Induction and Co-induction
(co-inductive definitions and the (co-)inductive method)

What we proved?

F(A) is the set of judgements that can be inferred in one step from the judgments in A by using the rules.

A is

closed if F(A) ⊆ A
consistent if A ⊆ F(A)

The rules operator has both a least fixed point and a greatest fixed point, which are the smallest closed set and the largest consistent set:

\[ \text{lp}(F) = \bigcap \{A \mid F(A) \subseteq A\} \]
\[ \text{gp}(F) = \bigcup \{A \mid A \subseteq F(A)\} \]

Inductive and co-inductive interpretation of rules

\[ \text{lp}(F) = \bigcap \{A \mid F(A) \subseteq A\} \]
\[ \text{gp}(F) = \bigcup \{A \mid A \subseteq F(A)\} \]

- If F(A) ⊆ A then F_{\text{ind}} ⊆ A --- Induction proof principle
- If A ⊆ F(A) then A ⊆ F_{\text{coind}} --- Co-induction proof principle

Reasoning on equivalence of programs

We observe the termination of the term placed in a closing context, i.e.,

\[ M \downarrow \text{ and say that } M \text{ converges if } \exists V \ M \downarrow V \]

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Motivating example

\[
\begin{align*}
\text{one} & \overset{\text{def}}{=} \lambda x, y. x \ y \\
\text{two} & \overset{\text{def}}{=} \lambda x, y. x \ (y \ y) \\
\text{succ} & \overset{\text{def}}{=} \lambda x, y. x \ (\text{one} \ y)
\end{align*}
\]

Is it the case that \(\text{one} \leq \text{two}\), two holds?

Simulation

We consider weak call-by-name \(\lambda\) calculus. We write \(\Downarrow\) for \(\Downarrow^\uparrow\).

**Definition.** \(\leq_S\) is the largest fixed point of the following function on binary relations

\[
f(S) = \{(M, N) \mid \Downarrow^\uparrow M \implies N \Downarrow^\uparrow \land \forall P \in S \ (MP, NP) \in S\}
\]

Ex. 3 Simulation

To prove that \(M \leq_S N\) (\(M, N\) closed) it suffices to find a relation \(S\) which is a simulation and such that \(M \leq_S N\).

**Ex.**

i. Show that \(\leq_S\) is a preorder over \(\Lambda\) i.e. a reflexive and transitive binary relation

ii. Is the union of two simulations a simulation?

iii. If \(M \Downarrow^\uparrow V\) and \(N \Downarrow^\uparrow V\), \(M, N\) closed, then \(M \equiv_S N\). Prove it.

Bisimulation where the idea comes from?

The Reference:
**Homework**

**Ex 1. Co-continuity**

1. Prove that if \( F \) is co-continuous (or continuous), then it is also monotone. (Hint: take \( x \geq y \), and the sequence \( x, y, y, y, \ldots \).)

**Ex 2. Consider the strings over an alphabet \( \Sigma \)**

- Consider the set \( S \) co-inductively defined by the following rules (where \( \Sigma \) is an alphabet)
  \[
  \begin{array}{c}
  e \in S \\
  s \in S \\
  \sigma_1 \leq \sigma_2 \\
  \sigma_1 \leq \sigma_2 \\
  P(\lambda) = \{(s, t) \mid s \leq \sigma_1 \land t \leq \sigma_2 \} \\
  \end{array}
  \]
  EX. Prove that \( \text{aaa...} \leq \text{bbb...} \) (the two strings are infinite)

**Ex 3. Simulation**

To prove that \( M \preceq \Sigma N \) (\( M, N \) closed) it suffices to find a relation \( S \) which is in a simulation and such that \( M \preceq S \).

EX

1. Show that \( \preceq \) is a preorder over \( \lambda \) (is a reflexive and transitive binary relation)

2. Is the union of two simulations a simulation?

3. If \( M \parallel V \) and \( N \parallel V \), then \( M \parallel S \). Prove it.

Recall that:

- A relation is a set. In particular, a simulation is a subset of \( \text{Terms} \times \text{Terms} \) (where \( \text{Terms} \) is the set of terms).
- \( \preceq \) is the largest fixed point of a function \( f \) on binary relations. The function \( f \) is monotone.
Ex. 4  Contextual pre-order

Prove the following properties

\( \mathcal{L}_O \) is a pre-order (reflexive and transitive):

\[ \forall x, y, z \in \mathcal{L}_O \quad x \preceq y \iff \forall z \in \mathcal{L}_O \quad (x \preceq z \Rightarrow z \preceq y). \]