial derivatives should be "trivial", the purpose of the next result is to state precisely this triviality. Given $i \in \{0, \dots, n\}$, we define the i-th "partial application" of the functor $\mathbf{S}$ to l as $\mathbf{S}_i\, l = (\mathbf{S}\, l)\ \varphi^i \in \mathcal{L}(X_0 \otimes \cdots \otimes \mathbf{S}\, X_i \otimes \cdots \otimes X_n, \mathbf{S}\, Y)$.

Theorem 4.26. *For each $i \in \{0, \dots, n\}$ we have $\mathsf{D}_i\, \widetilde{l} = \widetilde{\mathbf{S}_i\, l}$ and for $j \in \{0, 1\}$, we have $\pi_j \circ \mathsf{D}_i\, \widetilde{l} = \widetilde{l} \circ (X_0\ \&\ \cdots\ \&\ \pi_j\ \&\ \cdots\ \&\ X_n)$.*

Proof. For the sake of readability we take $n = 1$. The general case is conceptually not more difficult to deal with, just harder to read due to cumbersome notations. We first prove the second equation for $i = 1$ (the case $i = 0$ is similar), namely, in $\mathcal{L}$:

$$\pi_j\ \mathsf{D}_1\, \widetilde{l} = \widetilde{l}\, !(X_0\ \&\ \pi_j)$$

where $l \in \mathcal{L}(X_0 \otimes X_1, Y)$ so that $\widetilde{l} = l\, (\mathsf{der}_{X_0} \otimes \mathsf{der}_{X_1})\, (\mathsf{m}^2)^{-1}$ is a bilinear morphism in $\mathcal{L}_!$. We have

$$\pi_j\ \mathsf{D}_1\, \widetilde{l} = \pi_j\ \mathbf{S}\, \widetilde{l}\ \partial_{X_0 \& X_1}\ !\psi^1 = \widetilde{l}\ \pi_j\ \partial_{X_0 \& X_1}\ !\psi^1\,.$$

Therefore

$$\begin{aligned} \pi_0\ \mathsf{D}_1\, \widetilde{l} &= \widetilde{l}\, !(\pi_0\ \&\ \pi_0)\, !\psi^1 \quad \text{by } (\partial\textbf{-local}) \\ &= \widetilde{l}\, !(X_0\ \&\ \pi_0) \end{aligned}$$

since $(\pi_0\ \&\ \pi_0)\ \psi^1 = (X_0\ \&\ \pi_0)$. Next

$$\begin{aligned} \pi_1\ \mathsf{D}_1\, \widetilde{l} &= \widetilde{l}\, \mathsf{m}^2\, (((\pi_1\ \partial_{X_0}) \otimes !\pi_0) + (!\pi_0 \otimes (\pi_1\ \partial_{X_1})))\, (\mathsf{m}^2)^{-1}\, !\psi^1 \\ &\qquad \text{by } (\partial\text{-}\&) \text{ and def. of } \mathsf{L} \\ &= l\, (\mathsf{der}_{X_0} \otimes \mathsf{der}_{X_1})\, (\mathsf{m}^2)^{-1}\, \mathsf{m}^2\, (((\pi_1\ \partial_{X_0}) \otimes !\pi_0) + (!\pi_0 \otimes (\pi_1\ \partial_{X_1})))\, (\mathsf{m}^2)^{-1}\, !\psi^1 \\ &\qquad \text{by def. of } \widetilde{l} \\ &= l\, ((\mathsf{der}_{X_0}\ \pi_1\ \partial_{X_0}) \otimes (\mathsf{der}_{X_1}\ !\pi_0) + (\mathsf{der}_{X_0}\ !\pi_0) \otimes (\mathsf{der}_{X_1}\ \pi_1\ \partial_{X_1}))\ (\mathsf{m}^2)^{-1}\, !\psi^1 \\ &= l\, ((\pi_1\ \mathbf{S}\, \mathsf{der}_{X_0}\ \partial_{X_0}) \otimes (\mathsf{der}_{X_1}\ !\pi_0) + (\mathsf{der}_{X_0}\ !\pi_0) \otimes (\pi_1\ \mathbf{S}\, \mathsf{der}_{X_1}\ \partial_{X_1}))\ (\mathsf{m}^2)^{-1}\, !\psi^1 \\ &\qquad \text{by nat. of } \pi_1 \\ &= l\, ((\pi_1\ \mathsf{der}_{\mathbf{S}\, X_0}) \otimes (\mathsf{der}_{X_1}\ !\pi_0) + (\mathsf{der}_{X_0}\ !\pi_0) \otimes (\pi_1\ \mathsf{der}_{\mathbf{S}\, X_1}))\ (\mathsf{m}^2)^{-1}\, !\psi^1 \\ &\qquad \text{by } (\partial\textbf{-chain}) \\ &= l\, ((\mathsf{der}_{X_0}\ !\pi_1) \otimes (\mathsf{der}_{X_1}\ !\pi_0) + (\mathsf{der}_{X_0}\ !\pi_0) \otimes (\mathsf{der}_{X_1}\ !\pi_1))\ (\mathsf{m}^2)^{-1}\, !\psi^1 \\ &= \widetilde{l}\, \mathsf{m}^2\, (!\pi_0 \otimes !\pi_1 + !\pi_1 \otimes !\pi_0)\ (\mathsf{m}^2)^{-1}\, !\psi^1 \\ &= \widetilde{l}\, (!(\pi_0\ \&\ \pi_1) + !(\pi_1\ \&\ \pi_0))\, !\psi^1 \\ &= \widetilde{l}\, !(X_0\ \&\ \pi_1) + \widetilde{l}\, !(0\ \&\ \pi_0) \end{aligned}$$

since $(\pi_0 \mathbin{\&} \pi_1)\,\psi^1 = (X_0 \mathbin{\&} \pi_1)$ and $(\pi_1 \mathbin{\&} \pi_0)\,\psi^1 = (0 \mathbin{\&} \pi_0)$. Finally we have

$$\begin{aligned}\widetilde{l}\,!(0 \mathbin{\&} \pi_0) &= l\,(\mathsf{der}_{X_0} \otimes \mathsf{der}_{X_1})\,(\mathsf{m}^2)^{-1}\,!(0 \mathbin{\&} \pi_0)\\ &= l\,(\mathsf{der}_{X_0} \otimes \mathsf{der}_{X_1})\,(!0 \otimes !\pi_0)\,(\mathsf{m}^2)^{-1}\\ &= l\,(0 \otimes \pi_0)\,(\mathsf{m}^2)^{-1} = 0\end{aligned}$$

proving our contention. Now we prove the first equation for $i = 1$ (the case $i = 0$ is similar). For $j \in \{0, 1\}$ we have

$$\begin{aligned}\pi_j\ \widetilde{(\mathbf{S}\,l)\ \varphi^1} &= \pi_j\,(\mathbf{S}\,l)\ \varphi^1\,(\mathsf{der}_{X_0} \otimes \mathsf{der}_{\mathbf{S}\,X_1})\,(\mathsf{m}^2)^{-1}\\ &= l\ \pi_j\ \varphi^1\,(\mathsf{der}_{X_0} \otimes \mathsf{der}_{\mathbf{S}\,X_1})\,(\mathsf{m}^2)^{-1}\\ &= l\,(X_0 \otimes \pi_j)\,(\mathsf{der}_{X_0} \otimes \mathsf{der}_{\mathbf{S}\,X_1})\,(\mathsf{m}^2)^{-1} \quad \text{by def. of } \varphi^1\\ &= l\,(\mathsf{der}_{X_0} \otimes \mathsf{der}_{X_1})\,(!X_0 \otimes !\pi_j)\,(\mathsf{m}^2)^{-1}\\ &= l\,(\mathsf{der}_{X_0} \otimes \mathsf{der}_{X_1})\,(\mathsf{m}^2)^{-1}\,!(X_0 \mathbin{\&} \pi_j)\\ &= \widetilde{l}\,!(X_0 \mathbin{\&} \pi_j) \quad \text{by definition of } \widetilde{l}\\ &= \pi_j\ \mathsf{D}_1\,\widetilde{l} \quad \text{as we have just proven,}\end{aligned}$$

which proves our first equation by the fact that π_0, π_1 are jointly epic. □

4.5. **The basic multilinear constructs.** Now we introduce the multilinear operations which will interpret the basic constructs of Λ_{cd}. We make the following assumption about $\mathcal{L}$.

(**Int**) The functor $X \mapsto 1 \oplus X$ from $\mathcal{L}$ to $\mathcal{L}$ has an initial algebra N.

This means that there is a morphism $\chi \in \mathcal{L}(1 \oplus \mathsf{N}, \mathsf{N})$ such that for any $f \in \mathcal{L}(1 \oplus X, X)$ there is exactly one morphism $g \in \mathcal{L}(\mathsf{N}, X)$ such that $f\,(1 \oplus g) = g\,\chi$. We know that there is only one such morphism χ, and that this morphism is an iso (Lambek's Lemma). We assume that χ is the identity to simplify notations, so that $\mathsf{N} = 1 \oplus \mathsf{N}$ "on the nose". Given $f \in \mathcal{L}(1 \oplus X, X)$ we use $\overline{\mathsf{it}}(f)$ for the unique element of $\mathcal{L}(\mathsf{N}, X)$ such that $\overline{\mathsf{it}}(f) = f\,(1 \oplus \overline{\mathsf{it}}(f))$.

We set $\overline{\mathsf{suc}} = \overline{\pi}_1 \in \mathcal{L}(\mathsf{N}, 1 \oplus \mathsf{N}) = \mathcal{L}(\mathsf{N}, \mathsf{N})$ which represents the successor constructor on integers and $\overline{\mathsf{zero}} = \overline{\pi}_0 \in \mathcal{L}(1, \mathsf{N})$ which represents the zero constant. It follows that for each $n \in \mathbb{N}$ we can define the constants $\overline{n} \in \mathcal{L}(1, \mathsf{N})$ by $\overline{0} = \overline{\mathsf{zero}}$ and $\overline{n+1} = \overline{\mathsf{suc}}(\overline{n})$.

Next we define the predecessor morphism $\overline{\mathsf{pred}} = [\overline{\pi}_0, \mathsf{N}] \in \mathcal{L}(1 \oplus \mathsf{N}, \mathsf{N})$, that is $\overline{\mathsf{pred}} \in \mathcal{L}(\mathsf{N}, \mathsf{N})$. We have $\overline{\mathsf{pred}}\,\overline{0} = \overline{0}$ and $\overline{\mathsf{pred}}\,\overline{n+1} = \overline{n}$.

Next notice that we have a morphism $h_{\mathsf{N}} \in \mathcal{L}(\mathsf{N}, !\mathsf{N})$ which turns N into a $!\_$-coalgebra (that is, an object of $\mathcal{L}^!$, the Eilenberg-Moore category of the comonad $!\_$). This morphism is $h_{\mathsf{N}} = \overline{\mathsf{it}}(f)$ where $f = [\overline{\pi}_0\,h_1, !\mathsf{N}] \in \mathcal{L}(1 \oplus !\mathsf{N}, !\mathsf{N})$ where $h_1 = !(\mathsf{m}^0)^{-1}\ \mathsf{dig}_\top\ \mathsf{m}^0 \in \mathcal{L}(1, !1)$ is the canonical $!\_$-coalgebra structure of 1. This allows in particular to define an erasing morphism $\mathsf{w}_{\mathsf{N}} = \mathsf{weak}_{\mathsf{N}}\ h_{\mathsf{N}} \in \mathcal{L}(\mathsf{N}, 1)$ as well as a duplicating morphism $\mathsf{c}_{\mathsf{N}} = (\mathsf{der}_{\mathsf{N}} \otimes \mathsf{der}_{\mathsf{N}})\ \mathsf{contr}_{\mathsf{N}}\ h_{\mathsf{N}} \in \mathcal{L}(\mathsf{N}, \mathsf{N} \otimes \mathsf{N})$.

Given an object X we set $\overline{\mathsf{let}} = \mathsf{ev}\ \gamma\,(h_{\mathsf{N}} \otimes (!\mathsf{N} \multimap X)) \in \mathcal{L}(\mathsf{N} \otimes (!\mathsf{N} \multimap X), X)$.

Last we define $\overline{\mathsf{if}} = \mathsf{ev}(g \otimes \mathsf{N}) \in \mathcal{L}(\mathsf{N} \otimes (X \mathbin{\&} X), X)$ where

$$g = [\mathsf{cur}(\mathsf{pr}_0\,\lambda), \mathsf{cur}(\mathsf{pr}_1\,\lambda)\,\mathsf{w}_{\mathsf{N}}] \in \mathcal{L}(\mathsf{N}, X \mathbin{\&} X \multimap X)$$

where $\mathsf{pr}_j\,\lambda$ is typed as follows

$$1 \otimes (X \mathbin{\&} X) \xrightarrow{\lambda} (X \mathbin{\&} X) \xrightarrow{\mathsf{pr}_j} X$$

so that the two following diagrams commute

$$\begin{array}{ccc} 1 \otimes (X \mathbin{\&} X) & \xrightarrow{\overline{\mathsf{zero}} \otimes (X \& X)} & \mathsf{N} \otimes (X \mathbin{\&} X) \\ \lambda \downarrow & & \downarrow \widetilde{\overline{\mathsf{if}}} \\ X \mathbin{\&} X & \xrightarrow{\mathsf{pr}_0} & X \end{array} \qquad \begin{array}{ccccc} \mathsf{N} \otimes (X \mathbin{\&} X) & & \xrightarrow{\overline{\mathsf{suc}} \otimes (X \& X)} & & \mathsf{N} \otimes (X \mathbin{\&} X) \\ \mathsf{w}_{\mathsf{N}} \otimes (X \& X) \downarrow & & & & \downarrow \widetilde{\overline{\mathsf{if}}} \\ 1 \otimes (X \mathbin{\&} X) & \xrightarrow{\lambda} & X \mathbin{\&} X & \xrightarrow{\mathsf{pr}_1} & X \end{array} \tag{4.6}$$

4.6. **Syntactic constructs in the model.** We introduce now semantical constructs on morphisms which exactly mimic the syntax so as to make the translation from syntax to semantics straightforward.

First, given $n \in \mathbb{N}$ we also use the notation $\overline{n}$ for the morphism $\overline{n}\, \mathsf{weak}_Z \in \mathcal{L}_!(Z, \mathsf{N})$.

Given $f \in \mathcal{L}_!(Z, \mathsf{N})$ we define $\overline{\mathsf{suc}}(f) = \overline{\mathsf{suc}}\, f \in \mathcal{L}_!(Z, \mathsf{N})$ and similarly $\overline{\mathsf{pred}}(f) = \overline{\mathsf{pred}}\, f$.

More generally given $d \in \mathbb{N}$ and $f \in \mathcal{L}_!(Z, \mathsf{D}^d \mathsf{N})$ we set $\overline{\mathsf{suc}}^d(f) = (\mathsf{D}^d \overline{\mathsf{suc}}) \circ f \in \mathcal{L}_!(Z, \mathsf{D}^d \mathsf{N})$. We define similarly $\overline{\mathsf{pred}}^d(f) \in \mathcal{L}_!(Z, \mathsf{D}^d \mathsf{N})$.

We have defined $\widetilde{\overline{\mathsf{if}}} \in \mathcal{L}_!(\mathsf{N} \mathbin{\&} (X \mathbin{\&} X), X)$. So we have

$$\mathsf{D}_0^d \widetilde{\overline{\mathsf{if}}} \in \mathcal{L}_!(\mathsf{D}^d \mathsf{N} \mathbin{\&} (X \mathbin{\&} X), \mathsf{D}^d X)$$

(notice that this is not a trilinear morphism, but a bilinear one, separately linear in $\mathsf{D}^k \mathsf{N}$ and $X \mathbin{\&} X$). Let $g \in \mathcal{L}_!(Z, \mathsf{D}^d \mathsf{N})$ and $f_j \in \mathcal{L}_!(Z, X)$ for $j = 0, 1$. We set

$$\overline{\mathsf{if}}^d(g, f_0, f_1) = \mathsf{D}_0^d \widetilde{\overline{\mathsf{if}}} \circ \langle g, f_0, f_1 \rangle \in \mathcal{L}_!(Z, \mathsf{D}^k X)\,.$$

Notice that $\mathsf{D}_0^d \widetilde{\overline{\mathsf{if}}} = \widetilde{\mathbf{S}_0^d \overline{\mathsf{if}}}$.

We have defined $\widetilde{\overline{\mathsf{let}}} \in \mathcal{L}_!(\mathsf{N} \mathbin{\&} (\mathsf{N} \Rightarrow X), X)$ so that $\mathsf{D}_0^d \widetilde{\overline{\mathsf{let}}} \in \mathcal{L}_!(\mathsf{D}^d \mathsf{N} \mathbin{\&} (\mathsf{N} \Rightarrow X), \mathsf{D}^d X)$. Let $g \in \mathcal{L}_!(Z, \mathsf{D}^d \mathsf{N})$ and $f \in \mathcal{L}_!(Z \mathbin{\&} \mathsf{N}, X)$ so that $\mathsf{Cur}\, f \in \mathcal{L}_!(Z, \mathsf{N} \Rightarrow X)$, we set

$$\overline{\mathsf{let}}^d(g, f) = \mathsf{D}_0^d \widetilde{\overline{\mathsf{let}}} \circ \langle g, \mathsf{Cur}\, f \rangle \in \mathcal{L}_!(Z, \mathsf{D}^d X)\,.$$

If $f \in \mathcal{L}_!(Z, X \Rightarrow Y)$ and $g \in \mathcal{L}_!(Z, X)$ then we define $(f)g \in \mathcal{L}_!(Z, Y)$ as $(f)g = \mathsf{Ev} \circ \langle f, g \rangle$.

If $f \in \mathcal{L}_!(Z, X \Rightarrow Y)$ we have $\mathsf{Ev} \circ (f \mathbin{\&} X) \in \mathcal{L}_!(Z \mathbin{\&} X, Y)$ and hence $\mathsf{D}_1(\mathsf{Ev} \circ (f \mathbin{\&} X)) \in \mathcal{L}_!(Z \mathbin{\&} \mathsf{D}\, X, \mathsf{D}\, Y)$ so that we set

$$\mathsf{D}_{\mathsf{cur}}(f) = \mathsf{Cur}\,(\mathsf{D}_1(\mathsf{Ev} \circ (f \mathbin{\&} X))) \in \mathcal{L}_!(Z, \mathsf{D}\, X \Rightarrow \mathsf{D}\, Y)\,.$$

Notice that

$$\begin{aligned} \mathsf{D}_1(\mathsf{Ev} \circ (f \mathbin{\&} X)) &= \mathsf{D}\, \mathsf{Ev} \circ (\mathsf{D}\, f \mathbin{\&} \mathsf{D}\, X) \circ \psi^1_{Z,X} \\ &= \mathsf{D}\, \mathsf{Ev} \circ \psi^1_{X \Rightarrow Y, X} \circ (f \mathbin{\&} \mathsf{D}\, X) \end{aligned}$$

by naturality of ψ^1 and hence

$$\mathsf{D}_{\mathsf{cur}}(f) = \mathsf{D}_{\mathsf{int}}^{X,Y} \circ f \tag{4.7}$$

where

$$\mathsf{D}_{\mathsf{int}}^{X,Y} = \mathsf{Cur}\,(\mathsf{D}\, \mathsf{Ev} \circ \psi^1_{X \Rightarrow Y, X}) \in \mathcal{L}_!(X \Rightarrow Y, \mathsf{D}\, X \Rightarrow \mathsf{D}\, Y)$$

is the “internalization” of the functor D made possible by its strength.

Remember that a morphism $f \in \mathcal{L}_!(X, Y)$ is linear if $f = \mathsf{Lin}_! g$ for some $g \in \mathcal{L}(X, Y)$ and that, when this g exists, it is unique.

Lemma 4.27. *The morphism* $\mathsf{D}_{\mathsf{int}}^{X,Y} = \mathsf{Cur}\,(\mathsf{D}\,\mathsf{Ev} \circ \psi^1_{X\Rightarrow Y,X}) \in \mathcal{L}_!(X \Rightarrow Y, \mathsf{D}\,X \Rightarrow \mathsf{D}\,Y)$ *is linear.*

Proof. This results from the fact that Ev is left-linear and $\psi^1_{X\Rightarrow Y,X}$ is linear on $(X \Rightarrow Y) \,\&\, \mathsf{D}\,X$. □

So we shall also consider tacitly $\mathsf{D}_{\mathsf{int}}$ as an element of $\mathcal{L}(X \Rightarrow Y, \mathsf{D}\,X \Rightarrow \mathsf{D}\,Y)$.

If $f \in \mathcal{L}_!(Z, \mathsf{D}\,X)$ and $j \in \{0,1\}$ we set $\pi_j(f) = \pi_j\ f \in \mathcal{L}_!(Z, X)$, and if $f \in \mathcal{L}_!(Z, \mathsf{D}^2\,X)$ we set $\tau(f) = \tau\, f \in \mathcal{L}_!(Z, \mathsf{D}\,X)$ and $\mathsf{c}(f) = \mathsf{c}\ f \in \mathcal{L}_!(Z, \mathsf{D}^2\,X)$. Last if $f \in \mathcal{L}_!(Z, X)$ we set $\iota_j(f) = \iota_j\ f \in \mathcal{L}_!(Z, \mathsf{D}\,Y)$.

Lemma 4.28. *For any object* X *of* $\mathcal{L}$ *we have*

$$\mathsf{D}_1\,\widetilde{\overline{\mathsf{if}}}_X = \widetilde{\overline{\mathsf{if}}}_{\mathsf{D}\,X} \in \mathcal{L}_!(\mathsf{N} \,\&\, \mathsf{D}\,(X \,\&\, X)\,, \mathsf{D}\,X)$$
$$\mathsf{D}_1\,\widetilde{\overline{\mathsf{let}}}_X = \widetilde{\overline{\mathsf{let}}}_{\mathsf{D}\,X} \in \mathcal{L}_!(\mathsf{N} \,\&\, \mathsf{D}(\mathsf{N} \Rightarrow X), \mathsf{D}\,X)$$

Proof. We have

$$\begin{aligned}
\pi_0 \circ \mathsf{D}_1\,\widetilde{\overline{\mathsf{if}}}_X &= \pi_0 \circ \mathsf{D}\,\widetilde{\overline{\mathsf{if}}}_X \circ \psi^1_{\mathsf{N},X\&X} \\
&= \widetilde{\overline{\mathsf{if}}}_X \circ (\pi_0 \,\&\, (\pi_0 \,\&\, \pi_0)) \circ \psi^1_{\mathsf{N},X\&X} \quad \text{by Theorem 4.25} \\
&= \widetilde{\overline{\mathsf{if}}}_X \circ (\mathsf{N} \,\&\, (\pi_0 \,\&\, \pi_0)) \\
&= \pi_0 \circ \widetilde{\overline{\mathsf{if}}}_{\mathsf{D}\,X}
\end{aligned}$$

by naturality of $\overline{\mathsf{if}}_X$ with respect to X. Next

$$\begin{aligned}
\pi_1 \circ \mathsf{D}_1\,\widetilde{\overline{\mathsf{if}}}_X &= \pi_1 \circ \mathsf{D}\,\widetilde{\overline{\mathsf{if}}}_X \circ \psi^1_{\mathsf{N},X\&X} \\
&= \widetilde{\overline{\mathsf{if}}}_X \circ (\pi_1 \,\&\, (\pi_0 \,\&\, \pi_0)) \circ \psi^1_{\mathsf{N},X\&X} + \widetilde{\overline{\mathsf{if}}}_X \circ (\pi_0 \,\&\, (\pi_1 \,\&\, \pi_1)) \circ \psi^1_{\mathsf{N},X\&X} \\
&\qquad \text{by Theorem 4.25} \\
&= \widetilde{\overline{\mathsf{if}}}_X \circ (0 \,\&\, (\pi_0 \,\&\, \pi_0)) + \widetilde{\overline{\mathsf{if}}}_X \circ (\mathsf{N} \,\&\, (\pi_1 \,\&\, \pi_1)) \\
&= \widetilde{\overline{\mathsf{if}}}_X \circ (\mathsf{N} \,\&\, (\pi_1 \,\&\, \pi_1)) \quad \text{by bilinearity of } \widetilde{\overline{\mathsf{if}}} \\
&= \pi_1 \circ \widetilde{\overline{\mathsf{if}}}_{\mathsf{D}\,X} \quad \text{by naturality}
\end{aligned}$$

and the contention follows by joint monicity of π_0, π_1. The case of $\overline{\mathsf{let}}$ is completely similar. □

Lemma 4.29. *For* $k \in \mathbb{N}$ *we have* $\mathsf{D}_1\,\mathsf{D}_0^{k+1}\,\widetilde{\overline{\mathsf{if}}}_X = \mathsf{c}(k) \circ \mathsf{D}_0^{k+1}\,\widetilde{\overline{\mathsf{if}}}_{\mathsf{D}\,X}$ *and* $\mathsf{D}_1\,\mathsf{D}_0^{k+1}\,\widetilde{\overline{\mathsf{let}}}_X = \mathsf{c}(k) \circ \mathsf{D}_0^{k+1}\,\widetilde{\overline{\mathsf{let}}}_{\mathsf{D}\,X}$.

Proof. By Lemma 4.18 and Lemma 4.28. □

4.6.1. *Recursion.* Let $(\mathcal{L}, \mathbf{S})$ be a summable category. Let $f_0, f_1 \in \mathcal{L}(X,Y)$, we write $f_0 \leq f_1$ if there exists $h \in \mathcal{L}(X, \mathbf{S}\,Y)$ such that $\pi_0 = f_0$ and $\sigma\,h = f_1$. In other words: there is $g \in \mathcal{L}(X,Y)$ such that f_0, g is summable and $f_1 = f_0 + g$.

Lemma 4.30. *The relation* $\leq$ *on* $\mathcal{L}(X,Y)$ *is a preorder relation for which* 0 *is the least element.*

Definition 4.31. The summable category $(\mathcal{L}, \mathbf{S})$ is *Scott* if, equipped with $\leq$, any homset $\mathcal{L}(X, Y)$ is a poset (with 0 as least element as we have seen), and such that any monotone ω-sequence of elements of $\mathcal{L}(X, Y)$ has a least element[10]. Moreover, the functor $\mathbf{S}$ is locally continuous, morphism composition is continuous and the $\otimes$ operation on morphisms when $\mathcal{L}$ is an SMC. In the case where $\mathcal{L}$ is a resource category, the functor $!\_$ is also assumed to be locally continuous.

▶ **Example 4.1.** The summable categories **Coh** and **NCoh** (see [Ehr21]) are Scott resource categories, where the order relation $\leq$ on morphisms is set inclusion. The category **Pcoh** (see Section 6.1) is a Scott summable resource category where the order relation is the componentwise order on $\mathbb{R}_{\geq 0}$-valued vectors. ◀

From now on we assume that $\mathcal{L}$ is a differential summable resource category which is Scott. In the CCC $\mathcal{L}_!$, for any object X, we can define a sequence of morphisms $\mathcal{Y}_n^X \in \mathcal{L}_!(X \Rightarrow X, X)$ by induction as follows

$$\begin{aligned} \mathcal{Y}_0^X &= 0 \\ \mathcal{Y}_{n+1}^X &= \mathsf{Ev} \circ \langle X \Rightarrow X, \mathcal{Y}_n^X \rangle \end{aligned}$$

and an easy induction, using the minimality of 0 and the fact that all categorical operations are monotone wrt. $\leq$, shows that the sequence $(\mathcal{Y}_n^X)_{n \in \mathbb{N}}$ is monotone. We set

$$\mathcal{Y}^X = \sup_{n \in \mathbb{N}} \mathcal{Y}_n^X$$

and by continuity of all categorical operations we have

$$\mathcal{Y}^X = \mathsf{Ev} \circ \langle X \Rightarrow X, \mathcal{Y}^X \rangle \,. \tag{4.8}$$

Theorem 4.32. *For any object X we have*

$$\mathsf{D}\, \mathcal{Y}^X = \mathcal{Y}^{\mathsf{D}\, X} \circ \mathsf{Cur}\,(\mathsf{D}\,\mathsf{Ev})$$

Observe that this equation is well typed: we have $\mathsf{Ev} : (X \Rightarrow X) \,\&\, X \to X$ and hence $\mathsf{D}\,\mathsf{Ev} : (X \Rightarrow \mathsf{D}\,X) \,\&\, \mathsf{D}\,X \to \mathsf{D}\,X$ so that $\mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}) : (X \Rightarrow \mathsf{D}\,X) \to (\mathsf{D}\,X \Rightarrow \mathsf{D}\,X)$ and hence both sides of the equation are morphisms $(X \Rightarrow \mathsf{D}\,X) \to \mathsf{D}\,X$.

Proof. By induction on $n \in \mathbb{N}$ we prove that $\forall n \in \mathbb{N}\ \ \mathsf{D}\,\mathcal{Y}_n^X = \mathcal{Y}_n^{\mathsf{D}\,X} \circ \mathsf{Cur}\,(\mathsf{D}\,\mathsf{Ev})$ and the result follows by continuity. For $n = 0$ the equation is obvious so assume that it holds for some $n \in \mathbb{N}$.

[10]On purpose we do not ask for the existence of lubs for arbitrary directed sets because we have in mind probability based models where such a requirement would prevent us to use the crucial monotone convergence theorem of measure theory.

We have

$$\begin{aligned}
\mathcal{Y}_{n+1}^{\mathsf{D}\,X} \circ \mathsf{Cur}\,(\mathsf{D}\,\mathsf{Ev}) &= \mathsf{Ev}^{X,\mathsf{D}\,X} \circ \langle X \Rightarrow \mathsf{D}\,X, \mathcal{Y}_n^{\mathsf{D}\,X}\rangle \circ \mathsf{Cur}\,(\mathsf{D}\,\mathsf{Ev}^{X,X}) \\
&= \mathsf{Ev}^{X,\mathsf{D}\,X} \circ \langle \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{X,X}), \mathcal{Y}_n^{\mathsf{D}\,X} \circ \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{X,X})\rangle \\
&= \mathsf{Ev}^{X,\mathsf{D}\,X} \circ \langle \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{X,X}), \mathsf{D}\,\mathcal{Y}_n^{X}\rangle \quad \text{by ind. hyp.} \\
&= \mathsf{Ev}^{X,\mathsf{D}\,X} \circ \big(\mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{X,X}) \mathbin{\&} \mathsf{D}\,X\big) \circ \langle X \Rightarrow \mathsf{D}\,X, \mathsf{D}\,\mathcal{Y}_n^{X}\rangle \\
&= \mathsf{D}\,\mathsf{Ev}^{X,X} \circ \langle X \Rightarrow \mathsf{D}\,X, \mathsf{D}\,\mathcal{Y}_n^{X}\rangle \quad \text{by cart. closedness} \\
&= \mathsf{D}(\mathsf{Ev} \circ \langle X \Rightarrow X, \mathcal{Y}_n^{X}\rangle) \\
&= \mathsf{D}\,\mathcal{Y}_{n+1}^{X}
\end{aligned}$$

as contended, using also the fact that D is a functor which commutes with cartesian products. □

4.7. **Interpreting types and terms.** The translation of any type A into an object $[\![A]\!]$ of $\mathcal{L}_!$ (that is, of $\mathcal{L}$) is given by $[\![\mathsf{D}^d\iota]\!] = \mathsf{D}^d\,\mathsf{N}$ and $[\![A \Rightarrow B]\!] = ([\![A]\!] \Rightarrow [\![B]\!])$ so that $[\![\mathsf{D}^d A]\!] = \mathsf{D}^d[\![A]\!]$ holds for all type A and all $d \in \mathbb{N}$ thanks to our identification of $X \Rightarrow \mathsf{D}\,Y$ with $\mathsf{D}(X \Rightarrow Y)$.

A context $\Gamma = (x_1 : A_1, \dots, x_k : A_k)$ is interpreted as $[\![\Gamma]\!] = [\![A_1]\!] \mathbin{\&} \cdots \mathbin{\&} [\![A_k]\!]$ considered as an object of $\mathcal{L}_!$.

The next theorem also provides our definition of the interpretation of terms.

Theorem 4.33. *Given a term M, a type A and a context Γ such that $\Gamma \vdash M : A$ for some typing derivation δ (so that A is actually determined by M) one can associate $[\![M]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$ in such a way that*

- *$[\![M]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$ depends only on M and not on δ*
- *and if $M = M_0 + M_1$ then $[\![M_0]\!]_\Gamma, [\![M_1]\!]_\Gamma$ are summable in $\mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$ and $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$.*

Proof. By induction on $\mathsf{sz}(\delta)$ where δ is a derivation of the typing judgment $\Gamma \vdash M : A$. We proceed by cases, according to the last rule in δ.

▶ If $M = x_i$ for some $i \in \{1, \dots, k\}$ we set $[\![M]\!]_\Gamma = \mathsf{pr}_i$.

▶ If $M = \lambda x^B\, N$ then we have $A = (B \Rightarrow C)$ and $\Gamma, x : B \vdash N : C$ so that by inductive hypothesis $[\![N]\!]_{\Gamma,x:B} \in \mathcal{L}_!([\![\Gamma]\!] \mathbin{\&} [\![B]\!], [\![C]\!])$ and we set $[\![M]\!]_\Gamma = \mathsf{Cur}\,[\![N]\!]_{\Gamma,x:B} \in \mathcal{L}_!([\![\Gamma]\!], [\![B]\!] \Rightarrow [\![C]\!])$ by inductive hypothesis.

▶ If $M = (N)P$ with $\Gamma \vdash N : B \Rightarrow A$ and $\Gamma \vdash P : B$ then we have by inductive hypothesis $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![B]\!] \Rightarrow [\![A]\!])$ and $[\![P]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![B]\!])$ and hence we set $[\![M]\!]_\Gamma = \mathsf{Ev} \circ \langle [\![N]\!]_\Gamma, [\![P]\!]_\Gamma\rangle \in \mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$.

▶ If $M = \mathsf{Y}N$ with $\Gamma \vdash N : A \Rightarrow A$ so that by inductive hypothesis $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![A]\!] \Rightarrow [\![A]\!])$ and so we set $[\![M]\!]_\Gamma = \mathcal{Y}^{[\![A]\!]} \circ [\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$ as required.

▶ If $M = \underline{n}$ for some $n \in \mathbb{N}$, we we set $[\![M]\!]_\Gamma = \overline{n} \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{N})$.

▶ If $M = \mathsf{succ}^d(N)$ so that $\Gamma \vdash N : \mathsf{D}^d\iota$ and hence $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d\,\mathsf{N})$ by inductive hypothesis, we set $[\![M]\!]_\Gamma = \overline{\mathsf{suc}}^d([\![N]\!]_\Gamma) \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d\,\mathsf{N})$. Of course we set similarly $[\![\mathsf{pred}^d(N)]\!]_\Gamma = \overline{\mathsf{pred}}^d([\![N]\!]_\Gamma) \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d\,\mathsf{N})$.

▶ If $M = \mathsf{if}^d(N, P_0, P_1)$ with $\Gamma \vdash N : \mathsf{D}^d\iota$ and $\Gamma \vdash P_j : A$ for $j = 0, 1$ so that by inductive hypothesis $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d\,\mathsf{N})$ and $[\![P_j]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$ for $j = 0, 1$. So we set $[\![M]\!]_\Gamma = \overline{\mathsf{if}}^d([\![N]\!]_\Gamma, [\![P_0]\!]_\Gamma, [\![P_1]\!]_\Gamma) \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d[\![A]\!] = [\![\mathsf{D}^d A]\!])$ where we use the notation $\overline{\mathsf{if}}^d$ introduced in Section 4.6.

▶ If $M = \mathsf{let}^d(x, N, P)$ with $\Gamma \vdash N : \mathsf{D}^d\iota$ and $\Gamma, x : \iota \vdash P : A$ so that by inductive hypothesis $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d\,\mathsf{N})$ and $[\![P]\!]_{\Gamma,x:\iota} \in \mathcal{L}_!([\![\Gamma]\!] \ \&\ \mathsf{N}, [\![A]\!])$ and we set $[\![M]\!]_\Gamma = \overline{\mathsf{let}}^d([\![N]\!]_\Gamma, \mathsf{Cur}\,[\![P]\!]_{\Gamma,x:\iota}) \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d[\![A]\!]_\Gamma)$ where we use the notation $\overline{\mathsf{let}}^d$ introduced in Section 4.6.

▶ We set $[\![0^A]\!]_\Gamma = 0 \in \mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$.

▶ If $M = \pi_i^d(N)$ then we have $\Gamma \vdash N : \mathsf{D}^{d+1}B$ with $A = \mathsf{D}^d B$ so that $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{d+1}[\![B]\!])$ and we set $[\![M]\!]_\Gamma = \mathsf{D}^d\,\pi_i \circ [\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d[\![B]\!] = [\![A]\!])$.

▶ If $M = \iota_i^d(N)$ then we have $\Gamma \vdash N : \mathsf{D}^d B$ with $A = \mathsf{D}^{d+1}B$ so that $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d[\![B]\!])$ and we set $[\![M]\!]_\Gamma = \mathsf{D}^d\,\iota_i \circ [\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{d+1}[\![B]\!] = [\![A]\!])$.

▶ If $M = \theta^d(N)$ then we have $\Gamma \vdash N : \mathsf{D}^{d+1}B$ with $A = \mathsf{D}^d B$ so that $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{d+1}[\![B]\!])$ and we set $[\![M]\!]_\Gamma = \mathsf{D}^d\,\theta \circ [\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d[\![B]\!] = [\![A]\!])$.

▶ If $M = \mathsf{c}_l^d(N)$ then we have $\Gamma \vdash N : \mathsf{D}^{d+l+2}B$ with $A = \mathsf{D}^{d+l+2}B$ and $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{d+l+2}[\![B]\!])$ and we set $[\![M]\!]_\Gamma = \mathsf{D}^d\,\mathsf{c}(l) \circ [\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{d+l+2}[\![B]\!] = [\![A]\!])$.

▶ If $M = \mathsf{D}N$ then we have $\Gamma \vdash N : B \Rightarrow C$ and $A = (\mathsf{D}B \Rightarrow \mathsf{D}C)$ and hence $[\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![B]\!] \Rightarrow [\![C]\!])$ and we set $[\![M]\!]_\Gamma = \mathsf{D}_{\mathsf{int}}^{[\![B]\!],[\![C]\!]} \circ [\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], (\mathsf{D}[\![B]\!] \Rightarrow \mathsf{D}[\![C]\!]) = [\![A]\!])$.

Assume now that $M = M_0 + M_1$. We distinguish the same subcases as in the proof of Lemma 3.10.

▶ The last rule of δ is (**proj2**) so that $M_j = \pi_j^d(N)$, $A = \mathsf{D}^d B$ and $\Gamma \vdash N : \mathsf{D}^{d+1}B$ by a derivation δ' such that $\mathsf{sz}(\delta') = \mathsf{sz}(\delta) - 1$. By inductive hypothesis we have $f = [\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{d+1}[\![B]\!])$ and we know that the morphisms $\mathsf{D}^d\,\pi_0 \circ f = [\![M_0]\!]_\Gamma$ and $\mathsf{D}^d\,\pi_1 \circ f = [\![M_1]\!]_\Gamma$ are summable, with sum $\mathsf{D}^d\,\sigma \circ f$. We set $[\![M]\!]_\Gamma = \mathsf{D}^d\,\sigma \circ [\![N]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d[\![B]\!])$ so that actually $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$.

