Proof Nets

A graph syntax for proofs

Reference:
Notes on proof-nets by Olivier Laurent

Focus on MLL

The sequent calculus derivation rules of MLL are:

\[ \frac{\Gamma, A \quad \Delta}{\Gamma, A} \] (Cut)
\[ \frac{\Gamma, A \quad \Delta, B}{\Gamma, A \otimes B, \Delta} \] (\(\otimes\))
\[ \frac{\Gamma, A, B \quad \Delta}{\Gamma, A \land B, \Delta} \] (\(\land\))

Linear negation is defined by:

\[ X^{\bot} = X \]

\[ (A \otimes B)^{\bot} = A^{\bot} \otimes B^{\bot} \]
\[ (A \land B)^{\bot} = A^{\bot} \land B^{\bot} \]

Forgetting Sequential Structure

Proof Nets

A graph syntax for proofs
Proof structures

A proof structure $M$ is a labelled directed acyclic graph (DAG) with possibly pending edges (i.e., some edges may have no source and/or no target) built over the alphabet of nodes which is represented below. (Note: in figures, the edges orientation is always top-bottom.)

- The nodes are labelled by $\text{ax}$, $\text{cut}$, $\otimes$, $\bowtie$.
- The edges are labelled by MLL formulas.

For each node/link: premises = entering edges, conclusions = exiting edges.

The conclusions of $M$ is the set of pending edges of $M$.

In the graphical representation of a proof structure, we do not mention explicitly the direction of edges, but we draw them in such a way that direction in represented in a top-down way:

From proofs to proofs structures

Translate each of these sequent calculus proofs. Start from axioms...

Is every structure the image of an MLL proof?

A PROOF NET is a proof structure which is the image of an MLL proof.

Proof Nets

Internal condition!

Purely geometrical conditions (correction) characterize the proof structures which are proof nets.

Theorem 1. A proof structure is correct iff it is a proof net.

Correctness criteria:
- (UT) Long trip [Girard]
- (AC) Connected-Acyclic [Danos-Regnier]
- ...
Danos-Regnier Criterion

Correctness Criterion

Switching Graphs

Correctness
- Switching graphs are acyclic.
- Switching graphs are connected.

Definition 2 (Correctness criterion AC [Danos-Regnier]). Let $R$ be a proof structure.
A switching $s$ is a function on the nodes of $R$, which chooses, for each $\gamma$-link, either the left or the right premise.
A proof structure $R$ is correct if for each switching, the unoriented graph obtained by erasing for each $\gamma$-link of $R$ the edges not chosen by $s$ is:
- connected and acyclic

Acyclicity. A multiplicative proof structure is acyclic if its switching graphs do not contain any undirected cycle.

A proof structure with $p \gamma$ nodes induces $2^p$ switchings and thus $2^p$ switching graphs. A switching graph is not a proof structure in general since its $\gamma$ nodes have only one premise.
A connected component of a switching graph is a connected component of its underlying (undirected) multigraph.

Is this correct?

PN1:

PN2:

• Correctness guarantees:
  ✓ Graph is image of a proof (sequentialization)
  ✓ Normalization progresses (no deadlocks)
  ✓ Normalization terminates (no infinite cycles)
Sequentialization

Soundness

Proposition 4.1.1 (Soundness of Correctness). The translation of a sequent calculus proof of MLL is a connected multiplicative proof net.

\[ \frac{\Gamma}{\Delta} \quad A \quad A \quad \Delta \]

\[ \frac{\Gamma}{\Delta} \quad A \quad B \quad \Delta \]

\[ \frac{\Gamma}{\Delta} \quad A \quad B \quad \Delta \]

Theorem 4.1.1 (Sequentialization). Any connected multiplicative proof net is the translation of a sequent calculus proof of MLL.

Sequentialization answers the question:

We have a proof net. The problem: it is the image of a sequent calculus proof? And which?

The beauty of proof nets is normalization

Normalization (local graph reductions!)

Let us try out!

Write a proof net with this conclusion... and normalize it
How we write a proof net of these conclusions?

\[ A \land A \] must type an edge conclusion of a par link, with premisses ...

\[ A \otimes A \] must type an edge conclusion of a tensor link, with premisses ...

Then we have to choose the axiom links!

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Properties of normalization

1. Confluence?
2. Is normalization weakly/strongly normalizing?
3. Would you be able to define a normalizing strategy?
4. Would you be able to define a normalizing strategy which reaches normal form in a minimal number of steps?

Normalization of MLL proof-nets:

- Strongly normalizing
- Confluent
- Cut elimination: a proof-net in normal form contains no cuts

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Let us try out!

- Let us try out another example

Write a proof net with this conclusion... and normalize it
Let us try one more. First, write a proof net with this conclusion...

\[(X \otimes X) \rightarrow (X \otimes X) = (X \otimes X) \dual \neg (X \otimes X) = (X \otimes \neg X \dual X) \dual (X \otimes X)\]

TIP: How we write a proof net? As before, all proof nets with the same conclusion, start with the same nodes (the formula tree!).

What distinguishes different proofs are the axiom links.

