• RECAP:

**Cbn and Cbv Calculi.**

- The (pure) **Call-by-Name** calculus $\Lambda^{cbn} = (\Lambda, \rightarrow_{\beta})$ is the set of terms equipped with the contextual closure of the $\beta$-rule.

\[
(\lambda x. M)N \rightarrow_{\beta} M[N/x]
\]

- The (pure) **Call-by-Value** calculus $\Lambda^{cbv} = (\Lambda, \rightarrow_{\beta_v})$ is the same set equipped with the contextual closure of the $\beta_v$-rule.

\[
(\lambda x. M)V \rightarrow_{\beta_v} M[V/x] \quad \text{where } V \in \mathcal{V}
\]

**Head reduction in Cbn**

Head reduction is the closure of $\beta$ under head context

\[
\lambda x_1 \ldots x_n. (\| M_1 \ldots M_k)
\]

*Head normal forms (hnf)*, whose set is denoted by $\mathcal{H}$, are its normal forms.

- Given a rule $\rho$, we write $\rightarrow^v_\rho$ for its closure under head context.
- A step $\rightarrow^v_\rho$ is non-head, written $\rightarrow^h_\rho$ if it is not head.

**Weak reductions in Cbv**

The result of interest are **values** (i.e. functions).

In languages, in general the reduction is weak, that is, it does not reduce in the body of a function.

There are three main weak schemes: left, right and in arbitrary order.

**Left contexts $L$, right contexts $R$, and (arbitrary order) weak contexts $W$** are defined by

\[
L ::= \emptyset \mid L \cdot M \mid L
\]

\[
R ::= \emptyset \mid R \cdot M \mid R
\]

\[
W ::= \emptyset \mid W \cdot M \mid W
\]

Given a rule $\rightarrow^v_\rho$ on $\Lambda$, **weak reduction** $\rightarrow^v_\rho$ is the closure of $\rightarrow^v_\rho$ under context $W$.

A step $T \rightarrow S$ is non-weak, written $T \rightarrow \nexists S$ if it is not weak. Similarly for left ($\rightarrow^l_\rho$ and $\rightarrow^h_\rho$), and right ($\rightarrow^r_\rho$ and $\rightarrow^h_\rho$).

**Fact 3** (Weak normal forms). Given $M$ a closed term, $M$ is $\rightarrow^v_\rho$-normal iff $M$ is a value.

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• TD 1. We work on the properties and exercises which are highlighted

**BASIC PROPERTIES OF THE CONTEXTUAL CLOSURE**

If a step $T \rightarrow_{\beta} T'$ is obtained by closure under **non-empty context** of a rule $\rightarrow_{\gamma}$, then $T$ and $T'$ have the same shape, i.e. both terms are an application (resp. an abstraction, a variable).

**Fact 5** (Shape preservation).

- Assume $T = C[R] \rightarrow C[R'] = T'$ and that the context $C$ is non-empty. Then $T$ and $T'$ have the same shape.

- Hence, for any internal step $M \rightarrow M'$ ($s \in \{h, w, l, r, \ldots\}$) $M$ and $M$ have the same shape.
The following is an easy to verify consequence.

\textbf{Lemma 6} (Redexes preservation).
1. \textit{CbN}. Assume $T \xrightarrow{\beta} S$. $T$ is a $\beta$-redex iff so is $S$.
2. \textit{CbV}. Assume $T \xrightarrow{\beta_v} S$. $T$ is a $\beta_v$-redex iff so is $S$.

Fixed a set of redexes $\mathcal{R}$, $M$ is $w$-normal (resp. $h$-normal) if there is no redex $R \in \mathcal{R}$ such that $M = \mathbf{W}(\{R\})$ (resp. $M = \mathbf{H}(\{R\})$)

\textbf{Lemma 7} (Surface normal forms).
1. \textit{CbN}. Let $\mathcal{R}$ be the set of $\beta_v$-redexes.

Assume $M \xrightarrow{\mathcal{R}} \beta M'$. $M$ is $h$-normal $\iff$ $M'$ is $h$-normal.

2. \textit{CbV}. Let $\mathcal{R}$ be the set of $\beta_v$-redexes.