▶ The last rule of δ is (**projd**) so that $M_0 = \pi_1^d(N_0)$, $M_1 = \pi_0^d(N_1)$, $A = \mathsf{D}^d B$ and $\Gamma \vdash N_0 + N_1 : \mathsf{D}^{d+1}B$ by a derivation δ' such that $\mathsf{sz}(\delta') = \mathsf{sz}(\delta) - 1$. By inductive hypothesis we have defined two summable morphisms $[\![N_j]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{d+1}[\![B]\!])$ for $j = 0, 1$. It follows that the 4 morphisms $(\mathsf{D}^d\,\pi_i)\,[\![N_j]\!]_\Gamma$ (for $i, j \in \{0, 1\}$) are summable, and hence $\mathsf{D}^d\,\pi_1 \circ [\![N_0]\!]_\Gamma = [\![M_0]\!]_\Gamma$ and $\mathsf{D}^d\,\pi_0 \circ [\![N_1]\!]_\Gamma = [\![M_1]\!]_\Gamma$ are summable. We set $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$.

▶ The last rule of δ is (**lin**) so that there is a linear context L of height 1 and terms N_0, N_1 such that $M_j = L[N_j]$ and $\Gamma \vdash L[N_0 + N_1] : A$ by a derivation δ' such that $\mathsf{sz}(\delta') = \mathsf{sz}(\delta) - 1$. This implies (by a simple inspection of the various possibilities for L which has height 1) that for some context Δ and some type B one has $\Delta \vdash N_0 + N_1 : B$ by a derivation δ'' such that $\mathsf{sz}(\delta'') = \mathsf{sz}(\delta') - k_L$ where $k_L \in \mathbb{N}^+$ *depends only on* L (if for instance $L = \mathsf{if}^d([\,], P_0, P_1)$ then $k_L = 1 + k_0 + k_1$ where k_i is the size of the typing derivation of P_i). Now we consider the various possibilities for L.

- $L = \lambda x^C\,[\,]$ and we have $\Delta = (\Gamma, x : B)$, $A = (C \Rightarrow B)$. By inductive hypothesis we have $[\![N_j]\!]_{\Gamma,x:C} \in \mathcal{L}_!([\![\Gamma]\!] \ \&\ [\![C]\!], [\![B]\!])$ for $j = 0, 1$, $[\![N_0]\!]_{\Gamma,x:C}$ and $[\![N_1]\!]_{\Gamma,x:C}$ are summable and also that $[\![N_0 + N_1]\!]_{\Gamma,x:C} = [\![N_0]\!]_{\Gamma,x:C} + [\![N_1]\!]_{\Gamma,x:C}$. We have $[\![M_j]\!]_\Gamma = \mathsf{Cur}\,[\![N_j]\!]_{\Gamma,x:C}$ because we know that there is a derivation δ_j of $\Gamma, x : C \vdash N_j : B$ such that $\mathsf{sz}(\delta_j) \le \mathsf{sz}(\delta)$

by Lemma 3.10. It follows that $[\![M_0]\!]_\Gamma$ and $[\![M_1]\!]_\Gamma$ are summable and we can set $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$

- $L = ([\,])P$ and we have $\Delta = \Gamma$, $B = (C \Rightarrow A)$, $\Gamma \vdash P : C$ by a derivation of size $k_P > 0$ and the derivation δ' of $\Gamma \vdash N_0 + N_1 : C \Rightarrow A$ satisfies $\mathsf{sz}(\delta) = \mathsf{sz}(\delta') + k_P + 1$. So by inductive hypothesis $[\![P]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![C]\!])$, and $[\![N_j]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![C]\!] \Rightarrow [\![A]\!])$ (for $j = 0, 1$) are summable and we have $[\![N_0 + N_1]\!]_\Gamma = [\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma$. We have $[\![M_j]\!]_\Gamma = \mathsf{Ev} \circ \langle [\![N_j]\!]_\Gamma, [\![P]\!]_\Gamma \rangle$ because the derivation δ_j of $\Gamma \vdash M_j : A$ satisfies $\mathsf{sz}(\delta_j) \leq \mathsf{sz}(\delta)$ and hence $[\![M_0]\!]_\Gamma$ and $[\![M_1]\!]_\Gamma$ are summable with $[\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma = \mathsf{Ev} \circ \langle [\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma, [\![P]\!]_\Gamma \rangle$ by left-linearity of Ev. We set $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$.
- $L = \mathsf{if}^d([\,], P_0, P_1)$ and we have $\Delta = \Gamma$, $B = \mathsf{D}^d\iota$, $A = \mathsf{D}^d C$ and $\Gamma \vdash P_i : C$ for $i = 0, 1$ by derivations of sizes k_0 and k_1 respectively so that, denoting by δ' the derivation of $\Gamma \vdash N_0 + N_1 : \mathsf{D}^d\iota$, we have $\mathsf{sz}(\delta) = \mathsf{sz}(\delta') + k_0 + k_1 + 1$. It follows by inductive hypothesis that $[\![P_i]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![C]\!])$ for $i = 0, 1$, and that $[\![N_j]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^d\,\mathsf{N})$ for $j = 0, 1$ are summable with $[\![N_0 + N_1]\!]_\Gamma = [\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma$. For $j = 0, 1$ we have $[\![M_j]\!]_\Gamma = \mathsf{D}_0^d \widetilde{\widetilde{\mathsf{if}}} \circ \langle [\![N_j]\!]_\Gamma, \langle [\![M_0]\!]_\Gamma, [\![M_1]\!]_\Gamma \rangle \rangle$ because the derivation δ_j of $\Gamma \vdash M_j : A$ satisfies $\mathsf{sz}(\delta_j) \leq \mathsf{sz}(\delta)$ (by Lemma 3.10) and hence, by left-linearity of $\mathsf{D}_0^d \widetilde{\widetilde{\mathsf{if}}}$, $[\![M_0]\!]_\Gamma$ and $[\![M_1]\!]_\Gamma$ are summable and satisfy $[\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma = \mathsf{D}_0^d \widetilde{\widetilde{\mathsf{if}}} \circ \langle [\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma, \langle [\![P_0]\!]_\Gamma, [\![P_1]\!]_\Gamma \rangle \rangle$. We set $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$.
- $L = \mathsf{let}^d(x, [\,], P)$ and we have $\Delta = \Gamma$, $B = \mathsf{D}^d\iota$, $\Gamma, x : \iota \vdash P : C$ by a derivation of size k and $A = \mathsf{D}^d C$ so that denoting by δ' the derivation of $\Gamma \vdash N_0 + N_1 : \mathsf{D}^d\iota$ we have $\mathsf{sz}(\delta) = \mathsf{sz}(\delta') + k + 1$. This case is completely similar to the previous one. By inductive hypothesis we have $[\![P]\!]_{\Gamma, x:\mathsf{D}^d\iota} \in \mathcal{L}_!([\![\Gamma]\!] \,\&\, \mathsf{D}^d\,\mathsf{N}, [\![C]\!])$ and, for $j = 0, 1$ we have $[\![N_j]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{\mathsf{N}})$ which are summable with $[\![N_0 + N_1]\!]_\Gamma = [\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma$. For $j = 0, 1$ we have $[\![M_j]\!]_\Gamma = \mathsf{D}_0^d \widetilde{\widetilde{\mathsf{let}}} \circ \langle [\![N_j]\!]_\Gamma, \mathsf{Cur} [\![P]\!]_{\Gamma, x:\mathsf{D}^d\iota} \rangle$ because the derivation δ_j of $\Gamma \vdash M_j : A$ satisfies $\mathsf{sz}(\delta_j) \leq \mathsf{sz}(\delta)$ (by Lemma 3.10) and hence, by left-linearity of $\mathsf{D}_0^d \widetilde{\widetilde{\mathsf{let}}}$, $[\![M_0]\!]_\Gamma$ and $[\![M_1]\!]_\Gamma$ are summable and satisfy $[\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma = \mathsf{D}_0^d \widetilde{\widetilde{\mathsf{let}}} \circ \langle [\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma, \mathsf{Cur}\, [\![P]\!]_{\Gamma, x:C} \rangle$. We set $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$.
- $L = \mathsf{D}[\,]$ and we have $\Delta = \Gamma$, $B = (C \Rightarrow E)$ and $A = (\mathsf{D}C \Rightarrow \mathsf{D}E)$ and we use δ' for the derivation of $\Gamma \vdash N_0 + N_1 : C \Rightarrow E$ so that $\mathsf{sz}(\delta) = \mathsf{sz}(\delta') + 1$ and hence by inductive hypothesis we have $[\![N_j]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![C]\!] \Rightarrow [\![E]\!])$ for $j = 0, 1$, these two morphisms are summable and we have $[\![N_0 + N_1]\!]_\Gamma = [\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma$. For $j = 0, 1$ we have $[\![M_j]\!]_\Gamma = \mathsf{D}_{\mathsf{int}}^{[\![C]\!], [\![E]\!]} \circ [\![N_j]\!]_\Gamma$ because the derivation δ_j of $\Gamma \vdash M_j : A$ satisfies $\mathsf{sz}(\delta_j) \leq \mathsf{sz}(\delta)$ (by Lemma 3.10) and hence, by linearity of $\mathsf{D}_{\mathsf{int}}^{[\![C]\!], [\![E]\!]}$, $[\![M_0]\!]_\Gamma$ and $[\![M_1]\!]_\Gamma$ are summable and satisfy $[\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma = \mathsf{D}_{\mathsf{int}}^{[\![C]\!], [\![E]\!]} \circ ([\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma)$. We set $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$.
- $L = \pi_i^d([\,])$ and we have $\Delta = \Gamma$, $B = \mathsf{D}^{d+1} C$ and $A = \mathsf{D}^d C$ and we use δ' for the derivation of $\Gamma \vdash N_0 + N_1 : \mathsf{D}^{d+1} C$ so that $\mathsf{sz}(\delta) = \mathsf{sz}(\delta') + 1$ and hence by inductive hypothesis we have $[\![N_j]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], \mathsf{D}^{d+1} [\![C]\!])$ for $j = 0, 1$, these two morphisms are summable and we have $[\![N_0 + N_1]\!]_\Gamma = [\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma$. For $j = 0, 1$ we have $[\![M_j]\!]_\Gamma = \mathsf{D}^d\,\pi_i \circ [\![N_j]\!]_\Gamma$ because the derivation δ_j of $\Gamma \vdash M_j : A$ satisfies $\mathsf{sz}(\delta_j) \leq \mathsf{sz}(\delta)$ (by Lemma 3.10) and hence, by linearity of $\mathsf{D}^d\,\pi_i$, $[\![M_0]\!]_\Gamma$ and $[\![M_1]\!]_\Gamma$ are summable and satisfy $[\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma = \mathsf{D}^d\,\pi_i \circ ([\![N_0]\!]_\Gamma + [\![N_1]\!]_\Gamma)$. We set $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$.
- The remaining cases are similar: in each of them we see that we can sensibly set $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$. □

4.7.1. *Substitution lemmas.* The first substitution lemma is completely standard in a λ-calculus setting.

Lemma 4.34. *If* $\Gamma \vdash M : B$*, so that* $\Gamma, x : A \vdash M : B$*, then we have* $[\![\Gamma, x : A]\!]_M = [\![M]\!]_\Gamma \circ \mathsf{pr}_0$ *where* $\mathsf{pr}_0 \in \mathcal{L}_!([\![\Gamma]\!] \,\&\, [\![A]\!], [\![\Gamma]\!])$ *is the first projection.*

Proof. By induction on the typing derivation of M. □

Lemma 4.35 (Ordinary substitution)**.** *If* $\Gamma \vdash N : A$ *and* $\Gamma, x : A \vdash M : B$ *then* $[\![M\,[N/x]]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![B]\!])$ *satisfies*

$$[\![M\,[x/N]]\!]_\Gamma = [\![M]\!]_{\Gamma,x:A} \circ \langle [\![\Gamma]\!], [\![N]\!]_\Gamma \rangle\,.$$

Proof. By induction on the typing derivation M. □

Lemma 4.36 (Semantics of the differential)**.** *If* $\Gamma, x : A \vdash M : B$ *then* $[\![\partial(x, M)]\!]_{\Gamma,x:\mathsf{D}A} \in \mathcal{L}_!([\![\Gamma]\!] \,\&\, \mathsf{D}[\![A]\!], \mathsf{D}[\![B]\!])$ *satisfies*

$$[\![\partial(x, M)]\!]_{\Gamma,x:\mathsf{D}A} = \mathsf{D}_1 [\![M]\!]_{\Gamma,x:A}\,.$$

If $\Gamma = (A_1, \dots, A_k)$ and $\Delta = (A_1, \dots, A_{i-1}, \mathsf{D}A_i, A_{i+1}, \dots, A_k)$ for some $i \in \{1, \dots, k\}$ and $\Gamma \vdash M : B$, and then we have $\Delta \vdash \partial(x_i, M) : \mathsf{D}B$ and in this slightly more general situation the lemma states that $[\![\partial(x_i, M)]\!]_\Delta = \mathsf{D}_i^{Z_1,\dots,Z_k}\,[\![M]\!]_\Delta$ where $\mathsf{D}_i^{Z_1,\dots,Z_k}\, f = \mathsf{D}\, f \circ \psi^i_{Z_1,\dots,Z_k}$ is the "ith partial derivative of f". This slightly more general statement is equivalent to the lemma by the symmetry of the cartesian product &.

Proof. By induction on M, *and not on its typing derivation* δ. This is possible thanks to Theorem 4.33 which states that the interpretation does not depend on the typing derivation. This point is crucial when dealing with sums. We use the following notations: f for $[\![M]\!]_{\Gamma,x:A}$, Z for $[\![\Gamma]\!]$, U for $[\![A]\!]$, and X for $[\![B]\!]$. Sometimes we also write $\Gamma = (x_1 : A_1, \dots, x_k : A_k)$ and in that case we set $Z_i = [\![A_i]\!]$ so that $Z = Z_1 \,\&\, \cdots \,\&\, Z_k$.

▶ Assume that $M = x$ and hence δ must end with (**var**). We have $X = U$ and $f = \mathsf{pr}_1 \in \mathcal{L}_!(Z \,\&\, U, U)$, we have $\mathsf{D}_1\, f = (\mathsf{D}\,\mathsf{pr}_1) \circ \psi^1 = \mathsf{pr}_1 \circ (\iota_0 \,\&\, \mathsf{D}\,U) = \mathsf{pr}_1 \in \mathcal{L}_!(Z \,\&\, \mathsf{D}\,U, \mathsf{D}\,U)$ since D commutes with cartesian products in $\mathcal{L}_!$. We have used Lemma 4.5. It follows that $\mathsf{D}_1\, f = [\![\partial(x, M)]\!]_{\Gamma,x:\mathsf{D}A}$ since $\partial(x, M) = M$.

▶ Assume that $M = x_i$ for some $i \in \{1, \dots, k\}$ and hence δ must end with (**var**). So we have $f = \mathsf{pr}_i \in \mathcal{L}_!(Z_1 \,\&\, \cdots \,\&\, Z_k \,\&\, U, Z_i)$, $X = Z_i$ and we have $\mathsf{D}_1\, f = \mathsf{D}\, f \circ \psi^1_{Z,U} = \mathsf{D}\, f \circ \psi^{k+1}_{Z_1,\dots,Z_k,U} = \mathsf{D}\,\mathsf{pr}_i \circ (\iota_0 \,\&\, \cdots \,\&\, \iota_0 \,\&\, U) = \mathsf{pr}_i \circ (\iota_0 \,\&\, \cdots \,\&\, \iota_0 \,\&\, U) = \iota_0 \circ \mathsf{pr}_i$ using the fact that $\mathbf{S}$ preserves cartesian products and the expression (4.5) of the $k+1$-ary ψ^i. Therefore $\mathsf{D}_1\, f = [\![\partial(x, M)]\!]_{\Gamma,x:\mathsf{D}A}$ since $\partial(x, M) = \iota_0(x_i)$.

▶ Assume that $M = \lambda y^C\, P$ so that δ must end with (**abs**) applied to $\Gamma, x : A, y : C \vdash P : E$ and we have $B = (C \Rightarrow E)$ and hence $X = (V \Rightarrow Y)$ where $[\![C]\!] = V$ and $[\![E]\!] = Y$. Let $g = [\![P]\!]_{\Gamma,x:A,y:C} \in \mathcal{L}_!(Z \,\&\, U \,\&\, V, Y)$. Notice that $\mathsf{D}_1\, g = \mathsf{D}\, g \circ \psi^1_{Z,U,V} \in \mathcal{L}_!(Z \,\&\, \mathsf{D}\,U \,\&\, V, \mathsf{D}\,Y)$. We have

$$\begin{aligned}
\mathsf{D}_1\, f &= \mathsf{D}_1(\mathsf{Cur}\, g) \\
&= \mathsf{D}(\mathsf{Cur}\, g) \circ \psi^1_{Z,U} \\
&= \mathsf{Cur}((\mathsf{D}\, g) \circ \psi^0_{Z\&U,V}) \circ \psi^1_{Z,U} \quad \text{by Lemma 4.19}
\end{aligned}$$

where $\psi^0_{Z\&U,V} \in \mathcal{L}_!(\mathsf{D}\,Z \,\&\, \mathsf{D}\,U \,\&\, V, \mathsf{D}\,Z \,\&\, \mathsf{D}\,U \,\&\, \mathsf{D}\,V)$. So $\mathsf{D}_1\, f = \mathsf{Cur}((\mathsf{D}\, g) \circ \psi^0_{Z\&U,V} \circ (\psi^1_{Z,U} \,\&\, V)) = \mathsf{Cur}((\mathsf{D}\, g) \circ \psi^1_{Z,U,V}) = \mathsf{Cur}(\mathsf{D}_1\, g)$. By inductive hypothesis we have $\mathsf{D}_1\, g = [\![\partial(x, P)]\!]_{\Gamma,x:\mathsf{D}A,y:C}$ and hence $\mathsf{D}_1\, f = [\![\lambda y^C\, \partial(x, P)]\!]_{\Gamma,x:\mathsf{D}A}$ as required.

▶ Assume that $M = \mathsf{D}P$ so that δ must end with (**diff**) applied to $\Gamma, x : A \vdash P : C \Rightarrow E$ and we have $B = (C \Rightarrow E)$ and hence $X = (V \Rightarrow Y)$ where $[\![C]\!] = V$ and $[\![E]\!] = Y$. Let $g = [\![P]\!]_{\Gamma,x:A} \in \mathcal{L}_!(Z \,\&\, U, V \Rightarrow Y)$ so that $f = \mathsf{D}_{\mathsf{cur}}\, g = \mathsf{D}_{\mathsf{int}}^{V,Y} \circ g \in \mathcal{L}_!(Z \,\&\, U, \mathsf{D}\, V \Rightarrow \mathsf{D}\, Y)$ where $\mathsf{D}_{\mathsf{int}}^{V,Y} = \mathsf{Cur}\,(\mathsf{D}\,\mathsf{Ev} \circ \psi_1^{V\Rightarrow Y,V})$. Then

$$\begin{aligned}\mathsf{D}_1\, f &= \mathsf{D}\,\mathsf{D}_{\mathsf{cur}}\, g \circ \psi^1_{Z,U}\\ &= \mathsf{D}\,\mathsf{Cur}\,(\mathsf{D}\,\mathsf{Ev} \circ \psi_1^{V\Rightarrow Y,V}) \circ \mathsf{D}\, g \circ \psi^1_{Z,U}\\ &= \mathsf{Cur}(\mathsf{D}(\mathsf{D}\,\mathsf{Ev} \circ \psi_1^{V\Rightarrow Y,V}) \circ \psi^0_{V\Rightarrow Y,\mathsf{D}\,V}) \circ \mathsf{D}_1\, g \quad \text{by Lemma 4.19.}\end{aligned}$$

Next we have

$$\begin{aligned}\mathsf{D}^2\,\mathsf{Ev} \circ \mathsf{D}\,\psi^1_{V\Rightarrow Y,V} \circ \psi^0_{V\Rightarrow Y,\mathsf{D}\,V} &= \mathsf{D}^2\,\mathsf{Ev} \circ \mathsf{c} \circ \mathsf{D}\,\psi^0_{V\Rightarrow Y,V} \circ \psi^1_{V\Rightarrow \mathsf{D}\,Y,V}\\ &= \mathsf{c} \circ \mathsf{D}^2\,\mathsf{Ev} \circ \mathsf{D}\,\psi^0_{V\Rightarrow Y,V} \circ \psi^1_{V\Rightarrow \mathsf{D}\,Y,V}\end{aligned}$$

by Lemma 4.8 and naturality of c. By our identification of $\mathsf{D}(V \Rightarrow Y)$ with $V \Rightarrow \mathsf{D}\,Y$ through the iso $\mathsf{Cur}(\mathsf{D}_0\,\mathsf{Ev})$ we have $\mathsf{D}\,\mathsf{Ev} \circ \psi^0_{V\Rightarrow Y,V} = \mathsf{Ev} \in \mathcal{L}_!((V \Rightarrow \mathsf{D}\,Y) \,\&\, V, \mathsf{D}\,Y)$ and hence

$$\begin{aligned}\mathsf{D}_1\, f &= \mathsf{Cur}(\mathsf{c} \circ \mathsf{D}\,\mathsf{Ev} \circ \psi^1_{V\Rightarrow \mathsf{D}\,Y,V}) \circ \mathsf{D}_1\, g\\ &= \mathsf{c} \circ \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev} \circ \psi^1_{V\Rightarrow \mathsf{D}\,Y,V}) \circ \mathsf{D}_1\, g\\ &= \mathsf{c} \circ \mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\, g\\ &= [\![\mathsf{c}(\mathsf{D}\partial(x, P))]\!]_{\Gamma,x:\mathsf{D}A}\,.\end{aligned}$$

Notice that we have identified $\mathsf{c}_{V\Rightarrow Y}$ with $V \Rightarrow \mathsf{c}_Y$ in accordance with our convention of identifying $\mathsf{D}^2(V \Rightarrow Y)$ with $V \Rightarrow \mathsf{D}^2\,Y$.

▶ Assume that $M = (P)Q$ so that δ must end with (**app**) applied to $\Gamma, x : A \vdash P : C \Rightarrow B$ and $\Gamma, x : A \vdash Q : C$ and let $Y = [\![C]\!]$. Let $p = [\![P]\!]_\Gamma \in \mathcal{L}_!(Z \,\&\, U, Y \Rightarrow X)$ and $q = [\![Q]\!]_\Gamma \in \mathcal{L}_!(Z \,\&\, U, Y)$ so that $f = [\![M]\!]_\Gamma = (p)q = \mathsf{Ev} \circ \langle p, q\rangle \in \mathcal{L}_!(Z \,\&\, U, X)$, $\mathsf{D}_1\, p \in \mathcal{L}_!(Z \,\&\, \mathsf{D}\,U, Y \Rightarrow \mathsf{D}\,X)$ and $\mathsf{D}_1\, q \in \mathcal{L}_!(Z \,\&\, \mathsf{D}\,U, \mathsf{D}\,Y)$. We have $\mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\, p \in \mathcal{L}_!(Z \,\&\, \mathsf{D}\,U, \mathsf{D}\,Y \Rightarrow \mathsf{D}^2\,X)$ and hence $(\theta \circ \mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\, p)\mathsf{D}_1\, q \in \mathcal{L}_!(Z \,\&\, \mathsf{D}\,Y, \mathsf{D}^2\,X)$, we prove that

$$\mathsf{D}_1((p)q) = (\theta \circ \mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\, p)\mathsf{D}_1\, q\,.$$

We have

$$\begin{aligned}\mathsf{D}_1((p)q) &= (\mathsf{D}\,\mathsf{Ev}) \circ (\mathsf{D}\langle p, q\rangle) \circ \psi^1_{Z,U}\\ &= (\mathsf{D}\,\mathsf{Ev}) \circ \langle \mathsf{D}\,p, \mathsf{D}\,q\rangle \circ \psi^1_{Z,U} \quad \text{since } \mathsf{D} \text{ preserves cart. prod.}\\ &= (\mathsf{D}\,\mathsf{Ev}) \circ \langle \mathsf{D}_1\,p, \mathsf{D}_1\,q\rangle\,.\end{aligned}$$

where $\mathsf{D}\,\mathsf{Ev} \in \mathcal{L}_!((X \Rightarrow \mathsf{D}\,Y) \,\&\, \mathsf{D}\,X, \mathsf{D}\,Y)$ and we know by Lemma 4.21 that $\mathsf{D}\,\mathsf{Ev} = \theta \circ (\mathsf{D}\,\mathsf{Ev}') \circ \psi^1_{Y\Rightarrow \mathsf{D}\,X,Y}$ where Ev' is the evaluation morphism in $\mathcal{L}_!((Y \Rightarrow \mathsf{D}\,X) \,\&\, Y, \mathsf{D}\,X)$.

On the other hand

$$\begin{aligned}(\mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\, p)\mathsf{D}_1\, q &= \mathsf{Ev} \circ \langle \mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\, p, \mathsf{D}_1\, q\rangle\\ &= \mathsf{Ev} \circ \left(\mathsf{D}_{\mathsf{int}}^{Y,\mathsf{D}\,X} \,\&\, \mathsf{D}\,X\right) \circ \langle \mathsf{D}_1\, p, \mathsf{D}_1\, q\rangle\end{aligned}$$

where we recall that $\mathsf{D}_{\mathsf{int}}^{Y,\mathsf{D}\,X} = \mathsf{Cur}((\mathsf{D}\,\mathsf{Ev}') \circ \psi^1_{Y\Rightarrow \mathsf{D}\,X,Y}) \in \mathcal{L}_!(Y \Rightarrow \mathsf{D}\,X, \mathsf{D}\,Y \Rightarrow \mathsf{D}^2\,X)$ and therefore

$$(\mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\, p)\mathsf{D}_1\, q = \mathsf{D}\,\mathsf{Ev}' \circ \psi^1_{Y\Rightarrow \mathsf{D}\,X,Y} \circ \langle \mathsf{D}_1\, p, \mathsf{D}_1\, q\rangle\,.$$

It follows that

$$\begin{aligned}(\theta \circ \mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\,p)\mathsf{D}_1\,q &= \theta \circ (\mathsf{D}_{\mathsf{cur}}\,\mathsf{D}_1\,p)\mathsf{D}_1\,q\\ &= \theta \circ (\mathsf{D}\,\mathsf{Ev}') \circ \psi^1_{Y \Rightarrow \mathsf{D}\,X, Y} \circ \langle \mathsf{D}_1\,p, \mathsf{D}_1\,q\rangle\\ &= \mathsf{D}\,\mathsf{Ev} \circ \langle \mathsf{D}_1\,p, \mathsf{D}_1\,q\rangle \quad \text{by Lemma 4.21}\\ &= \mathsf{D}_1\,f\end{aligned}$$

where the first equation results from our identification of $\mathsf{D}\,Y \Rightarrow \theta$ with θ and we have also used the fact that $f = (p)q = \mathsf{Ev} \circ \langle p, q\rangle$. Finally, using Equation (4.7),

$$\begin{aligned}\mathsf{D}_1\,f &= (\theta \circ \mathsf{D}_{\mathsf{int}} \circ \mathsf{D}_1\,p)\mathsf{D}_1\,q\\ &= [\![(\theta(\mathsf{D}\partial(x, P)))\partial(x, Q)]\!]_{\Gamma, x:\mathsf{D}A}\end{aligned}$$

as contended, using of course also the inductive hypothesis.

▶ Assume that $M = \mathsf{succ}^d(P)$ so that the last rule of δ is (**suc**) and that we have $\Gamma, x : A \vdash P : \mathsf{D}^d\iota = B$ and hence $X = \mathsf{D}^d\,\mathsf{N}$. Let $g = [\![P]\!]_{\Gamma,x:A}$ so that $f = \mathsf{D}^d\,\widetilde{\mathsf{suc}} \circ g$, we have

$$\begin{aligned}\mathsf{D}_1\,f &= \mathsf{D}_1(\mathsf{D}^d\,\widetilde{\mathsf{suc}} \circ f)\\ &= \mathsf{D}^{d+1}\,\widetilde{\mathsf{suc}} \circ \mathsf{D}\,g \circ \psi^1_{Z,U}\\ &= \mathsf{D}^{d+1}\,\widetilde{\mathsf{suc}} \circ \mathsf{D}_1\,g\\ &= \mathsf{D}^{d+1}\,\widetilde{\mathsf{suc}} \circ [\![\partial(x, P)]\!]_{\Gamma,x:\mathsf{D}A} \quad \text{by inductive hypothesis}\\ &= [\![\mathsf{succ}^{d+1}(\partial(x, P))]\!]_{\Gamma,x:\mathsf{D}A}\end{aligned}$$

The cases where M is of shape $\mathsf{pred}^d(P)$, $\pi^d_i(P)$, $\iota^d_i(P)$, $\theta^d(P)$ and $\mathsf{c}^d(P)$ are similarly dealt with.

▶ Assume that $M = \mathsf{if}^d(P, Q_0, Q_1)$ so that δ ends with (**if**) and that we have $\Gamma, x : A \vdash P : \mathsf{D}^d\iota$, $\Gamma, x : A \vdash Q_i : C$ for $i = 0, 1$ so that $B = \mathsf{D}^dC$ and $X = \mathsf{D}^d\,Y$ where $Y = [\![C]\!]$. Let $p = [\![P]\!]_{\Gamma,x:A} \in \mathcal{L}_!(Z \mathbin{\&} U, \mathsf{D}^d\,\iota)$ and $q_i = [\![Q_i]\!]_{\Gamma,x:A} \in \mathcal{L}_!(Z \mathbin{\&} U, Y)$ for $i = 0, 1$. We have $\mathsf{D}^d_0\,\widetilde{\widetilde{\mathsf{if}}}_X \in \mathcal{L}(\mathsf{D}^d\,\mathsf{N} \mathbin{\&} (Y \mathbin{\&} Y), \mathsf{D}^d\,Y = X)$ and $f = [\![M]\!]_{\Gamma,x:A} = \mathsf{D}^d_0\,\widetilde{\widetilde{\mathsf{if}}}_X \circ \langle p, \langle q_0, q_1\rangle\rangle \in \mathcal{L}_!(Z \mathbin{\&} U, X)$. We have

$$\begin{aligned}\mathsf{D}_1(\mathsf{D}^d_0\,\widetilde{\widetilde{\mathsf{if}}}_X \circ \langle p, \langle q_0, q_1\rangle\rangle) &= \mathsf{D}\,\mathsf{D}^d_0\,\widetilde{\widetilde{\mathsf{if}}}_X \circ \mathsf{D}\langle p, \langle q_0, q_1\rangle\rangle \circ \psi^1_{Z,U}\\ &\qquad \text{by def. of } \mathsf{D}_1 \text{ and functoriality of } \mathsf{D}\\ &= \theta \circ \mathsf{D}_1\,\mathsf{D}^{d+1}_0\,\widetilde{\widetilde{\mathsf{if}}}_X \circ \langle \mathsf{D}_1\,p, \langle \mathsf{D}_1\,q_0, \mathsf{D}_1\,q_1\rangle\rangle \quad \text{by Theorem 4.12}\\ &= \theta \circ \mathsf{c}(d) \circ \mathsf{D}^{d+1}_0\,\widetilde{\widetilde{\mathsf{if}}}_{\mathsf{D}\,X} \circ \langle \mathsf{D}_1\,p, \langle \mathsf{D}_1\,q_0, \mathsf{D}_1\,q_1\rangle\rangle \quad \text{by Lemma 4.29}\end{aligned}$$

therefore

$$\begin{aligned}\mathsf{D}_1\,f &= \theta \circ \mathsf{c}(d) \circ \mathsf{D}^{d+1}_0\,\widetilde{\widetilde{\mathsf{if}}}_{\mathsf{D}\,X} \circ \langle \mathsf{D}_1\,p, \langle \mathsf{D}_1\,q_0, \mathsf{D}_1\,q_1\rangle\rangle\\ &= \theta \circ \mathsf{c}(d) \circ \mathsf{D}^{d+1}_0\,\widetilde{\widetilde{\mathsf{if}}}_{\mathsf{D}\,X} \circ \langle [\![\partial(x, P)]\!]_{\Gamma,x:\mathsf{D}A}, \langle [\![\partial(x, Q_0)]\!]_{\Gamma,x:\mathsf{D}A}, [\![\partial(x, Q_1)]\!]_{\Gamma,x:\mathsf{D}A}\rangle\rangle\\ &\qquad \text{by ind. hyp.}\\ &= [\![\theta(\mathsf{c}_d(\mathsf{if}^{d+1}(\partial(x, P), \partial(x, Q_0), \partial(x, Q_1))))]\!]_{\Gamma,x:\mathsf{D}A}\\ &= [\![\partial(x, M)]\!]_{\Gamma,x:\mathsf{D}A}\end{aligned}$$

as required.

The case where M is of shape $\mathsf{let}^d(y, P, Q)$ is completely similar.

▶ Assume that $M = M_0 + M_1$. Whatever be the last rule of δ, we know that $\Gamma, x : A \vdash M_i : B$ for $i = 0, 1$ and, by Theorem 4.33, that $g_i = [\![M_i]\!]_{\Gamma,x:A} \in \mathcal{L}_!(Z \,\&\, U, X)$ are well defined and summable and that $f = g_0 + g_1$, where $f = [\![M]\!]_{\Gamma,x:A}$. By inductive hypothesis we have $[\![\partial(x, M_i)]\!]_\Gamma = g_i$ for $i = 0, 1$. By left-linearity of composition in $\mathcal{L}_!$, we have

$$\begin{aligned} f &= g_0 + g_1 \\ &= [\![\partial(x, M_0)]\!]_{\Gamma,x:\mathsf{D}A} + [\![\partial(x, M_0)]\!]_{\Gamma,x:\mathsf{D}A} \\ &= [\![\partial(x, M_0) + \partial(x, M_1)]\!]_{\Gamma,x:\mathsf{D}A} \quad \text{by Theorem 4.33} \\ &= [\![\partial(x, M)]\!]_{\Gamma,x:\mathsf{D}A} \end{aligned}$$

as required.

▶ Assume that $M = \mathsf{Y}P$ so that δ ends with (**fix**) and that we have $\Gamma, x : A \vdash P : B \Rightarrow B$ so that, setting $g = [\![M]\!]_{\Gamma,x:A}$ we have $g \in \mathcal{L}_!(Z \,\&\, U, X \Rightarrow X)$ and $f = [\![M]\!]_{\Gamma,x:A} = \mathcal{Y}^X \circ g$. We have

$$\begin{aligned} \mathsf{D}_1\, f &= \mathsf{D}\,\mathcal{Y}^X \circ \mathsf{D}_1\, g \\ &= \mathcal{Y}^{\mathsf{D}\,X} \circ \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{X,X}) \circ \mathsf{D}_1\, g \quad \text{by Theorem 4.32} \\ &= \mathcal{Y}^{\mathsf{D}\,X} \circ \mathsf{Cur}(\theta \circ \mathsf{D}\,\mathsf{Ev}^{X,\mathsf{D}\,X} \circ \psi^1_{X \Rightarrow \mathsf{D}\,X,X}) \circ \mathsf{D}_1\, g \quad \text{by Lemma 4.21} \\ &= \mathcal{Y}^{\mathsf{D}\,X} \circ (\mathsf{D}\,X \Rightarrow \theta) \circ \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{X,\mathsf{D}\,X} \circ \psi^1_{X \Rightarrow \mathsf{D}\,X,X}) \circ \mathsf{D}_1\, g \quad \text{by cartesian closedness} \\ &= \mathcal{Y}^{\mathsf{D}\,X} \circ (\mathsf{D}\,X \Rightarrow \theta) \circ \mathsf{D}^{X,\mathsf{D}\,X}_{\mathsf{int}} \circ \mathsf{D}_1\, g \quad \text{by definition of } \mathsf{D}^{X,\mathsf{D}\,X}_{\mathsf{int}} \\ &= \mathcal{Y}^{\mathsf{D}\,X} \circ \theta \circ \mathsf{D}^{X,\mathsf{D}\,X}_{\mathsf{int}} \circ \mathsf{D}_1\, g \end{aligned}$$

and hence

$$\begin{aligned} \mathsf{D}_1\, f &= \mathcal{Y}^{\mathsf{D}\,X} \circ \theta \circ \mathsf{D}^{X,\mathsf{D}\,X}_{\mathsf{int}} \circ \mathsf{D}_1\, g \\ &= \mathcal{Y}^{\mathsf{D}\,X} \circ \theta \circ \mathsf{D}^{X,\mathsf{D}\,X}_{\mathsf{int}} \circ [\![\partial(x, P)]\!]_{\Gamma,x:\mathsf{D}A} \\ &= [\![\mathsf{Y}(\theta(\mathsf{D}\partial(x, P)))]\!]_{\Gamma,x:\mathsf{D}A} \\ &= [\![\partial(x, M)]\!]_{\Gamma,x:\mathsf{D}A} \end{aligned}$$

as required. □

4.7.2. *Invariance theorem.*

Theorem 4.37 (Invariance of the semantics)**.** *Assume that $\Gamma \vdash M : A$ and $M \to_{\Lambda_{\mathsf{cd}}} M'$ (so that $\Gamma \vdash M' : A$). Then we have $[\![M]\!]_\Gamma = [\![M']\!]_\Gamma$.*

Proof. We consider the various cases in the definition of $\to_{\Lambda_{\mathsf{cd}}}$. We set $Z = [\![\Gamma]\!]$ and $X = [\![A]\!]$.

▶ Assume first that $M \to_{\mathsf{lin}} M'$. The fact that $[\![M]\!]_\Gamma = [\![M']\!]_\Gamma$ results simply from the linearity of the semantical constructs corresponding to the linear contexts. As an example, consider the situation where $M = \mathsf{let}^d(x, M_0 + M_1, N)$ and $M' = \mathsf{let}^d(x, M_0, N) + \mathsf{let}^d(x, M_1, N)$ (so that we have $\Gamma \vdash M_i : \mathsf{D}^d\iota$ and $\Gamma \vdash N : B$ for a type B, and we have $A = \mathsf{D}^d B$). Then we have

$$\begin{aligned}[\![M]\!]_\Gamma &= \mathsf{D}_0^d\,\widetilde{\overline{\mathsf{let}}} \circ \langle [\![M_0 + M_1]\!]_\Gamma, [\![N]\!]_\Gamma\rangle \\ &= \mathsf{D}_0^d\,\widetilde{\overline{\mathsf{let}}} \circ \langle [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma, [\![N]\!]_\Gamma\rangle \\ &= \mathsf{D}_0^d\,\widetilde{\overline{\mathsf{let}}} \circ \langle [\![M_0]\!]_\Gamma, [\![N]\!]_\Gamma\rangle + \mathsf{D}_0^d\,\widetilde{\overline{\mathsf{let}}} \circ \langle [\![M_1]\!]_\Gamma, [\![N]\!]_\Gamma\rangle\end{aligned}$$

by the bilinearity of $\overline{\mathsf{let}}$.

▶ Assume that $M = (\lambda x^A\, P)Q$ and $M' = P\,[Q/x]$, we directly apply Lemma 4.35.

▶ Assume that $M = \mathsf{Y}N$ and $M' = (N)M$. We directly apply Equation (4.8).

