The transitive-reflexive closure of a relation is a closure operator, i.e. satisfies
\[ \rightarrow \subseteq \rightarrow^*, \quad (\rightarrow^*)^* = \rightarrow^*, \quad \rightarrow_1 \subseteq \rightarrow_2 \implies \rightarrow_1^* \subseteq \rightarrow_2^* \]

As a consequence
\[ (\rightarrow_1 \cup \rightarrow_2)^* \subseteq (\rightarrow_1^* \cup \rightarrow_2^*)^* \]

**EX1.** Prove it!

**CONFLUENCE**

- **Confluent**

```
x * y1
```

- **Locally confluent**

```
x * y2
```

- **Strongly confluent**

```
x * y
```

- **Diamond**

```
x * y
```

**EX 2**

(a) Prove that strongly confluent implies confluent
(b) As a preliminary step, prove: \( \leftrightarrow^* \rightarrow \subseteq \rightarrow \leftrightarrow^* \) implies confluence

**EX 4.** Two relations commute if

\[ \alpha \text{ and } \beta \text{ commute} \]

```
\alpha
```

```
\beta
```

```
\alpha
```

```
\beta
```
Prove that

- **Lemma (Hindley-Rosen).** Let $\rightarrow_1$ and $\rightarrow_2$ be relations on the set $A$. If $\rightarrow_1$ and $\rightarrow_2$ are confluent and commute with each other, then $\rightarrow_1 \cup \rightarrow_2$ is confluent.

---

**EX. 5** Two relations strongly commute if

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\beta \\
\end{array}
\quad
\begin{array}{c}
\beta \\
\uparrow \\
\alpha \\
\end{array}
\]

Prove that strong commutation implies commutation

---

**TERMINATION**

- The element $s$ is $\mathcal{R}$-weakly normalising (WN) iff $s$ has at least one normal form
- The element $s$ is $\mathcal{R}$-strongly normalising (SN) iff there is no infinite sequence

---

Consider

\[
\begin{array}{c}
\ldots \\
\end{array}
\]

---

- **EX** Say which properties hold
  1. Confluent
  2. Locally confluent
  3. Normalising (weakly normalising, WN)
  4. Terminating (strongly normalising, SN)

---

- **EX. 8**

**Newman’s Lemma.** Every terminating and locally confluent ARS is confluent.

**A second Proof.**
It suffices to show that every element has unique normal forms
• suppose $B = \{ a \in A \mid \neg UN(a) \} \neq \emptyset$

• let $b \in B$ be minimal element (with respect to $\rightarrow$)

• $b \rightarrow^{1} n_{1}$ and $b \rightarrow^{1} n_{2}$ with $n_{1} \neq n_{2}$

➢ Conclude by showing that it is impossible (absurd)