Assume $M \xrightarrow{\mathcal{R}} \beta_v M'$. $M$ is $w$-normal $\iff$ $M'$ is $w$-normal.

\begin{itemize}
\item Using FACTORIZATION
\end{itemize}

\textbf{CbN}:

\begin{itemize}
\item Head Factorization: $\xrightarrow{\beta'} \subseteq \xrightarrow{h} \cdot \xrightarrow{\beta} \cdot \xrightarrow{h}$.
\item EX. A Prove that $M$ has hnf if and only if head reduction from $M$ terminates.
\end{itemize}

\textbf{CbV}:

Left contexts $\mathbf{L}$, right contexts $\mathbf{R}$, and (arbitrary order) weak contexts $\mathbf{W}$ are def by

\begin{align*}
\mathbf{L} & ::= \emptyset \mid \mathbf{L} M \mid V \mathbf{L} \\
\mathbf{R} & ::= \emptyset \mid \mathbf{M} R \mid R V \\
\mathbf{W} & ::= \emptyset \mid \mathbf{W} M \mid M \mathbf{W}
\end{align*}

The closure under $\mathbf{L}$ (resp. $\mathbf{W}, \mathbf{R}$) context is noted $\xrightarrow{\mathcal{I}}$ (resp $\xrightarrow{w}$, $\xrightarrow{r}$)

Let $s \in \{w,l,r\}$

\begin{itemize}
\item weak factorization of $\xrightarrow{\beta_v}$:

$\xrightarrow{\beta_v} \subseteq \xrightarrow{\mathcal{I}} \cdot \xrightarrow{\beta_v} \cdot \xrightarrow{\mathcal{I}}$.
\end{itemize}

\textbf{Fact 7} (?). Let $M$ be a closed term. We say that $M$ returns a value when $M \xrightarrow{\beta}$ for some $V$.

\begin{itemize}
\item EX. B Prove any of the following
\item 1. $M$ returns a value, if and only if $\xrightarrow{\beta_v}$-reduction from $M$ terminates.
\item 2. $M$ returns value, if and only if $\xrightarrow{w}$-reduction from $M$ terminates.
\end{itemize}
EX. C. Normalization.
1. Give an inductive definition of **leftmost reduction**, completing the following b

Consider \((\Lambda, \to)\), where \(\to = \\to_\beta\). The relation \(\to_\lo \subseteq \to\) is induc

- If \(M \rightarrow M'\) then \(M \rightarrow_\lo M'\).
- If \(M \rightarrow_\h (i.e., M is h-normal) then:

\[
\begin{align*}
\frac{P \rightarrow_\lo P'}{M := (\lambda x. P) \rightarrow_\lo (\lambda x. P')} & \quad \frac{P \rightarrow_\lo P'}{M := PQ \rightarrow_\lo P'Q} & \quad \frac{Q \rightarrow_\lo Q'}{M := PQ \rightarrow_\lo PQ'}
\end{align*}
\]

2. Prove that **it is a normalizing strategy**, using head factorization.

Normalization (or, make your own Normalizing strategy)

**Definition 8** (Iteration of surface reduction). **Given** \((\Lambda, \to)\), where \(\to\) is the context closure of a rule \(b \in \{\beta, \beta_c\}\), let \(\rightarrow_\s \subseteq \to\) be as follows:

\(\rightarrow_\s = \rightarrow\) if \(b = \beta\) (CbN) \quad \rightarrow_\s \in \{\rightarrow_\w, \rightarrow_\i, \rightarrow_\g\} if \(b = \beta_c\) (CbN).

The relation \(\rightarrow_\s \subseteq \to\) is inductively defined by

- If \(M \rightarrow_\s M'\) then \(M \rightarrow_\s M'\).
- If \(M \rightarrow_\s (i.e., M is s-normal) then:

\[
\begin{align*}
\frac{P \rightarrow_\s P'}{M := (\lambda x. P) \rightarrow_\s (\lambda x. P')} & \quad \frac{P \rightarrow_\s P'}{M := PQ \rightarrow_\s P'Q} & \quad \frac{Q \rightarrow_\s Q'}{M := PQ \rightarrow_\s PQ'}
\end{align*}
\]

**Theorem 10** (Normalization).

CbN: \(\rightarrow_\beta\) is a normalizing strategy for \(\rightarrow_\beta\)

CbV: \(\rightarrow_\beta_c\) is a normalizing strategy for \(\rightarrow_\beta_c\).