▶ Assume that $M = \mathsf{D}(\lambda x^B\, P)$ and $M' = \lambda x^{\mathsf{D}B}\,\partial(x, P)$ so that $A = (\mathsf{D}B \Rightarrow \mathsf{D}C)$ and $\Gamma, x : B \vdash P : C$, so that setting $U = [\![B]\!]$ and $Y = [\![C]\!]$ we have $f = [\![P]\!]_{\Gamma,x:B} \in \mathcal{L}_!(Z \,\&\, U, Y)$. Then we have

$$\begin{aligned}[\![M]\!]_\Gamma &= \mathsf{D}_{\mathsf{int}} \circ \mathsf{Cur}\, f \\ &= \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{U,Y} \circ \psi^1_{U\Rightarrow Y,U}) \circ \mathsf{Cur}\, f \\ &= \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{U,Y} \circ \psi^1_{U\Rightarrow Y,U} \circ (\mathsf{Cur}\, f \,\&\, \mathsf{D}\,U)) \\ &= \mathsf{Cur}(\mathsf{D}\,\mathsf{Ev}^{U,Y} \circ (\mathsf{D}(\mathsf{Cur}\, f) \,\&\, \mathsf{D}\,U) \circ \psi^1_{Z,U}) \quad \text{by nat. of } \psi^1 \\ &= \mathsf{Cur}(\mathsf{D}(\mathsf{Ev}^{U,Y} \circ (\mathsf{Cur}\, f \,\&\, U)) \circ \psi^1_{Z,U}) \\ &= \mathsf{Cur}(\mathsf{D}\, f \circ \psi^1_{Z,U}) \\ &= \mathsf{Cur}(\mathsf{D}_1\, f) \\ &= \mathsf{Cur}([\![\partial(x, P)]\!]_{\Gamma,x:\mathsf{D}B}) \quad \text{by Lemma 4.36} \\ &= [\![\lambda x^{\mathsf{D}B}\,\partial(x, P)]\!]_\Gamma\end{aligned}$$

▶ Assume that $M = \mathsf{if}^0(\underline{n+1}, P_0, P_1)$ and $M' = P_1$ so that, setting $p_i = [\![P_i]\!]_\Gamma \in \mathcal{L}_!(Z, X)$ for $i = 0, 1$, we have $[\![M]\!]_\Gamma = \widetilde{\overline{\mathsf{if}}} \circ \langle \overline{n+1} \circ 0, \langle p_0, p_1\rangle\rangle = p_1$ by the second diagram of (4.6), where 0 is the unique element of $\mathcal{L}_!(Z, \top)$. The cases where $M = \mathsf{succ}^0(\underline{n}) \to_{\Lambda_{\mathsf{cd}}} \underline{n+1}$, $M = \mathsf{pred}^0(\underline{0}) \to_{\Lambda_{\mathsf{cd}}} \underline{0}$, $M = \mathsf{pred}^0(\underline{n+1}) \to_{\Lambda_{\mathsf{cd}}} \underline{n}$, and $M = \mathsf{if}^0(\underline{0}, P, Q) \to_{\Lambda_{\mathsf{cd}}} P$ are similar.

▶ Assume that $M = \mathsf{let}^0(x, \underline{n}, P)$ and $M' = P\,[\underline{n}/x]$. Let $f = [\![P]\!]_{\Gamma,x:\iota} \in \mathcal{L}_!(Z \,\&\, \mathsf{N}, X)$. We have $\overline{n} \in \mathcal{L}_!(Z, \mathsf{N})$ and $h_{\mathsf{N}}\,\overline{n} = \overline{n}^!$ (actually $\overline{n}$ is a morphism in the Eilenberg-Moore category of $!\_$) from which it follows that

$$\begin{aligned}[\![M]\!]_\Gamma &= \overline{\mathsf{let}}^0(\overline{n}, f) \quad \text{using the notation of Section 4.6} \\ &= \widetilde{\overline{\mathsf{let}}} \circ \langle \overline{n}, \mathsf{Cur}\, f\rangle \\ &= \overline{\mathsf{let}}\,(\overline{n} \otimes \mathsf{Cur}\, f)\ \mathsf{contr}_Z \\ &= \mathsf{ev}\ \gamma\,(h_{\mathsf{N}} \otimes (!\mathsf{N} \multimap X))(\overline{n} \otimes \mathsf{Cur}\, f)\ \mathsf{contr}_Z \\ &= \mathsf{ev}\,(f \otimes \overline{n}^!)\ \mathsf{contr}_Z \\ &= f \circ \langle Z, \overline{n}\rangle \\ &= [\![P\,[\underline{n}/x]]\!]_\Gamma\end{aligned}$$

by Lemma 4.35.

▶ Assume that $M = \pi^d_j(\lambda x^B\, P)$ and $M' = \lambda x^B\, \pi^d_i(P)$ so that we must have $\Gamma, x : B \vdash P : \mathsf{D}^{d+1}C$ with $A = \mathsf{D}^d(B \Rightarrow C)$, $X = (U \Rightarrow \mathsf{D}^d\, Y)$ where $U = [\![B]\!]$ and $Y = [\![C]\!]$. Let

$f = [\![P]\!]_{\Gamma,x:B} \in \mathcal{L}_!(Z \,\&\, U, \mathsf{D}^{d+1}\,Y)$ so that $[\![\lambda x^B\, P]\!]_\Gamma = \mathsf{Cur}\, f \in \mathcal{L}_!(Z, U \Rightarrow \mathsf{D}^{d+1}\,Y)$ and $\mathsf{D}^d\,\pi_i \circ f \in \mathcal{L}_!(Z \,\&\, U, \mathsf{D}^d\,Y)$. Remember that, for any object V, $\varphi^0_{U\Rightarrow V,!U} \in \mathcal{L}(\mathsf{D}\,(!U \multimap V)\otimes\, !X, \mathsf{D}((!X \multimap U) \otimes !X))$ and that by (**S⊗-fun**) the morphism $\varphi^{\multimap} = \mathsf{Cur}((\mathsf{D}\,\mathsf{ev})\ \varphi^0_{U\Rightarrow V,!U}) \in \mathcal{L}(\mathsf{D}(U \Rightarrow V), U \Rightarrow \mathsf{D}\,V)$ is an iso. Thanks to this iso we identify the objects $\mathsf{D}^{d+1}(U \Rightarrow Y)$ and $U \Rightarrow \mathsf{D}^{d+1}\,Y$. Under this identification the morphisms $\mathsf{D}^d\,\pi_j \in \mathcal{L}(\mathsf{D}^{d+1}(U \Rightarrow Y), \mathsf{D}^d(U \Rightarrow Y))$ and $U \Rightarrow \mathsf{D}^d\,\pi_j \in \mathcal{L}(U \Rightarrow \mathsf{D}^{d+1}\,Y, U \Rightarrow \mathsf{D}^d\,Y)$ are identified as well. Since $(U \Rightarrow \mathsf{D}^d\,\pi_j) \circ \mathsf{Cur}\, f = \mathsf{Cur}(\mathsf{D}^d\,\pi_j \circ f)$ we have

$$\begin{aligned}
[\![M]\!]_\Gamma &= \mathsf{D}^d\,\pi_i([\![\lambda x^B\, P]\!]_\Gamma) \\
&= (U \Rightarrow \mathsf{D}^d\,\pi_j) \circ \mathsf{Cur}\, f \\
&= \mathsf{Cur}(\mathsf{D}^d\,\pi_j \circ f) \\
&= \mathsf{Cur}(\mathsf{D}^d\,\pi_j \circ [\![P]\!]_{\Gamma,x:B}) \\
&= \mathsf{Cur}([\![\pi^d_i(P)]\!]_{\Gamma,x:B}) \\
&= [\![\lambda x^B\, \pi^d_i(P)]\!]_\Gamma
\end{aligned}$$

as required.

▶ Assume that $M = \pi^d_i((P)Q)$ and $M' = (\pi^d_i(P))Q$ so that we must have $\Gamma \vdash P : B \Rightarrow \mathsf{D}^{d+1}C$ and $\Gamma \vdash Q : B$ with $A = \mathsf{D}^d C$. Setting $U = [\![B]\!]$, $Y = [\![C]\!]$, $p = [\![P]\!]_\Gamma$ and $q = [\![Q]\!]_\Gamma$ we have $[\![M]\!]_\Gamma = \mathsf{D}^d\,\pi_i \circ (p)q \in \mathcal{L}_!(Z, \mathsf{D}^d Y)$. Then, by naturality of evaluation, we have $[\![M]\!]_\Gamma = (\mathsf{D}^d\,\pi_j) \circ \mathsf{Ev} \circ \langle p, q\rangle = \mathsf{Ev} \circ \big((U \Rightarrow (\mathsf{D}^d\,\pi_j)) \,\&\, U\big) \circ \langle p, q\rangle = \mathsf{Ev} \circ \langle (\mathsf{D}^{d+1}\,\pi_j) \circ p, q\rangle$ under the same identification as in the previous case. That is $[\![M]\!]_\Gamma = [\![M']\!]_\Gamma$.

▶ Assume $M = \pi^d_j(\mathsf{succ}^e(P))$ and $M' = \mathsf{succ}^{e-1}(\pi_j(P))$ so that we must have $d < e$, $\Gamma \vdash M : \mathsf{D}^e\iota$ and $A = \mathsf{D}^{e-1}\iota$. we have $\overline{\mathsf{suc}} \in \mathcal{L}(\mathsf{N}, \mathsf{N})$ that we consider as usual as a morphism in $\mathcal{L}_!(\mathsf{N}, \mathsf{N})$ so that $\mathsf{D}^e\,\overline{\mathsf{suc}} \in \mathcal{L}_!(\mathsf{D}^e\,\mathsf{N}, \mathsf{D}^e\,\mathsf{N})$ and since $d < e$ we can write $\mathsf{D}^e\,\mathsf{N} = \mathsf{D}^{d+1}\,\mathsf{D}^{e-d-1}\,\mathsf{N}$ so that $\mathsf{D}^d\,\pi_i \in \mathcal{L}_!(\mathsf{D}^e\,\mathsf{N}, \mathsf{D}^{e-1}\,\mathsf{N})$ and we have

$$\begin{aligned}
\mathsf{D}^d\,\pi_i \circ \mathsf{D}^e\,\overline{\mathsf{suc}} &= \mathsf{D}^d(\pi_j \circ \mathsf{D}^{e-d}\,\overline{\mathsf{suc}}) \\
&= \mathsf{D}^d(\mathsf{D}^{e-d-1}\,\overline{\mathsf{suc}} \circ \pi_j) \quad \text{by nat. of } \pi_j \\
&= \mathsf{D}^{e-1}\,\overline{\mathsf{suc}} \circ \mathsf{D}^d\,\pi_j
\end{aligned}$$

so that $[\![M]\!]_\Gamma = [\![M']\!]_\Gamma$. The case $M = \pi^d_j(\mathsf{pred}^e(P))$ and $M' = \mathsf{pred}^{e-1}(\pi_j(P))$ (still with $d < e$) is completely similar.

▶ Assume that $M = \pi^d_i(\mathsf{if}^e(N, P_0, P_1))$ and $M' = \mathsf{if}^{e-1}(\pi^d_i(N), P_0, P_1)$ with $d < e$. We must have $\Gamma \vdash N : \mathsf{D}^e\iota$ and $\Gamma \vdash P_j : B$ for $j = 0, 1$, and we have $A = \mathsf{D}^d B$, let $Y = [\![Y]\!]$, we have $X = \mathsf{D}^d\,Y$. Let $f = [\![N]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^e\,\mathsf{N})$ and $p_j = [\![P_j]\!]_\Gamma \in \mathcal{L}_!(Z, Y)$ for $j = 0, 1$. We have

$$\begin{aligned}
\mathsf{D}^d\,\pi_i \circ \mathsf{D}^e_0\,\widetilde{\overline{\mathsf{if}}} \circ \langle f, \langle p_0, p_1\rangle\rangle &= \mathsf{D}^d\,\pi_i \circ \mathsf{D}^e\,\widetilde{\overline{\mathsf{if}}} \circ \psi^0_{\mathsf{N},Y\&Y}(e) \circ \langle f, \langle p_0, p_1\rangle\rangle \quad \text{by Lemma 4.15} \\
&= \mathsf{D}^d(\pi_i \circ \mathsf{D}^{e-d}\,\widetilde{\overline{\mathsf{if}}}) \circ \psi^0_{\mathsf{N},Y\&Y}(e) \circ \langle f, \langle p_0, p_1\rangle\rangle\,.
\end{aligned}$$

Now we have to consider the two cases $i = 0$ and $i = 1$. The first case is dealt with as follows.

$$\begin{aligned}
\mathsf{D}^d\,\pi_0 \circ \mathsf{D}_0^e\,\widetilde{\widetilde{\mathsf{if}}} \circ \langle f, \langle p_0, p_1\rangle\rangle &= \mathsf{D}^d(\mathsf{D}^{e-d-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ (\pi_0 \mathbin{\&} \pi_0)) \circ \psi^0_{\mathsf{N},Y\&Y}(e) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&\qquad \text{by Theorems 4.23 and 4.25} \\
&= \mathsf{D}^{e-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ \left(\mathsf{D}^d\,\pi_0 \mathbin{\&} \mathsf{D}^d\,\pi_0\right) \circ (\mathsf{D}^e\,\mathsf{N} \mathbin{\&} \iota_0(e)) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&\qquad \text{by Lemma 4.14} \\
&= \mathsf{D}^{e-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ \left(\mathsf{D}^d\,\pi_0 \mathbin{\&} \iota_0(e-1)\right) \circ \langle f, \langle p_0, p_1\rangle\rangle \quad \text{by Lemma 4.16} \\
&= \mathsf{D}^{e-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ \left(\mathsf{D}^{e-1}\,\mathsf{N} \mathbin{\&} \iota_0(e-1)\right) \circ \langle \mathsf{D}^d\,\pi_0 \circ f, \langle p_0, p_1\rangle\rangle \\
&= \mathsf{D}_0^{e-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ \langle \mathsf{D}^d\,\pi_0 \circ f, \langle p_0, p_1\rangle\rangle\,.
\end{aligned}$$

Let us deal with the second case.

$$\begin{aligned}
\mathsf{D}^d\,\pi_1 \circ \mathsf{D}_0^e\,\widetilde{\widetilde{\mathsf{if}}} \circ \langle f, \langle p_0, p_1\rangle\rangle &= \mathsf{D}^d(\mathsf{D}^{e-d-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ (\pi_0 \mathbin{\&} \pi_1)) \circ \psi^0_{\mathsf{N},X\&X}(e) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&\quad + \mathsf{D}^d(\mathsf{D}^{e-d-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ (\pi_1 \mathbin{\&} \pi_0)) \circ \psi^0_{\mathsf{N},X\&X}(e) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&\qquad \text{by Theorems 4.23 and 4.25.}
\end{aligned}$$

By Lemma 4.14 and Lemma 4.16 we have $\left(\mathsf{D}^d\,\pi_1 \mathbin{\&} \mathsf{D}^d\,\pi_0\right) \circ \psi_0^{\mathsf{N},X\&X}(e) = \left(\mathsf{D}^d\,\pi_1 \mathbin{\&} \iota_0(e-1)\right)$ and $\left(\mathsf{D}^d\,\pi_0 \mathbin{\&} \mathsf{D}^d\,\pi_1\right) \circ \psi_0^{\mathsf{N},X\&X}(e) = \left(\mathsf{D}^d\,\pi_0 \mathbin{\&} 0\right)$. It follows by bilinearity of $\mathsf{D}^{e-1}\,\widetilde{\widetilde{\mathsf{if}}}$ that

$$\begin{aligned}
\mathsf{D}^d\,\pi_1 \circ \mathsf{D}_0^e\,\widetilde{\widetilde{\mathsf{if}}} \circ \langle f, \langle p_0, p_1\rangle\rangle &= \mathsf{D}^{e-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ \left(\mathsf{D}^d\,\pi_1 \mathbin{\&} \iota_0(e-1)\right) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&= \mathsf{D}_0^{e-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ \langle \mathsf{D}^d\,\pi_1 \circ f, \langle p_0, p_1\rangle\rangle\,.
\end{aligned}$$

To summarize, for $i = 0, 1$ and if $d < e$ we have

$$\mathsf{D}^d\,\pi_i \circ \mathsf{D}_0^e\,\widetilde{\widetilde{\mathsf{if}}} \circ \langle f, \langle p_0, p_1\rangle\rangle = \mathsf{D}_0^{e-1}\,\widetilde{\widetilde{\mathsf{if}}} \circ \langle \mathsf{D}^d\,\pi_i \circ f, \langle p_0, p_1\rangle\rangle\,.$$

It follows that $[\![M]\!]_\Gamma = [\![M']\!]_\Gamma$.

The case $M = \pi_i^d(\mathsf{let}^e(x, M, P))$ and $M' = \mathsf{let}^{e-1}(x, \pi_i^d(M), P)$ with $d < e$ is handled similarly.

► Assume that $M = \pi_i^d(\mathsf{if}^e(N, P_0, P_1))$ and $M' = \mathsf{if}^e(N, \pi_i^{d-e}(P_0), \pi_i^{d-e}(P_1))$, with $e \le d$. We must have $\Gamma \vdash N : \mathsf{D}^e\iota$ and there must be a type B such that $\Gamma \vdash P_j : B$ for $j = 0, 1$ so that $\Gamma \vdash M : \mathsf{D}^e B$ and hence $A = \mathsf{D}^e B$ (of course B is determined by A). Since $\Gamma \vdash M : A$ the type B must be of shape $\mathsf{D}^{d-e+1}C$ for a (uniquely determined) type C, and then $A = \mathsf{D}^d C$ and $\Gamma \vdash \mathsf{if}^e(N, P_0, P_1) : \mathsf{D}^{d+1}C$. We set $U = [\![C]\!]$ and $Y = [\![B]\!] = \mathsf{D}^{d-e+1}\,U$,

$f = [\![N]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^e\,\mathsf{N})$ and $p_j = [\![P_j]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^{d-e+1}\,U)$. We have

$$\begin{aligned}
\mathsf{D}^d\,\pi_i \circ \mathsf{D}^e_0\,\widetilde{\overline{\mathsf{if}}}_{\mathsf{D}^{d-e+1}\,U} \circ \langle f, \langle p_0, p_1\rangle\rangle &= \mathsf{D}^d\,\pi_i \circ \mathsf{D}^e\,\widetilde{\overline{\mathsf{if}}}_{\mathsf{D}^{d-e+1}\,U} \circ \psi^0_{\mathsf{N},\mathsf{D}^{d-e+1}(U\&U)}(e) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&= \mathsf{D}^e(\mathsf{D}^{d-e}\,\pi_i \circ \widetilde{\overline{\mathsf{if}}}_{\mathsf{D}^{d-e+1}\,U}) \circ \psi^0_{\mathsf{N},\mathsf{D}^{d-e+1}(U\&U)}(e) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&= \mathsf{D}^e(\widetilde{\overline{\mathsf{if}}}_{\mathsf{D}^{d-e}\,X} \circ \left(\mathsf{N} \,\&\, \mathsf{D}^{d-e}\,\pi_i\right)) \circ \psi^0_{\mathsf{N},\mathsf{D}^{d-e+1}(U\&U)}(e) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&\qquad\qquad \text{by naturality of } \widetilde{\overline{\mathsf{if}}} \\
&= \mathsf{D}^e\,\widetilde{\overline{\mathsf{if}}}_U \circ \left(\mathsf{D}^e\,\mathsf{N} \,\&\, \mathsf{D}^d\,\pi_i\right) \circ (\mathsf{D}^e\,\mathsf{N} \,\&\, \iota_0(e)) \circ \langle f, \langle p_0, p_1\rangle\rangle \\
&= \mathsf{D}^e\,\widetilde{\overline{\mathsf{if}}}_U \circ (\mathsf{D}^e\,\mathsf{N} \,\&\, \iota_0(e)) \circ \left(\mathsf{D}^e\,\mathsf{N} \,\&\, \mathsf{D}^{d-e}\,\pi_i\right) \circ \langle f, \langle p_0, p_1\rangle\rangle \quad \text{by nat. of } \iota_0(e) \\
&= \mathsf{D}^e_0\,\widetilde{\overline{\mathsf{if}}} \circ \langle f, \langle (\mathsf{D}^{d-e}\,\pi_i) \circ p_0, (\mathsf{D}^{d-e}\,\pi_i) \circ p_1\rangle\rangle
\end{aligned}$$

and it follows that $[\![M]\!]_\Gamma = [\![M']\!]_\Gamma$ since $(\mathsf{D}^{d-e}\,\pi_i) \circ p_j = [\![\pi_i^{d-e}(P_j)]\!]_\Gamma$ for $j = 0, 1$.

The case where $M = \pi_i^d(\mathsf{let}^e(x, M, P))$ and $M' = \mathsf{let}^e(x, M, \pi_i^{d-e}(P))$ with $e \leq d$ is handled similarly.

▶ Assume that $M = \pi_0^d(\theta^d(N))$ and $M' = \pi_0^d(\pi_0^d(N))$ so that we must have $\Gamma \vdash N : \mathsf{D}^{d+2}B$ for a (uniquely determined) type B such that $A = \mathsf{D}^d B$. Let $Y = [\![B]\!]$ so that $X = \mathsf{D}^d\,Y$ and $f = [\![N]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^{d+2}\,Y)$. We have

$$\begin{aligned}
(\mathsf{D}^d\,\pi_0) \circ (\mathsf{D}^d\,\theta) \circ f &= \mathsf{D}^d(\pi_0 \circ \theta) \circ f \\
&= \mathsf{D}^d(\pi_0 \circ \pi_0) \circ f \\
&= (\mathsf{D}^d\,\pi_0) \circ (\mathsf{D}^d\,\pi_0) \circ f \\
&= [\![M']\!]_\Gamma\,.
\end{aligned}$$

Notice that we are using the fact that we are composing *linear* morphisms in $\mathcal{L}_!$ so that the equation $\pi_0 \circ \theta = \pi_0 \circ \pi_0$ holds in $\mathcal{L}_!$ because $\pi_0\ \tau = \pi_0\ \pi_0$ holds in $\mathcal{L}$.

The case where $M = \pi_1^d(\theta^d(N))$ and $M' = \pi_1^d(\pi_0^d(N)) + \pi_0^d(\pi_1^d(N))$ is similar, using the equation $\pi_1\ \tau = \pi_1\ \pi_0 + \pi_0\ \pi_1$ in $\mathcal{L}$.

▶ Assume that $M = \pi_i^d(\theta^e(N))$ and $M' = \theta^{e-1}(\pi_i^d(N))$ with $d < e$ so that we must have $\Gamma \vdash N : \mathsf{D}^{e+2}B$ for a (uniquely determined) type B such that $A = \mathsf{D}^e B$. Let $Y = [\![B]\!]$ so that $X = \mathsf{D}^e\,Y$. Let $f = [\![N]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^{e+2}\,Y)$ so that $(\mathsf{D}^e\,\theta) \circ f \in \mathcal{L}_!(Z, \mathsf{D}^{e+1}\,X)$ and hence $(\mathsf{D}^d\,\pi_i) \circ (\mathsf{D}^e\,\theta) \circ f \in \mathcal{L}_!(Z, \mathsf{D}^e\,X)$. We have

$$\begin{aligned}
(\mathsf{D}^d\,\pi_i) \circ (\mathsf{D}^e\,\theta) \circ f &= \mathsf{D}^d(\pi_i \circ (\mathsf{D}^{e-d}\,\theta)) \circ f \\
&= \mathsf{D}^d(\mathsf{D}^{e-d-1}\,\theta \circ \pi_i) \circ f \quad \text{by naturality of } \pi_i \\
&= (\mathsf{D}^{e-1}\,\theta) \circ (\mathsf{D}^d\,\pi_i) \circ f \\
&= [\![M']\!]_\Gamma\,.
\end{aligned}$$

▶ Assume that $M = \pi_i^d(\theta^e(N))$ and $M' = \theta^e(\pi_i^{d+1}(N))$ with $e < d$ so that we must have $\Gamma \vdash N : \mathsf{D}^{e+2}B$ for some type B and we have $\Gamma \vdash \theta^e(N) : \mathsf{D}^{e+2}B$. There must be a type C such that $\mathsf{D}^{e+2}B = \mathsf{D}^{d+1}C$ and $A = \mathsf{D}^d C$. In other words $B = \mathsf{D}^{d-e-1}C$ and of course C is uniquely determined by A. Let $Y = [\![C]\!]$ so that we have $f = [\![M]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^{d+1}Y)$ and hence $(\mathsf{D}^d\,\pi_i)\circ(\mathsf{D}^e\,\theta)\circ f\in\mathcal{L}_!(Z,\mathsf{D}^{d-1}\,Y)$. We have

$$\begin{aligned}(\mathsf{D}^d\,\pi_i)\circ(\mathsf{D}^e\,\theta)\circ f&=\mathsf{D}^e(\mathsf{D}^{d-e}\,\pi_i\circ\theta)\circ f\\&=\mathsf{D}^e(\theta\circ\mathsf{D}^{d-e+1}\,\pi_i)\circ f\quad\text{by nat. of }\theta\text{, observing that }d-e\geq 1\\&=\mathsf{D}^e\,\theta\circ\mathsf{D}^{d+1}\,\pi_i\circ f\\&=[\![M']\!]_\Gamma\,.\end{aligned}$$

▶ Assume that $M=\pi_i^d(\mathsf{c}_l^e(N))$ and $M'=\mathsf{c}_l^{e-1}(\pi_i^d(N))$ with $d<e$. Then there must be a type B such that $\Gamma\vdash N:\mathsf{D}^{e+l+2}B$ and hence $\Gamma\vdash\mathsf{c}_l^e(N):\mathsf{D}^{e+l+2}B$, and $\Gamma\vdash M:\mathsf{D}^{e+l+1}B$ since $d<e+l+2$. So we have $A=\mathsf{D}^{e+l+1}B$. Let $Y=[\![B]\!]$ so that $f=[\![N]\!]_\Gamma\in\mathcal{L}_!(Z,\mathsf{D}^{e+l+2}\,Y)$ and we have

$$\begin{aligned}[\![M]\!]_\Gamma&=(\mathsf{D}^d\,\pi_i)\circ(\mathsf{D}^e\,\mathsf{c}(l))\circ f\\&=\mathsf{D}^d(\pi_i\circ\mathsf{D}^{e-d}\,\mathsf{c}(l))\circ f\\&=\mathsf{D}^d(\mathsf{D}^{e-d-1}\,\mathsf{c}(l)\circ\pi_i)\circ f\quad\text{by naturality of }\pi_i\\&=(\mathsf{D}^{e-1}\,\mathsf{c}(l))\circ(\mathsf{D}^d\,\pi_i)\circ f\\&=[\![M']\!]_\Gamma\,.\end{aligned}$$

▶ Assume that $M=\pi_i^d(\mathsf{c}_l^e(N))$ and $M'=\mathsf{c}_l^e(\pi_i^d(M))$ with $e+l+2\leq d$. Then there must be a type B such that $\Gamma\vdash N:\mathsf{D}^{e+l+2}B$ and hence $\Gamma\vdash\mathsf{c}_l^e(N):\mathsf{D}^{e+l+2}B$ and therefore there must be a type C such that $\mathsf{D}^{d+1}C=\mathsf{D}^{e+l+2}B$, that is $B=\mathsf{D}^{d-e-l-1}C$. Setting $Y=[\![C]\!]$ we have $f=[\![N]\!]_\Gamma\in\mathcal{L}_!(Z,\mathsf{D}^{d+1}\,Y)$ and

$$\begin{aligned}[\![M]\!]_\Gamma&=(\mathsf{D}^d\,\pi_i)\circ(\mathsf{D}^e\,\mathsf{c}(l))\circ f\\&=\mathsf{D}^e((\mathsf{D}^{d-e}\,\pi_i)\circ\mathsf{c}(l))\circ f\\&=\mathsf{D}^e((\mathsf{D}^{l+2}\,\mathsf{D}^{d-e-l-2}\,\pi_i)\circ\mathsf{c}(l))\circ f\\&=\mathsf{D}^e(\mathsf{c}(l)\circ(\mathsf{D}^{d-e}\,\pi_i))\circ f\quad\text{by naturality of }\mathsf{c}(l)\\&=(\mathsf{D}^e\,\mathsf{c}(l))\circ(\mathsf{D}^d\,\pi_i)\circ f\\&=[\![M']\!]_\Gamma\,.\end{aligned}$$

▶ Assume that $M=\pi_{i_{l+1}}^d(\cdots\pi_{i_0}^d(\mathsf{c}_l^d(N)))$ and $M'=\pi_{i_0}^d(\pi_{i_{l+1}}^d(\cdots\pi_{i_1}^d(N)))$ so that there must be a type B such that $\Gamma\vdash N:\mathsf{D}^{d+l+2}B$ and hence $\Gamma\vdash M:\mathsf{D}^dB=A$. Let $Y=[\![B]\!]$ and $f=[\![N]\!]_\Gamma\in\mathcal{L}_!(Z,\mathsf{D}^{d+l+2}\,Y)$. We have $(\mathsf{D}^d\,\pi_{i_{l+1}})\circ\cdots\circ(\mathsf{D}^d\,\pi_{i_0})\in\mathcal{L}_!(\mathsf{D}^{d+l+2}\,X,\mathsf{D}^d\,X)$ and then

$$\begin{aligned}[\![M]\!]_\Gamma&=(\mathsf{D}^d\,\pi_{i_{l+1}})\circ\cdots\circ(\mathsf{D}^d\,\pi_{i_0})\circ(\mathsf{D}^d\,\mathsf{c}(l))\circ f\\&=(\mathsf{D}^d\,\pi_{i_0})\circ(\mathsf{D}^d\,\pi_{i_{l+1}})\circ\cdots\circ(\mathsf{D}^d\,\pi_{i_1})\circ f\\&=[\![M']\!]_\Gamma\end{aligned}$$

by Lemma 4.17.

▶ Assume that $M=\pi_i^d(\iota_j^d(N))$, $M'=N$ if $i=j$ and $M'=0$ if $i\neq j$. We must have $\Gamma\vdash N:\mathsf{D}^dB$ for a type B such that $A=\mathsf{D}^dB$. Setting $Y=[\![B]\!]$ we have $f=[\![N]\!]_\Gamma\in$

$\mathcal{L}_!(Z, \mathsf{D}^d\, Y)$ and

$$\begin{aligned}(\mathsf{D}^d\, \pi_i) \circ (\mathsf{D}^d\, \iota_j) \circ f &= \mathsf{D}^d(\pi_i \circ \iota_j) \circ f \\ &= \boldsymbol{\delta}_{i,j} f\end{aligned}$$

since $\pi_i \circ \iota_j = \boldsymbol{\delta}_{i,j}\, \mathsf{Id}$.

▶ Assume that $M = \pi_i^d(\iota_j^e(N))$ and $M' = \iota_j^{e-1}(\pi_i^d(N))$ with $d < e$ so that we must have $\Gamma \vdash N : \mathsf{D}^e B$ and $A = \mathsf{D}^d C$ with $\mathsf{D}^{d+1} C = \mathsf{D}^{e+1} B$ so that $C = \mathsf{D}^{e-d} B$. Let $Y = [\![B]\!]$ so that we have $f = [\![N]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^e\, Y)$ and

$$\begin{aligned}[\![M]\!]_\Gamma &= (\mathsf{D}^d\, \pi_i) \circ (\mathsf{D}^e\, \iota_j) \circ f \\ &= \mathsf{D}^d(\pi_i \circ \mathsf{D}^{e-d}\, \iota_j) \circ f \\ &= \mathsf{D}^d((\mathsf{D}^{e-d-1}\, \iota_j) \circ \pi_i) \circ f \quad \text{by naturality of } \pi_i \\ &= (\mathsf{D}^{e-1}\, \iota_j) \circ (\mathsf{D}^d\, \pi_i) \circ f \\ &= [\![M']\!]_\Gamma\, .\end{aligned}$$

▶ Assume that $M = \pi_i^d(\iota_j^e(N))$ and $M' = \iota_j^e(\pi_i^{d-1}(N))$ with $e < d$ so that we must have $\Gamma \vdash N : \mathsf{D}^e B$ and $A = \mathsf{D}^d C$ with $\mathsf{D}^{d+1} C = \mathsf{D}^{e+1} B$ so that $B = \mathsf{D}^{d-e} C$. Let $Y = [\![C]\!]$ so that we have $f = [\![N]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^d\, Y)$ and

$$\begin{aligned}[\![M]\!]_\Gamma &= (\mathsf{D}^d\, \pi_i) \circ (\mathsf{D}^e\, \iota_j) \circ f \\ &= \mathsf{D}^e((\mathsf{D}^{d-e}\, \pi_i) \circ \iota_j) \circ f \\ &= \mathsf{D}^e(\iota_j \circ (\mathsf{D}^{d-e-1}\, \pi_i)) \circ f \quad \text{by naturality of } \iota_j \\ &= (\mathsf{D}^e\, \iota_j) \circ (\mathsf{D}^{d-1}\, \pi_i) \circ f \\ &= [\![M']\!]_\Gamma\, .\end{aligned}$$

▶ Assume that $M = \pi_i^d(\pi_j^e(N))$ and $M' = \pi_j^{e-1}(\pi_i^d(N))$ with $d < e$. We must have $\Gamma \vdash N : \mathsf{D}^{e+1} B$ for a type B such that $A = \mathsf{D}^{e-1} B$. Let $Y = [\![B]\!]$ so that we have $f = [\![N]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^{e+1}\, Y)$ and

$$\begin{aligned}[\![M]\!]_\Gamma &= (\mathsf{D}^d\, \pi_i) \circ (\mathsf{D}^e\, \pi_j) \circ f \\ &= \mathsf{D}^d(\pi_i \circ (\mathsf{D}^{e-d}\, \pi_j)) \circ f \\ &= \mathsf{D}^d((\mathsf{D}^{e-d-1}\, \pi_j) \circ \pi_i) \circ f \quad \text{by naturality of } \pi_i \\ &= (\mathsf{D}^{e-1}\, \pi_j) \circ (\mathsf{D}^d\, \pi_i) \circ f \\ &= [\![M']\!]_\Gamma\, .\end{aligned}$$

▶ Assume that $M = \pi_i^d(\pi_j^e(N))$ and $M' = \pi_j^e(\pi_i^{d+1}(N))$ with $e \leq d$. We must have $\Gamma \vdash N : \mathsf{D}^{e+1} B$ for a type B and then $\Gamma \vdash \pi_j^e(N) : \mathsf{D}^e B$ so that $B = \mathsf{D}^{d-e+1} C$ and we have

$A = \mathsf{D}^d C$. Let $Y = [\![C]\!]$ so that we have $f = [\![N]\!]_\Gamma \in \mathcal{L}_!(Z, \mathsf{D}^{d+1} Y)$ and

$$\begin{aligned}[\![M]\!]_\Gamma &= (\mathsf{D}^d\,\pi_i) \circ (\mathsf{D}^e\,\pi_j) \circ f \\ &= \mathsf{D}^e((\mathsf{D}^{d-e}\,\pi_i) \circ \pi_j) \circ f \\ &= \mathsf{D}^e(\pi_j \circ (\mathsf{D}^{d-e+1}\,\pi_i)) \circ f \quad \text{by naturality of } \pi_j \\ &= (\mathsf{D}^e\,\pi_j) \circ (\mathsf{D}^{d+1}\,\pi_i) \circ f \\ &= [\![M']\!]_\Gamma\,.\end{aligned}$$

□

Let $S = [M_1, \dots, M_k] \in \underline{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})}$ and assume that $\Gamma \vdash S : A$, that is $(\Gamma \vdash M_i : A)_{i=1}^k$. We say that S is *$\mathcal{L}$-summable* if the family $([\![M_i]\!]_\Gamma)_{i=1}^k$ is summable in $\mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$. When this property holds we set $[\![S]\!]_\Gamma = \sum_{i=1}^k [\![M_i]\!]_\Gamma \in \mathcal{L}_!([\![\Gamma]\!], [\![A]\!])$. Notice that this is not a purely syntactic notion, it depends on the notion of summability of $\mathcal{L}$.

Remark 4.38. We can introduce an absolute notion of summability by quantifying universally on $\mathcal{L}$ but this will not be really useful here.

Theorem 4.39 (multiset invariance)**.** *If $S \in \underline{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})}$ is such that $\Gamma \vdash S : A$ and S is $\mathcal{L}$-summable and if $S \to_{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})} S'$ then S' is $\mathcal{L}$-summable and $[\![S']\!]_\Gamma = [\![S]\!]_\Gamma$.*

Remember that $\Gamma \vdash S' : A$ by Theorem 3.11.

Proof. The following cases are possible, for some $S_0 = [N_1, \dots, N_k]$.

▶ $S = S_0 + [M]$, $M \to_{\Lambda_{\mathsf{cd}}} 0$ and $S' = S_0$. Since $([\![N_1]\!]_\Gamma, \dots, [\![N_k]\!]_\Gamma, [\![M]\!]_\Gamma)$ is a summable family, we know that the family $([\![N_1]\!]_\Gamma, \dots, [\![N_k]\!]_\Gamma)$ is summable by Theorem 2.1 of [Ehr21]. Moreover $[\![S']\!]_\Gamma = [\![S]\!]_\Gamma$ because $[\![M]\!]_\Gamma = [\![0]\!]_\Gamma = 0$ by Theorem 4.37.

▶ $S = S_0 + [M]$, $M \to_{\Lambda_{\mathsf{cd}}} M'$ and $S' = S_0 + [M']$. Then we have $[\![M']\!]_\Gamma = [\![M]\!]_\Gamma$ by Theorem 4.37 and hence S' is $\mathcal{L}$-summable and $[\![S']\!]_\Gamma = [\![S]\!]_\Gamma$.

▶ $S = S_0 + [M]$, $M \to_{\Lambda_{\mathsf{cd}}} M_0 + M_1$ and $S' = S_0 + [M_0, M_1]$. Then by Theorem 4.37 we know that $[\![M_0]\!]_\Gamma, [\![M_1]\!]_\Gamma$ are summable and $[\![M]\!]_\Gamma = [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$. By Theorem 2.1 of [Ehr21] we know that $([\![N_1]\!]_\Gamma, \dots, [\![N_k]\!]_\Gamma, [\![M_0]\!]_\Gamma, [\![M_1]\!]_\Gamma)$ is a summable family with $[\![S]\!]_\Gamma = [\![N_1]\!]_\Gamma + \cdots + [\![N_k]\!]_\Gamma + [\![M_0]\!]_\Gamma + [\![M_1]\!]_\Gamma$, that is S' is $\mathcal{L}$-summable and $[\![S']\!]_\Gamma = [\![S]\!]_\Gamma$. □

5. Completeness of the reduction rules

5.1. **A differential abstract machine.** Our goal now is to define a specific reduction strategy within the rewriting system Λ_{cd}. We do this by means of an "abstract machine" which has no environment (just as in the work of Krivine on classical realizability). In further papers we will certainly develop an environment machine more suitable for implementations.

So a state of our machine will consist of a term and a stack. It will also contain an *access word* which will be a finite sequence of 0 and 1 describing which component of the output is sought: the machine will allow to evaluate terms of type $\mathsf{D}^d\iota$ and the word, of length d, specifies which leaf of the associated tree of height d has to be computed by the machine.