To distinguish the different occurrences of atoms, let us write indices:

\[(X_1 \dual X_2 \dual \neg X_3 \dual (X_4 \otimes X_5)\]

In this case, we have two possible proofs, corresponding to two possible ways to write axioms:

1. 1,3 and 2,4
2. 1,4 and 2,3

In sequent calculus, they correspond to these two proofs (one uses exchange, one no):

\[\begin{align*}
&\vdash X_1 \dual X_1, X_1 \\
&\vdash X_1 \dual X_2, X_2 \otimes X_4 \\
&\vdash X_1 \dual \neg X_3 \dual X_3 \otimes X_4 \\
&\vdash X_4 \dual X_4 \dual \neg X_5 \dual (X_5 \otimes X_6) \\
&\vdash X_6 \dual X_6 \dual X_6 \dual (X_6 \otimes X_7) \\
&\vdash (X_1 \dual X_2) \dual \neg X_3 \dual (X_3 \otimes X_4) \\
&\vdash (X_1 \dual X_2) \dual (X_2 \otimes X_4) \dual (X_3 \otimes X_4) \\
&\vdash (X_3 \dual X_4) \dual \neg X_5 \dual (X_5 \otimes X_6) \\
&\vdash (X_3 \dual X_4) \dual (X_4 \otimes X_7) \dual (X_6 \otimes X_7) \\
&\vdash (X_4 \dual X_5) \dual X_6 \dual (X_6 \otimes X_7) \\
&\vdash (X_5 \dual X_6) \dual X_6 \dual (X_6 \otimes X_7) \\
&\vdash (X_6 \dual X_7) \dual (X_7 \otimes X_8) \\
&\vdash (X_7 \dual X_8) \dual (X_7 \otimes X_8) \\
&\vdash (X_8 \dual X_8) \dual (X_8 \otimes X_8)
\end{align*}\]

Try to normalize one of the proofs of

\[(X_1 \dual X_2) \dual (X_3 \dual X_4) \dual (X_5 \dual X_6) \dual (X_7 \dual X_8) \dual (X_8 \dual X_8)\]

with the proof net which has conclusions

\[(X_1 \dual X_2) \dual (X_3 \dual X_4) \dual (X_5 \dual X_6) \dual (X_7 \dual X_8) \dual (X_8 \dual X_8)\]

and axiom links: (1,6) (2,5) (3,7) (4,8)

What is the function coded by this proof net?

Correctness criterion, think again
Correctness: if we focus on acyclicity, Danos-Regnier criterion can be reformulated (in equivalent way)

Let $R$ be a proof structure; a switching path of $R$ is a path which does not use any two edges entering on the same red link (such edges are called switching edges); a switching cycle is a switching path which is a cycle.

Definition 3 (Correctness criterion). A proof structure is correct if it does not contain any switching cycle.

- $\vdash A, A \vdash \text{(Ax)}$
- $\vdash A, A, \Delta \vdash \text{(Cut)}$
- $\vdash \Gamma, A \vdash \Delta, B \vdash \text{(0)}$
- $\vdash \Gamma, A, B \vdash \Delta \vdash \text{($\otimes$)}$
- $\vdash \Gamma \vdash \Delta, \text{(Mix)}$

we can throw away MIX later
By requiring connectness

Exponentials

From proofs to proofs structures

- $\vdash \Gamma, A \vdash \text{?d}$
- $\vdash \Gamma \vdash \text{?A}$
- $\vdash \Gamma, \text{?A}, \text{?A} \vdash \text{?c}$
- $\vdash \Gamma \vdash \text{?A}$
- $\vdash \Gamma, \text{?A} \vdash \text{?w}$
- $\vdash \Gamma \vdash \text{?A}$

From proofs to proofs structures

- $\vdash \Gamma, A \vdash \text{?A} \vdash \text{?A}$
- $\vdash \Gamma \vdash \text{?A}$
- $\vdash \Gamma \vdash \text{?A}$
- $\vdash \Gamma \vdash \text{?A}$
- $\vdash \Gamma \vdash \text{?A}$

- Boxes never overlap.
- Boxes are sequential (as rules in sequent calculus).
- Correctness: box by box.
- Boxes permit duplication and erasure.

What is the associated proof-net?

- $\vdash A, A \vdash \text{ax}$
- $\vdash A, A \vdash \text{ax}$
- $\vdash A \otimes A \vdash A$
- $\vdash A \downarrow A \vdash A$
- $\vdash A \downarrow A \vdash A$

Reduction steps

Can you write the proof so that all axioms are atomic (ie on atomic formulas)?
Properties of MELL reduction

1. Is confluent?
2. Is weakly normalizing?

Tip for WN.
Given a proof-net $R$, try to make decrease a size $S(R)$.

For example:

- **Size of a cut**: pair $(s,t)$ where $s$ is the size of the cut and $t$ is the size of the ?-tree above the ? premise of the cut if any, or 0.

- **Size $S(R)$ of the proof-net $R$**: the multiset of the sizes of all its cuts.

Weak Normalization

- cut relation: $c_1 \prec c_2$ (exponential cuts only)

- correctness $\implies$ no cycle in $\prec$ $\implies$ maximal cuts

- reduction of a maximal cut $\implies$ $\left( \text{[formula]}, \text{[?-tree]} \right)$ decreases

Confluence

Is the reduction confluent?
A proof-nets is polarized if every edge is labelled by a positive or a negative formula.

Let $M$ be a MELL polarized proof structure. We denote by $\text{Pol}(M)$ the graph which has the same nodes and edges as $M$, but where the edges are directed downward if positive, upwards if negative.

Do you see any simple way to show that the following are equivalent?

(1) $M$ is acyclic correct, (2) $\text{Pol}(M)$ is a DAG.