Nonstandard analysis in Krivine realizability

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Introduced by Robinson in the 60s [8], nonstandard analysis brought the first formal account for the notion of infinitesimal. To do so, he presented a construction to extend usual mathematical sets ($\mathbb{N}$, $\mathbb{R}$, etc.) witnessing the existence of new elements, the so-called *nonstandard* individuals. Later in the 70s, Nelson developed a syntactic approach to nonstandard analysis, introducing in particular three key principles (dubbed idealization, standardization and transfer) [7]. The validity of these principles for constructive mathematics has been studied in different settings, in particular several recent works lead to interpretations of nonstandard theories in intuitionistic realizability models [1, 3, 2].

In this talk, we propose to contemplate a different approach to nonstandard interpretations in realizability. On the one hand, we would like to deal with nonstandard analysis in the context of Krivine classical realizability [5]. By completely reformulating Kleene's intuitionistic realizability to take into account both terms (the realizers) and stacks (*i.e.* evaluation contexts for terms), Krivine managed to defined a notion of realizability compatible with classical logic that highlighted how new reasoning principles can be obtained by considering new programming primitives in the underlying operational semantics [4, 6]. On the other hand, we focus on Robinson's construction of a model for nonstandard analysis via an ultraproduct rather than on Nelson's syntactic approach (as is done in the aforementioned work in intuitionistic realizability). In particular, we add a memory cell to the Krivine machine in order to indicate in which slice of the ultraproduct $\mathcal{M}^{\mathbb{N}}$ the machine is. We will then explain how this product can be quotiented to mimick Robinson's construction, and we shall pay attention to the nonstandard principles (and their computational content) that we can obtain in this setting.

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