Let us say that a type A is *sharp* if it is not of shape $\mathsf{D}B$; in our type language, this is equivalent to $A = (A_1 \Rightarrow \cdots A_k \Rightarrow \iota)$ for some uniquely determined types $A_1, \dots, A_k$. Any

$$\frac{}{() : \iota \vdash \iota} \qquad \frac{s : \iota \vdash \iota}{\mathsf{succ} \cdot s : \iota \vdash \iota} \qquad \frac{s : \iota \vdash \iota}{\mathsf{pred} \cdot s : \iota \vdash \iota}$$

$$\frac{s : E \vdash \iota \qquad \vdash M_0 : \mathsf{D}^d E \qquad \vdash M_1 : \mathsf{D}^d E \qquad \delta \in \mathsf{I}^d}{\mathsf{if}(\delta, M_0, M_1) \cdot s : \iota \vdash \iota}$$

$$\frac{s : E \vdash \iota \qquad x : \iota \vdash M : \mathsf{D}^d E \qquad \delta \in \mathsf{I}^d}{\mathsf{let}(\delta, x, M) \cdot s : \iota \vdash \iota} \qquad \frac{\vdash M : A \qquad s : E \vdash \iota}{\mathsf{arg}(M) \cdot s : A \Rightarrow E \vdash \iota}$$

$$\frac{s : \mathsf{D}A \Rightarrow E \vdash \iota \qquad i \in \mathsf{I}}{\mathsf{D}(i) \cdot s : A \Rightarrow E \vdash \iota}$$

FIGURE 6. Typing rules for stacks

type A can be written uniquely $A = \mathsf{D}^d E$ where E is sharp and $d \in \mathbb{N}$. We use the letters $E, F, \dots$ for sharp types.

We say that a term is *sum-implicit* if it does not contain the following constructs: 0 and $M_0 + M_1$.

A *word* is an element α of $\{0,1\}^{<\omega}$, we refer to Section 2.1 for notations concerning words. Then we define the stacks as follows:

$$s, t, \dots := () \mid \mathsf{arg}(M) \cdot s \mid \mathsf{succ} \cdot s \mid \mathsf{pred} \cdot s \mid \mathsf{if}(\alpha, M_0, M_1) \cdot s \mid \mathsf{let}(\alpha, x, M) \cdot s \mid \mathsf{D}(i) \cdot s$$

where α is a word and $i \in \{0,1\}$. Stacks are typed by judgments of shape $s : E \vdash \iota$ where E is a sharp type and M, M_0 and M_1 are sum-implicit. The typing rules for stacks are given in Figure 6.

5.1.1. *The machine.* We define the states of our machine as follows:

$$c, c_0, c_1, \dots := (\delta, M, s) \mid 0 \mid c_0 + c_1$$

where δ is a word, M is a sum-implicit term and s is a stack. The state (δ, M, s) is well typed if

$$s : E \vdash \iota \quad \text{and} \quad \vdash M : \mathsf{D}^{\mathsf{len}(\delta)} E$$

for some sharp type E (uniquely determined by M). The state 0 is always well typed and $c_0 + c_1$ is well typed if c_0 and c_1 are. We define a rewriting system Θ_{cd} such that $\underline{\Theta_{\mathsf{cd}}}$ is the set of all states. A state is sum-implicit if it is not of shape 0 or $c_1 + c_2$.

The associated reduction relation $\to_{\Theta_{\mathsf{cd}}}$ is defined in Figure 7. It is a deterministic reduction relation on states: determinism results from the fact that the rule to be applied on (α, M, s) is completely determined by the shape of M (actually, by the outermost construct of M). Notice that states which are not sum-implicit (that is, which are of shape 0 or $c_0 + c_1$) cannot be reduced by the system Θ_{cd}. The associated reduction system $\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})$, see Section 2.2, is precisely designed to reduce such sums.

Proposition 5.1. *If $c \to_{\Theta_{\mathsf{cd}}} c'$ and c is a well typed state, then c' is a well typed state.*

Proof. Since $c \to_{\Theta_{\mathsf{cd}}} c'$ we must have $c = (\alpha, M, s)$ with $s : E \vdash \iota$ and $\vdash M : \mathsf{D}^{\mathsf{len}(\alpha)} E$. We have to consider each rewrite rule of Figure 7 so we reason by cases on the shape of M, we focus on the most interesting cases, the other ones are similar and easier.

$$
\begin{array}{rlrl}
(\delta,(M)N,s) &\to_{\Theta_{\mathsf{cd}}} (\delta,M,\mathsf{arg}(N)\cdot s) & (\delta,\lambda x^A\,M,\mathsf{arg}(N)\cdot s) &\to_{\Theta_{\mathsf{cd}}} (\delta,M\,[N/x]\,,s)\\
(\delta i,\mathsf{D}M,s) &\to_{\Theta_{\mathsf{cd}}} (\delta,M,\mathsf{D}(i)\cdot s) & (\delta,\lambda x^A\,M,\mathsf{D}(i)\cdot s) &\to_{\Theta_{\mathsf{cd}}} (\delta i,\lambda x^{\mathsf{D}A}\,\partial(x,M),s)\\
(\delta,\mathsf{Y}M,s) &\to_{\Theta_{\mathsf{cd}}} (\delta,M,\mathsf{arg}(\mathsf{Y}M)\cdot s) & &\\
(\delta,\mathsf{succ}^d(M),s) &\to_{\Theta_{\mathsf{cd}}} (\delta,M,\mathsf{succ}\cdot s) & (\delta,\mathsf{pred}^d(M),s) &\to_{\Theta_{\mathsf{cd}}} (\delta,M,\mathsf{pred}\cdot s)\\
(\langle\,\rangle,\underline{n},\mathsf{succ}\cdot s) &\to_{\Theta_{\mathsf{cd}}} (\langle\,\rangle,\underline{n+1},s) & (\langle\,\rangle,\underline{0},\mathsf{pred}\cdot s) &\to_{\Theta_{\mathsf{cd}}} (\langle\,\rangle,\underline{0},s)\\
(\langle\,\rangle,\underline{n+1},\mathsf{pred}\cdot s) &\to_{\Theta_{\mathsf{cd}}} (\langle\,\rangle,\underline{n},s) & &\\
(\varepsilon\delta,\mathsf{if}^d(M,P_0,P_1),s) &\to_{\Theta_{\mathsf{cd}}} (\delta,M,\mathsf{if}(\varepsilon,P_0,P_1)\cdot s) & (\langle\,\rangle,\underline{0},\mathsf{if}(\varepsilon,P_0,P_1)\cdot s) &\to_{\Theta_{\mathsf{cd}}} (\varepsilon,P_0,s)\\
(\langle\,\rangle,\underline{n+1},\mathsf{if}(\varepsilon,P_0,P_1)\cdot s) &\to_{\Theta_{\mathsf{cd}}} (\varepsilon,P_1,s) & &\\
(\varepsilon\delta,\mathsf{let}^d(x,N,M),s) &\to_{\Theta_{\mathsf{cd}}} (\delta,N,\mathsf{let}(\varepsilon,x,N)\cdot s) & (\langle\,\rangle,\underline{n},\mathsf{let}(\varepsilon,x,N)\cdot s) &\to_{\Theta_{\mathsf{cd}}} (\varepsilon,N\,[\underline{n}/x]\,,s)\\
(\varepsilon i\delta,\iota_i^d(M),s) &\to_{\Theta_{\mathsf{cd}}} (\varepsilon\delta,M,s) & (\varepsilon i\delta,\iota_{1-i}^d(M),s) &\to_{\Theta_{\mathsf{cd}}} 0\\
(\varepsilon\alpha\delta,\mathsf{c}_l^d(M),s) &\to_{\Theta_{\mathsf{cd}}} (\varepsilon\underset{\to}{\alpha}\delta,M,s) & (\varepsilon\delta,\pi_i^d(M),s) &\to_{\Theta_{\mathsf{cd}}} (\varepsilon i\delta,M,s)\\
(\varepsilon 0\delta,\theta^d(M),s) &\to_{\Theta_{\mathsf{cd}}} (\varepsilon 00\delta,M,s) & (\varepsilon 1\delta,\theta^d(M),s) &\to_{\Theta_{\mathsf{cd}}} (\varepsilon 10\delta,M,s)+(\varepsilon 01\delta,M,s)
\end{array}
$$

FIGURE 7. Reduction rules for states. Convention: $d = \mathsf{len}(\delta)$, $e = \mathsf{len}(\varepsilon)$.

▶ Assume that $M = \mathsf{if}^d(N, P_0, P_1)$ so that we must have $\vdash N : \mathsf{D}^d\iota$, $\vdash P_i : \mathsf{D}^e E$ for $i = 0, 1$ and hence $\vdash M : \mathsf{D}^{d+e}E$, $s : E \vdash \iota$ and $\mathsf{len}(\alpha) = e + d$ so that we can write $\alpha = \varepsilon\delta$ with $\mathsf{len}(\delta) = d$ and $\mathsf{len}(\varepsilon) = e$. Then we have $\mathsf{if}(\varepsilon, P_0, P_1) \cdot s : \iota \vdash \iota$ and hence $c' = (\delta, N, \mathsf{if}(\varepsilon, P_0, P_1) \cdot s)$ is well typed.

▶ Assume $M = \mathsf{D}N$ so that we have $\mathsf{len}(\alpha) > 0$ and $\vdash N : \mathsf{D}^{\mathsf{len}(\alpha)-1}E = (A \Rightarrow \mathsf{D}^{\mathsf{len}(\alpha)-1}F)$ for some type A, and $E = (A \Rightarrow F)$ where F is sharp. We have

$$\vdash M : (\mathsf{D}A \Rightarrow \mathsf{D}^{\mathsf{len}(\alpha)}F) = \mathsf{D}^{\mathsf{len}(\alpha)}(A \Rightarrow F)$$

and we can write $\alpha = \delta i$ and with this notation $c' = (\delta, N, \mathsf{D}(i) \cdot s)$ is well typed since $s : \mathsf{D}A \Rightarrow F \vdash \iota$ and hence $\mathsf{D}(i) \cdot s : A \Rightarrow F \vdash \iota$.

▶ Assume that $M = \theta^d(N)$ then we have $\vdash N : \mathsf{D}^{d+2}B$ for some type $B = \mathsf{D}^e E$ where E is sharp and $s : E \vdash \iota$. And $\vdash M : \mathsf{D}^{d+1}B = \mathsf{D}^{d+e+1}E$ and hence we can write $\alpha = \varepsilon i\delta$ with $i \in \{0, 1\}$, $\mathsf{len}(\varepsilon) = e$ and $\mathsf{len}(\delta) = d$. If $i = 0$ we have $c' = (\varepsilon 00\delta, N, s)$ which is well typed since $\mathsf{D}^{d+2}B = \mathsf{D}^{d+e+2}E = \mathsf{len}(\varepsilon 00\delta)$. If $i = 1$ we have $c' = c_0+c_1$ with $c_j = (\varepsilon(1-j)j\delta, N, s)$ for $j = 0, 1$, and c_0 and c_1 are well typed for the same reason.

▶ Assume that $M = \iota_i^d(N)$ with $i \in \{0, 1\}$. Then we have $\vdash N : \mathsf{D}^d B$ for some type $B = \mathsf{D}^e E$ where E is sharp, $s : E \vdash \iota$ and $\vdash M : \mathsf{D}^{e+d+1}E$. So we can write $\alpha = \varepsilon j\delta$ where $j \in \{0, 1\}$. If $j \neq i$ we have $c' = 0$ and hence is the sum of the empty family of well typed states. If $j = i$ then $c' = (\varepsilon\delta, N, s)$ which is well typed since $\mathsf{D}^d B = \mathsf{D}^{e+d}E$. □

Proposition 5.2. *If c is a sum-implicit well typed state which is $\to_{\Theta_{\mathsf{cd}}}$-normal then there is $\nu \in \mathbb{N}$ such that $c = (\langle\,\rangle, \underline{\nu}, ())$.*

Proof. We have $c = (\alpha, M, s)$ with $\vdash M : \mathsf{D}^{\mathsf{len}(\alpha)}E$ and $s : E \vdash \iota$, we reason by cases on the last typing rule of M which is sum-implicit, that is, on the structure of M.

▶ M cannot be a variable because it is closed.

▶ If $M = \underline{\nu}$ then $E = \iota$ and $\alpha = \langle\,\rangle$ and hence $s : \iota \vdash \iota$. According to the typing rule for stacks, s must be of one of the following shapes: $()$, $\mathsf{succ} \cdot t$, $\mathsf{pred} \cdot t$, $\mathsf{if}(\varepsilon, P_0, P_1) \cdot t$, $\mathsf{let}(\varepsilon, x, P) \cdot t$ for some stack t and the first case only is possible since c is normal.

$$\langle () \rangle = [\;] \qquad \langle \mathsf{succ} \cdot s \rangle = \langle s \rangle [\mathsf{succ}^0([\;])]$$
$$\langle \mathsf{pred} \cdot s \rangle = \langle s \rangle [\mathsf{pred}^0([\;])] \qquad \langle \mathsf{if}(\delta, M_0, M_1) \cdot s \rangle = \langle s \rangle [\pi_\delta(\mathsf{if}^0([\;], M_0, M_1))]$$
$$\langle \mathsf{let}(\delta, x, M) \cdot s \rangle = \langle s \rangle [\pi_\delta(\mathsf{let}^0(x, [\;], M))] \qquad \langle \mathsf{arg}(M) \cdot s \rangle = \langle s \rangle [([\;])M]$$
$$\langle \mathsf{D}(i) \cdot s \rangle = \langle s \rangle [\pi_i(\mathsf{D}[\;])]$$

FIGURE 8. Context associated with a stack

▶ M cannot be $(N)P$ or $M = \mathsf{Y}N$ since c is normal.

▶ Assume that $M = \lambda x^A N$ so that $E = (A \Rightarrow F)$ and we must have $s : A \Rightarrow F \vdash \iota$. According to the typing rule for stacks we must have $s = \mathsf{arg}(P) \cdot t$ or $s = \mathsf{D}(i) \cdot t$. In both cases a reduction rule applies, contradicting the assumption that c is normal. This case is impossible.

▶ Assume that $M = \mathsf{if}^d(N, P_0, P_1)$ so that we must have $\vdash N : \iota$ and $\vdash P_i : B = \mathsf{D}^e E$ for $i = 0, 1$ where E is a sharp type, and $s : E \vdash \iota$, and $\vdash M : \mathsf{D}^{e+d} E$. Since c is well typed we must have $\mathsf{len}(\alpha) = e + d$ with $e = \mathsf{len}(\varepsilon)$, $d = \mathsf{len}(\delta)$ and $\alpha = \varepsilon\delta$. It follows that c is not normal, reducing to $(\delta, N, \mathsf{if}(\varepsilon, P_0, P_1) \cdot s)$.

▶ The case $M = \mathsf{let}^d(x, P, M)$ is completely similar to the previous one.

▶ The remaining cases are dealt with as in the proof of Proposition 5.1: in each case it appears that, because c is well typed, it cannot be normal. □

5.1.2. *Context associated with a stack, term associated with a state.* Given $\delta \in \{0,1\}^d$, $e \in \mathbb{N}$ and a term M, we define a term $\pi^e_\delta(M)$ by $\pi^e_{\langle\rangle}(M) = M$ and $\pi^e_{\delta i}(M) = \pi^e_\delta(\pi^e_i(M))$.

Given a stack s such that $s : E \vdash \iota$ we define a context $\langle s \rangle$ in Figure 8. Notice that this context is closed (it has no free occurrences of variables). Remember that the notion of linear context is defined in Equation (3.1).

Lemma 5.3. *Let s be a well typed state such that $s : E \vdash \iota$. Then the context $\langle s \rangle$ is linear and satisfies $x : E \vdash \langle s \rangle [x] : \iota$.*

Proof. Straightforward induction on the typing derivation of s. □

Lemma 5.4. *If $\Gamma \vdash M : \mathsf{D}^{d+e} A$ then $\Gamma \vdash \pi^e_\delta(M) : \mathsf{D}^e A$ if $d = \mathsf{len}(\delta)$.*

The proof is straightforward. We set $\pi_\delta(M) = \pi^0_\delta(M)$. Given a state c of shape $c = (\delta, M, s)$ with $s : E \vdash \iota$ we set $\langle c \rangle = \langle s \rangle [\pi_\delta(M)]$. We extend this definition to all states by

$$\langle 0 \rangle = 0 \qquad \langle c_0 + c_1 \rangle = \langle c_0 \rangle + \langle c_1 \rangle .$$

Lemma 5.5. *If c is a well typed state which is not of shape $c_0 + c_1$ then $\vdash \langle c \rangle : \iota$.*

Theorem 5.6. *If c is a well typed state and $c \rightarrow_{\Theta_{\mathsf{cd}}} c'$ then $\langle c \rangle \rightarrow_{\Lambda_{\mathsf{cd}}}{}^* \langle c' \rangle$.*

Proof. We must have $c = (\delta, Q, s)$ with $s : E \vdash \iota$ and $\vdash Q : \mathsf{D}^d E$ where $\mathsf{len}(\delta) = d$ so that $\langle c \rangle = \langle s \rangle [\pi_\delta(Q)]$ and we reason by considering the various cases in the transition $c \rightarrow_{\Theta_{\mathsf{cd}}} c'$ as listed in Figure 7.

▶ $Q = (M)N$ so that $\langle c\rangle = \langle s\rangle[\pi_\delta((M)N)]$, and $c' = (\delta, M, \mathsf{arg}(N)\cdot s)$ and hence $\langle c'\rangle = \langle \mathsf{arg}(N)\cdot s\rangle[\pi_\delta(M)] = \langle s\rangle[(\pi_\delta(M))N]$. We have $\pi_\delta((M)N) \to_{\Lambda_{\mathsf{cd}}}{}^* (\pi_\delta(M))N$ (in $\mathsf{len}(\delta)$ steps) and hence $\langle c\rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c'\rangle$ since $\langle s\rangle$ is an evaluation context.

▶ $Q = \lambda x^A\, M$ and $s = \mathsf{arg}(N)\cdot t$ so that $\langle c\rangle = \langle \mathsf{arg}(N)\cdot t\rangle[\pi_\delta(\lambda x^A\, M)] = \langle t\rangle[(\pi_\delta(\lambda x^A\, M))N]$. And $c' = (\delta, M\,[N/x]\,, t)$ so that $\langle c'\rangle = \langle t\rangle[\pi_\delta(M\,[N/x])]$. We have

$$\begin{aligned}(\pi_\delta(\lambda x^A\, M))N &\to_{\Lambda_{\mathsf{cd}}}{}^* (\lambda x^A\, \pi_\delta(M))N\\ &\to_{\Lambda_{\mathsf{cd}}} \pi_\delta(M)\,[N/x] = \pi_\delta(M\,[N/x])\end{aligned}$$

and hence $\langle c\rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c'\rangle$.

▶ $Q = \mathsf{D}M$ and $\delta = \varepsilon i$ so that $\langle c\rangle = \langle s\rangle[\pi_{\varepsilon i}(\mathsf{D}M)]$. And $c' = (\varepsilon, M, \mathsf{D}(i)\cdot s)$ so that $\langle c'\rangle = \langle \mathsf{D}(i)\cdot s\rangle[\pi_\varepsilon(M)] = \langle s\rangle[\pi_i(\mathsf{D}\pi_\varepsilon(M))]$. We have

$$\begin{aligned}\pi_{\varepsilon i}(\mathsf{D}M) &\to_{\Lambda_{\mathsf{cd}}}{}^* \pi_i(\pi^1_\varepsilon(\mathsf{D}M))\\ &\to_{\Lambda_{\mathsf{cd}}} \pi_i(\mathsf{D}\pi_\varepsilon(M))\end{aligned}$$

and hence $\langle c\rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c'\rangle$.

▶ $Q = \lambda x^A\, M$ and $s = \mathsf{D}(i)\cdot t$ with $i \in \{0,1\}$ so that we have $\langle c\rangle = \langle \mathsf{D}(i)\cdot t\rangle[\pi_\delta(\lambda x^A\, M)] = \langle t\rangle[\pi_i(\mathsf{D}(\pi_\delta(\lambda x^A\, M)))]$. And $c' = (\delta i, \lambda x^{\mathsf{D}A}\, \partial(x, M), t)$, so $\langle c'\rangle = \langle t\rangle[\pi_{\delta i}(\lambda x^{\mathsf{D}A}\, \partial(x, M))]$. We have

$$\begin{aligned}\pi_i(\mathsf{D}(\pi_\delta(\lambda x^A\, M))) &\to_{\Lambda_{\mathsf{cd}}}{}^* \pi_i(\pi^1_\delta(\mathsf{D}(\lambda x^A\, M)))\\ &\to_{\Lambda_{\mathsf{cd}}}{}^* \pi_\delta(\pi_i(\mathsf{D}(\lambda x^A\, M)))\\ &\to_{\Lambda_{\mathsf{cd}}} \pi_\delta(\pi_i(\lambda x^{\mathsf{D}A}\, \partial(x, M))) = \pi_{\delta i}(\lambda x^{\mathsf{D}A}\, \partial(x, M))\end{aligned}$$

and hence $\langle c\rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c'\rangle$.

▶ $Q = \mathsf{Y}M$ so that $\langle c\rangle = \langle s\rangle[\pi_\delta(\mathsf{Y}M)]$ and $\langle c'\rangle = \langle \mathsf{arg}(\mathsf{Y}M)\cdot s\rangle[\pi_\delta(M)] = \langle s\rangle[(\pi_\delta(M))\mathsf{Y}M]$. We have

$$\begin{aligned}\pi_\delta(\mathsf{Y}M) &\to_{\Lambda_{\mathsf{cd}}} \pi_\delta((M)\mathsf{Y}M)\\ &\to_{\Lambda_{\mathsf{cd}}}{}^* (\pi_\delta(M))\mathsf{Y}M\end{aligned}$$

and hence $\langle c\rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c'\rangle$.

▶ $Q = \mathsf{succ}^d(M)$ with $d = \mathsf{len}(\delta)$, so $\langle c\rangle = \langle s\rangle[\pi_\delta(\mathsf{succ}^d(M))]$. And $\langle c'\rangle = \langle \mathsf{succ}\cdot s\rangle[\pi_\delta(M)] = \langle s\rangle[\mathsf{succ}^0(\pi_\delta(M))]$. We have $\pi_\delta(\mathsf{succ}^d(M)) \to_{\Lambda_{\mathsf{cd}}}{}^* \mathsf{succ}^0(\pi_\delta(M))$ in d steps and hence $\langle c\rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c'\rangle$.

▶ The case $Q = \mathsf{pred}^d(M)$ is similar and the cases $Q = \underline{n}$ are straightforward (and then $\delta = \langle\,\rangle$ and there are several cases to consider as to the shape of s, they are all easy).

▶ $Q = \mathsf{if}^e(M, P_0, P_1)$ and $\delta = \eta\varepsilon$ with $e = \mathsf{len}(\varepsilon)$. We have $\langle c\rangle = \langle s\rangle[\pi_{\eta\varepsilon}(\mathsf{if}^e(M, P_0, P_1))]$ and $\langle c'\rangle = \langle \mathsf{if}(\eta, P_0, P_1)\cdot s\rangle[\pi_\varepsilon(M)] = \langle s\rangle[\pi_\eta(\mathsf{if}^0(\pi_\varepsilon(M), P_0, P_1))]$. We have

$$\pi_{\eta\varepsilon}(\mathsf{if}^e(M, P_0, P_1)) = \pi_\eta(\pi_\varepsilon(\mathsf{if}^e(M, P_0, P_1))) \to_{\Lambda_{\mathsf{cd}}}{}^* \pi_\eta(\mathsf{if}^0(\pi_\varepsilon(M), P_0, P_1))$$

and hence $\langle c\rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c'\rangle$.

▶ The case $Q = \mathsf{let}^d(x, N, M)$ is similar.

▶ $Q = \iota_i^e(M)$ and $\delta = \alpha i \varepsilon$ so that $\langle c \rangle = \langle s \rangle[\pi_{\alpha i \varepsilon}(\iota_i^e(M))]$ and $\langle c' \rangle = \langle s \rangle[\pi_{\alpha\varepsilon}(M)]$. We have

$$\begin{aligned}\pi_{\alpha i\varepsilon}(\iota_i^e(M)) &= \pi_\alpha(\pi_i(\pi_\varepsilon(\iota_i^e(M)))) \\ &\to_{\Lambda_{\mathsf{cd}}}{}^* \pi_\alpha(\pi_i(\iota_i^0(\pi_\varepsilon(M)))) \\ &\to_{\Lambda_{\mathsf{cd}}} \pi_\alpha(\pi_\varepsilon(M)) = \pi_{\alpha\varepsilon}(M)\end{aligned}$$

and hence $\langle c \rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c' \rangle$.

▶ $Q = \iota_i^e(M)$ and $\delta = \alpha(1-i)\varepsilon$ so that $\langle c \rangle = \langle s \rangle[\pi_{\alpha(1-i)\varepsilon}(\iota_i^e(M))]$ and $\langle c' \rangle = \langle 0 \rangle = 0$. We have

$$\begin{aligned}\pi_{\alpha(1-i)\varepsilon}(\iota_i^e(M)) &= \pi_\alpha(\pi_{1-i}(\pi_\varepsilon(\iota_i^e(M)))) \\ &\to_{\Lambda_{\mathsf{cd}}}{}^* \pi_\alpha(\pi_{1-i}(\iota_i^0(\pi_\varepsilon(M)))) \to_{\Lambda_{\mathsf{cd}}} 0\end{aligned}$$

and hence $\langle c \rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c' \rangle$.

▶ $Q = \mathsf{c}_l^e(M)$ and $\delta = \alpha i_{l+1} \cdots i_0 \varepsilon$ with $\mathsf{len}(\varepsilon) = e$. Then we have $\langle c \rangle = \langle s \rangle[\pi_{\alpha i_{l+1}\cdots i_0\varepsilon}(\mathsf{c}_l^e(M))]$ and $\langle c' \rangle = \langle s \rangle[\pi_{\alpha i_0 i_{l+1}\cdots i_1\varepsilon}(M)]$. Next observe that

$$\begin{aligned}\pi_{\alpha i_{l+1}\cdots i_0\varepsilon}(\mathsf{c}_l^e(M)) &\to_{\Lambda_{\mathsf{cd}}}{}^* \pi_{\alpha i_{l+1}\cdots i_0}(\mathsf{c}_l^0(\pi_\varepsilon(M))) \\ &\to_{\Lambda_{\mathsf{cd}}} \pi_{\alpha i_0 i_{l+1}\cdots i_1}(\pi_\varepsilon(M)) = \pi_{\alpha i_0 i_{l+1}\cdots i_1\varepsilon}(M)\end{aligned}$$

and hence $\langle c \rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c' \rangle$.

▶ $Q = \theta^e(M)$ and $\delta = \alpha 0 \varepsilon$ with $\mathsf{len}(\varepsilon) = e$. Then we have $\langle c \rangle = \langle s \rangle[\pi_{\alpha 0\varepsilon}(\theta^e(M))]$ and $\langle c' \rangle = \langle s \rangle[\pi_{\alpha 00\varepsilon}(M)]$. And

$$\begin{aligned}\pi_{\alpha 0\varepsilon}(\theta^e(M)) &\to_{\Lambda_{\mathsf{cd}}}{}^* \pi_\alpha(\pi_0(\theta^0(\pi_\varepsilon(M)))) \\ &\to_{\Lambda_{\mathsf{cd}}} \pi_{\alpha 00\varepsilon}(M)\end{aligned}$$

and hence $\langle c \rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c' \rangle$.

▶ $Q = \theta^e(M)$ and $\delta = \alpha 1 \varepsilon$ with $\mathsf{len}(\varepsilon) = e$. Then we have $\langle c \rangle = \langle s \rangle[\pi_{\alpha 1\varepsilon}(\theta^e(M))]$ and $\langle c' \rangle = \langle s \rangle[\pi_{\alpha 10\varepsilon}(M)] + \langle s \rangle[\pi_{\alpha 01\varepsilon}(M)]$. And

$$\begin{aligned}\pi_{\alpha 1\varepsilon}(\theta^e(M)) &\to_{\Lambda_{\mathsf{cd}}}{}^* \pi_\alpha(\pi_1(\theta^0(\pi_\varepsilon(M)))) \\ &\to_{\Lambda_{\mathsf{cd}}} \pi_{\alpha 10\varepsilon}(M) + \pi_{\alpha 01\varepsilon}(M)\end{aligned}$$

and hence $\langle c \rangle \to_{\Lambda_{\mathsf{cd}}}{}^* \langle c' \rangle$. □

For $C = [c_1, \dots, c_k] \in \mathcal{M}_{\mathrm{fin}}(\underline{\Theta_{\mathsf{cd}}})$ we define $\langle C \rangle = [\langle c_1 \rangle, \dots, \langle c_k \rangle] \in \mathcal{M}_{\mathrm{fin}}(\underline{\Lambda_{\mathsf{cd}}})$. We say that C is well typed if $c_1, \dots, c_k$ are well typed.

Theorem 5.7. *If $C \in \mathcal{M}_{\mathrm{fin}}(\underline{\Theta_{\mathsf{cd}}})$ is well typed and $C \to_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})} C'$ then $\langle C \rangle \to_{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})} \langle C' \rangle$.*

This is an immediate corollary of Theorem 5.6 and of the definition of $\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})$ and $\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})$ follow the same pattern. Let us say that $C = [c_1, \dots, c_k] \in \underline{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})}$ is $\mathcal{L}$-summable if C is well typed and the family of terms $(\langle c_1 \rangle, \dots, \langle c_k \rangle)$ is $\mathcal{L}$-summable.

Theorem 5.8. *If $C \in \underline{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})}$ is $\mathcal{L}$-summable and $C \to_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})} C'$ then C' is $\mathcal{L}$-summable.*

5.2. **Soundness.** Let s be a stack such that $s : E \vdash \iota$ and a variable x we know that $\langle s\rangle[x]$ is a term such that $x : E \vdash \langle s\rangle[x] : \iota$. So we can define the semantics of s by $[\![s]\!] = [\![\langle s\rangle[x]]\!]_{x:E} \in \mathcal{L}_!([\![E]\!], \mathsf{N})$.

Lemma 5.9. *If $s : E \vdash \iota$ then $[\![s]\!]$ is linear.*

This is due to the fact that the context $\langle s\rangle$ is always linear. The proof is straightforward.

If c is a well typed state, we set $[\![c]\!] = [\![\langle c\rangle]\!] \in \mathcal{L}(1, \mathsf{N})$ and if $C = [c_1, \dots, c_k] \in \underline{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})}$ is $\mathcal{L}$-summable we set $[\![C]\!] = \sum_{i=1}^{k} \in [\![c_i]\!]$ which is a well defined element of $\mathcal{L}(1, \mathsf{N})$ by definition of $\mathcal{L}$-summability.

Theorem 5.10. *If $C \in \underline{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})}$ is $\mathcal{L}$-summable and $C \to_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})} C'$ then $[\![C']\!] = [\![C]\!]$.*

This makes sense by Theorem 5.8 which entails that C' is $\mathcal{L}$-summable. The proof is a straightforward application of the definition of $\to_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})}$.

5.3. **The relational model and its associated intersection typing system.** The category **Rel** of sets and relations is a well known model of classical linear logic. In [Ehr21] it is proved that it is also a differential canonically summable Seely category; actually we prove this result for the category **nCoh** of non-uniform coherence spaces, but in that category the operations on morphisms are exactly the same as in **Rel** and the operations on objects are the same as in **Rel** as far as the webs are concerned[11].

Let us briefly recall the definition of this model of LL. An object of **Rel** is a set and $\mathbf{Rel}(X, Y) = \mathcal{P}(X \times Y)$, composition being the usual composition of relations denoted $v\,u$ when $u \in \mathbf{Rel}(X, Y)$ and $v \in \mathbf{Rel}(Y, Z)$. The identity morphism is the diagonal relation. The isos in this category are the bijections.

This category **Rel** is symmetric monoidal with $1 = \{*\}$ as tensorial unit and $X_0 \otimes X_1 = X_0 \times X_1$, and given $u_i \in \mathbf{Rel}(X_i, Y_i)$ for $i = 0, 1$, one sets $u_0 \otimes u_1 = \{((a_0, a_1), (b_0, b_1)) \mid (a_i, b_i) \in u_i \text{ for } i = 0, 1)\}$ defining a functor $\mathbf{Rel}^2 \to \mathbf{Rel}$ which has obvious natural isos $\lambda_X \in \mathbf{Rel}(1 \otimes X, X)$, $\rho_X \in \mathbf{Rel}(X \otimes 1, X)$, $\alpha_{X_0,X_1,X_2} \in \mathbf{Rel}((X_0 \otimes X_1) \otimes X_2, X_0 \otimes (X_1 \otimes X_2))$ and $\gamma_{X_0,X_1} \in \mathbf{Rel}(X_0 \otimes X_1, X_1 \otimes X_0)$. This SMC is closed with internal hom from X to Y the pair $(X \multimap Y, \mathsf{ev})$ where $X \multimap Y = X \times Y$ and $\mathsf{ev} = \{(((a, b), a), b) \mid a \in X \text{ and } b \in Y\} \in \mathbf{Rel}((X \multimap Y) \otimes X, Y)$. Given any morphism $u \in \mathbf{Rel}(Z \otimes X, Y)$, the associated morphism (Curry transpose of u) $\mathsf{cur}\,u \in \mathbf{Rel}(Z, X \multimap Y)$ is simply $\mathsf{cur}\,u = \{(a, (b, c)) \mid ((a, b), c) \in u\}$. This SMCC is $*$-autonomous with dualizing object $\perp = 1$, so that the "linear negation" of an object X is simply X.

The category **Rel** is cartesian: the cartesian product of a family $(X_i)_{i\in I}$ of objects of **Rel** is $(\&_{i\in I} X_i, (\mathsf{pr}_i)_{i\in I})$ where $\&_{i\in I} X_i = \bigcup_{i\in I}\{i\} \times X_i$ and $\mathsf{pr}_i = \{((i, a), a) \mid i \in I \text{ and } a \in X_i\} \in \mathbf{Rel}(\&_{j\in I} X_j, X_i)$ is the ith projection. The tupling of a family of morphisms $(u_i \in \mathbf{Rel}(Y, X_i))_{i\in I}$ is the morphism $\langle u_i\rangle_{i\in I} \in \mathbf{Rel}(Y, \&_{i\in I} X_i)$ given by $\langle u_i\rangle_{i\in I} = \{(b, (i, a)) \mid i \in I \text{ and } (b, a) \in u_i\}$. The coproduct $(\oplus_{i\in I} X_i, (\overline{\pi}_i)_{i\in I})$ also exists and is given by $\oplus_{i\in I} X_i = \&_{i\in I} X_i$ and $\overline{\pi}_i \in \mathbf{Rel}(X_i, \oplus_{j\in I} X_j)$ is given by $\overline{\pi}_i = \{(a, (i, a)) \mid i \in I \text{ and } a \in X_i\}$; it is the ith injection. The cotupling of morphisms $(u_i \in \mathbf{Rel}(X_i, Y))_{i\in I}$ is $[u_i]_{i\in I} \in \mathbf{Rel}(\oplus_{i\in I} X_i, Y)$ given by $[u_i]_{i\in I} = \{((i, a), b) \mid (a, b) \in u_i\}$. Notice that the terminal (and initial) object of **Rel** is $\top = 0 = \emptyset$.

[11]A non-uniform coherence space X is a triple consisting of a set $|X|$, the web of X, and two disjoint binary symmetric relations on that web, the strict coherence and the strict incoherence relations.

The SMC **Rel** is a Lafont category [Mel09, Ehr21]. The exponential functor is given by $!X = \mathcal{M}_{\mathrm{fin}}(X)$ and, if $s \in \mathbf{Rel}(X,Y)$ then $!s = \{([a_1,\dots,a_k],[b_1,\dots,b_k]) \mid k \in \mathbb{N} \text{ and } \forall i\ (a_i,b_i) \in s)\}$, defining a functor $\mathbf{Rel} \to \mathbf{Rel}$. The comonad structure of that functor is given by the natural transformations $\mathsf{der}_X = \{([a],a) \mid a \in X\} \in \mathbf{Rel}(!X,X)$ and $\mathsf{dig}_X = \{([m_1,\dots,m_n], m_1+\cdots+m_n) \mid n \in \mathbb{N} \text{ and } m_1,\dots,m_n \in !X\} \in \mathbf{Rel}(!X,!!X)$. The Seely isomorphisms are $\mathsf{m}^0 = \{(*,[\,])\}$ and

$$\mathsf{m}^2_{X,Y} = \{(([a_1,\dots,a_n],[b_1,\dots,b_k]),[(1,a_1),\dots,(1,a_n)],(2,b_1),\dots,(2,b_k)) \mid \\ a_1,\dots,a_n \in X \text{ and } b_1,\dots,b_k \in Y\} \in \mathbf{Rel}(!X \otimes !Y, !(X \mathbin{\&} Y))\,.$$

The Kleisli category $\mathbf{Rel}_!$ can be directly described as $\mathbf{Rel}_!(X,Y) = \mathcal{M}_{\mathrm{fin}}(X) \times Y$, the identity is $\mathsf{Id}_X = \{([a],a) \mid a \in X\} \in \mathbf{Rel}_!(X,X)$ and, given $u \in \mathbf{Rel}_!(X,Y)$ and $v \in \mathbf{Rel}_!(Y,Z)$, composition is given by $v \circ u = \{(m_1+\cdots+m_k, c) \mid k \in \mathbb{N} \text{ and } \exists b_1,\dots,b_k \in Y\ ([b_1,\dots,b_k],c) \in v \text{ and } (m_i,b_i) \in u \text{ for } i=1,\dots,k\}$.

It is easy to see that **Rel** is canonically summable (see [Ehr21]) and has therefore exactly one coherent differential structure by Theorem 4.9 of [Ehr21]. More explicitly the object $\mathsf{I} = 1 \mathbin{\&} 1 = \{0,1\}$ has a commutative $\otimes$-comonoid structure with counit $\mathsf{pr}_0 = \{(0,*)\} \in \mathbf{Rel}(\mathsf{I},1)$ and comultiplication

$$\widetilde{\mathsf{L}} = \{(0,(0,0)),(1,(1,0)),(1,(0,1))\} \\ = \{(r,(r_0,r_1)) \mid r,r_0,r_1 \in \mathsf{I} \text{ and } r = r_0+r_1)\} \in \mathbf{Rel}(\mathsf{I},\mathsf{I}\otimes\mathsf{I})$$

as explained in Section 4.1 of [Ehr21] which, by the Lafont property, induces the !-coalgebra structure $\tilde\partial \in \mathbf{Rel}(\mathsf{I},!\mathsf{I})$ given by $\tilde\partial = \{(0,k[0]) \mid k \in \mathbb{N}\} \cup \{(1,[1]+k[0]) \mid k \in \mathbb{N}\} = \{(r,[r_1,\dots,r_k]) \in \mathsf{I}\times\mathcal{M}_{\mathrm{fin}}(\mathsf{I}) \mid r = \sum_{i=1}^k r_i\}$. The associated summability functor $\mathbf{S}_\mathsf{I} = \mathsf{I} \multimap \_$ is given by $\mathbf{S}_\mathsf{I}\, X = \{0,1\}\times X$ and, for $u \in \mathbf{Rel}(X,Y)$, $\mathbf{S}_\mathsf{I}\, u = \{((i,a),(i,b)) \mid (a,b) \in u\}$. The associated natural transformations are $\pi_i = \{((i,a),a) \mid a \in X\}$, $\sigma = \pi_0 \cup \pi_1 = \{((i,a),a) \mid i \in \mathsf{I} \text{ and } a \in X\} \in \mathbf{Rel}(\mathbf{S}_\mathsf{I}\, X, X)$ and the two injections are $\iota_i = \{(a,(i,a)) \mid a \in X\} \in \mathbf{Rel}(X, \mathbf{S}_\mathsf{I}\, X)$ for $i = 0,1$.

