Proof Nets
A graph syntax for proofs

Reference:
Notes on proof-nets by Olivier Laurent
(Note: most slides are taken from the notes of Olivier Laurent)

Recall that linear negation is defined:

$$(X^\perp)^\perp = X$$
$$(A \otimes B)^\perp = A^\perp \nabla B^\perp$$
$$(A \nabla B)^\perp = A^\perp \otimes B^\perp$$
Proof Nets

A graph syntax for proofs

Proof structures

A proof structure $M$ is a labelled directed acyclic graph (DAG) with possibly pending edges (i.e., some edges may have no source and/or no target) built over the alphabets of nodes which is represented below.

( Note: in figures, the edges orientation is always top-bottom. )

- The nodes are labelled by ax. cut. $\otimes$, $\Rightarrow$
- The edges are labelled by MLL formulas.

For each node/link, premises = entering edges, conclusions = exiting edges

The conclusions of $M$ is the set of pending edges of $M$.

In the graphical representation of a proof structure, we do not mention explicitly the direction of edges, but we draw them in such a way that direction is represented in a top-down way:

From proofs to proofs structures

example

Translate each of these sequent calculus proofs. Start from axioms...
**Proof Nets**

A PROOF NET is a proof structure which is the image of an MLL proof.

Internal condition!  
Purely geometrical conditions (correction) characterize the proof structures which are proof nets.

Theorem 1. A proof structure is correct iff it is a proof net.

Correctness criteria:
- (LT) Long trip [Girard]
- (AC) Connected-Acyclic [Danos-Regnier]
- ...

---

**Danos-Regnier Criterion**

Correctness Criterion

Switching Graphs

![Diagram of Switching Graphs]

Correctness
- Switching graphs are acyclic.
- Switching graphs are connected.

---

**Acyclicity**. A multiplicative proof structure is acyclic if its switching graphs do not contain any undirected cycle.

A proof structure with $p \Rightarrow q$ nodes induces $2^p$ switchings and thus $2^p$ switching graphs. A switching graph is not a proof structure in general since its $\Rightarrow$ nodes have only one premise.

A connected component of a switching graph is a connected component of its underlying (undirected) multigraph.

---

**Is this correct?**

PN1:

![Diagram of PN1]

PN2:

![Diagram of PN2]
• Correctness guarantees:
  ✔ Graph is image of a proof (sequentialization)
  ✔ Normalization progresses (no deadlocks)
  ✔ Normalization terminates (no infinite cycles)

**Soundness**

**Proposition 4.1.1 (Soundness of Correctness),** The translation of a sequent calculus proof of $MLL$ is a connected multiplicative proof net.

**Sequentialization**

**Theorem 4.1.1 (Sequentialization),** Any connected multiplicative proof net is the translation of a sequent calculus proof of $MLL$.

**The beauty of proof nets is normalization**
Let us try out!

How we write a proof net of these conclusions?

\[ A \otimes A \vdash A \]

must type an edge conclusion of a par link, with premisses ....

\[ A \& A \vdash A \]

must type an edge conclusion of a tensor link, with premisses ....

Then we have to choose the axiom links!

Properties of normalization

1. Confluence?
2. Is normalization weakly/strongly normalizing?
3. Would you be able to define a normalizing strategy?
4. Would you be able to define a normalizing strategy which reaches normal form in a minimal number of steps?

Normalization of MLL proof-nets:

- Strongly normalizing
- Confluent
- Cut elimination:
  a proof-net in normal form contains no cuts

Let us try out!

Write a proof net with this conclusion... and normalize it
• Let us try out another example

Let us try one more. First, write a proof net with this conclusion...

\[
\begin{align*}
(X \otimes X) \rightarrow & (X \otimes X) \\
(X \otimes X) \rightarrow & (X \otimes X) = (X^1 \rightarrow X^1) \rightarrow (X \otimes X)
\end{align*}
\]

TIP: How we write a proof net? As before, all proof nets with the same conclusion start with the same nodes (the formula tree!) What distinguishes different proofs are the axiom links

To distinguish the different occurrences of atoms, let us write indices:

\[(X^1 \otimes X^2) \equiv (X_3 \otimes X_4)\]

In this case, we have two possible proofs, corresponding to two possible ways to write axioms:

1. 3, and 2, 4
2. 1, 4, and 2, 3

In sequent calculus, they correspond to these two proofs (one uses exchange, one no)

In sequent calculus, they correspond to these two proofs (one uses exchange, one no)

\[
\begin{align*}
\vdash & X_1 \vdash X_2 \\
\vdash & X_1 \vdash X_2 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4 \\
\vdash & X_1 \vdash X_2, X_3 \otimes X_4
\end{align*}
\]

Let us indicate the formula \((X^1 \otimes X^2) \equiv (X_3 \otimes X_4)\) with B (for boolean)

We call one proof true, and the other false...

We can feed one of our two values to a proof which takes a boolean, and return a boolean.

We know that the normal form (i.e., the result of computation) will be of type B... Hence one of our two values.

Try to normalize one of the proofs of (\(X^1 \otimes X^2) \equiv (X_3 \otimes X_4)\)

with the proof net which has conclusions

\[(X_1 \otimes X_2) \otimes (X_3 \otimes X_4) \rightarrow (X_1 \otimes X_4) \rightarrow (X_7 \otimes X_4)\]

and axiom links: (1, 6) (2, 5) (3, 7) (4, 8)

What is the function coded by this proof net?
Correctness: If we focus on acyclicity, Danos-Regnier criterion can be reformulated (in equivalent way).

Let $R$ be a proof structure; a switching path of $R$ is a path which does not use any two edges entering on the same $\Rightarrow$ link (such edges are called switching edges); a switching cycle is a switching path which is a cycle.

Definition 3 (Correctness criterion). A proof structure is correct if it does not contain any switching cycle.

Exponentials

- Boxes never overlap.
- Boxes are sequential (as rules in sequent calculus).
- Correctness: box by box.
- Boxes permit duplication and erasure.

Cut-elimination steps
What is the associated proof-net?

\[
\begin{align*}
\frac{\Gamma, A, A^- \alpha}{\Gamma} \\
\frac{\Gamma, A \supset A^+ \gamma}{\Gamma} \\
\frac{\Gamma, A \supset A^+, ?B}{\Gamma} \\
! A \otimes ! A \vdash ! A \\
? A \supset ? A \vdash ? A
\end{align*}
\]

Can you write the proof so that all axioms are atomic (i.e., on atomic formulas)?

Reduction steps

Reduction Steps: ?d

Reduction Steps: ?w

Reduction Steps: ?e

Reduction Steps: ?w
Confluence?
Weak Normalization?
Strong Normalization?