The monad structure of $\mathbf{S}_\mathsf{I}$ has $\iota_0 = \{(a,(0,a)) \mid a \in X\} \in \mathbf{Rel}(X,\mathbf{S}_\mathsf{I}\, X)$ as unit and

$$\tau = \{((0,0,a),(0,a)) \mid a \in X\} \cup \{((1,0,a),(1,a)) \mid a \in X\} \cup \{((0,1,a),(1,a)) \mid a \in X\} \\ = \{((r_0,r_1,a),(r,a)) \mid a \in X,\ r,r_0,r_1 \in \mathsf{I} \text{ and } r = r_0+r_1\} \in \mathbf{Rel}(\mathbf{S}_\mathsf{I}^2\, X, \mathbf{S}_\mathsf{I}\, X)$$

as multiplication. The standard flip is $\mathsf{c} = \{((r_0,r_1,a),(r_1,r_0,a)) \mid r_0,r_1 \in \mathsf{I} \text{ and } a \in X\} \in \mathbf{Rel}(\mathbf{S}_\mathsf{I}^2\, X, \mathbf{S}_\mathsf{I}^2\, X)$. More generally for $l \in \mathbb{N}$, the cyclic flip of length $l+2$ is $\mathsf{c}(l) = \{((\alpha,a),(\underrightarrow{\alpha},a)) \mid \alpha \in \mathsf{I}^{l+2} \text{ and } a \in X\} \in \mathbf{Rel}(\mathbf{S}_\mathsf{I}^{l+2}\, X, \mathbf{S}_\mathsf{I}^{l+2}\, X)$. We prove this property this by induction on l. For $l = 0$ this is due to the fact that $\mathsf{c}(0) = \mathsf{c}$. Assume that this property holds for l and remember that by definition $\mathsf{c}(l+1) = \mathsf{c}\,(\mathbf{S}_\mathsf{I}\, \mathsf{c}(l))$. We have $\mathbf{S}_\mathsf{I}\, \mathsf{c}(l) = \{(r\alpha,a),(r\,\underrightarrow{\alpha},a)) \mid r \in \mathsf{I},\ a \in X \text{ and } \alpha \in \mathsf{I}^{l+2}\}$ and hence $\mathsf{c}\,(\mathbf{S}_\mathsf{I}\, \mathsf{c}(l)) = \{((r\alpha,a),(\underrightarrow{r\alpha},a) \mid r \in I, \alpha \in \mathsf{I}^{l+1} \text{ and } a \in X\} = \mathsf{c}(l+1)$.

Notice that there is a natural bijection between $!\mathbf{S}_\mathsf{I}\, X$ and $\mathcal{M}_{\mathrm{fin}}(X)\times\mathcal{M}_{\mathrm{fin}}(X)$, mapping the multiset $[(0,a_1),\dots,(0,a_l),(1,b_1),\dots,(1,b_r)]$ to $([a_1,\dots,a_l],[b_1,\dots,b_r])$. The natural transformation $\partial_X \in \mathbf{Rel}(!\mathbf{S}_\mathsf{I}\, X, \mathbf{S}_\mathsf{I}\, !X)$ is defined as the Curry transpose of the following composition of morphisms

$$!(\mathsf{I} \multimap X) \otimes \mathsf{I} \xrightarrow{!(\mathsf{I}\multimap X)\otimes\tilde\partial} !(\mathsf{I} \multimap X) \otimes !\mathsf{I} \xrightarrow{\mu^2_{\mathsf{I}\multimap X,\mathsf{I}}} !((\mathsf{I} \multimap X) \otimes \mathsf{I}) \xrightarrow{!\mathsf{ev}} !X \quad .$$

(see [Ehr21], Theorem 4.8) and hence

$$\partial_X = \{((m,[\,]),(0,m)) \mid m \in \mathcal{M}_{\mathrm{fin}}(X)\} \\ \cup \{((m,[a]),(1,m+[a])) \mid m \in \mathcal{M}_{\mathrm{fin}}(X) \text{ and } a \in X\}\,.$$

It will also be convenient to write equivalently

$$\partial_X = \{(m',(r,m)) \in \mathcal{M}_{\mathrm{fin}}(\mathsf{I} \times X) \times (\mathsf{I} \times \mathcal{M}_{\mathrm{fin}}(X)) \mid \\ m = [a_1,\dots,a_k],\ m' = [(r_1,a_1),\dots,(r_k,a_k)] \text{ and } r = r_1 + \cdots + r_k\}\,.$$

Therefore the functor $\mathsf{D} : \mathbf{Rel}_! \to \mathbf{Rel}_!$, which is defined on objects by $\mathsf{D}\,X = \mathbf{S}_\mathsf{I}\,X$ and on morphisms by $\mathsf{D}\,u = (\mathbf{S}_\mathsf{I}\,u)\ \partial_X \in \mathbf{Rel}(!\mathbf{S}_\mathsf{I}\,X, \mathbf{S}_\mathsf{I}\,Y)$ for $u \in \mathbf{Rel}_!(X,Y)$ satisfies

$$\begin{aligned}\mathsf{D}\,u &= \{((m,[\,]),(0,b)) \mid (m,b) \in u\} \cup \{((m,[a]),(1,b)) \mid (m+[a],b) \in u\} \\ &= \{(m',(r,b)) \mid m' = [(r_1,a_1),\dots,(r_k,a_k))] \in \mathcal{M}_{\mathrm{fin}}(\mathsf{I} \times X), \\ &\qquad ([a_1,\dots,a_k],b) \in u \text{ and } r = r_1 + \cdots + r_k\}\,.\end{aligned}$$

Then the monad structure of $\mathbf{S}_\mathsf{I}$ can be extended to D by $\zeta_X = \{([a],(0,a)) \mid a \in X\} \in \mathbf{Rel}_!(X, \mathsf{D}\,X)$ and

$$\begin{aligned}\theta_X &= \{([(0,0,a)],(0,a)) \mid a \in X\} \\ &\quad \cup \{([(1,0,a)],(1,a)) \mid a \in X\} \cup \{([(0,1,a)],(1,a)) \mid a \in X\} \\ &= \{([(r_0,r_1,a)],(r,a)) \mid a \in X,\ r,r_0,r_1 \in \mathsf{I} \text{ and } r = r_0 + r_1\} \in \mathbf{Rel}_!(\mathsf{D}^2\,X, \mathsf{D}\,X)\,.\end{aligned}$$

The additive strength of that monad $\psi^i_{X_0,\dots,X_n} \in \mathcal{L}_!(X_0 \mathbin{\&} \cdots \mathbin{\&} \mathsf{D}\,X_i \mathbin{\&} \cdots \mathbin{\&} X_n, \mathsf{D}(X_0 \mathbin{\&} \cdots \mathbin{\&} X_n)$ is given by

$$\begin{aligned}\psi^i_{X_0,\dots,X_n} = \{([(j,a)],(0,j,a)) \mid j \in \{0,\dots,n\} \setminus \{i\} \text{ and } a \in X_j\} \\ \cup \{([(i,0,a)],(0,i,a)) \mid a \in X_i\} \\ \cup \{([(i,1,a)],(1,i,a)) \mid a \in X_i\}\,.\end{aligned}$$

Using the notations of Section 4.5 we set $\mathsf{N} = \mathbb{N}$ and then $\overline{\mathsf{zero}} = \{(*,0)\} \in \mathbf{Rel}(1,\mathsf{N})$, $\overline{\mathsf{suc}} = \{(\nu,\nu+1) \mid n \in \mathbb{N}\} \in \mathbf{Rel}(\mathsf{N},\mathsf{N})$ and[12] $\overline{\mathsf{pred}} = \{(0,0)\} \cup \{(\nu+1,\nu) \mid \nu \in \mathbb{N}\}$. The !-coalgebra structure of N is given by $h_\mathsf{N} = \{(\nu,k[\nu]) \mid k,\nu \in \mathbb{N}\} \in \mathbf{Rel}(\mathsf{N},!\mathsf{N})$. Given a set X, the morphism $\overline{\mathsf{let}} \in \mathbf{Rel}(\mathsf{N} \otimes (!\mathsf{N} \multimap X), X)$ is given by $\overline{\mathsf{let}} = \{(\nu,(k[\nu],a),a) \mid \nu,k \in \mathbb{N} \text{ and } a \in X\}$ and the morphism $\overline{\mathsf{if}} \in \mathcal{L}(\mathsf{N} \otimes (X \mathbin{\&} X), X)$ is given by $\overline{\mathsf{if}} = \{(0,(0,a),a) \mid a \in X\} \cup \{(\nu+1,(1,a),a) \mid a \in X\}$.

Concerning fixpoints, observe that $(\mathbf{Rel},\mathbf{S}_\mathsf{I})$ is Scott (in the sense of Section 4.6.1) with $\subseteq$ as associated order relation on morphisms. One checks easily that the $(\mathcal{Y}^X_n \in \mathbf{Rel}_!(X \Rightarrow X, X))_{n\in\mathbb{N}}$ are given by $\mathcal{Y}^X_0 = \emptyset$ and $\mathcal{Y}^X_{n+1} = \{(m_1 + \cdots + m_k + [([a_1,\dots,a_k],a)],a) \mid k \in \mathbb{N} \text{ and } (m_1,a_1),\dots,(m_k,a_k) \in \mathcal{Y}^X_n\}$ so that $\mathcal{Y}^X \in \mathbf{Rel}_!(X \Rightarrow X, X)$ is the least set such that for all $k \in \mathbb{N}$, $(m_1,a_1),\dots,(m_k,a_k) \in \mathcal{Y}^X$ and $a \in X$, one has $(m_1 + \cdots + m_k + [([a_1,\dots,a_k],a)],a) \in \mathcal{Y}^X$.

[12]To avoid confusions with integers denoting indices, we use Greek letters κ,ν to denote numerals.

5.3.1. *Intersection typing system for terms.* The interpretation $[\![A]\!]^{\mathbf{Rel}}$ of a type A in **Rel** is given by $[\![\mathsf{D}^d\iota]\!]^{\mathbf{Rel}} = \mathsf{I}^d \times \mathbb{N}$ and $[\![A \Rightarrow B]\!]^{\mathbf{Rel}} = \mathcal{M}_{\mathrm{fin}}([\![A]\!]^{\mathbf{Rel}}) \times [\![B]\!]^{\mathbf{Rel}}$. We consider the elements of I^d as sequences of length d of elements of $\mathsf{I} = 1 \,\&\, 1 = \{0,1\}$, that is, as sequences of bits of length d. Given $\delta \in \mathsf{I}^d$ and $a \in [\![A]\!]^{\mathbf{Rel}}$ one defines $\delta \cdot a \in [\![\mathsf{D}^d A]\!]^{\mathbf{Rel}}$ by induction on A: if $A = \mathsf{D}^e\iota$ then $a = (\varepsilon, \nu)$ where $\nu \in \mathbb{N}$ and $\varepsilon \in \mathsf{I}^e$ and we set $\delta \cdot a = (\delta\varepsilon, \nu) \in \mathsf{D}^{d+e}\iota$, and if $A = (B \Rightarrow C)$ then $a = (p, c)$ where $p \in \mathcal{M}_{\mathrm{fin}}([\![B]\!]^{\mathbf{Rel}})$ and $c \in [\![C]\!]^{\mathbf{Rel}}$ and we set $\delta \cdot a = (p, \delta \cdot c) \in [\![B \Rightarrow \mathsf{D}^d C]\!]^{\mathbf{Rel}} = [\![\mathsf{D}^d(B \Rightarrow C)]\!]^{\mathbf{Rel}}$. Any type A can be written uniquely $A = \mathsf{D}^d F$ where F is sharp, and then any element $a \in [\![A]\!]^{\mathbf{Rel}}$ can be written uniquely $a = \delta \cdot f$ where $\delta \in \mathsf{I}^d$ and $f \in [\![F]\!]^{\mathbf{Rel}}$.

An intersection typing context is a sequence $\Phi = (x_1 : m_1 : A_1, \dots, x_n : m_n : A_n)$ where the x_i's are pairwise distinct variables, and $m_i \in \mathcal{M}_{\mathrm{fin}}([\![A_i]\!]^{\mathbf{Rel}})$ for each i. Then we use $\underline{\Phi}$ for the underlying typing context defined as $\underline{\Phi} = (x_1 : A_1, \dots, x_n : A_n)$. If $\underline{\Phi} = \underline{\Phi'}$, so that $\Phi = (x_1 : m_1 : A_1, \dots, x_n : m_n : A_n)$ and $\Phi = (x_1 : m'_1 : A_1, \dots, x_n : m'_n : A_n)$, then we set $\Phi + \Phi' = (x_1 : m_1 + m'_1 : A_1, \dots, x_n : m_n + m'_n : A_n)$ and when we use this notation we always implicitly assume that all the intersection typing contexts involved have the same underlying typing context. Given a typing context $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$ we set $0_\Gamma = (x_1 : [\,] : A_1, \dots, x_n : [\,] : A_n)$.

The intersection typing rules are given in Figure 9. We also define an intersection typing system for stacks in Figure 10. An intersection stack typing judgment is an expression $s : f : F \vdash \nu : \iota$ where s is a stack, F is a sharp type, $f \in [\![F]\!]^{\mathbf{Rel}}$ and $\nu \in \mathbb{N}$. The intuition is that s produces ν if it receives f as a linear argument.

The main feature of this system is the following result which is obtained by a simple induction structured exactly as the proof of Theorem 4.33.

Theorem 5.11. *For any term M such that $x_1 : A_1, \dots, x_n : A_n \vdash M : B$ and any $(m_i \in \mathcal{M}_{\mathrm{fin}}([\![A_i]\!]^{\mathbf{Rel}}))_{i=1}^n$ and $b \in [\![B]\!]^{\mathbf{Rel}}$, the two following properties are equivalent.*

- $(m_1, \dots, m_n, b) \in [\![M]\!]^{\mathbf{Rel}}_\Gamma$
- *the judgment $x_1 : m_1 : A_1, \dots, x_n : m_n : A_n \vdash M : b : B$ is provable in the system of Figure 9.*

For any stack s such that $s : F \vdash \iota$ and any $f \in [\![F]\!]^{\mathbf{Rel}}$ one has $(f, n) \in [\![s]\!]^{\mathbf{Rel}}$ iff $s : f : F \vdash n : \iota$. For any well-typed state c and $n \in \mathbb{N}$, one has $\vdash c : n : \iota$ iff $n \in [\![c]\!]^{\mathbf{Rel}}$.

Of course $[\![M]\!]^{\mathbf{Rel}}_\Gamma$ is the interpretation of M in **Rel**, and similarly for stacks and states.

Given a typing context $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$ and $j \in \{1, \dots, n\}$ we define $\partial(j, \Gamma)$ as the set of all triples (Φ', r, Φ) where $r \in \mathsf{I}$, $\underline{\Phi} = \underline{\Phi'} = \Gamma$ and, if we set $\Phi = (x_i : m_i : A_i)_{i=1}^n$ and $\Phi' = (x_i : m'_i : A'_i)_{i=1}^n$ then we have

$$\begin{cases} A'_i = \mathsf{D}A_i \text{ and } (m'_j, (r, m_j)) \in \partial_{[\![A_j]\!]} \\ A'_i = A_i \text{ and } m'_i = m_i & \text{if } i \neq j\,. \end{cases}$$

By Lemma 4.35 we have

$$[\![\partial(x, M)]\!]^{\mathbf{Rel}}_{\Gamma, x:\mathsf{D}A} = \mathsf{D}_1 [\![M]\!]^{\mathbf{Rel}}_{\Gamma, x:A} \in \mathcal{L}_!([\![\Gamma]\!]^{\mathbf{Rel}} \,\&\, \mathsf{D}\,[\![A]\!]^{\mathbf{Rel}}, \mathsf{D}\,[\![B]\!]^{\mathbf{Rel}})\,.$$

If we rephrase this property in the model **Rel**, we get the following.

Theorem 5.12. *Assume that $\Phi \vdash M : b : B$ and $(\Phi', r, \Phi) \in \partial(i, \Gamma)$. Then $\Phi' \vdash \partial(x_i, M) : r \cdot b : \mathsf{D}B$.*

Proof. By Theorem 5.11. □

$$\frac{}{0_\Gamma, x : [a] : A \vdash x : a : A}\ (\textbf{i-var}) \qquad \frac{\Phi, x : m : A \vdash M : b : B}{\Phi \vdash \lambda x^A\, M : (m, b) : A \Rightarrow B}\ (\textbf{i-abs})$$

$$\frac{\Phi^0 \vdash M : ([a_1, \dots, a_n], b) : A \Rightarrow B \qquad (\Phi^j \vdash N : a_j : A)_{j=1}^n}{\sum_{j=0}^n \Phi^j \vdash (M)N : b : B}\ (\textbf{i-app})$$

$$\frac{\Phi^0 \vdash M : ([a_1, \dots, a_n], a) : A \Rightarrow A \qquad (\Phi^j \vdash \mathsf{Y}M : a_j : A)_{j=1}^n}{\sum_{j=0}^n \Phi^j \vdash \mathsf{Y}M : a : A}\ (\textbf{i-fix})$$

$$\frac{\nu \in \mathbb{N}}{0_\Gamma \vdash \underline{\nu} : \nu : \iota}\ (\textbf{i-num}) \qquad \frac{\Phi \vdash M : \delta \cdot \nu : \mathsf{D}^d\iota \qquad \mathsf{len}(\delta) = d}{\Phi \vdash \mathsf{succ}^d(M) : \delta \cdot (\nu + 1) : \mathsf{D}^d\iota}\ (\textbf{i-suc})$$

$$\frac{\Phi \vdash M : \delta \cdot 0 : \mathsf{D}^d\iota \qquad \mathsf{len}(\delta) = d}{\Phi \vdash \mathsf{pred}^d(M) : \delta \cdot 0 : \mathsf{D}^d\iota}\ (\textbf{i-prd}0)$$

$$\frac{\Phi \vdash M : \delta \cdot (\nu + 1) : \mathsf{D}^d\iota \qquad \mathsf{len}(\delta) = d}{\Phi \vdash \mathsf{pred}^d(M) : \delta \cdot \nu : \mathsf{D}^d\iota}\ (\textbf{i-prd}+)$$

$$\frac{\Phi^0 \vdash M : \delta \cdot 0 : \mathsf{D}^d\iota \qquad \Phi^1 \vdash P : a : A \qquad \underline{\Phi^0} \vdash Q : A \qquad \mathsf{len}(\delta) = d}{\Phi^0 + \Phi^1 \vdash \mathsf{if}_A^d(M, P, Q) : \delta \cdot a : \mathsf{D}^d A}\ (\textbf{i-if}0)$$

$$\frac{\Phi^0 \vdash M : \delta \cdot (\nu + 1) : \mathsf{D}^d\iota \qquad \underline{\Phi^0} \vdash P : A \qquad \Phi^1 \vdash Q : a : A \qquad \mathsf{len}(\delta) = d}{\Phi^0 + \Phi^1 \vdash \mathsf{if}_A^d(M, P, Q) : \delta \cdot a : \mathsf{D}^d A}\ (\textbf{i-if}+)$$

$$\frac{\Phi \vdash M : \delta r \cdot a : \mathsf{D}^{d+1}A \qquad \mathsf{len}(\delta) = d}{\Phi \vdash \pi_r^d(M) : \delta \cdot a : \mathsf{D}^d A}\ (\textbf{i-proj})$$

$$\frac{\Phi \vdash M_i : a : A \qquad \underline{\Phi} \vdash M_0 + M_1 : A \qquad i \in \mathsf{I}}{\Phi \vdash M_0 + M_1 : a : A}\ (\textbf{i-add})$$

$$\frac{\Phi \vdash M : \delta \cdot a : \mathsf{D}^d A \qquad \mathsf{len}(\delta) = d \qquad r \in \mathsf{I}}{\Phi \vdash \iota_r^d(M) : \delta r \cdot a : \mathsf{D}^{d+1}A}\ (\textbf{i-inj})$$

$$\frac{\Phi \vdash M : \delta r_0 r_1 \cdot a : \mathsf{D}^{d+2}A \qquad r, r_0, r_1 \in \mathsf{I} \text{ and } r = r_0 + r_1 \qquad \mathsf{len}(\delta) = d}{\Phi \vdash \theta^d(M) : \delta r \cdot a : \mathsf{D}^{d+1}A}\ (\textbf{i-sum})$$

$$\frac{\Phi \vdash M : (m, b) : A \Rightarrow B \qquad r \in \mathsf{I} \text{ and } (m', (r, m)) \in \partial_{[\![A]\!]}}{\Phi \vdash \mathsf{D}M : (m', r \cdot b) : \mathsf{D}A \Rightarrow \mathsf{D}B}\ (\textbf{i-diff})$$

$$\frac{\Phi \vdash M : \delta\alpha \cdot a : \mathsf{D}^{d+l+2}A \qquad \mathsf{len}(\delta) = d,\ \mathsf{len}(\alpha) = l + 2}{\Phi \vdash \mathsf{c}_l^d(M) : \delta\, \underrightarrow{\alpha} \cdot a : \mathsf{D}^{d+l+2}A}\ (\textbf{i-circ})$$

$$\frac{\Phi^0 \vdash M : \delta \cdot \nu : \mathsf{D}^d\iota \qquad \Phi^1, x : k[\nu] : \iota \vdash N : b : B \qquad n, k \in \mathbb{N} \text{ and } \mathsf{len}(\delta) = d}{\Phi^0 + \Phi^1 \vdash \mathsf{let}_B^d(x, M, N) : \delta \cdot b : \mathsf{D}^d B}\ (\textbf{i-let})$$

FIGURE 9. Intersection typing rules for terms

We set $\partial^0(x, M) = M$ and $\partial^{d+1}(x, M) = \partial(x, \partial^d(x, M))$.

$$\frac{}{():\nu:\iota\vdash\nu:\iota}\qquad\frac{s:\kappa+1:\iota\vdash\nu:\iota}{\mathsf{succ}\cdot s:\kappa:\iota\vdash\nu:\iota}$$

$$\frac{s:0:\iota\vdash\nu:\iota}{\mathsf{pred}\cdot s:0:\iota\vdash\nu:\iota}\qquad\frac{s:\kappa:\iota\vdash\nu:\iota}{\mathsf{pred}\cdot s:\kappa+1:\iota\vdash\nu:\iota}$$

$$\frac{s:f:F\vdash\nu:\iota\qquad\vdash M_0:\delta\cdot f:\mathsf{D}^dF\qquad\vdash M_1:\mathsf{D}^dF\qquad\mathsf{len}(\delta)=d}{\mathsf{if}(\delta,M_0,M_1)\cdot s:0:\iota\vdash\nu:\iota}$$

$$\frac{s:f:F\vdash\nu:\iota\qquad\vdash M_0:\mathsf{D}^dF\qquad\vdash M_1:\delta\cdot f:\mathsf{D}^dF\qquad\mathsf{len}(\delta)=d}{\mathsf{if}(\delta,M_0,M_1)\cdot s:\kappa+1:\iota\vdash\nu:\iota}$$

$$\frac{s:f:F\vdash\nu:\iota\qquad x:k[\kappa]:\iota\vdash M:\delta\cdot f:\mathsf{D}^dF\qquad\mathsf{len}(\delta)=d}{\mathsf{let}(\delta,x,M)\cdot s:\kappa:\iota\vdash\nu:\iota}$$

$$\frac{(\vdash M:a_i:A)_{i=1}^k\qquad s:f:F\vdash\nu:\iota}{\mathsf{arg}(M)\cdot s:([a_1,\dots,a_k],f):A\Rightarrow F\vdash\nu:\iota}$$

$$\frac{s:(m',f):\mathsf{D}A\Rightarrow F\vdash\nu:\iota\qquad r\in\mathsf{I}\text{ and }(m',(r,m))\in\partial_{[\![A]\!]}}{\mathsf{D}(r)\cdot s:(m,f):A\Rightarrow F\vdash\nu:\iota}$$

$$\frac{\vdash M:\delta\cdot f:\mathsf{D}^dF\qquad s:f:F\vdash\nu:\iota}{\vdash(\delta,M,s):\nu:\iota}$$

FIGURE 10. Intersection typing rules for stacks and states

5.3.2. *Normalization.* Given $\nu\in\mathbb{N}$ we define $\perp\!\!\!\perp(\nu)$ as the set of all well-typed states c such that $[c]\to^*_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})} C+[(\langle\,\rangle,\underline{\nu},())]$ for some $C\in\mathcal{M}_{\mathrm{fin}}(\underline{\Theta_{\mathsf{cd}}})$.

With any type A and $a\in[\![A]\!]^{\mathbf{Rel}}$, we associate a set $|a|_A$ of terms M such that $\vdash M:A$. If F is a sharp type, $f\in[\![F]\!]^{\mathbf{Rel}}$ and $\nu\in\mathbb{N}$ we also define a set $\|f\vdash\nu\|_F$ of stacks s such that $s:F\vdash\iota$. The definition is by mutual induction on types, and more precisely on the number of "$\Rightarrow$" in types. Remember that $\overline{\delta}$ is the reversed word of δ.

- $|\delta\cdot f|_{\mathsf{D}^dF}=\{M\mid\ \vdash M:\mathsf{D}^dF\text{ and }\forall\nu\in\mathbb{N}\,\forall s\in\|f\vdash\nu\|_F\ (\overline{\delta},M,s)\in\perp\!\!\!\perp(\nu)\}$
- $\|\kappa\vdash\nu\|_\iota=\{s\mid s:\iota\vdash\iota\text{ and }(\langle\,\rangle,\underline{\kappa},s)\in\perp\!\!\!\perp(\nu)\}$
- let $m\in\mathcal{M}_{\mathrm{fin}}([\![A]\!]^{\mathbf{Rel}})$ and $f\in[\![F]\!]^{\mathbf{Rel}}$, then

$$\begin{aligned}\|(m,f)\vdash\nu\|_{A\Rightarrow F}=\{&\mathsf{D}(r_1)\cdot\cdots\mathsf{D}(r_d)\cdot\mathsf{arg}(M)\cdot s\mid r_1,\dots,r_d\in\mathsf{I},\\&\exists(m_i\in\mathcal{M}_{\mathrm{fin}}([\![\mathsf{D}^iA]\!]^{\mathbf{Rel}}))_{i=0}^d\ m_0=m\\&\text{and }((m_i,(r_i,m_{i-1}))\in\partial_{[\![\mathsf{D}^{i-1}A]\!]^{\mathbf{Rel}}})_{i=1}^d,\\&M\in|m_d|_{\mathsf{D}^dA}\text{ and }s\in\|f\vdash\nu\|_F\}.\end{aligned}$$

In this definition, we use the following notation: if $m=[a_1,\dots,a_k]\in\mathcal{M}_{\mathrm{fin}}([\![A]\!]^{\mathbf{Rel}})$ with $a_1,\dots,a_n\in[\![A]\!]^{\mathbf{Rel}}$, then $|m|_A=\bigcap_{j=1}^k|a_j|_A$.

Given an intersection typing context $\Phi=(x_i:m_i:A_i)_{i=1}^k$ we define $|\Phi|$ as the set of all substitutions $\vec{P}=(P_1/x_1,\dots,P_k/x_k)$ where $(P_i\in|m_i|_{A_i})_{i=1}^k$. Given also $b\in[\![B]\!]^{\mathbf{Rel}}$ we define a set $|\Phi\vdash b:B|^{(n)}$ of terms M such that $\underline{\Phi}\vdash M:B$, the definition is by induction on n:

$$|\Phi \vdash b : B|^{(0)} = \{M \mid \underline{\Phi} \vdash M : B \text{ and } \forall \vec{P} \in |\Phi|\ M[\vec{P}] \in |b|_B\}$$

$$\begin{aligned}|\Phi \vdash b : B|^{(n+1)} = \{M \mid \underline{\Phi} \vdash M : B \text{ and } \forall r \in \mathsf{I} \forall i \in \{1, \dots, k\},\\ \forall \Phi'\ (\Phi', (r, \Phi)) \in \partial(i, \underline{\Phi}) \Rightarrow \partial(x_i, M) \in |\Phi' \vdash r \cdot b : \mathsf{D}B|^{(n)}\}\end{aligned}$$

and last we set

$$|\Phi \vdash b : B| = \bigcap_{n \in \mathbb{N}} |\Phi \vdash b : B|^{(n)}.$$

Remark 5.13. Given M, Γ and B such that $\Gamma \vdash M : B$, saying that $M \in |\Phi \vdash b : B|^{(n)}$ for all Φ such that $\underline{\Phi} = \Gamma$ and all $b \in [\![B]\!]^{\mathbf{Rel}}$ intuitively means that M is n times differentiable with respect to its free variables, and $M \in |\Phi \vdash b : B|$ for all Φ and b means that M is infinitely differentiable. This definition bears some similarity with the definition of C^∞ maps in Analysis.

Lemma 5.14. *Let X be a set, let $r \in \mathsf{I}$ and let $(m', (r, m)) \in \partial_X$. Let $m_1, \dots, m_k \in \mathcal{M}_{\mathrm{fin}}(X)$ be such that $m = m_1 + \cdots + m_k$. There are $r_1, \dots, r_k \in \mathsf{I}$ and $m'_1, \dots, m'_k \in \mathcal{M}_{\mathrm{fin}}(X)$ such that $r = r_1 + \cdots + r_k$, $m' = m'_1 + \cdots + m'_k$ and $((m'_i, (r_i, m_i)) \in \partial_X)_{i=1}^k$.*

Proof. Categorically this is a simple consequence of the fact that the morphism $\tilde{\partial} \in \mathbf{Rel}(\mathsf{I}, !\mathsf{I})$ of Section 5.3 is a !-coalgebra structure, and can also be checked directly as we do now. Setting $m = [a_j \mid j \in J]$, the assumption $(m', (r, m)) \in \partial_X$ means that $m' = [(r'_j, a_j) \mid j \in J]$ with $(r'_j \in \mathsf{I})_{j=1}^k$ and $r = \sum_{j \in J} r'_j$. Since $m = m_1 + \cdots + m_k$ we can find pairwise disjoint $(J_i)_{i=1}^k$ with $J = \bigcup_{i=1}^k J_i$ and $m_i = [a_j \mid j \in J_i]$. Then we take $m'_i = [(r'_j, a_j) \mid j \in J_i]$ and $r_i = \sum_{j \in J_i} r'_i$ for $i = 1, \dots, k$. We have $r_i \in \mathsf{I}$ because $\sum_{j \in J} r'_j = r \in \mathsf{I}$. □

Now we prove a series of lemmas (from 5.16 to 5.34) which express that the sets $|\Phi \vdash b : B|^{(n)}$ have some stability properties with respect to the syntactic constructs of the language. They will make the proof of Theorem 5.35 essentially trivial.

Remark 5.15. These lemmas are proven by induction on n (the superscript in $|\Phi \vdash b : B|^{(n)}$) and it is essential to notice that the statement we prove by induction on n is *universally quantified* on Φ, B and b because, when proving the implication for M, we have to apply the inductive hypothesis to $\partial(x, M)$ for all the variables x in the context.

Lemma 5.16. *If $M \in |\Phi \vdash \delta\alpha \cdot b : \mathsf{D}^{d+h+2}B|^{(n)}$ with $d = \mathsf{len}(\delta)$ and $h + 2 = \mathsf{len}(\alpha)$ then we have $\mathsf{c}^d_h(M) \in |\Phi \vdash \delta \underset{\rightarrow}{\alpha} \cdot b : \mathsf{D}^{d+h}B|^{(n)}$.*

Proof. By induction on n. For $n = 0$: let $\vec{P} \in |\Phi|$ and let us set $M' = M[\vec{P}]$. We can write $B = \mathsf{D}^e F$ where F is sharp and $b = \varepsilon \cdot f$ where $\mathsf{len}(\varepsilon) = e$ and $f \in [\![F]\!]^{\mathbf{Rel}}$. Let $\nu \in \mathbb{N}$. Let $s \in \|f \vdash \nu\|_F$, we have

$$\begin{aligned}(\overline{\delta \underset{\rightarrow}{\alpha} \varepsilon}, \mathsf{c}^d_h(M'), s) &= (\overline{\varepsilon} \underset{\leftarrow}{(\overline{\alpha})} \overline{\delta}, \mathsf{c}^d_h(M'), s)\\ &\to_{\Theta_{\mathsf{cd}}} (\overline{\varepsilon} \underset{\overset{\leftarrow}{\rightarrow}}{(\overline{\alpha})} \overline{\delta}, M', s)\\ &= (\overline{\delta\alpha\varepsilon}, M', s) \in \perp\!\!\!\perp(\nu)\end{aligned}$$

since $M \in |\Phi \vdash \delta\alpha\varepsilon \cdot f : \mathsf{D}^{d+h+e+2}F|^{(0)}$.

For the inductive step, we assume that the implication holds for n and we prove it for $n+1$ so assume that $M \in |\Phi \vdash \delta\alpha \cdot b : \mathsf{D}^{d+h+2} B|^{(n+1)}$ and let $r \in \mathsf{I}$, $l \in \{1, \dots, \mathsf{len}(\Phi)\}$ and Φ' be such that $(\Phi', r, \Phi) \in \partial(l, \underline{\Phi})$, we have $\partial(x_l, M) \in |\Phi' \vdash r\delta\alpha \cdot b : \mathsf{D}^{d+h+3} b|^{(n)}$ and hence, by inductive hypothesis, $\partial(x_l, \mathsf{c}^d_h(M)) = \mathsf{c}^{d+1}_h(\partial(x_l, M)) \in |\Phi' \vdash r\delta \underrightarrow{\alpha} \cdot b : \mathsf{D}^{d+h+3} b|^{(n)}$. Since we have this property for all l and r, Φ' such that $(\Phi', r, \Phi) \in \partial(l, \underline{\Phi})$, we have proven that $\mathsf{c}^d_h(M) \in |\Phi \vdash \delta \underrightarrow{\alpha} \cdot b : \mathsf{D}^{d+h+2} B|^{(n+1)}$ as required. □

Lemma 5.17. *If $M \in |\Phi \vdash \delta \cdot b : \mathsf{D}^d B|^{(n)}$ and $r \in \mathsf{I}$ then $\iota^d_r(M) \in |\Phi \vdash \delta r \cdot b : \mathsf{D}^{d+1} B|^{(n)}$ where $d = \mathsf{len}(\delta)$.*

Proof. By induction on n. For $n = 0$: let $\vec{P} \in |\Phi|$ and let us set $M' = M[\vec{P}]$. We can write $B = \mathsf{D}^e F$ where F is sharp and $b = \varepsilon \cdot f$ where $\mathsf{len}(\varepsilon) = e$ and $f \in [\![F]\!]^{\mathbf{Rel}}$. Let $\nu \in \mathbb{N}$. Let $s \in \|f \vdash \nu\|_F$, we have $(\overline{\delta r \varepsilon}, \iota^d_r(M'), s) = (\bar{\varepsilon} r \bar{\delta}, \iota^d_r(M'), s) \to_{\Theta_{\mathsf{cd}}} (\bar{\varepsilon}\bar{\delta}, \iota^d_r(M'), s) = (\overline{\delta\varepsilon}, M', s) \in \perp\!\!\!\perp(\nu)$ by our assumption about M, hence $\iota^d_r(M') \in |\delta r \cdot b|_{\mathsf{D}^{d+1} B}$ as required.

For the inductive step we assume that $M \in |\Phi \vdash \delta \cdot b : \mathsf{D}^d B|^{(n+1)}$. Let $r' \in \mathsf{I}$, $l \in \{1, \dots, \mathsf{len}(\Phi)\}$, and Φ' be such that $(\Phi', r', \Phi) \in \partial(l, \underline{\Phi})$. We have $\partial(x_l, M) \in |\Phi' \vdash r'\delta \cdot b : \mathsf{D}^{d+1} B|^{(n)}$ and hence, by inductive hypothesis, $\partial(x_l, \iota^d_r(M)) = \iota^{d+1}_r(\partial(x_l, M)) \in |\Phi' \vdash r'\delta r \cdot b : \mathsf{D}^{d+2} B|^{(n)}$ since $\mathsf{len}(r'\delta) = d + 1$. □

Lemma 5.18. *If $M \in |\Phi \vdash \delta r_0 r_1 \cdot b : \mathsf{D}^{d+2} B|^{(n)}$ with $d = \mathsf{len}(\delta)$ then $\theta^d(M) \in |\Phi \vdash \delta r \cdot b : \mathsf{D}^{d+2} B|^{(n)}$ if $r = r_0 + r_1 \in \mathsf{I}$.*

Proof. By induction on n. For $n = 0$: let $\vec{P} \in |\Phi|$ and let us set $M' = M[\vec{P}]$. We can write $B = \mathsf{D}^e F$ where F is sharp and $b = \varepsilon \cdot f$ where $\mathsf{len}(\varepsilon) = e$ and $f \in [\![F]\!]^{\mathbf{Rel}}$. Let $\nu \in \mathbb{N}$. Let $s \in \|f \vdash \nu\|_F$. Assume first $r = 0$ so that $r_0 = r_1 = 0$, we have $(\overline{\delta 0 \varepsilon}, \theta^d(M'), s) \to_{\Theta_{\mathsf{cd}}} (\bar{\delta} 00 \varepsilon, M', s) \in \perp\!\!\!\perp(\nu)$ by our assumption about M, hence $\theta^d(M') \in |\delta 0 \cdot b|_{\mathsf{D}^{d+1} B}$ as required. Assume now $r = 1$, we have $(\overline{\delta 1 \varepsilon}, \theta^d(M'), s) \to_{\Theta_{\mathsf{cd}}} (\overline{\delta 01 \varepsilon}, M', s) + (\overline{\delta 10 \varepsilon}, M', s)$ and since one of these summands belongs to $\perp\!\!\!\perp(\nu)$ by our assumption about M, we get $(\overline{\delta 1 \varepsilon}, \theta^d(M'), s) \in \perp\!\!\!\perp(\nu)$.

For the inductive step we assume that $M \in |\Phi \vdash \delta r_0 r_1 \cdot b : \mathsf{D}^{d+2} B|^{(n+1)}$. Let $r' \in \mathsf{I}$, $l \in \{1, \dots, \mathsf{len}(\Phi)\}$, and Φ' be such that $(\Phi', r', \Phi) \in \partial(l, \underline{\Phi})$. We have $\partial(x_l, M) \in |\Phi' \vdash r'\delta r_0 r_1 \cdot b : \mathsf{D}^{d+3} B|^{(n)}$ and hence, by inductive hypothesis, $\partial(x_l, \theta^d(M)) = \theta^{d+1}(\partial(x_l, M)) \in |\Phi' \vdash r'\delta r_0 r_1 \cdot b : \mathsf{D}^{d+3} B|^{(n)}$ since $\mathsf{len}(r'\delta) = d + 1$. □

Lemma 5.19. *For all $n \in \mathbb{N}$, if $r \in \mathsf{I}$ and $M \in |\Phi \vdash \delta r \cdot b : \mathsf{D}^{d+1} B|^{(n)}$ then $\pi^d_r(M) \in |\Phi \vdash \delta \cdot b : \mathsf{D}^d B|^{(n)}$*

Proof. By induction on n. For $n = 0$: let $\vec{P} \in |\Phi|$ and let us set $M' = M[\vec{P}]$. We can write $B = \mathsf{D}^e F$ where F is sharp and $b = \varepsilon \cdot f$ where $\mathsf{len}(\varepsilon) = e$ and $f \in [\![F]\!]^{\mathbf{Rel}}$. Let $\nu \in \mathbb{N}$. Let $s \in \|f \vdash \nu\|_F$, we have $(\overline{\delta\varepsilon}, \pi^d_r(M'), s) \to_{\Theta_{\mathsf{cd}}} (\overline{\delta r \varepsilon}, M', s) \in \perp\!\!\!\perp(\nu)$ by our assumption about M, hence $\pi^d_r(M') \in |\delta \cdot b|_{\mathsf{D}^{d+1} B}$ as required.

For the inductive step we assume that $M \in |\Phi \vdash \delta r \cdot b : \mathsf{D}^{d+1} B|^{(n+1)}$. Let $r' \in \mathsf{I}$, $l \in \{1, \dots, \mathsf{len}(\Phi)\}$, and Φ' be such that $(\Phi', r', \Phi) \in \partial(l, \underline{\Phi})$. We have $\partial(x_l, M) \in |\Phi' \vdash r'\delta r \cdot b : \mathsf{D}^{d+2} B|^{(n)}$ and hence, by inductive hypothesis, $\partial(x_l, \pi^d_r(M)) = \pi^{d+1}_r(\partial(x_l, M)) \in |\Phi' \vdash r'\delta \cdot b : \mathsf{D}^{d+1} B|^{(n)}$ since $\mathsf{len}(r'\delta) = d + 1$. □

Lemma 5.20. *If $M \in |\Phi \vdash (m, b) : A \Rightarrow B|^{(n)}$ and $(m', (r, m)) \in \partial_{[\![A]\!]^{\mathbf{Rel}}}$ then $\mathsf{D}M \in |\Phi \vdash (m', r \cdot b) : \mathsf{D}A \Rightarrow \mathsf{D}B|^{(n)}$.*

Proof. By induction on n. For $n = 0$: let $\vec{P} \in |\Phi|$ and let us set $M' = M[\vec{P}]$. We can write $B = \mathsf{D}^e F$ where F is sharp and $b = \varepsilon \cdot f$ where $\mathsf{len}(\varepsilon) = e$ and $f \in [\![F]\!]^{\mathbf{Rel}}$. Let $\nu \in \mathbb{N}$. Let $s \in \|(m', f) \vdash \nu\|_{\mathsf{D}A \Rightarrow F}$, we have $\mathsf{D}(r) \cdot s \in \|(m, f) \vdash \nu\|_{A \Rightarrow F}$ since $(m', (r, m)) \in \partial_{[\![A]\!]^{\mathbf{Rel}}}$, and hence $(\overline{\varepsilon}, M', \mathsf{D}(r) \cdot s) \in \perp\!\!\!\perp(\nu)$ and therefore $(\overline{r\varepsilon}, \mathsf{D}M', s) \in \perp\!\!\!\perp(\nu)$ since $(\overline{r\varepsilon}, \mathsf{D}M', s) = (\overline{\varepsilon}r, \mathsf{D}M', s) \to_{\Theta_{\mathsf{cd}}} (\overline{\varepsilon}, M', \mathsf{D}(r) \cdot s)$. It follows that $\mathsf{D}M' \in |(m', r\varepsilon \cdot F)|_{\mathsf{D}^{e+1}F}$ as required.

For the inductive step, assume that $M \in |\Phi \vdash (m, b) : A \Rightarrow B|^{(n+1)}$ and let $r' \in \mathsf{I}$, $l \in \{1, \dots, \mathsf{len}(\Phi)\}$ and Φ' be such that $(\Phi', r', \Phi) \in \partial(l, \underline{\Phi})$, we have $\partial(x_l, M) \in |\Phi' \vdash (m, r' \cdot b) : A \Rightarrow \mathsf{D}B|^{(n)}$ and hence, by inductive hypothesis, $\mathsf{D}\partial(x_l, M) \in |\Phi' \vdash (m', rr' \cdot b) : A \Rightarrow \mathsf{D}^2 B|^{(n)}$ since $(m', (r, m)) \in \partial_{[\![A]\!]^{\mathbf{Rel}}}$, and hence $\partial(x_l, \mathsf{D}M) = \mathsf{c}(\mathsf{D}\partial(x_l, M)) \in |\Phi' \vdash (m', r'r \cdot b) : A \Rightarrow \mathsf{D}^2 B|^{(n)}$ by Lemma 5.16. Since we have this property for all l and r', Φ' such that $(\Phi', r', \Phi) \in \partial(l, \underline{\Phi})$, we have proven that $\mathsf{D}M \in |\Phi \vdash (m', r \cdot b)) : \mathsf{D}A \Rightarrow \mathsf{D}B|^{(n+1)}$ as required. □

Lemma 5.21. *Given $\Phi = (x_1 : m_1 : A_1, \dots, x_k : m_k : A_k)$ and $l \in \{1, \dots, k\}$ such that $m_j = [\,]$ if $j \neq l$ and $m_l = [a]$, one has $x_l \in |\Phi \vdash a : A|^{(n)}$.*

Proof. By induction on n. For $n = 0$, let $\vec{P} \in |\Phi|$ so that $P_l \in |a|_{A_l}$ so that $x_l \in |\Phi \vdash a : A_l|^{(0)}$. We prove the property for $n+1$ assuming that it holds for n. So with the notations of the lemma we must prove that $x_l \in |\Phi \vdash a : A_l|^{(n+1)}$. Let $j \in \{1, \dots, k\}$, $r \in \mathsf{I}$ and Φ' be such that $(\Phi', r, \Phi) \in \partial(j, \underline{\Phi})$. We set $\Phi' = (x_p : m'_p : A'_p)_{p=1}^k$, we must prove that $\partial(x_j, x_l) \in |\Phi' \vdash r \cdot a : \mathsf{D}A_l|^{(n)}$. There are two cases.

- If $j \neq l$ then $m_j = [\,]$. We have $m'_j = [\,]$ and $A'_j = \mathsf{D}A_j$, and $m'_p = m_p$ and $A'_p = A_p$ for $p \neq j$. It follows that $r = 0$. We have $x_l \in |\Phi' \vdash a_l : A_l|^{(n)}$ by inductive hypothesis and hence $\partial(x_j, x_l) = \iota_0(x_l) \in |\Phi' \vdash 0 \cdot a : \mathsf{D}A_l|^{(n)}$ by Lemma 5.17, that is $\partial(x_j, x_l) \in |\Phi' \vdash r \cdot a : \mathsf{D}A_i|^{(n)}$ as required.
- Assume now that $j = l$. We have $m'_p = [\,]$ and $A'_p = A_p$ if $p \neq l$, and $m'_l = [r \cdot a_l]$ and $A'_l = \mathsf{D}A_l$. Then by inductive hypothesis we have $x_l \in |\Phi' \vdash r \cdot a_l : \mathsf{D}A_l|^{(n)}$ which is exactly what we need since $\partial(x_j, x_l) = x_l$ in that case. □

Lemma 5.22. *Given $\Phi = (x_1 : [\,] : A_1, \dots, x_k : [\,] : A_k)$ and $\kappa \in \mathbb{N}$ one has $\underline{\kappa} \in |\Phi \vdash \kappa : \iota|^{(n)}$.*

Proof. By induction on n. For $n = 0$ it suffices to observe that $\underline{\kappa} \in |\kappa|_\iota$ which results obviously from the definition of $\|\kappa \vdash \nu\|_\iota$. We prove the property for $n+1$ assuming that it holds for n. So with the notations of the lemma we must prove that $\underline{\kappa} \in |\Phi \vdash a : A_l|^{(n+1)}$. Let $l \in \{1, \dots, k\}$, $r \in \mathsf{I}$ and Φ' be such that $(\Phi', r, \Phi) \in \partial(j, \underline{\Phi})$. Setting $\Phi' = (x_j : m'_j : A'_j)_{j=1}^k$, this means that $m'_j = [\,]$ for each j, that $A'_j = A_j$ for $j \neq l$ and that $A'_l = \mathsf{D}A_l$. This also implies that $r = 0$. By inductive hypothesis we have $\underline{\kappa} \in |\Phi' \vdash \kappa : \iota|^{(n)}$ and hence $\partial(x_l, \underline{\kappa}) = \iota_0(\underline{\kappa}) \in |\Phi' \vdash (0, \kappa) : \mathsf{D}\iota|^{(n)}$ by Lemma 5.17. □

Lemma 5.23. *For all $n \in \mathbb{N}$, if $M \in |\Phi \vdash \delta \cdot \kappa : \mathsf{D}^d \iota|^{(n)}$ then $\mathsf{succ}^d(M) \in |\Phi \vdash \delta \cdot (\kappa + 1) : \mathsf{D}^d \iota|^{(n)}$ where $d = \mathsf{len}(\delta)$.*

Proof. By induction on n. Assume that $n = 0$, let $\vec{P} \in |\Phi|$, $M' = M[\vec{P}]$. Let $\nu \in \mathbb{N}$ and $s \in \|\kappa + 1 \vdash \nu\|_\iota$. We have $(\langle\,\rangle, \underline{\kappa}, \mathsf{succ} \cdot s) \to_{\Theta_{\mathsf{cd}}} (\langle\,\rangle, \underline{\kappa + 1}, s) \in \perp\!\!\!\perp(\nu)$ and hence $\mathsf{succ} \cdot s \in \|\kappa \vdash \nu\|_\iota$. Since $\vec{P} \in |\Phi|$ we have $M' \in |\delta \cdot \kappa|_{\mathsf{D}^d \iota}$ and hence $(\overline{\delta}, \mathsf{succ}^d(M'), s) \to_{\Theta_{\mathsf{cd}}} (\overline{\delta}, M', \mathsf{succ} \cdot s) \in \perp\!\!\!\perp(\nu)$. This shows that $\mathsf{succ}^d(M') \in |\delta \cdot (\kappa + 1)|_{\mathsf{D}^d \iota}$ as required.

For the inductive step, we assume that the implication holds for n and we prove it for $n+1$, so we assume that $M \in |\Phi \vdash \delta \cdot \kappa : \mathsf{D}^d \iota|^{(n+1)}$. Let $r \in \mathsf{I}$, $l \in \{1, \dots, \mathsf{len}(\Phi)\}$ and

Φ' such that $(\Phi', r, \Phi) \in \partial(l, \underline{\Phi})$. We have $\partial(x_l, M) \in |\Phi \vdash r\delta \cdot \kappa : \mathsf{D}^{d+1}\iota|^{(n)}$ and hence, by inductive hypothesis, $\partial(x_l, \mathsf{succ}^d(M)) = \mathsf{succ}^{d+1}(\partial(x_l, M)) \in |\Phi' \vdash r\delta \cdot (\kappa+1) : \mathsf{D}^{d+1}\iota|^{(n)}$ and so we have shown that $\mathsf{succ}^d(M) \in |\Phi \vdash \delta \cdot (\kappa+1) : \mathsf{D}^d\iota|^{(n+1)}$. ☐

Lemma 5.24. *For all $n \in \mathbb{N}$, if $M \in |\Phi \vdash \delta \cdot 0 : \mathsf{D}^d\iota|^{(n)}$ then $\mathsf{pred}^d(M) \in |\Phi \vdash \delta \cdot 0 : \mathsf{D}^{d+1}\iota|^{(n)}$ where $d = \mathsf{len}(\delta)$.*

Proof. Similar to that of Lemma 5.23. ☐

Lemma 5.25. *For all $n \in \mathbb{N}$, if $M \in |\Phi \vdash \delta \cdot (\kappa+1) : \mathsf{D}^d\iota|^{(n)}$ then $\mathsf{pred}^d(M) \in |\Phi \vdash \delta \cdot \kappa : \mathsf{D}^{d+1}\iota|^{(n)}$ where $d = \mathsf{len}(\delta)$.*

Proof. Similar to that of Lemma 5.23. ☐

Lemma 5.26. *For all $n \in \mathbb{N}$, if $M \in |\Phi_0 \vdash \delta \cdot 0 : \mathsf{D}^d\iota|^{(n)}$, $Q_0 \in |\Phi_1 \vdash b : B|^{(n)}$ and $\underline{\Phi} \vdash Q_1 : B$ then $\mathsf{if}^d(M, Q_0, Q_1) \in |\Phi \vdash \delta \cdot b : \mathsf{D}^d B|^{(n)}$, where $d = \mathsf{len}(\delta)$ and $\Phi = \Phi_0 + \Phi_1$.*

Proof. By induction on n. Assume that $n = 0$, let $\vec{P} \in |\Phi|$, $M' = M[\vec{P}]$ and $Q'_i = Q_i[\vec{P}]$ for $i = 0, 1$. We can write $B = \mathsf{D}^e F$ where F is sharp and $b = \varepsilon \cdot f$ where $f \in [\![F]\!]^{\mathbf{Rel}}$ and $e = \mathsf{len}(\varepsilon)$. Let $\nu \in \mathbb{N}$ and $s \in \|f \vdash \nu\|_F$. We have $(\langle\,\rangle, \underline{0}, \mathsf{if}(\overline{\varepsilon}, Q'_0, Q'_1) \cdot s) \rightarrow_{\Theta_{\mathsf{cd}}} (\overline{\varepsilon}, Q'_0, s)$. Since $\vec{P} \in |\Phi| \subseteq |\Phi_1|$ we have $Q'_0 \in |\varepsilon \cdot f|_{\mathsf{D}^e F}$ and hence $(\overline{\varepsilon}, Q'_0, s) \in \perp\!\!\!\perp(\nu)$ and therefore $\mathsf{if}(\overline{\varepsilon}, Q'_0, Q'_1) \cdot s \in \|0 \vdash \nu\|_\iota$. Since $\vec{P} \in |\Phi| \subseteq |\Phi_0|$ we have $M' \in |\delta \cdot 0|_{\mathsf{D}^d\iota}$ and hence $(\overline{\delta\varepsilon}, \mathsf{if}^d(M', Q'_0, Q'_1), s) = (\overline{\varepsilon}\overline{\delta}, \mathsf{if}^d(M', Q'_0, Q'_1), s) \rightarrow_{\Theta_{\mathsf{cd}}} (\overline{\delta}, M', \mathsf{if}(\overline{\varepsilon}, Q'_0, Q'_1) \cdot s) \in \perp\!\!\!\perp(\nu)$. This shows that $\mathsf{if}^d(M', Q'_0, Q'_1) \in |\delta\varepsilon \cdot f|_{\mathsf{D}^{d+e}F}$ as required.

For the inductive step, we assume that the implication holds for n and we prove it for $n+1$, so we assume that $M \in |\Phi_0 \vdash \delta \cdot 0 : \mathsf{D}^d\iota|^{(n+1)}$, $Q_0 \in |\Phi_1 \vdash b : B|^{(n+1)}$ and $\underline{\Phi} \vdash Q_1 : B$. Let $r \in \mathsf{I}$, $l \in \{1, \dots, \mathsf{len}(\Phi)\}$ and Φ' such that $(\Phi', r, \Phi) \in \partial(l, \underline{\Phi})$. By Lemma 5.14, since $\Phi = \Phi_0 + \Phi_1$, we can find $r_0, r_1 \in \mathsf{I}$ such that $r = r_0 + r_1$ as well as Φ'_0, Φ'_1 such that $(\Phi'_i, r_i, \Phi_i) \in \partial(l, \underline{\Phi})$ for $i = 0, 1$, and $\Phi' = \Phi'_0 + \Phi'_1$. If follows that $\partial(x_l, M) \in |\Phi'_0 \vdash r_0\delta \cdot 0 : \mathsf{D}^{d+1}\iota|^{(n)}$ and $\partial(x_l, Q_0) \in |\Phi'_1 \vdash r_1 \cdot b : \mathsf{D}B|^{(n)}$. Since $\underline{\Phi'} \vdash \partial(x_l, Q_1) : \mathsf{D}B$, we have $\mathsf{if}^{d+1}(\partial(x_l, M), \partial(x_l, Q_0), \partial(x_l, Q_1)) \in |\Phi' \vdash r_0\delta r_1 \cdot b : \mathsf{D}^{d+2}B|^{(n)}$ by inductive hypothesis. By Lemma 5.16 we have $\mathsf{c}_d(\mathsf{if}^{d+1}(\partial(x_l, M), \partial(x_l, Q_0), \partial(x_l, Q_1))) \in |\Phi' \vdash \overrightarrow{r_0\delta r_1} \cdot b : \mathsf{D}^{d+2}B|^{(n)}$, that is $\mathsf{c}_d(\mathsf{if}^{d+1}(\partial(x_l, M), \partial(x_l, Q_0), \partial(x_l, Q_1))) \in |\Phi' \vdash r_1 r_0 \delta \cdot b : \mathsf{D}^{d+2}B|^{(n)}$. Therefore

$$\partial(x_l, \mathsf{if}^d(M, Q_0, Q_1)) = \theta(\mathsf{c}_d(\mathsf{if}^{d+1}(\partial(x_l, M), \partial(x_l, Q_0), \partial(x_l, Q_1)))) \in |\Phi' \vdash r\delta \cdot b : \mathsf{D}^{d+2}B|^{(n)}$$

by Lemma 5.18, since $r = r_0 + r_1$. ☐

Remark 5.27. In some sense this proof motivates syntactically the introduction of the cyclic permutation combinator $\mathsf{c}_d(\_)$ in the definition of $\partial(x, \mathsf{if}^d(M, Q_0, Q_1))$ in Figure 3: it allows the $\theta(\_)$ combinator to act at the right level. We have already motivated it denotationally by Theorem 4.37.

Lemma 5.28. *For all $n \in \mathbb{N}$, if $M \in |\Phi_0 \vdash \delta \cdot (\kappa+1) : \mathsf{D}^d\iota|^{(n)}$, $Q_1 \in |\Phi_1 \vdash b : B|^{(n)}$ for some $\kappa \in \mathbb{N}$ and $\underline{\Phi} \vdash Q_0 : B$ then $\mathsf{if}^d(M, Q_0, Q_1) \in |\Phi \vdash \delta \cdot b : \mathsf{D}^d B|^{(n)}$, where $d = \mathsf{len}(\delta)$ and $\Phi = \Phi_0 + \Phi_1$.*

Proof. Similar to that of Lemma 5.26. ☐

Lemma 5.29. *For all $n \in \mathbb{N}$, if $M \in |\Phi_0 \vdash \delta \cdot \kappa : \mathsf{D}^d\iota|^{(n)}$ and $N \in |\Phi_1, x : k[\kappa] \vdash b : B|^{(n)}$ where $k, \kappa \in \mathbb{N}$, then $\mathsf{let}^d(x, M, N) \in |\Phi \vdash \delta \cdot b : \mathsf{D}^d B|^{(n)}$ where $d = \mathsf{len}(\delta)$ and $\Phi = \Phi_0 + \Phi_1$.*

Proof. By induction on n. Assume that $n=0$, let $\vec{P}\in|\Phi|$, $M'=M[\vec{P}]$ and $N'=N[\vec{P}]$. We can write $B=\mathsf{D}^eF$ where F is sharp and $b=\varepsilon\cdot f$ where $f\in[\![F]\!]^{\mathbf{Rel}}$ and $e=\mathsf{len}(\varepsilon)$. Let $\nu\in\mathbb{N}$ and $s\in\|f\vdash\nu\|_F$. We have $(\langle\rangle,\underline{\kappa},\mathsf{let}(\overline{\varepsilon},x,N')\cdot s)\to_{\Theta_{\mathsf{cd}}}(\overline{\varepsilon},N'\left[\underline{\kappa}/x\right],s)$. Since $\vec{P}\in|\Phi|\subseteq|\Phi_0|$ and $\underline{\kappa}\in|k[\kappa]|_\iota$ we have $N'\left[\underline{\kappa}/x\right]\in|\varepsilon\cdot f|_{\mathsf{D}^eF}$ and hence $(\overline{\varepsilon},N'\left[\underline{\kappa}/x\right],s)\in\perp\!\!\!\perp(\nu)$ and therefore $\mathsf{let}(\overline{\varepsilon},x,N')\cdot s\in\|\kappa\vdash\nu\|_\iota$. Since $\vec{P}\in|\Phi|\subseteq|\Phi_1|$ we have $M'\in|\delta\cdot\kappa|_{\mathsf{D}^d\iota}$ and hence $(\overline{\delta\varepsilon},\mathsf{let}^d(x,M',N'),s)=(\overline{\varepsilon}\overline{\delta},\mathsf{let}^d(x,M',N'),s)\to_{\Theta_{\mathsf{cd}}}(\overline{\delta},M',\mathsf{let}(\overline{\varepsilon},x,N')\cdot s)\in\perp\!\!\!\perp(\nu)$. This shows that $\mathsf{let}^d(x,M',N')\in|\delta\varepsilon\cdot f|_{\mathsf{D}^{d+e}F}$ as required.

For the inductive step, we assume that the implication holds for n and we prove it for $n+1$, so we assume that $M\in|\Phi_0\vdash\delta\cdot\kappa:\mathsf{D}^d\iota|^{(n+1)}$, and $N\in|\Phi_1,x:k[\kappa]\vdash b:B|^{(n+1)}$. Let $r\in\mathsf{I}$, $l\in\{1,\ldots,\mathsf{len}(\Phi)\}$ and Φ' be such that $(\Phi',r,\Phi)\in\partial(l,\underline{\Phi})$. By Lemma 5.14, since $\Phi=\Phi_0+\Phi_1$, we can find $r_0,r_1\in\mathsf{I}$ such that $r=r_0+r_1$ as well as Φ'_0,Φ'_1 such that $(\Phi'_0,r_0,\Phi_0)\in\partial(l,\underline{\Phi})$ so that $((\Phi'_0,x:k[\kappa]:\iota),r_0,(\Phi_0,x:k[\kappa]:\iota))\in\partial(l,(\underline{\Phi},x:\iota))$, and $(\Phi'_1,r_1,\Phi_1)\in\partial(l,\underline{\Phi})$ and $\Phi'=\Phi'_0+\Phi'_1$. If follows that $\partial(x_l,M)\in|\Phi'_0\vdash r_0\delta\cdot\kappa:\mathsf{D}^{d+1}\iota|^{(n)}$ and $\partial(x_l,N)\in|\Phi'_1,x:k[\kappa]:\iota\vdash r_1\cdot b:\mathsf{D}B|^{(n)}$. We have $\mathsf{let}^{d+1}(x,\partial(x_l,M),\partial(x_l,N))\in|\Phi'\vdash r_0\delta r_1\cdot b:\mathsf{D}^{d+2}B|^{(n)}$ by inductive hypothesis. By Lemma 5.16 we have $\mathsf{c}_d(\mathsf{let}^{d+1}(x,\partial(x_l,M),\partial(x_l,N)))\in|\Phi'\vdash\overrightarrow{r_0\delta r_1}\cdot b:\mathsf{D}^{d+2}B|^{(n)}$, that is $\mathsf{c}_d(\mathsf{let}^{d+1}(x,\partial(x_l,M),\partial(x_l,N)))\in|\Phi'\vdash r_1r_0\delta\cdot b:\mathsf{D}^{d+2}B|^{(n)}$. Therefore

$$\partial(x_l,\mathsf{let}^d(x,M,N))=\theta(\mathsf{c}_d(\mathsf{let}^{d+1}(x,\partial(x_l,M),\partial(x_l,N))))\in|\Phi'\vdash r\delta\cdot b:\mathsf{D}^{d+2}B|^{(n)}$$

by Lemma 5.18, since $r=r_0+r_1$. □

Lemma 5.30. *If $M\in|\Phi_0\vdash(m,b):A\Rightarrow B|^{(n)}$ with $m=[a_1,\ldots,a_k]$ and $(N\in|\Phi_j\vdash a_j:A|^{(n)})_{j=1}^k$ then $(M)N\in|\Phi\vdash b:B|^{(n)}$ where $\Phi=\sum_{j=0}^k\Phi_j$ (so that $\forall j\ \underline{\Phi_j}=\underline{\Phi}$).*

Proof. By induction on n. Assume that $n=0$. So let $\vec{P}\in|\Phi|$. Since for each $j=0,\ldots,k$ we have $|\Phi|\subseteq|\Phi_j|$ by the assumption that $\Phi=\sum_{j=0}^k\Phi_j$, we have $M'\in|(m,b)|_{A\Rightarrow B}$ (setting $M'=M[\vec{P}]$, remember that we use this convention systematically) and $N'\in\bigcap_{j=1}^k|a_j|_A$. We can write uniquely $B=\mathsf{D}^eF$ with F sharp and $b=\varepsilon\cdot f$ with $\varepsilon\in\mathsf{I}^e$ and $f\in[\![F]\!]^{\mathbf{Rel}}$. Let $\nu\in\mathbb{N}$ and $s\in\|f\vdash\nu\|_F$, we have $(\overline{\varepsilon},(M')N',s)\to_{\Theta_{\mathsf{cd}}}(\overline{\varepsilon},M',\mathsf{arg}(N')\cdot s)\in\perp\!\!\!\perp(\nu)$ since $\mathsf{arg}(N')\cdot s\in\|f\vdash\nu\|_F$.

Now assume that the implication holds for n and let us prove it for $n+1$ so assume that $M\in|\Phi_0\vdash(m,b):A\Rightarrow B|^{(n+1)}$ and $(N\in|\Phi_j\vdash a_j:A|^{(n+1)})_{j=1}^k$. Let $r\in\mathsf{I}$ and let Φ' be such that $(\Phi',r,\Phi)\in\partial(l,\underline{\Phi})$ for some $1\leq l\leq\mathsf{len}(\Phi)$. By Lemma 5.14 we can find $(r_i\in\mathsf{I})_{i=0}^k$ such that $r=\sum_{i=0}^k r_i$ as well as $(\Phi'_i)_{i=0}^k$ such that $\Phi'=\sum_{i=0}\Phi'_i$ and $((\Phi'_i,r_i,\Phi_i)\in\partial(l,\underline{\Phi}))_{i=0}^k$. So by our assumptions we have $\partial(x_l,M)\in|\Phi'_0\vdash(m,r_0\cdot b):A\Rightarrow\mathsf{D}B|^{(n)}$ and $\partial(x_l,N)\in\bigcap_{i=1}^k|\Phi'_i\vdash r_j\cdot a_j:\mathsf{D}A|^{(n)}$. Let $r'=\sum_{i=1}^k r_i\in\mathsf{I}$ and $m'=[r_1\cdot a_1,\ldots,r_k\cdot a_k]$. By Lemma 5.20 we have $\mathsf{D}\partial(x_l,M)\in|\Phi'_0\vdash(m',r'r_0\cdot b):\mathsf{D}A\Rightarrow\mathsf{D}^2B|^{(n)}$ and hence by Lemma 5.18 we have $\theta(\mathsf{D}\partial(x_l,M))\in|\Phi'_0\vdash(m',(r'+r_0)\cdot b):\mathsf{D}A\Rightarrow\mathsf{D}B|^{(n)}$. By inductive hypothesis we get $\partial(x_l,(M)N)=(\theta(\mathsf{D}\partial(x_l,M)))\partial(x_l,N)\in|\Phi'\vdash r\cdot b:\mathsf{D}B|^{(n)}$ since $r=r'+r_0$. Since we have proven this for all choices of l, r and Φ', our contention follows. □

Lemma 5.31. *For any $n\in\mathbb{N}$, if $(M)\mathsf{Y}M\in|\Phi\vdash b:B|^{(n)}$ then $\mathsf{Y}M\in|\Phi\vdash b:B|^{(n)}$.*

Proof. By induction on n. For $n=0$, let $\vec{P}\in|\Phi|$ and let $M'=M[\vec{P}]$. We write $B=\mathsf{D}^eF$ where F is sharp and $b=\varepsilon\cdot f$ where $f\in[\![F]\!]^{\mathbf{Rel}}$. Let $\nu\in\mathbb{N}$ and $s\in\|f\vdash\nu\|_F$, we have $(\overline{\varepsilon},\mathsf{Y}M',s)\to_{\Theta_{\mathsf{cd}}}(\overline{\varepsilon},M',\mathsf{arg}(\mathsf{Y}M')\cdot s)\in\perp\!\!\!\perp(\nu)$ since we have $(\overline{\varepsilon},(M')\mathsf{Y}M',s)\to_{\Theta_{\mathsf{cd}}}$

$(\overline{\varepsilon}, M', \mathsf{arg}(\mathsf{Y}M') \cdot s)$ and we have $(M')\mathsf{Y}M' \in |b|_B$ by our assumption. So we have shown that $\mathsf{Y}M \in |\Phi \vdash b : B|^{(0)}$.

Assume that the implication holds for n and let us prove it for $n+1$. So we assume that $(M)\mathsf{Y}M \in |\Phi \vdash b : B|^{(n+1)}$. Let $l \in \{1, \dots, \mathsf{len}(\Phi)\}$, $r \in \mathsf{I}$ and Φ' be such that $(\Phi', r, \Phi) \in \partial(l, \underline{\Phi})$. We know that $\partial(x_l, (M)\mathsf{Y}M) \in |\Phi' \vdash r \cdot b : \mathsf{D}B|^{(n)}$. On the other hand $\partial(x_l, (M)\mathsf{Y}M) = (\theta(\mathsf{D}\partial(x_l, M)))\partial(x_l, \mathsf{Y}M)$, see Figure 3. On the same figure we see that $\partial(x_l, \mathsf{Y}M) = \mathsf{Y}(\theta(\mathsf{D}\partial(x_l, M)))$. So we have $(\theta(\mathsf{D}\partial(x_l, M)))\mathsf{Y}(\theta(\mathsf{D}\partial(x_l, M))) \in |\Phi' \vdash r \cdot b : \mathsf{D}B|^{(n)}$ and hence $\mathsf{Y}(\theta(\mathsf{D}\partial(x_l, M))) \in |\Phi' \vdash r \cdot b : \mathsf{D}B|^{(n)}$ by inductive hypothesis. □

Remark 5.32. The next lemma has a different structure: the hypothesis in the implication we prove by induction is stronger. This feature is used only in the base case where it is absolutely crucial.

Lemma 5.33. *For any $n \in \mathbb{N}$, if $M \in \bigcap_{h \geq n} |\Phi, x : m : A \vdash b : B|^{(h)}$ then $\lambda x^A\, M \in |\Phi \vdash (m, b) : A \Rightarrow B|^{(n)}$.*

Proof. We prove the statement by induction on n. For $n = 0$ our assumption is $\forall h \in \mathbb{N}\ M \in |\Phi, x : m : A \vdash b : B|^{(h)}$ and we prove $\lambda x^A\, M \in |\Phi \vdash (m, b) : A \Rightarrow B|^{(0)}$. So let $\vec{P} \in |\Phi|$ and let $M' = M[\vec{P}]$. We can write uniquely $B = \mathsf{D}^e F$ with F sharp and $b = \varepsilon \cdot f$ with $f \in [\![F]\!]^{\mathbf{Rel}}$. Let $\nu \in \mathbb{N}$ and $s \in \|(m, f) \vdash \nu\|_{A \Rightarrow F}$. Then, by the typing rules for stacks, we have $s = \mathsf{D}(r_1) \cdot \dots \cdot \mathsf{D}(r_d) \cdot \mathsf{arg}(P) \cdot t$ where $d \in \mathbb{N}$, $r_1, \dots, r_d \in \mathsf{I}$ and there are $m_1 \in \mathcal{M}_{\mathrm{fin}}([\![\mathsf{D}A]\!]^{\mathbf{Rel}}), \dots, m_d \in \mathcal{M}_{\mathrm{fin}}([\![\mathsf{D}^d A]\!]^{\mathbf{Rel}})$ such that $((m_i, (r_i, m_{i-1})) \in \partial_{[\![\mathsf{D}^{i-1}A]\!]^{\mathbf{Rel}}})_{i=1}^d$, where we set $m_0 = m$. And last $P \in |m_d|_{\mathsf{D}^d A}$ and $t \in \|f \vdash \nu\|_F$. By our assumption[13] about M we have in particular $M \in |\Phi, x : m : A \vdash b : B|^{(d)}$ and hence $\partial^d(x, M) \in |\Phi, x : m_d : \mathsf{D}^d A \vdash r_d \cdots r_1 \cdot b : \mathsf{D}^d B|^{(0)}$ so that $\partial^d(x, M')\,[P/x] = \partial^d(x, M)[\vec{P}, P/x] \in |r_d \cdots r_1 \cdot b|_{\mathsf{D}^d B}$. So we have

$$\begin{aligned}(\overline{\varepsilon}, \lambda x^A\, M', s) &\to_{\Theta_{\mathsf{cd}}} (\overline{\varepsilon} r_1, \lambda x^{\mathsf{D}A}\, \partial(x, M'), \mathsf{D}(r_2) \cdot \dots \cdot \mathsf{D}(r_d) \cdot \mathsf{arg}(P) \cdot t) \\ &\to_{\Theta_{\mathsf{cd}}} \cdots \to_{\Theta_{\mathsf{cd}}} (\overline{\varepsilon} r_1 \cdots r_d, \lambda x^{\mathsf{D}^d A}\, \partial^d(x, M'), \mathsf{arg}(P) \cdot t) \\ &\to_{\Theta_{\mathsf{cd}}} (\overline{r_d \cdots r_1 \varepsilon}, \partial^d(x, M')\,[P/x]\,, t) \in \perp\!\!\!\perp(\nu)\end{aligned}$$

which entails that $(\overline{\varepsilon}, \lambda x^A\, M', s) \in \perp\!\!\!\perp(\nu)$ and hence $\lambda x^A\, M' \in |(m, b)|_{A \Rightarrow B}$ as required.

For the inductive step we assume that the implication holds for n and prove it for $n+1$. Remember that in this implication, M, Φ, m, A, b and B are universally quantified. So we assume that $\forall h \geq n+1\ M \in |\Phi, x : m : A \vdash b : B|^{(h)}$ and prove that $\lambda x^A\, M \in |\Phi \vdash (m, b) : A \Rightarrow B|^{(n+1)}$. We set $\Phi = (x_1 : m_1 : A_1, \dots, x_k : m_k : A_k)$. Let $r \in \mathsf{I}$, $l \in \{1, \dots, k\}$ and Φ' be such that $(\Phi', r, \Phi) \in \partial(l, \underline{\Phi})$. We have $((\Phi', x : m : A), r, (\Phi, x : m : A)) \in \partial(l, (\underline{\Phi}, x : A))$ and hence $\forall h \geq n\ \partial(x_l, M) \in |\Phi', x : A \vdash (m, r \cdot b) : A \Rightarrow \mathsf{D}B|^{(h)}$ by our assumption about M. It follows by inductive hypothesis that $\partial(x_l, \lambda x^A\, M) = \lambda x^A\, \partial(x_l, M) \in |\Phi' \vdash (m, r \cdot b) : A \Rightarrow \mathsf{D}B|^{(n)}$. Therefore $\lambda x^A\, M \in |\Phi \vdash (m, b) : A \Rightarrow B|^{(n+1)}$ as required. □

Lemma 5.34. *If $M \in |\Phi, x : m : A \vdash b : B|$ then $\lambda x^A\, M \in |\Phi \vdash (m, b) : A \Rightarrow B|$.*

Proof. Apply Lemma 5.33. □

Theorem 5.35. *If $\Phi \vdash M : b : B$ then $M \in |\Phi \vdash b : B|$.*

Proof. By induction on the derivation of $\Phi \vdash M : b : B$. We write $\Phi = (x_1 : m_1 : A_1, \dots, x_k : m_k : A_k)$.

[13] It is here that the special form of the hypothesis is crucial.

▶ Assume that $M = x_l$ for some $l \in \{1, \dots, k\}$ so that the derivation consists of a rule (**i-var**), $B = A_l$ and $m_l = [b]$ and we have $m_j = [\,]$ for $j \neq l$. So we can apply Lemma 5.21 which gives us $x_l \in |\Phi \vdash b : B|^{(n)}$ for all $n \in \mathbb{N}$.

▶ Assume that $M = \underline{\kappa}$ so that the derivation consists of a rule (**i-num**) and hence $B = \iota$ and $b = \kappa$, and we have $m_j = [\,]$ for each j. By Lemma 5.21 we get $M \in |\Phi \vdash \underline{\kappa} : \iota|^{(n)}$ for each $n \in \mathbb{N}$.

▶ Assume that $M = \lambda x^A\, N$ so that $B = (A \Rightarrow C)$ and $b = (m, c)$ and we have $\Phi, x : m : A \vdash M : c : C$ and hence $M \in |\Phi, x : m : A \vdash c : C|$ by inductive hypothesis from which we get $\lambda x^A\, M \in |\Phi \vdash (m, c) : A \Rightarrow C|$ by Lemma 5.34.

▶ Assume that $M = (N)Q$ with $\Phi_0 \vdash N : (m, b) : A \Rightarrow B$ with $m = [a_1, \dots, a_n]$ and $(\Phi_j \vdash Q : a_j : A)_{j=1}^n$, and $\Phi = \sum_{j=0}^n \Phi_j$. By inductive hypothesis we have $N \in |\Phi_0 \vdash (m, b) : A \Rightarrow B|$ and $(Q \in |\Phi_j \vdash a_j : A|)_{j=1}^n$ so we get $(N)Q \in |\Phi \vdash b : B|$ by Lemma 5.30.

▶ Assume that $M = \mathsf{D}N$ with $B = (\mathsf{D}A \Rightarrow \mathsf{D}C)$, $b = (m', r \cdot c)$, $(m', (r, m)) \in \partial_{[\![A]\!]^{\mathbf{Rel}}}$, $\Phi \vdash N : (m, c) : A \Rightarrow C$. By inductive hypothesis we have $N \in |\Phi \vdash (m, c) : A \Rightarrow C|$ and hence $M \in |\Phi \vdash (m', r \cdot c) : \mathsf{D}A \Rightarrow \mathsf{D}C|$ by Lemma 5.20.

▶ Assume that $M = \mathsf{Y}N$ with $\Phi_0 \vdash N : (m, b) : B \Rightarrow B$ with $m = [b_1, \dots, b_n]$, $(\Phi_j \vdash \mathsf{Y}N : b_j : B)_{j=1}^n$ and $\Phi = \sum_{j=0}^n \Phi_j$. By inductive hypothesis we get $N \in |\Phi_0 \vdash (m, b) : B \Rightarrow B|$ and $(\mathsf{Y}N \in |\Phi_j \vdash b_j : B|)_{j=1}^n$ and hence $(N)\mathsf{Y}N \in |\Phi \vdash b : B|$ by Lemma 5.30. It follows that $M = \mathsf{Y}N \in |\Phi \vdash b : B|$ by Lemma 5.31.

▶ Assume that $M = \mathsf{succ}^d(N)$ with $\Phi \vdash N : \delta \cdot \kappa : \mathsf{D}^d\iota$ where $\kappa \in \mathbb{N}$ and $d = \mathsf{len}(\delta)$. By inductive hypothesis we have $N \in |\Phi \vdash \delta \cdot \kappa : \mathsf{D}^d\iota|$ and hence $M \in |\Phi \vdash \delta \cdot \kappa + 1 : \mathsf{D}^d\iota|$ by Lemma 5.23.

▶ The cases where $M = \mathsf{pred}^d(N)$ are similar, using Lemmas 5.24 and 5.25.

▶ Assume that $M = \mathsf{if}^d(N, Q_0, Q_1)$ and that $\Phi_0 \vdash N : \delta \cdot 0 : \mathsf{D}^d\iota$ (with $d = \mathsf{len}(\delta)$), $B = \mathsf{D}^dC$, $b = \delta \cdot c$ with $\Phi_1 \vdash Q_0 : c : C$, $\underline{\Phi} \vdash Q_1 : C$ and $\Phi = \Phi_0 + \Phi_1$. By inductive hypothesis we have $N \in |\Phi_0 \vdash \delta \cdot 0 : \mathsf{D}^d\iota|$ and $Q_0 \in |\Phi_1 \vdash c : C|$ and hence $M \in |\Phi \vdash \delta \cdot c : \mathsf{D}^dC|$ by Lemma 5.26.

▶ The case $M = \mathsf{if}^d(N, Q_0, Q_1)$, $\Phi_0 \vdash N : \delta \cdot (\kappa + 1) : \mathsf{D}^d\iota$ (with $\kappa \in \mathbb{N}$ and $d = \mathsf{len}(\delta)$), $B = \mathsf{D}^dC$, $b = \delta \cdot c$ with $\Phi_1 \vdash Q_1 : c : C$, $\underline{\Phi} \vdash Q_0 : C$ and $\Phi = \Phi_0 + \Phi_1$ is similar to the previous one, using Lemma 5.28.

▶ Assume that $M = \mathsf{let}^d(x, N, Q)$ with $\Phi_0 \vdash N : \delta \cdot \kappa : \mathsf{D}^d\iota$ and $\Phi_1, x : k[\kappa] : \iota \vdash Q : b : B$, $d = \mathsf{len}(\delta)$ and $\Phi = \Phi_0 + \Phi_1$. By inductive hypothesis we have $N \in |\Phi_0 \vdash \delta \cdot \kappa : \mathsf{D}^d\iota|$ and $Q \in |\Phi_1, x : k[\kappa] : \iota \vdash b : B|$ and hence by Lemma 5.29 we have $\mathsf{let}^d(x, N, Q) \in |\Phi \vdash \delta \cdot b : \mathsf{D}^dB|$.

▶ Assume that $M = \iota_r^d(N)$ with $\Phi \vdash N : \delta \cdot c : \mathsf{D}^dC$ so that $B = \mathsf{D}^{d+1}C$ and $b = \delta r \cdot c$ (of course $d = \mathsf{len}(\delta)$). By inductive hypothesis we have $N \in |\Phi \vdash \delta \cdot c : \mathsf{D}^dC|$ and hence $M \in |\Phi \vdash \delta r \cdot c : \mathsf{D}^{d+1}C|$ by Lemma 5.17.

▶ Assume that $M = \theta^d(N)$ with $\Phi \vdash N : \delta r_0 r_1 \cdot c : \mathsf{D}^{d+2}C$ so that $B = \mathsf{D}^{d+1}C$ and $b = \delta r \cdot c$ with $r = r_0 + r_1 \in \mathsf{I}$ (of course $d = \mathsf{len}(\delta)$). By inductive hypothesis we have $N \in |\Phi \vdash \delta r_0 r_1 \cdot c : \mathsf{D}^{d+2}C|$ and hence $M \in |\Phi \vdash \delta r \cdot c : \mathsf{D}^{d+1}C|$ by Lemma 5.18.

▶ Assume that $M = \pi_r^d(N)$ with $\Phi \vdash N : \delta r \cdot c : \mathsf{D}^{d+1}C$ so that $B = \mathsf{D}^dC$ and $b = \delta \cdot c$ (of course $d = \mathsf{len}(\delta)$). By inductive hypothesis we have $N \in |\Phi \vdash \delta r \cdot c : \mathsf{D}^{d+1}C|$ and hence $M \in |\Phi \vdash \delta \cdot c : \mathsf{D}^dC|$ by Lemma 5.19.

▶ Assume that $M = \mathsf{c}_l^d(N)$ with $\Phi \vdash N : \delta\alpha \cdot c : \mathsf{D}^{d+l+2}C$ so that $B = \mathsf{D}^{d+l+2}C$ and $b = \delta \underset{\rightarrow}{\alpha} \cdot c$ (of course $\mathsf{len}(\delta) = d$ and $\mathsf{len}(\alpha) = l + 2$). By inductive hypothesis we have $N \in |\Phi \vdash \delta\alpha \cdot c : \mathsf{D}^{d+l+2}C|$ and hence $M \in |\Phi \vdash \delta \underset{\rightarrow}{\alpha} \cdot c : \mathsf{D}^{d+l+2}C|$ by Lemma 5.16. □

Theorem 5.36. *Let M be a closed term and let $\nu \in \mathbb{N}$ be such that $\vdash M : \nu : \iota$. Then $[(\langle\,\rangle, M, ())] \to^*_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})} C = C_0 + [(\langle\,\rangle, \underline{\nu}, ())]$ for some multiset of well typed states C_0 such that C is $\mathcal{L}$-summable in any model $\mathcal{L}$.*

6. Determinism and probabilities

Our goal is to refine Theorem 5.36 by showing that none of the elements of C_0 reduces to a value. This will be also the opportunity to present the model which motivated this whole investigation, the model of probabilistic coherence spaces (PCS [DE11]), and explain why it is a canonical model of coherent differentiation.

6.1. Probabilistic coherence spaces as a model of LL.

Given an at most countable set A and $u, u' \in \overline{\mathbb{R}_{\geq 0}}^A$, we set $\langle u, u' \rangle = \sum_{a \in A} u_a u'_a \in \overline{\mathbb{R}_{\geq 0}}$ where $\overline{\mathbb{R}_{\geq 0}}$ is the completed half real line. Given $P \subseteq \overline{\mathbb{R}_{\geq 0}}^A$, we define $P^\perp \subseteq \overline{\mathbb{R}_{\geq 0}}^A$ as

$$P^\perp = \{u' \in \overline{\mathbb{R}_{\geq 0}}^A \mid \forall u \in P\ \langle u, u' \rangle \leq 1\}\,.$$

Observe that if P satisfies $\forall a \in A\, \exists x \in P\ x_a > 0$ and $\forall a \in A\, \exists m \in \mathbb{R}_{\geq 0} \forall x \in P\ x_a \leq m$ then $P^\perp \in (\mathbb{R}_{\geq 0})^I$ and $P^\perp$ satisfies the same two properties that we call *local boundedness*.

A probabilistic pre-coherence space (pre-PCS) is a pair $X = (|X|, \mathsf{P}X)$ where $|X|$ is an at most countable set[14] and $\mathsf{P}X \subseteq \overline{\mathbb{R}_{\geq 0}}^{|X|}$ satisfies $\mathsf{P}X^{\perp\perp} = \mathsf{P}X$. A probabilistic coherence space (PCS) is a pre-PCS X such that $\forall a \in |X|\, \exists x \in \mathsf{P}X\ x_a > 0$ and $\forall a \in |X|\, \exists m \in \mathbb{R}_{\geq 0} \forall x \in \mathsf{P}X\ x_a \leq m$ or equivalently

$$\forall a \in |X| \quad 0 < \sup_{x \in \mathsf{P}X} x_a < \infty$$

so that $\mathsf{P}X \subseteq (\mathbb{R}_{\geq 0})^{|X|}$ and is locally bounded.

Given a PCS X and $x \in \mathsf{P}X$ we set $\|x\|_X = \sup_{x' \in \mathsf{P}X^\perp} \langle x, x' \rangle \in [0, 1]$. This operation obeys the usual properties of a norm: $\|x\| = 0 \Rightarrow x = 0$, $\|x_0 + x_1\| \leq \|x_0\| + \|x_1\|$ and $\|\lambda x\| = \lambda \|x\|$ for all $\lambda \in [0, 1]$.

Remark 6.1. Given $x \in \mathsf{P}X$ and $a \in |X|$ we use the notations x_a or $x(a)$ for the corresponding element of $\mathbb{R}_{\geq 0}$, depending on the context. In some situations x_i can denote an element of $\mathsf{P}X$ and in such a situation we will prefer the notation $x_i(a)$ to denote the a-component of x_i to avoid the ugly x_{ia}.

Given $t \in \overline{\mathbb{R}_{\geq 0}}^{A \times B}$ considered as a matrix (where A and B are at most countable sets) and $u \in \overline{\mathbb{R}_{\geq 0}}^A$, we define $t \cdot u \in \overline{\mathbb{R}_{\geq 0}}^B$ by $(t \cdot u)_b = \sum_{a \in A} t_{a,b} u_a$ (usual formula for applying a matrix to a vector), and if $s \in \overline{\mathbb{R}_{\geq 0}}^{B \times C}$ we define the product $s\,t \in \overline{\mathbb{R}_{\geq 0}}^{A \times C}$ of the matrix s and t as usual by $(s\,t)_{a,c} = \sum_{b \in B} t_{a,b} s_{b,c}$. This is an associative operation.

[14] This restriction is not technically necessary, but very meaningful from a philosophic point of view; the non countable case should be handled via measurable spaces and then one has to consider more general objects as in [EPT18b] for instance.

Let X and Y be PCSs, a morphism from X to Y is a matrix $t \in (\mathbb{R}_{\geq 0})^{|X| \times |Y|}$ such that $\forall x \in \mathsf{P}X\ t \cdot x \in \mathsf{P}Y$. It is clear that the identity (diagonal) matrix is a morphism from X to X and that the matrix product of two morphisms is a morphism and therefore, PCSs equipped with this notion of morphism form a category **Pcoh**.

The condition $t \in \mathbf{Pcoh}(X, Y)$ is equivalent to $\forall x \in \mathsf{P}X\, \forall y' \in \mathsf{P}Y^\perp\ \langle t \cdot x, y' \rangle \leq 1$ and observe that $\langle t \cdot x, y' \rangle = \langle t, x \otimes y' \rangle$ where $(x \otimes y')_{(a,b)} = x_a y'_b$. We define $X \multimap Y = (|X| \times |Y|, \{t \in (\mathbb{R}_{\geq 0})^{|X \multimap Y|} \mid \forall x \in \mathsf{P}X\ t \cdot x \in \mathsf{P}Y\})$: this is a pre-PCS by this observation, and checking that it is indeed a PCS is easy.

We define then $X \otimes Y = \left(X \multimap Y^\perp\right)^\perp$; this is a PCS which satisfies $\mathsf{P}(X \otimes Z) = \{x \otimes z \mid x \in \mathsf{P}X \text{ and } z \in \mathsf{P}Z\}^{\perp\perp}$ where $(x \otimes z)_{(a,c)} = x_a z_c$. Then it is easy to see that we have equipped in that way the category **Pcoh** with a symmetric monoidal structure for which it is $*$-autonomous wrt. the dualizing object $\perp = 1 = (\{*\}, [0,1])$ which is also the unit of $\otimes$. The $*$-autonomy follows easily from the observation that $(X \multimap \perp) \simeq X^\perp$.

Lemma 6.2. *Given $s, t \in \mathbf{Pcoh}(X_1 \otimes \cdots \otimes X_k, Y)$, if for all $(x_i \in \mathsf{P}X_i)_{i=1}^k$ one has $s \cdot (x_1 \otimes \cdots \otimes x_k) = t \cdot (x_1 \otimes \cdots \otimes x_k)$ then $s = t$.*

The category **Pcoh** is cartesian: if $(X_j)_{j \in J}$ is an at most countable family of PCSs, then $(\&_{j \in J} X_j, (\mathsf{pr}_j)_{j \in J})$ is the cartesian product of the X_j's, with $|\&_{j \in J} X_j| = \cup_{j \in J}\{j\} \times |X_j|$, $(\mathsf{pr}_j)_{(k,a),a'} = 1$ if $j = k$ and $a = a'$ and $(\mathsf{pr}_j)_{(k,a),a'} = 0$ otherwise, and $x \in \mathsf{P}(\&_{j \in J} X_j)$ if $\mathsf{pr}_j \cdot x \in \mathsf{P}X_j$ for each $j \in J$ (for $x \in (\mathbb{R}_{\geq 0})^{|\&_{j \in J} X_j|}$). Given $(t_j \in \mathbf{Pcoh}(Y, X_j))_{j \in J}$, the unique morphism $t = \langle t_j \rangle_{j \in J} \in \mathbf{Pcoh}(Y, \&_{j \in J} X_j)$ such that $\mathsf{pr}_j\, t = t_j$ is simply defined by $t_{b,(j,a)} = (t_j)_{a,b}$. The dual operation $\oplus_{j \in J} X_j$, which is a coproduct, is characterized by $|\oplus_{j \in J} X_j| = \cup_{j \in J}\{j\} \times |X_j|$ and $x \in \mathsf{P}(\oplus_{j \in J} X_j)$ if $x \in \mathsf{P}(\&_{j \in J} X_j)$ and $\sum_{j \in J} \|\mathsf{pr}_j \cdot x\|_{X_j} \leq 1$.

A particular case is $\mathsf{N} = \oplus_{\nu \in \mathbb{N}} X_\nu$ where $X_\nu = 1$ for each $\nu \in \mathbb{N}$. So that $|\mathsf{N}| = \mathbb{N}$ and $x \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ belongs to PN if $\sum_{\nu \in \mathbb{N}} x_\nu \leq 1$ (that is, x is a sub-probability distribution on $\mathbb{N}$). For each $\nu \in \mathbb{N}$ we have $\mathsf{e}_\nu \in \mathsf{PN}$ which is the distribution concentrated on the integer ν. There are successor and predecessor morphisms $\overline{\mathsf{suc}}, \overline{\mathsf{pred}} \in \mathbf{Pcoh}(\mathsf{N}, \mathsf{N})$ given by $\overline{\mathsf{suc}}_{\nu,\nu'} = \boldsymbol{\delta}_{\nu+1,\nu'}$ and $\overline{\mathsf{pred}}_{\nu,\nu'} = 1$ if $\nu = \nu' = 0$ or $\nu = \nu' + 1$ (and $\overline{\mathsf{pred}}_{\nu,\nu'} = 0$ in all other cases). An element of $\mathbf{Pcoh}(\mathsf{N}, \mathsf{N})$ is a (sub)stochastic matrix and the very idea of this model is to represent programs as transformations of this kind, and their generalizations.

As to the exponentials, one sets $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and $\mathsf{P}(!X) = \{x^! \mid x \in \mathsf{P}X\}^{\perp\perp}$ where, given $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$, $x^!_m = x^m = \prod_{a \in |X|} x_a^{m(a)}$. A morphism $t \in \mathbf{Pcoh}(!X, Y) = \mathsf{P}(!X \multimap Y)$ is completely characterized by the associated function

$$\begin{aligned} \widehat{t} : \mathsf{P}X &\to \mathsf{P}Y \\ x &\mapsto t \cdot x^! = \sum_{m \in |!X|, b \in |Y|} t_{m,b} x^m\, \mathsf{e}_b \end{aligned}$$

Lemma 6.3. *Let $t \in (\mathbb{R}_{\geq 0})^{|!X_1 \otimes \cdots \otimes !X_k \multimap Y|}$. One has $t \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, Y)$ iff for all $(x_i \in \mathsf{P}X_i)_{i=1}^k$ one has $t \cdot (x_1^! \otimes \cdots \otimes x_k^!) \in \mathsf{P}Y$.*

Lemma 6.4. *If $s, t \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, Y)$ satisfy $s \cdot (x_1^! \otimes \cdots \otimes x_k^!) = t \cdot (x_1^! \otimes \cdots \otimes x_k^!)$ for all $(x_i \in \mathsf{P}X_i)_{i=1}^k$ then $s = t$.*

This very useful property uses crucially the local boundedness property of PCSs.

Then given $t \in \mathbf{Pcoh}(X, Y)$, we explain now how to define $!t \in \mathbf{Pcoh}(!X, !Y)$. Let $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$ and $p \in \mathcal{M}_{\mathrm{fin}}(|Y|)$. We use $\mathsf{L}(m, p)$ for the set of all $r \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y|)$

such that

$$\forall a \in |X| \; m(a) = \sum_{b \in |Y|} r(a,b) \quad \text{and} \quad \forall b \in |Y| \; p(b) = \sum_{a \in |X|} r(a,b)\,.$$

Notice that if $r \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y|)$ then $\#r = \#m = \#p$ so that $\mathsf{L}(m,p)$ is non-empty iff $\#m = \#p$. When $r \in \mathsf{L}(n,p)$ we set

$$\begin{bmatrix} p \\ r \end{bmatrix} = \prod_{b \in |Y|} \frac{p(b)!}{\prod_{a \in |X|} r(a,b)!}$$

which belongs to $\mathbb{N} \setminus \{0\}$; this is a generalized multinomial coefficient. Then we have

$$(!t)_{m,p} = \sum_{r \in \mathsf{L}(m,p)} \begin{bmatrix} p \\ r \end{bmatrix} t^r$$

where we recall that $t^r = \prod_{(a,b) \in |X| \times |Y|} t_{a,b}^{r(a,b)}$. The main feature of this definition is that for all $x \in \mathsf{P}X$ one has $\widehat{!t}(x) = !t \cdot x^! = (t \cdot x)^!$. This property fully characterizes $!t$. The comonad structure is given by $\mathsf{der}_X \in (\mathbb{R}_{\geq 0})^{|!X \multimap X|}$ given by $(\mathsf{der}_X)_{m,a} = \boldsymbol{\delta}_{m,[a]}$ so that $\forall x \in \mathsf{P}X \; \mathsf{der}_X \cdot x^! = x \in \mathsf{P}X$ and therefore $\mathsf{der}_X \in \mathsf{P}(!X, X)$. Similarly one defines $\mathsf{dig}_X \in (\mathbb{R}_{\geq 0})^{|!X \multimap !!X|}$ so that $\forall x \in \mathsf{P}X \; \mathsf{dig}_X \cdot x^! = x^{!!}$ and hence, again, $\mathsf{dig}_X \in \mathsf{P}(!X, !!X)$. The equations required to prove that $(!\_, \mathsf{der}, \mathsf{dig})$ is indeed a comonad are proven using Lemma 6.4. For instance, let $t \in \mathsf{P}(X,X)$, we have $(\mathsf{dig}_Y \; !t) \cdot x^! = \mathsf{dig}_Y \cdot (t \cdot x^!) = \mathsf{dig}_X \cdot (t \cdot x)^! = (t \cdot x)^{!!}$ and $(!!t \; \mathsf{dig}_X) \cdot x^! = !!t \cdot (\mathsf{dig}_X \cdot x^!) = !!t \cdot x^{!!} = (!t \cdot x^!)^! = (t \cdot x)^{!!}$ which shows that dig is a natural transformation. As another example, we have $(\mathsf{dig}_{!X} \; \mathsf{dig}_X) \cdot x^! = \mathsf{dig}_{!X} \cdot x^{!!} = x^{!!!}$ and $(!\mathsf{dig}_X \; \mathsf{dig}_X) \cdot x^! = !\mathsf{dig}_X \cdot x^{!!} = (\mathsf{dig}_X \cdot x^!)^! = (x^{!!})^! = x^{!!!}$ and hence $\mathsf{dig}_{!X} \; \mathsf{dig}_X = !\mathsf{dig}_X \; \mathsf{dig}_X$ which is one of the required comonad commutations. The others are proven similarly.

The monoidality Seely isomorphisms $\mathsf{m}^0 \in \mathbf{Pcoh}(1, !\top)$ and $\mathsf{m}^2_{X_1,X_2} \in \mathbf{Pcoh}(!X_1 \otimes !X_2, !(X_1 \mathbin{\&} X_2))$ are given by $\mathsf{m}^0_{*,[]} = 1$ and $\mathsf{m}^2_{((m_1,m_2),m)} = \boldsymbol{\delta}_{1*m_1+2*m_2,m}$ where, for a multiset $m = [a_1, \dots, a_k]$ we set $i * m = [(i,a_1), \dots, (i,a_k)]$, see Section 2.1. It is obvious that m^0 is an iso. To check that $\mathsf{m}^2_{X_1,x_2}$ is a morphism we use Lemma 6.3: let $x_i \in \mathsf{P}X_i$ for $i = 1,2$, one has $\mathsf{m}^2_{X_1,X_2} \cdot (x_1^! \otimes x_2^!) = \langle x_1, x_2 \rangle^! \in \mathsf{P}!(X_1 \mathbin{\&} X_2)$. Conversely, defining $s \in (\mathbb{R}_{\geq 0})^{!(X_1 \& X_2) \multimap (!X_1 \otimes !X_2)}$ by $s_{m,(m_1,m_2)} = \boldsymbol{\delta}_{1*m_1+2*m_2,m}$ we have $s \cdot \langle x_1, x_2 \rangle^! = x_1^! \otimes x_2^! \in \mathsf{P}(!X_1 \otimes !X_2)$ for all $x_i \in \mathsf{P}X_i$ $(i = 1,2)$, and hence $s \in \mathbf{Pcoh}(!(X_1 \mathbin{\&} X_2), (!X_1 \otimes !X_2))$. It is obvious that s is the inverse of $\mathsf{m}^2_{X_1,X_2}$ which is therefore an iso in $\mathbf{Pcoh}$. Proving that it is natural and that it satisfies all the required commutations for turning $\mathbf{Pcoh}$ into a model of LL is routine (using crucially Lemma 6.4).

The induced lax monoidality $\mu^k \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, !(X_1 \otimes \cdots \otimes X_k))$ is such that $(\mu^k)_{(m_1,\dots,m_k),m} = 1$ if $m = [(a_1^1, \dots, a_k^1), \dots, (a_1^n, \dots, a_k^n))]$ and $(m_i = [a_i^1, \dots, a_i^n])_{i=1}^k$, and $(\mu^k)_{(m_1,\dots,m_k),m} = 0$ otherwise.

Theorem 6.5. *The SMC* $\mathbf{Pcoh}$ *is a Lafont category.*

This means that the above defined exponential functor $!\_$ is the free exponential, that is, for each object X, $!X$ is the free commutative comonoid generated by X. See [Ehr21] for a precise definition.

Proof. See [CEPT17]. □

6.2. **Canonical differential structure of Pcoh.** The category **Pcoh** has 0-morphisms (we have the 0 matrix in $\mathbf{Pcoh}(X,Y)$ for any two objects X and Y).

The object $\mathsf{I} = 1 \mathbin{\&} 1$ can be described as $|\mathsf{I}| = \{0,1\}$ and $\mathsf{PI} = [0,1]^2$. Then the morphisms $(\overline{\pi}^{\&}_r \in \mathbf{Pcoh}(1,\mathsf{I}))_{r=0,1}$ are characterized by $\overline{\pi}^{\&}_0 \cdot u = (u,0)$ and $\overline{\pi}^{\&}_1 \cdot u = (0,u)$ for $u \in \mathsf{P}1 = [0,1]$. These two morphisms are jointly epic because, for any $t \in \mathbf{Pcoh}(\mathsf{I},X)$ and $(u_0,u_1) \in \mathsf{PI}$ one has $t \cdot (u_0,u_1) = t \cdot (u_0,0) + t \cdot (0,u_1)$ by linearity of t.

Given a PCS X, the PCS $\mathbf{S}_\mathsf{I}\, X = (\mathsf{I} \multimap X)$ is characterized by $|\mathbf{S}_\mathsf{I}\, X| = \{0,1\} \times |X|$ and $\mathsf{P}\mathbf{S}_\mathsf{I}\, X = \{(x_0,x_1) \in \mathsf{P}X^2 \mid x_0 + x_1 \in \mathsf{P}X\}$ (to be more precise, an element $x \in (\mathbb{R}_{\geq 0})^{|\mathbf{S}_\mathsf{I}\, X|}$ belongs to $\mathsf{P}\mathbf{S}_\mathsf{I}\, X$ if $x_0 + x_1 \in \mathsf{P}X$, where $x_r \in \mathsf{P}X$ is given by $x_r(a) = x(r,a)$. We refer to Remark 6.1 for the notation. Given $x \in \mathsf{P}\mathbf{S}_\mathsf{I}\, X$, the morphism $x\,\Delta^{\&} \in \mathbf{Pcoh}(1,X)$, considered as an element of $\mathsf{P}X$, is simply x_0+x_1. So the natural transformations $\pi_0, \pi_1, \sigma \in \mathbf{Pcoh}(\mathbf{S}_\mathsf{I}\, X, X)$ are characterized by $\pi_r \cdot x = x_r$ and $\sigma \cdot x = x_0 + x_1$.

Therefore two morphisms $s_0, s_1 \in \mathbf{Pcoh}(X,Y)$ are summable iff $\forall x \in \mathsf{P}X$ $s_0 \cdot x + s_1 \cdot x \in \mathsf{P}Y$ which is equivalent to $s_0 + s_1 \in \mathbf{Pcoh}(X,Y)$ since $s_0 \cdot x + s_1 \cdot = (s_0 + s_1) \cdot x$. Then the witness of summability is $\langle s_0, s_1 \rangle_\mathbf{S} \in \mathbf{Pcoh}(X, \mathbf{S}_\mathsf{I}\, Y)$ characterized by $\langle s_0, s_1 \rangle_\mathbf{S} \cdot x = (s_0 \cdot x, s_1 \cdot x)$. Let $s_{00}, s_{01}, s_{10}, s_{11} \in \mathbf{Pcoh}(X,Y)$ be morphisms such that (s_{00}, s_{01}) and (s_{10}, s_{11}) are summable, and moreover $(s_{00}+s_{01}, s_{10}+s_{11})$ is summable. Then the witnesses $\langle s_{00}, s_{01} \rangle_\mathbf{S}, \langle s_{10}, s_{11} \rangle_\mathbf{S} \in \mathbf{Pcoh}(X, \mathbf{S}\, X)$ are summable because $\langle s_{00}, s_{01} \rangle_\mathbf{S} + \langle s_{10}, s_{11} \rangle_\mathbf{S} = \langle s_{00} + s_{10}, s_{01} + s_{11} \rangle_\mathbf{S}$ as easily checked. So (**S-witness**) holds (see [Ehr21]) which shows that **Pcoh** is a canonical summable category.

As explained in [Ehr21] Section 4.1, I is equipped with a commutative comonoid structure given by the two **Pcoh** morphisms $\mathsf{pr}_0 \in \mathbf{Pcoh}(\mathsf{I},1)$ and $\widetilde{\mathsf{L}} \in \mathbf{Pcoh}(\mathsf{I}, \mathsf{I} \otimes \mathsf{I})$, which are given by $(\mathsf{pr}_0)_{r,*} = \boldsymbol{\delta}_{r,0}$ and $\widetilde{\mathsf{L}}_{r,(r_0,r_1)} = \boldsymbol{\delta}_{r,r_0+r_1}$ for $r, r_0, r_1 \in \mathsf{I}$. Therefore, by Theorem 6.5, I has an induced !-coalgebra structure $\tilde{\partial} \in \mathbf{Pcoh}(\mathsf{I}, !\mathsf{I})$, which is given by

$$\tilde{\partial}_{r,[r_1,\dots,r_k]} = \boldsymbol{\delta}_{r,\sum_{i=1}^k r_k}\,,$$

in other words $\tilde{\partial}_{0,[r_1,\dots,r_k]}$ is equal to 1 if all the r_i's are $= 0$ and to 0 otherwise. And $\tilde{\partial}_{1,[r_1,\dots,r_k]}$ is equal to 1 if exactly one among the r_i's is equal to 1 an all the others are equal to 0, and to 0 otherwise.

By Theorem 4.9 of [Ehr21] we know that $\tilde{\partial}$ (denoted δ in that paper) defines a coherent differential structure on **Pcoh**. Let us describe explicitly the associated natural $\partial_X \in \mathbf{Pcoh}(!\mathsf{D}\, X, \mathsf{D}\, !X)$. We know that $\partial_X = \mathsf{cur}\, d$ where $d \in \mathbf{Pcoh}(!(\mathsf{I} \multimap X) \otimes \mathsf{I}, !X)$ is defined as the following composition of morphisms in **Pcoh**:

$$!(\mathsf{I} \multimap X) \otimes \mathsf{I} \xrightarrow{!(\mathsf{I} \multimap X) \otimes \tilde{\partial}} !(\mathsf{I} \multimap X) \otimes !\mathsf{I} \xrightarrow{\mu^2_{\mathsf{I} \multimap X, \mathsf{I}}} !((\mathsf{I} \multimap X) \otimes \mathsf{I}) \xrightarrow{!\mathsf{ev}} !X \ .$$

Let $d' = \mu^2_{\mathsf{I} \multimap X, \mathsf{I}}\,(!(\mathsf{I} \multimap X) \otimes \tilde{\partial}) \in \mathbf{Pcoh}(!(\mathsf{I} \multimap X) \otimes \mathsf{I}, !((\mathsf{I} \multimap X) \otimes \mathsf{I}))$, we have

$$d'_{([(r_1,a_1),\dots,(r_n,a_n))],r),m} = \begin{cases} 1 & \text{if } m = [((r_1,a_1),r'_1),\dots,((r_n,a_n),r'_n)] \text{ and } r = r'_1 + \cdots + r'_n \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, to have $(!\mathsf{ev})_{[((r_1,a_1),r'_1),\dots,((r_n,a_n),r'_n)],p} \neq 0$ we need $(r_i = r'_i)_{i=1}^n$ and $p = [a_1,\dots,a_n]$. So to have $d_{([(r_1,a_1),\dots,(r_n,a_n))],r),p} \neq 0$ we need $r = r_1 + \cdots + r_n$ and $p = [a_1,\dots,a_n]$, and then

$$d_{([(r_1,a_1),\dots,(r_n,a_n))],r),p} = (!\mathsf{ev})_{[((r_1,a_1),r_1),\dots,((r_n,a_n),r_n)],[a_1,\dots,a_n]}$$

Notice that $\mathsf{L}([((r_1,a_1),r_1),\dots,((r_n,a_n),r_n)],[a_1,\dots,a_n])$ contains exactly one element h such that $\mathsf{ev}^h \neq 0$, namely the multiset $h = [(((r_1,a_1),r_1),a_1),\dots,(((r_n,a_n),r_n),a_n)]$, and of course $\mathsf{ev}^h = 1$. If $r = 0$ we have $r_1 = \dots = r_n = 0$ and hence

$$d_{([(r_1,a_1),\dots,(r_n,a_n))],r),p} = \begin{bmatrix} p \\ h \end{bmatrix} = 1\,.$$

If $r = 1$, there is exactly one $i \in \{1,\dots,n\}$ such that $r_i = 1$, and we have $r_j = 0$ for $j \neq i$. Then we have

$$d_{([(r_1,a_1),\dots,(r_n,a_n))],r),p} = \begin{bmatrix} p \\ h \end{bmatrix} = p(a_i)\,.$$

To summarize

$$(\partial_X)_{m',(r,m)} = \begin{cases} 1 & \text{if } r = 0 \text{ and } m' = 0 * m \\ m(a) & \text{if } r = 1,\ a \in \mathsf{supp}(m) \text{ and } m' = 0 * (m - [a]) + [(1,a)] \\ 0 & \text{otherwise.} \end{cases}$$

Let $t \in \mathbf{Pcoh}_!(X,Y) = \mathsf{P}(!X \multimap Y)$. Then $\mathsf{D}\,t \in \mathbf{Pcoh}_!(\mathsf{D}\,X, \mathsf{D}\,Y)$ is defined as $(\mathbf{S}\,t)\ \partial_X$, so we have

$$(\mathsf{D}\,t)_{m',(r,b)} = \begin{cases} t_{m,b} & \text{if } r = 0 \text{ and } m' = 0 * m \\ (m(a)+1)t_{m+[a],b} & \text{if } r = 1 \text{ and } m' = 0 * m + [(1,a)] \\ 0 & \text{otherwise.} \end{cases}$$

Notice that in the above trichotomy the multiset m is completely determined by the condition on m': in the first case $m' = [(0,a_1),\dots,(0,a_n]$ and then $m = [a_1,\dots,a_n]$. In the second case $m' = [(r_1,a_1),\dots,(r_n,a_n)]$ and there is exactly one index i such that $r_i = 1$, and we have $r_j = 0$ for $j \neq i$. Then we have $m = [a_1,\dots,a_{i-1},a_{i+1},\dots,a_n]$ and $a = a_i$.

Let $t \in \mathbf{Pcoh}_!(X,Y)$, to describe the function $\widehat{\mathsf{D}\,t} : \mathsf{P}(\mathsf{D}\,X) \to \mathsf{P}(\mathsf{D}\,Y)$, remember first that $\mathsf{P}(\mathsf{D}\,X)$ can be identified with the set of all pairs $(x,u) \in \mathsf{P}X^2$ such that $x + u \in \mathsf{P}X$. With this identification, the element $(x,u)^!$ of $\mathsf{P}(!\mathbf{S}_{\mathsf{l}}\,X)$ is given by

$$(x,u)^!_{m'} = \prod_{a\in|X|} x_a^{m'(0,a)} \prod_{a\in|X|} u_a^{m'(1,a)}\,.$$

If $m' = 0 * m$ then $(x,u)^!_{m'} = x^m$ and if $m' = 0 * m + [(1,a)]$ then $(x,u)^!_{m'} = x^m u_a$. It follows that

$$\widehat{\mathsf{D}\,t}(x,u) = \left(\widehat{t}(x), \sum_{m\in\mathcal{M}_{\mathrm{fin}}(X),a\in|X|,b\in|Y|} (m(a)+1)t_{m+[a],b}x^m u_a\,\mathsf{e}_b\right)$$

and notice that the second component of this tuple is nothing but the u-linear component of the powerseries $\widehat{t}(x+u)$ (see [Ehr19]). So, as expected, if we set $f = \widehat{t}$ then $\widehat{\mathsf{D}\,t}(x,u) = (f(x), f'(x)\cdot u)$ in the ordinary sense of mathematical differentiation.

▶ **Example 6.1.** There is a morphism $t \in \mathbf{Pcoh}_!(1,1)$ such that, identifying $\mathsf{P}1$ with $[0,1]$, one has $\widehat{t}(x) = 1 - \sqrt{1-x} = \sum_{n\in\mathbb{N}} t_n x^n = f(x)$ for a sequence $(t_n)_{n\in\mathbb{N}}$ of non-negative real numbers that we could write explicitly. Then $\widehat{\mathsf{D}\,t}(x,u) = (\widehat{t}(x), \sum_{n\in\mathbb{N}}(n+1)t_{n+1}x^n u) = (f(x), f'(x)u)$. In this case it is interesting to notice that $f'(x) = \frac{1}{2\sqrt{1-x}}$ is not defined for $x = 1$ but that $f'(x)u$ is defined even for $x = 1$ (and takes value 0) because of the constraint that $x + u = 1$. And indeed we know that $\mathsf{D}\,t \in \mathbf{Pcoh}_!(\mathsf{D}\,1, \mathsf{D}\,1)$. The function f is entirely defined by the equation $f(x) = \frac{1}{2}x + \frac{1}{2}f(x)^2$ and by the fact that the corresponding series must have only non-negative coefficients. It is easy to write in a probabilistic version of PCF with a unit type a recursive program which is interpreted as t. ◀

6.3. **Integers and fixpoints.** The category **Pcoh** is Scott in the sense of Section 4.6.1. The order relation $\leq$ on morphisms in $\mathbf{Pcoh}(X,Y)$ is given by $s \leq t$ iff $\forall (a,b) \in |X \multimap Y|\ s_{a,b} \leq t_{a,b}$. It is a standard fact that for any PCS X the poset $\mathsf{P}X$ (equipped with the pointwise order of $|X|$-indexed families of non-negative real numbers) is a cpo and that all the operations (composition of morphisms, tensor product, $!\_$ functor) preserve the lubs of directed families of morphisms, that is **Pcoh** is Scott, see for instance [DE11].

As a consequence for any PCS X we have a fixpoint operator $\mathcal{Y} \in \mathbf{Pcoh}(X \Rightarrow X, X)$ which is characterized by $\widehat{\mathcal{Y}}(t) = \sup_{n\in\mathbb{N}} \widehat{t}^n(0)$. Concerning derivatives, this operator satisfies Theorem 4.32.

For the categorical axiomatization of integers we refer to Section 4.5. We define N by $|\mathsf{N}| = \mathbb{N}$ and $u \in \mathsf{PN}$ if $u \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ satisfies $\sum_{\nu\in\mathbb{N}} u_\nu \leq 1$. This structure N is a PCS such that $\mathsf{PN}^\perp = [0,1]^{\mathbb{N}}$ as easily checked. Notice that $\|u\|_{\mathsf{N}} = \sum_{\nu\in\mathbb{N}} u_\nu$. This is an ℓ^1 norm whereas in $\mathsf{N}^\perp$ the norm is $\|u'\|_{\mathsf{N}^\perp} = \sup_{\nu\in\mathbb{N}} u'_\nu$ which is an ℓ^∞ norm. The matrix $\chi \in (\mathbb{R}_{\geq 0})^{|1\oplus\mathsf{N}\multimap\mathsf{N}|}$ defined by $\chi_{(0,*),n} = \boldsymbol{\delta}_{0,n}$ and $\chi_{(1,n'),n} = \boldsymbol{\delta}_{n'+1,n}$ satisfies $\chi \in \mathbf{Pcoh}(1\oplus\mathsf{N},\mathsf{N})$ and is an isomorphism between these two PCSs. Then, given $t \in \mathbf{Pcoh}(1\oplus X, X)$, let $(s_n \in \mathbf{Pcoh}(\mathsf{N}, X))_{n\in\mathbb{N}}$ be the sequence of morphisms defined by

$$s_0 = 0 \quad \text{and} \quad s_{n+1} = t\,(1\oplus s_n)\,\chi^{-1}\,.$$

An easy induction shows that $\forall n \in \mathbb{N}\ s_n \leq s_{n+1}$ and so $s = \sup_{n\in\mathbb{N}} s_n \in \mathbf{Pcoh}(\mathsf{N}, X)$ satisfies $s = t\,(1\oplus s)\,\chi^{-1}$. This means that (N,χ) is an initial algebra for the functor $1\oplus\_ : \mathbf{Pcoh} \to \mathbf{Pcoh}$ and so **Pcoh** satisfies (**Int**).

The associated morphisms $\overline{\mathsf{suc}}, \overline{\mathsf{pred}} \in \mathbf{Pcoh}(\mathsf{N},\mathsf{N})$ are characterized by $\overline{\mathsf{suc}} \cdot u = \sum_{\nu\in\mathbb{N}} u_\nu \mathsf{e}_{\nu+1}$ and $\overline{\mathsf{pred}} \cdot u = u_0\mathsf{e}_0 + \sum_{\nu\in\mathbb{N}} u_{\nu+1}\mathsf{e}_\nu$. The morphism $\overline{\mathsf{if}} \in \mathbf{Pcoh}(\mathsf{N}\otimes(X \mathbin{\&} X), X)$ is characterized by

$$\overline{\mathsf{if}} \cdot (u \otimes \langle x^0, x^1\rangle) = u_0 x^0 + \big(\sum_{\nu=1}^{\infty} u_\nu\big) x^1\,.$$

Last the morphism $\overline{\mathsf{let}} \in \mathbf{Pcoh}(\mathsf{N}\otimes(!\mathsf{N}\multimap X), X)$ is characterized by

$$\overline{\mathsf{let}} \cdot (u \otimes t) = \sum_{\nu\in\mathbb{N}} u_\nu \widehat{t}(\mathsf{e}_\nu)\,.$$

So we have an interpretation of terms and states in **Pcoh** which is invariant by reduction. More precisely, following the general pattern of Section 4.7, we associate with each type A an object $[\![A]\!]^{\mathbf{Pcoh}}$ of **Pcoh** in such a way that $[\![\mathsf{D}^d\iota]\!]^{\mathbf{Pcoh}} = \mathsf{D}^d\,\mathsf{N}$, $[\![A \Rightarrow B]\!]^{\mathbf{Pcoh}} = ([\![A]\!]^{\mathbf{Pcoh}} \Rightarrow [\![B]\!]^{\mathbf{Pcoh}})$.

And with any term M such that $x_1 : A_1, \dots, x_k : A_k \vdash M : B$ we can associate the morphism $[\![M]\!]_\Gamma^{\mathbf{Pcoh}} \in \mathbf{Pcoh}_!(\&_{i=1}^k [\![A_i]\!]^{\mathbf{Pcoh}}, [\![B]\!]^{\mathbf{Pcoh}})$ and this interpretation satisfies that if $M \to_{\Lambda_{\mathsf{cd}}} M'$ then $[\![M]\!]_\Gamma^{\mathbf{Pcoh}} = [\![M']\!]_\Gamma^{\mathbf{Pcoh}}$. Remember that the reduction relation $\to_{\Lambda_{\mathsf{cd}}}$ can be extended into the reduction relation $\to_{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})}$ on **Pcoh**-summable multisets of terms, that is to multisets $S = [M_1, \dots, M_n]$ such that $(\Gamma \vdash M_j : B)_{j=1}^n$ and $[\![S]\!]_\Gamma = \sum_{j=1}^n [\![M_j]\!]_\Gamma^{\mathbf{Pcoh}} \in \mathbf{Pcoh}_!(\&_{i=1}^k [\![A_i]\!]^{\mathbf{Pcoh}}, [\![B]\!]^{\mathbf{Pcoh}})$ and this extended relation satisfies $S \to_{\mathcal{M}_{\mathrm{fin}}(\Lambda_{\mathsf{cd}})} S' \Rightarrow [\![S]\!]_\Gamma^{\mathbf{Pcoh}} = [\![M']\!]_\Gamma^{\mathbf{Pcoh}}$.

6.4. **A forgetful functor.** Given $s \in \mathbf{Pcoh}(X,Y)$, we set $\mathsf{Q}s = \{(a,b) \in |X| \times |Y| \mid s_{a,b} \neq 0\} \in \mathbf{Rel}(|X|,|Y|)$.

Theorem 6.6. *The operation* Q *extended to objects by* $\mathsf{Q}X = |X|$ *is a functor* $\mathbf{Pcoh} \to \mathbf{Rel}$ *which preserves all the structure of model of* LL*.*

Proof. This is essentially trivial. Let us prove functoriality: let $s \in \mathbf{Pcoh}(X,Y)$ and $t \in \mathbf{Pcoh}(Y,Z)$. If $(a,c) \in \mathsf{Q}(t\,s)$ then $\sum_{b\in|Y|} s_{a,b}t_{b,c} \neq 0$ and so there must be $b \in |Y|$ such that $s_{a,b} \neq 0$ and $t_{b,c} \neq 0$ and hence $(a,c) \in (\mathsf{Q}t)\,(\mathsf{Q}s)$. Conversely assume that $(a,c) \in (\mathsf{Q}t)\,(\mathsf{Q}s)$ and let $b \in |Y|$ be such that $s_{a,b} \neq 0$ and $t_{b,c} \neq 0$. Since all coefficients are non-negative we have $\sum_{b'\in|Y|} s_{a,b'}t_{b',c} \geq s_{a,b}t_{b,c} > 0$ and hence $(a,c) \in \mathsf{Q}(t\,s)$.

As another example, let us prove that if $s \in \mathbf{Pcoh}(X,Y)$ then $\mathsf{Q}(!s) = !(\mathsf{Q}s)$. Let $(m,p) \in \mathsf{Q}(!s)$, so let $r \in \mathsf{L}(m,p)$ be such that $s^r = \prod_{(a,b)\in|X|\times|Y|} s_{a,b}^{r(a,b)} \neq 0$. This implies that $\mathsf{supp}(r) \subseteq \mathsf{Q}s$ and hence we have $r \in !(\mathsf{Q}s)$. Conversely if $(m,p) \in !(\mathsf{Q}s)$ we can write $m = [a_1,\dots,a_n]$ and $p = [b_1,\dots,b_n]$ is such a way that $((a_i,b_i) \in \mathsf{Q}s)_{i=1}^k$ which means that $\prod_{i=1}^k s_{a_i,b_i} \neq 0$. Now setting $r = [(a_1,b_1),\dots,(a_k,b_k)]$ we have $r \in \mathsf{L}(m,p)$ and $s^r \neq 0$ and hence $(m,p) \in \mathsf{Q}(!s)$.

To prove that Q applied to the $\mathcal{Y}$ operator of $\mathbf{Pcoh}$ yields the $\mathcal{Y}$ operator of $\mathbf{Rel}$, it suffices to observe that the functor Q is locally continuous. ☐

Theorem 6.7. *For any type* A *we have* $\mathsf{Q}[\![A]\!]^{\mathbf{Pcoh}} = [\![A]\!]^{\mathbf{Rel}}$ *and for any term* M *such that* $\Gamma \vdash M : B$ *we have* $\mathsf{Q}[\![M]\!]_\Gamma^{\mathbf{Pcoh}} = [\![M]\!]_\Gamma^{\mathbf{Rel}}$*, and similarly for stacks and states.*

Theorem 6.8. *For any term* M *such that* $(x_i : A_i)_{1=1}^k \vdash M : B$ *all coefficients of the matrix* $[\![M]\!]_\Gamma^{\mathbf{Pcoh}} \in \mathbf{Pcoh}(\&_{i=1}^k[\![A_i]\!]^{\mathbf{Pcoh}}, [\![B]\!]^{\mathbf{Pcoh}})$ *belong to* $\mathbb{N}$*. The same holds for stacks and states.*

Proof. It suffices to observe that if $t \in \mathbf{Pcoh}(X,Y)$ belongs to $\mathbb{N}^{|X|\times|Y|}$, then $!t \in \mathbb{N}^{|!X|\times|!Y|}$ and that $\partial_X \in \mathbb{N}^{|!\mathbf{S}\,X|\times|\mathbf{S}\,!X|}$. ☐

Remark 6.9. Of course the above property is lost if we extend the Λ_{cd} with probabilistic choice, which is perfectly possible and compatible with the $\mathbf{Pcoh}$ semantics.

Theorem 6.10. *If* $[c_1,\dots,c_n]$ *is a* $\mathbf{Pcoh}$*-summable multiset of states, we have only the two following possibilities:*

- *either* $[\![c_i]\!]^{\mathbf{Pcoh}} = 0$ *for all* $i = 1,\dots,n$
- *or there is exactly one* $i \in \{1,\dots,n\}$ *and one* $\nu \in \mathbb{N}$ *such that* $[\![c_i]\!]^{\mathbf{Pcoh}} = \mathsf{e}_\nu$*, and* $[\![c_j]\!]^{\mathbf{Pcoh}} = 0$ *for* $j \neq i$*.*

Proof. Observe that if $u \in \mathsf{P}\mathbb{N}$ belongs to $\mathbb{N}^{\mathbb{N}}$ then we have either $u = 0$ of $u = \mathsf{e}_\nu$ for a uniquely determined $\nu \in \mathbb{N}$. ☐

Theorem 6.11. *Let* M *be a closed term and let* $\nu \in \mathbb{N}$ *be such that* $\vdash M : \nu : \iota$*. Then* $[(\langle\,\rangle, M, ())] \to^*_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})} C = C_0 + [(\langle\,\rangle, \underline{\nu}, ())]$ *for some multiset of well typed states* C_0 *such that* C *is* $\mathcal{L}$*-summable in any model* $\mathcal{L}$*, and all the elements* c *of* C_0 *satisfy* $[\![c]\!]^{\mathbf{Rel}} = \emptyset$*.*

Proof. Using Theorem 5.36 we have $[(\langle\,\rangle, M, ())] \to^*_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})} C = C_0 + [(\langle\,\rangle, \underline{\nu}, ())]$ and by Theorem 5.8 we know that C is $\mathbf{Pcoh}$-summable and hence by Theorem 6.10 we know that all the elements c of C_0 satisfy $[\![c]\!]^{\mathbf{Pcoh}} = 0$ and hence $[\![c]\!]^{\mathbf{Rel}} = \emptyset$ by Theorem 6.7. ☐

$$(\zeta u, k, \mathsf{D}M, \sigma) \to_{\mathsf{w}} (\delta, k, M, \mathsf{D}(u) \cdot \sigma) \tag{6.1}$$

$$(\zeta, k, \lambda x^{A}\, M, \mathsf{D}(u) \cdot \sigma) \to_{\mathsf{w}} (\zeta u, k, \lambda x^{\mathsf{D}A}\, \partial(x, M), \sigma) \tag{6.2}$$

$$(\eta\check{n}\zeta, k, \iota^{d}_{0}(M), s) \to_{\mathsf{w}} \gamma = (\eta\zeta, k, M, \sigma) \quad \text{if } n \in \mathsf{ws}(\gamma) \tag{6.3}$$

$$(\eta\check{n}\zeta, k, \iota^{d}_{1}(M), \sigma) \to_{\mathsf{w}} (\eta\zeta, k, M, \sigma)\,[0/\check{n}] \tag{6.4}$$

$$(\eta\check{n}\zeta, k, \theta^{d}(M), \sigma) \to_{\mathsf{w}} (\eta\check{n}\check{n}\zeta, k, M, \sigma) \tag{6.5}$$

$$(\eta 1\zeta, k, \theta^{d}(M), \sigma) \to_{\mathsf{w}} (\eta\check{k}\check{k}\zeta, k+1, M, \sigma) \tag{6.6}$$

FIGURE 11. Deterministic reduction rules, with the convention that $d = \mathsf{len}(\zeta)$

6.5. **A deterministic machine.** We slightly modify the Krivine machine presented in Section 5.1 so as to make it fully deterministic. Guillaume Geoffroy must be credited for the key idea of this determinization which consists in making the access word component of the Krivine machine writable. To this end we introduce the set $\check{\mathsf{I}} = \mathsf{I} \uplus \{\check{n} \mid n \in \mathbb{N}\}$ where for each $n \in \mathbb{N}$ the symbol $\check{n}$ represents a "writable cell of name n".

A state of the deterministic machine is 0 or a tuple

$$\gamma = (\zeta, k, M, \sigma)$$

where $\zeta \in \check{\mathsf{I}}^{d}$ for some $d \in \mathbb{N}$, $k \in \mathbb{N}$, M is a Λ_{cd}-term and σ is a stack (defined exactly as in Section 5.1 apart that words are now taken in $\check{\mathsf{I}}^{<\omega}$). We use $\mathsf{ws}(\gamma)$ for the set of all $n \in \mathbb{N}$ such that $\check{n}$ occurs in ζ or σ. When k is $>$ than all the elements of $\mathsf{ws}(\gamma)$, we say that γ is *well-formed*.

We use letters u, v to denote general elements of $\check{\mathsf{I}}$, letters i, j for elements of $\mathsf{I} \subset \check{\mathsf{I}}$ and $\check{\Theta}_{\mathsf{cd}}$ for the set of states of this new machine. The main novelty is that that the $+$ operation on states is no more required. The second component of a state of the machine is used as a `gensym` for creating cell names on request, which are fresh by the well-formedness condition.

The typing rules for stacks and states are exactly the same as in Figure 6, replacing I with $\check{\mathsf{I}}$. Most transition rules are the same as in Figure 7. The modified transition rules are given in Figure 11. Notice that the rule producing a 0 is still present ($(\eta i\zeta, \iota^{d}_{1-i}(M), \sigma) \to_{\Theta_{\mathsf{cd}}} 0$ for $i \in \mathsf{I}$ and $d = \mathsf{len}(\zeta)$) but that the rule producing a sum as been replaced by the deterministic rule (6.6) so that the machine $\check{\Theta}_{\mathsf{cd}}$ equipped with the transitions $\to_{\mathsf{w}}$ is fully deterministic. In rule (6.4), namely $(\eta\check{n}\zeta, k, \iota^{d}_{1}(M), \sigma) \to_{\mathsf{w}} (\eta\zeta, k, M, \sigma)\,[0/\check{n}]$, notice that the cell $\check{n}$ can occur in $\eta\zeta$ and in σ. The intuition behind this rule is that we give the value 1 to the occurrence of $\check{n}$ singled out in its left member, so we know that all the other occurrences must take value 0, see Remark 6.12 below.

Notice that if $\gamma \to_{\mathsf{w}} \gamma'$ and γ is well-formed then γ' is also well-formed thanks to rule (6.6) which is the only one which introduces a new writable cell: it advances the gensym counter by 1.

Given $\zeta \in \check{\mathsf{I}}^{<\omega}$ we use $\mathsf{w}_0(\zeta)$ for the set of all $\delta \in \mathsf{I}^{\mathsf{len}(\zeta)}$ such that, for all $l \in \{1, \dots, \mathsf{len}(\zeta)\}$, $\zeta_l \in \mathsf{I} \Rightarrow \delta_l = \zeta_l$. For a stack σ of $\check{\Theta}_{\mathsf{cd}}$ we define a set of stacks $\mathsf{w}_0(\sigma)$ of $\underline{\Theta_{\mathsf{cd}}}$ in Figure 12. Last given a $\check{\Theta}_{\mathsf{cd}}$-state $\gamma = (\zeta, k, M, \sigma)$ we define a set of $\underline{\Theta_{\mathsf{cd}}}$-states $\mathsf{w}_0(\gamma)$ by

$$\mathsf{w}_0(\gamma) = \{(\delta, M, s) \mid \delta \in \mathsf{w}_0(\zeta) \text{ and } s \in \mathsf{w}_0(\sigma)\}\,.$$

$$\begin{aligned}
\mathsf{w}_0(\mathsf{arg}(M)\cdot\sigma) &= \{\mathsf{arg}(M)\cdot s \mid s\in\mathsf{w}_0(\sigma)\}\\
\mathsf{w}_0(\mathsf{succ}\cdot\sigma) &= \{\mathsf{succ}\cdot s \mid s\in\mathsf{w}_0(\sigma)\}\\
\mathsf{w}_0(\mathsf{pred}\cdot\sigma) &= \{\mathsf{pred}\cdot s \mid s\in\mathsf{w}_0(\sigma)\}\\
\mathsf{w}_0(\mathsf{if}(\zeta,M_1,M_2)\cdot\sigma) &= \{\mathsf{if}(\delta,M_1,M_2)\cdot s \mid \delta\in\mathsf{w}_0(\zeta)\text{ and } s\in\mathsf{w}_0(\sigma)\}\\
\mathsf{w}_0(\mathsf{let}(\zeta,x,M)\cdot\sigma) &= \{\mathsf{let}(\delta,x,M)\cdot s \mid \delta\in\mathsf{w}_0(\zeta)\text{ and } s\in\mathsf{w}_0(\sigma)\}\\
\mathsf{w}_0(\mathsf{D}(i)\cdot\sigma) &= \{\mathsf{D}(i)\cdot s \mid s\in\mathsf{w}_0(\sigma)\}\\
\mathsf{w}_0(\mathsf{D}(\check{n})\cdot\sigma) &= \{\mathsf{D}(i)\cdot s \mid i\in\mathsf{I}\text{ and } s\in\mathsf{w}_0(\sigma)\}
\end{aligned}$$

FIGURE 12. The set of $\underline{\Theta_{\mathsf{cd}}}$-stacks associated with a $\check{\Theta}_{\mathsf{cd}}$-stack

Next given $n\in\mathbb{N}$ and $\delta\in\mathsf{w}_0(\zeta)$ we define $\Sigma_n(\delta:\zeta)\in\mathbb{N}$ by

$$\begin{aligned}
\Sigma_n(\langle\rangle:\langle\rangle) &= 0\\
\Sigma_n(\delta i:\zeta i) &= \Sigma_n(\delta:\zeta)\\
\Sigma_n(\delta i:\zeta\check{k}) &= \boldsymbol{\delta}_{n,k} i+\Sigma_n(\delta:\zeta)\,.
\end{aligned}$$

Given $s\in\mathsf{w}_0(\sigma)$ we define $\Sigma_n(s:\sigma)\in\mathbb{N}$ as follows.

$$\begin{aligned}
\Sigma_n(\mathsf{arg}(M)\cdot s:\mathsf{arg}(M)\cdot\sigma) &= \Sigma_n(s:\sigma)\\
\Sigma_n(\mathsf{succ}\cdot s:\mathsf{succ}\cdot\sigma) &= \Sigma_n(s:\sigma)\\
\Sigma_n(\mathsf{pred}\cdot s:\mathsf{pred}\cdot\sigma) &= \Sigma_n(s:\sigma)\\
\Sigma_n(\mathsf{if}(\delta,M_1,M_2)\cdot s:\mathsf{if}(\zeta,M_1,M_2)\cdot\sigma) &= \Sigma_n(\delta:\zeta)+\Sigma_n(s:\sigma)\\
\Sigma_n(\mathsf{let}(\delta,x,M)\cdot s:\mathsf{let}(\zeta,x,M)\cdot\sigma) &= \Sigma_n(\delta:\zeta)+\Sigma_n(s:\sigma)\\
\Sigma_n(\mathsf{D}(i)\cdot s:\mathsf{D}(i)\cdot\sigma) &= \Sigma_n(s:\sigma)\\
\Sigma_n(\mathsf{D}(i)\cdot s:\mathsf{D}(\check{k})\cdot\sigma) &= \boldsymbol{\delta}_{k,n} i+\Sigma_n(s:\sigma)\,.
\end{aligned}$$

Last, for states $\gamma=(\zeta,M,\sigma,)$ and $c=(\delta,M,s)$ such that $c\in\mathsf{w}_0(\gamma)$ we set

$$\Sigma_n(c:\gamma)=\Sigma_n(\delta:\zeta)+\Sigma_n(s:\sigma)\,.$$

Finally we define

$$\mathsf{w}(\gamma)=\{c\in\mathsf{w}_0(\gamma)\mid\forall n\in\mathsf{ws}(\gamma)\quad\Sigma_n(c:\gamma)=1\}\,.$$

Remark 6.12. Intuitively $\mathsf{w}(\gamma)$ is the set of all states of the machine $\underline{\Theta_{\mathsf{cd}}}$ obtained from $\gamma\in\check{\Theta}_{\mathsf{cd}}$ as follows: for each n such that $\check{n}$ occurs at some place in γ, we choose one occurrence of $\check{n}$ and replace it with 1, and we replace all the other occurrences of $\check{n}$ with 0's.

Let $\to_{\mathsf{nd}}$ be the transition relation on Θ^0_{cd}, the set of all sum-implicit elements of $\underline{\Theta_{\mathsf{cd}}}$, defined as follows:

$$c\to_{\mathsf{nd}} c'\quad\text{if}\quad\begin{cases} & c\to_{\Theta_{\mathsf{cd}}} c'\\ \text{or} & c\to_{\Theta_{\mathsf{cd}}} c'+c''\\ \text{or} & c\to_{\Theta_{\mathsf{cd}}} c''+c'\end{cases}$$

so that now

$$(\varepsilon 1\delta, \theta^d(M), s) \to_{\mathsf{nd}} (\varepsilon 10\delta, M, s)$$
$$(\varepsilon 1\delta, \theta^d(M), s) \to_{\mathsf{nd}} (\varepsilon 01\delta, M, s)$$

and $\to_{\mathsf{nd}}$ is defined exactly as $\to_{\Theta_{\mathsf{cd}}}$ in the other cases.

Lemma 6.13. *Assume that $c \in \mathsf{w}_0(\gamma)$ and that $\Sigma_n(c : \gamma) = 0$ for some $n \in \mathbb{N}$. Then $c \in \mathsf{w}_0(\gamma\,[0/\check{n}])$.*

Sketch of the proof. By the assumption that $\Sigma_n(c : \gamma) = 0$ we know that at all the places in c corresponding to occurrences of $\check{n}$ in γ we have the value 0. The conclusion follows readily. □

Lemma 6.14. *Let $\gamma \in \check{\Theta}_{\mathsf{cd}}$ and $c \in \mathsf{w}(\gamma)$. If $c \to_{\mathsf{nd}} c' \neq 0$ then $\gamma \to_{\mathsf{w}} \gamma'$ for γ' such that $c' \in \mathsf{w}(\gamma')$.*

Proof. We use our convention that $d = \mathsf{len}(\delta) = \mathsf{len}(\zeta)$. Most cases are straightforward, we deal first with one of these to illustrate its triviality.

▶ $c = (\varepsilon\delta, \mathsf{if}^d(M, N_1, N_2), s)$ and $c' = (\delta, M, \mathsf{if}(\varepsilon, N_1, N_2) \cdot s)$. Since $c \in \mathsf{w}(\gamma)$ we can write $\gamma = (\eta\zeta, k, \mathsf{if}^d(M, N_1, N_2), \sigma)$ and we have $\gamma \to_{\mathsf{w}} \gamma' = (\zeta, k, M, \mathsf{if}(\eta, N_1, N_2) \cdot \sigma)$ so that $c' \in \mathsf{w}(\gamma')$ because for each $n \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma')$ we have $\Sigma_n(c' : \gamma') = \Sigma_{n'}(c : \gamma) = 1$.

We consider now the more interesting cases.

▶ $c = (\delta i, \mathsf{D}M, s)$ and $c' = (\delta, M, \mathsf{D}(i) \cdot s)$. Then $\gamma = (\zeta u, k, \mathsf{D}M, \sigma)$ and we have $\gamma \to_{\mathsf{w}} \gamma' = (\zeta, M, k, \mathsf{D}(u) \cdot \sigma)$ and hence $c' \in \mathsf{w}(\gamma')$. To be more explicit we should consider the two possible cases for u.

- If $u \in \mathsf{I}$ we have $u = i$ because $c \in \mathsf{w}_0(\gamma)$ and hence $\gamma' = (\zeta, k, M, \mathsf{D}(i) \cdot \sigma)$. For each $n \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma')$ we have $\Sigma_n(c' : \gamma') = \Sigma_n(\delta : \zeta) + \Sigma_n(s : \sigma) = \Sigma_n(c : \gamma) = 1$ so that $c' \in \mathsf{w}(\gamma')$.
- If $u = \check{n}$ for some $n \in \mathbb{N}$ we know that $\Sigma_{n'}(c : \delta) = \boldsymbol{\delta}_{n',n} i + \Sigma_{n'}(\delta : \zeta) + \Sigma_{n'}(s : \sigma) = 1$ for each $n' \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma')$ and since we also have $\Sigma_{n'}(c' : \gamma') = \boldsymbol{\delta}_{n',n} i + \Sigma_{n'}(\delta : \zeta) + \Sigma_{n'}(s : \sigma)$ it follows that $c' \in \mathsf{w}(\gamma')$.

▶ $c = (\delta, \lambda x^A\, M, \mathsf{D}(i) \cdot s)$ and $c' = (\delta i, \lambda x^{\mathsf{D}A}\, \partial(x, M), s)$. Then $\gamma = (\zeta, k, \lambda x^A\, M, \mathsf{D}(u) \cdot \sigma)$ and we have $\gamma \to_{\mathsf{w}} \gamma' = (\zeta u, k, \lambda x^{\mathsf{D}A}\, \partial(x, M), \sigma)$ so that $c' \in \mathsf{w}_0(\gamma')$. We check easily as above that $\Sigma_n(c' : \gamma') = \Sigma_n(c : \gamma)$ for all $n \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma')$ and hence $c' \in \mathsf{w}(\gamma')$.

▶ $c = (\varepsilon i\delta, \iota^d_i(M), s)$ and $c' = (\varepsilon\delta, M, s)$ with $i = 0$. Assume that $\gamma = (\eta\check{n}\zeta, k, \iota^d_0(M), \sigma)$ and let $\gamma' = (\eta\zeta, k, M, \sigma)$. Clearly $c' \in \mathsf{w}_0(\gamma')$. Moreover $\Sigma_{n'}(c' : \gamma') = \Sigma_{n'}(c : \gamma) = 1$ for all $n' \in \mathsf{ws}(\gamma') \subseteq \mathsf{ws}(\gamma)$, including when $n' = n$ in which case we use the fact that $i = 0$ — assuming that $n \in \mathsf{ws}(\gamma')$ which is not only possible but necessary as we see now —. Since $\Sigma_n(c : \gamma) = 1$ and $i = 0$, $\check{n}$ must occur in η, ζ or σ so that $n \in \mathsf{ws}(\gamma')$ and hence $\gamma \to_{\mathsf{w}} \gamma'$ (see in Figure 11 the restrictive condition for this reduction (6.3)).

The case where $\gamma = (\eta 0\zeta, k, \iota^d_0(M), \sigma)$ is dealt with straightforwardly.

▶ $c = (\varepsilon i\delta, \iota^d_i(M), s)$ and $c' = (\varepsilon\delta, M, s)$ with $i = 1$. Assume that $\gamma = (\eta\check{n}\zeta, k, \iota^d_1(M), \sigma)$ so that we have $\gamma \to_{\mathsf{w}} \gamma' = (\eta\zeta, k, M, \sigma)\,[0/\check{n}]$ and we clearly have $c' \in \mathsf{w}_0(\gamma')$. Since $i = 1$, for $n' \in \mathsf{ws}(\gamma)$ we have $\Sigma_{n'}(c : \gamma) = \boldsymbol{\delta}_{n,n'} + \Sigma_{n'}(\varepsilon\delta : \eta\zeta) + \Sigma_{n'}(s : \sigma)$ and we know that $\Sigma_{n'}(c : \gamma) = 1$. It follows that $\Sigma_{n'}(c' : \gamma') = \Sigma_{n'}(\varepsilon\delta : \eta\zeta) + \Sigma_{n'}(s : \sigma) = 1$ for all $n' \in \mathsf{ws}(\gamma') = \mathsf{ws}(\gamma) \setminus \{n\}$ and moreover that $\Sigma_n(c' : \gamma') = \Sigma_n(\varepsilon\delta : \eta\zeta) + \Sigma_n(s : \sigma) = 0$. By Lemma 6.13 it follows that $c' \in \mathsf{w}_0(\gamma')$, and hence that $c' \in \mathsf{w}(\gamma')$ as expected.

The case where $\gamma = (\eta 1\zeta, k, \iota^d_0(M), \sigma)$ is dealt with straightforwardly.

▶ $c = (\varepsilon 1\delta, \theta^d(M), s)$ and $c' = (\varepsilon i_1 i_2 \delta, M, s)$ for some $i_1, i_2 \in \mathsf{I}$ such that $i_1 + i_2 = 1$.

Assume first that $\gamma = (\eta\check{n}\zeta, k, \theta^d(M), \sigma)$. Then $\gamma' = (\eta\check{n}\check{n}\delta, k, M, \sigma)$. We clearly have $c' \in \mathsf{w}_0(\gamma')$, and if $n' \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma')$ we have

$$\begin{aligned}\Sigma_{n'}(c' : \gamma') &= \Sigma_{n'}(\varepsilon\delta : \eta\zeta) + \boldsymbol{\delta}_{n',n}(i_1 + i_2) + \Sigma_{n'}(s : \sigma)\\ &= \Sigma_{n'}(\varepsilon\delta : \eta\zeta) + \boldsymbol{\delta}_{n',n} + \Sigma_{n'}(s : \sigma)\\ &= \Sigma_{n'}(c : \gamma) = 1.\end{aligned}$$

Assume next that $\gamma = (\eta 1\zeta, k, \theta^d(M), \sigma)$. Then $\gamma \to_{\mathsf{w}} \gamma' = (\eta\check{k}\check{k}\zeta, k+1, M, \sigma)$ so that we clearly have $c' \in \mathsf{w}_0(\gamma')$. For $n' \in \mathsf{ws}(\gamma') = \mathsf{ws}(\gamma) \uplus \{k\}$ (since γ is well-formed) we have

$$\Sigma_{n'}(c' : \gamma') = \Sigma_{n'}(\varepsilon\delta : \eta\zeta) + \Sigma_{n'}(s : \sigma) = \Sigma_{n'}(c : \gamma) = 1 \text{ if } n' \neq k$$

and

$$\Sigma_{n'}(c' : \gamma') = \Sigma_{n'}(\varepsilon\delta : \eta\zeta) + i_1 + i_2 + \Sigma_{n'}(s : \sigma) = 0 + 1 + 0 \text{ if } n' = k$$

since $i_1 + i_2 = 1$ and by our assumption that γ is well-formed which implies that $\Sigma_k(c : \gamma) = 0$.

The case where $c = (\varepsilon 0\delta, \theta^d(M), s)$, so that $c' = (\varepsilon 00\delta, M, s)$, is easy. □

Lemma 6.15. *Let $\gamma, \gamma' \in \check{\Theta}_{\mathsf{cd}}$ with $\gamma \to_{\mathsf{w}} \gamma'$. If $c' \in \mathsf{w}(\gamma')$ then there is $c \in \mathsf{w}(\gamma)$ such that $c \to_{\mathsf{nd}} c'$.*

Proof. We consider only a few cases, the other ones being straightforward.

▶ $\gamma = (\zeta u, k, \mathsf{D}M, \sigma)$ and $\gamma' = (\zeta, M, k, \mathsf{D}(u) \cdot \sigma)$. Then we have $c' = (\delta, M, \mathsf{D}(i) \cdot s)$ for some $i \in \mathsf{I}$ (with $i = u$ is $u \in \mathsf{I}$) and then $c = (\delta i, \mathsf{D}M, s)$ satisfies the required conditions.

▶ $\gamma = (\zeta, k, \lambda x^A\, M, \mathsf{D}(u) \cdot \sigma)$ and $\gamma' = (\zeta u, k, \lambda x^{\mathsf{D}A}\, \partial(x, M), \sigma)$. Then we have $c' = (\delta i, \lambda x^{\mathsf{D}A}\, \partial(x, M), s)$ for some $i \in \mathsf{I}$ (with $i = u$ is $u \in \mathsf{I}$) and then $c = (\delta, \lambda x^A\, M, \mathsf{D}(i) \cdot s)$ satisfies the required conditions.

▶ $\gamma = (\eta\check{n}\zeta, k, \iota_0^d(M), \sigma)$ and $\gamma' = (\eta\zeta, k, M, \sigma)$ with $n \in \mathsf{ws}(\gamma')$, applying rule (6.3). Then we have $c' = (\varepsilon\delta, M, s)$. Taking $c = (\varepsilon 0\delta, \iota_i^d(M), s)$ we have $c \to_{\mathsf{nd}} c'$ and $c \in \mathsf{w}(\gamma)$ since for all $n' \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma')$ we have $\Sigma_{n'}(c : \gamma) = \Sigma_{n'}(c' : \gamma') = 1$. It is here that the restrictive condition in the rule (6.3) of Figure 11 is quite important.

▶ $\gamma = (\eta\check{n}\zeta, k, \iota_1^d(M), \sigma)$ and $\gamma' = (\eta\zeta, k, M, \sigma)\,[0/\check{n}]$. Then we have $c' = (\varepsilon\delta, M, s)$ and we take $c = (\varepsilon 1\delta, \iota_i^d(M), s)$ so that $c \to_{\mathsf{nd}} c'$. For $n' \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma') \uplus \{n\}$ we have $\Sigma_{n'}(c : \gamma) = \Sigma_{n'}(c' : \gamma') = 1$ if $n' \neq n$. Since $c' \in \mathsf{w}_0((\eta\zeta, k, M, \sigma)\,[0/\check{n}])$ we have $\Sigma_n(c' : (\eta\zeta, k, M, \sigma)) = 0$ (that is, all the occurrences of $\check{n}$ in $(\eta\zeta, k, M, \sigma)$ are filled with 0's in c') and hence $\Sigma_n(c : \gamma) = 1$.

▶ $\gamma = (\eta\check{n}\zeta, k, \theta^d(M), \sigma)$ and $\gamma' = (\eta\check{n}\check{n}\delta, k, M, \sigma)$. Then $c' = (\varepsilon i_1 i_2 \delta, M, s)$ for some $i_1, i_2 \in \mathsf{I}$ such that $i = i_1 + i_2 \in \mathsf{I}$. Let $c = (\varepsilon i\delta, \theta^d(M), s)$ so that $c \to_{\mathsf{nd}} c'$ and $c \in \mathsf{w}_0(\gamma)$. Then for $n' \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma')$ we have

$$\begin{aligned}\Sigma_{n'}(c : \gamma) &= \Sigma_{n'}(\varepsilon\delta : \eta\zeta) + \boldsymbol{\delta}_{n',n} i + \Sigma_{n'}(s : \sigma)\\ &= \Sigma_{n'}(\varepsilon\delta : \eta\zeta) + \boldsymbol{\delta}_{n',n} i_1 + \boldsymbol{\delta}_{n',n} i_2 + \Sigma_{n'}(s : \sigma)\\ &= \Sigma_{n'}(c' : \gamma') = 1\end{aligned}$$

and hence $c \in \mathsf{w}(\gamma)$.

▶ $\gamma = (\eta 1\zeta, k, \theta^d(M), \sigma)$ and $\gamma' = (\eta\check{k}\check{k}\delta, k+1, M, \sigma)$. Then $c' = (\varepsilon i_1 i_2 \delta, M, s)$ for some $i_1, i_2 \in \mathsf{I}$ such that $i_1 + i_2 = 1$: this is due to the fact that we must have $\Sigma_k(c' : \gamma') = 1$ and we know that $\check{k}$ does not occur in η, ζ and σ since γ is well-formed. Let $c = (\varepsilon 1\delta, \theta^d(M), s)$ so that $c \to_{\mathsf{nd}} c'$ and $c \in \mathsf{w}_0(\gamma)$. Then for $n' \in \mathsf{ws}(\gamma) = \mathsf{ws}(\gamma') \setminus \{k\}$ we have $\Sigma_{n'}(c : \gamma) = \Sigma_{n'}(c' : \gamma') = 1$ and hence $c \in \mathsf{w}(\gamma)$. □

Theorem 6.16. *Let M be a term such that $\vdash M : \iota$ and let $\nu \in \mathbb{N}$. Then*

$$[\exists k \in \mathbb{N}\ (\langle\,\rangle, 0, M, ()) \to_{\mathsf{w}}^* (\langle\,\rangle, k, \underline{\nu}, ())] \Leftrightarrow (\langle\,\rangle, M, ()) \to_{\mathsf{nd}}^* (\langle\,\rangle, \underline{\nu}, ())$$

Moreover when one of these two reduction converges, the other one does, with the same number of steps.

Proof. ▶ $\Leftarrow$. By an obvious induction on the length of the $\to_{\mathsf{nd}}$-reduction using Lemma 6.14 one proves the following statement: if $(\langle\,\rangle, M, ()) \to_{\mathsf{nd}}^* c \neq 0$ then $(\langle\,\rangle, 0, M, ()) \to_{\mathsf{w}}^* \gamma$ with $c \in \mathsf{w}(\gamma)$. We apply this statement to the case where $c = (\langle\,\rangle, \underline{\nu}, ())$ and obtain that $(\langle\,\rangle, 0, M, ()) \to_{\mathsf{w}}^* \gamma$ with $(\langle\,\rangle, \underline{\nu}, ()) \in \mathsf{w}(\gamma)$, which means that $\gamma = (\langle\,\rangle, k, \underline{\nu}, ())$ for some $k \in \mathbb{N}$.

▶ $\Rightarrow$. By an obvious induction on the length of the $\to_{\mathsf{w}}$-reduction using Lemma 6.15 one proves the following statement: if $\gamma \to_{\mathsf{w}}^* (\langle\,\rangle, k, \underline{\nu}, ())$ for some $k \in \mathbb{N}$ then there is $c \in \mathsf{w}(\gamma)$ such that $c \to_{\mathsf{nd}}^* (\langle\,\rangle, \underline{\nu}, ())$. We apply this statement to the case where $\gamma = (\langle\,\rangle, 0, M, ())$ and obtain c such that $c \to_{\mathsf{nd}}^* (\langle\,\rangle, \underline{\nu}, ())$ and $c \in \mathsf{w}(\gamma)$. This latter property means that $c = (\langle\,\rangle, M, ())$. □

Remark 6.17. The fact that the lengths of the deterministic $\to_{\mathsf{w}}$-reduction and of the non-deterministic $\to_{\mathsf{nd}}$-reduction in Theorem 6.16 are equal is of course essential since it is always possible to similate a non-deterministic reduction by a deterministic one using interleaving techniques. One can use the simulation and co-simulation Lemmas 6.14 and 6.15 for proving various generalizations of that theorem, using the fact that if $c \in \mathsf{w}(\gamma)$ and γ contains no $\check{n}$'s then c and γ are essentially the same thing (the only difference is the counter contained in γ).

Theorem 6.18. *Let M be a term such that $\vdash M : \iota$ and let $\nu \in \mathbb{N}$. Then we have $\vdash M : \nu : \iota$ iff $(\langle\,\rangle, 0, M, ()) \to_{\mathsf{w}}^* (\langle\,\rangle, k, \underline{\nu}, ())$ for some $k \in \mathbb{N}$.*

Proof. By Theorems 6.16, 5.10 and 6.11, observing that $c \to_{\mathsf{nd}}^* c'$ is equivalent to the existence of C such that $[c] \to_{\mathcal{M}_{\mathrm{fin}}(\Theta_{\mathsf{cd}})}^* [c'] + C$. □

Conclusion

Building on the categorical axiomatization of coherent differentiation introduced in [Ehr21] we have defined a differential extension Λ_{cd} of the standard Turing complete functional programming language PCF.

The rewriting system of Λ_{cd} has many reduction rules and therefore it would be probably rather difficult to prove this determinism property syntactically (as a Church-Rosser property) so our use of denotational semantics for this purpose seems really crucial. We also use semantics for proving that our rewriting system is complete in the sense that it allows to reduce to $\underline{\nu}$ any closed term of type ι whose interpretation in the relational model contains ν. This proof is based on the use of a Krivine machine which is a way of extracting from the general Λ_{cd} rewriting system a fairly simple though sufficiently expressive subsystem. The completeness proof is based on a reducibility method that we have been obliged to modify drastically in order to adapt it to this differential setting. To make the proof more readable, we present the relational semantics of Λ_{cd} as a non-idempotent intersection typing system and completeness can be understood as a normalization property for this typing system.

One major novelty[15] of coherent differentiation wrt. DiLL is the fact that it is deterministic. This was already clear in [Ehr21] where we showed that coherent differentiation admit deterministic models, that is models where the superposition of the values **t** and **f** in the type **Bool** is rejected. The present article provides a syntactic evidence of this determinism via the fully deterministic version of our machine which is shown to be sound and complete wrt. the execution of closed Λ_{cd}-terms of type ι.

Future work. Our Krivine machine has no environment and uses actual substitutions in terms for implementing β-reduction, as well as a syntactic differential operation $\partial(x, M)$ defined by induction on M to implement the differential reduction of Λ_{cd}. From the viewpoint of efficiency this is of course not satisfactory and we will present in a forthcoming paper a machine using a stack as well as a (differential) environment not invoking any external operation defined by induction on terms for executing the expressions of our language.

The most puzzling questions however remain of a theoretical nature and concern the exact operational meaning of our language Λ_{cd}[16], which has now fully satisfactory deterministic operational and denotational semantics. From a programming point of view, what is exactly the meaning of the type construction $\mathsf{D}A$ and in what kind of programming situation could it be useful, as well as the syntactic term construction $\mathsf{D}M$? One way to address this question could be to consider a probabilistic extension of Λ_{cd}, for which differentiation has a clear mathematical meaning easily expressed in **Pcoh** as we have seen in Section 6. Then we could expect to use our language to approximate such a derivative by means of a Monte Carlo method, doing statistics on a number of runs of a Λ_{cd} program expressing it.

Another interesting direction, which might require the extension of Λ_{cd} with richer types[17], would be to understand if it has connection with incremental programming where syntactic constructs of a differential nature are also used. Such a connection remains however highly conjectural. More specifically, in [EL10, EL07], we have suggested possible connections between DiLL and various process calculi, it might be worthwhile to understand if such connections could be related to incremental computing and benefit from coherent differentiation.

Last the fact that Λ_{cd} has at the same time a general fixpoint construct and a differential construct means that it is possible to define programs by some kind of "differential equations" (recursive definitions of functions whose body contains the possibly higher derivatives of the functions being defined) and that such programs can be executed in our Krivine machine(s); this is a very exciting feature of our setting which justifies investigations *per se.*

[15]And improvement in some sense.

[16]Or of its variants, we can of course expect to design more syntactically elegant versions of Λ_{cd} in the next few months; the version presented in this paper has been chosen for its relatively straightforward denotational semantics.

[17]Most models of LL support inductive and coinductive definitions so such extensions should not be problematic.

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