LUDICS WITH REPETITIONS (EXPONENTIALS, INTERACTIVE TYPES AND COMPLETENESS)

MICHELE BASALDELLA AND CLAUDIA FAGGIAN

Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan.
e-mail address: mbasalde@kurims.kyoto-u.ac.jp

Équipe PPS, CNRS and Université Paris 7, 2 place Jussieu, case 7017, 75251 Paris cedex 05, France.
e-mail address: faggian@pps.jussieu.fr

Abstract. Ludics is peculiar in the panorama of game semantics: we first have the definition of interaction-composition and then we have semantical types, as a set of strategies which "behave well" and react in the same way to a set of tests. The semantical types which are interpretation of logical formulas enjoy a fundamental property, called internal completeness, which characterizes ludics and sets it apart also from realizability. Internal completeness entails standard full completeness as a consequence.

A growing body of works start to explore the potentiality of this specific interactive approach. However, ludics has two main limitations, which are consequence of the fact that in the original formulation, strategies are abstractions of **MALL** proofs. On one side, no repetitions are allowed. On the other side, there is a huge amount of structure, and the proofs tend to rely on the very specific properties of the **MALL** proof-like strategies, making it difficult to transfer the approach to semantical types into different settings.

In this paper, we provide an extension of ludics which allows repetitions and show that one can still have interactive types and internal completeness. From this, we obtain full completeness w.r.t. polarized **MELL**. In our extension, we use less structure than in the original formulation, which we believe is of independent interest. We hope this may open the way to applications of ludics approach to larger domains and different settings.

1. Introduction

Ludics is a research program started by Girard [20] with the aim of providing a foundation for logic based on interaction. It can be seen as a form of game semantics where *first* we have the definition of *interaction* (equivalently called composition, normalization), and *then* we have semantical *types*, as sets of strategies which "behave well" with respect to composition. This role of interaction in the definition of types is where lies the specificity of ludics in the panorama of game semantics.

1998 ACM Subject Classification: F.4.1 Mathematical Logic (Proof theory), F.3 Logics and Meanings of Programs.

Key words and phrases: linear logic, ludics, game semantics, internal completeness.

This paper is a completely revised and extended version of [4].

LOGICAL METHODS IN COMPUTER SCIENCE

DOI:10.2168/LMCS-???

Recently, a growing body of work is starting to explore and to develop the potential of this specific approach, and to put at work the more general notion of type offered by ludics: the notion of *type defined through interaction*.

We mention in particular work by Saurin on interactive proof-search as a logic programming paradigm [29], and work by Terui on computability [30]. Terui gives an especially interesting use of the notion of *orthogonality* ("to interact well"): if the *strategy* $\mathcal{D}$ describes an automaton, $\{\mathcal{D}\}^\perp$ (the set of all strategies which "interact well" with it) consists of the languages accepted by that automaton. Moreover, interactive types seem to be a natural domain for giving models of process calculi, Faggian and Piccolo [14] have shown a close correspondence of ludics with the linear π-calculus [31].

More recently, in [5] Basaldella and Terui have studied the traditional logical duality between proofs and models in the setting of computational ludics [30] enriched with exponentials (following our approach to exponentials [4], this paper). Both proofs and models live in an homogeneous setting, both *are strategies*, which are related by *orthogonality*.

Interactive types. The computational objects of ludics — *designs* — can be seen as a linear form of Hyland-Ong (HO) innocent strategies (as shown in [12]) or as Curien's abstract Böhm trees [7, 10].

However, in game semantics, we first define the types (*arenas*, *games*), and then the composition of strategies; the type guarantees that strategies compose well. In ludics, *strategies are untyped*, in the sense that all strategies are given on a universal arena (the arena of all possible moves); strategies can always interact with each other, and the interaction may *terminate well* (the two strategies "accept each other", and are said *orthogonal*) or not (they *deadlock*). An *interactive type* is a set of strategies which "compose well", and reacts in the same way to a set of tests (see Section 4). A semantical type $\mathbf{G}$ is any set of strategies which reacts well to the same set of tests $\mathbf{E}$, which are themselves strategies (counter-strategies), that is $\mathbf{G} = \mathbf{E}^\perp$.

Internal completeness. With ludics, Girard also introduces a new notion of completeness, which is called *internal completeness* (see Section 5). This is a key — *really characterizing* — element of ludics. We have already said that a semantical type is a set of strategy closed by biorthogonal ($\mathbf{G} = \mathbf{G}^{\perp\perp}$). Internal completeness is the property which says that *the constructions on semantical types do not require any closure,* i.e.*, are already closed by biorthogonal*.

While it is standard in realizability that a semantical type is a set $\mathbf{S}$ of terms closed by biorthogonal ($\mathbf{S} = \mathbf{S}^{\perp\perp}$), when interpreting types one has to perform some kind of closure, and this operation can *introduce new terms*. For example, the interpretation of $A \oplus B$ is $(\mathbf{A} \cup \mathbf{B})^{\perp\perp}$. This set of terms could be in general strictly greater than $\mathbf{A} \cup \mathbf{B}$. We have internal completeness whenever $\mathbf{A} \cup \mathbf{B}$ is *proved* to be equal to $(\mathbf{A} \cup \mathbf{B})^{\perp\perp}$. Since the closure by biorthogonal does not introduce new terms, $\mathbf{A} \cup \mathbf{B}$ already gives a *complete description of what inhabits the semantical type.*

In Girard's paper [20], the semantical types which are interpretation of formulas enjoy internal completeness. This is really the key property (and the one used in [29, 30]). Full completeness (for multiplicative-additive-linear logic **MALL**, in the case of [20]) directly *follows* from it.

1.1. **Contributions of the paper.** The purpose of this paper is two-fold.

On the one hand, we show that it is possible to overcome the main limitation of ludics, namely the constraint of linearity, hence the lack of exponentials: we show that internal completeness (and from that full completeness) can be obtained also when having repetitions, if one extends in a rather natural way the setting of ludics.

On the other hand, we provide proofs which use less structure than the original ones given by Girard. Not only we believe this improve the understanding of the results, but — more fundamentally — we hope this opens the way to the application of the approach of ludics to a larger domain.

We now give more details on the content of the paper.

1.1.1. *Ludics architecture.* A difficulty in [20] is that there is a huge amount of structure. Strategies are an abstraction of **MALL** proofs, and enjoy many good properties (analytical theorems). In [20], all proofs of the high level structure of ludics make essential use of these properties. Since some of those properties are very specific to the particular nature of the objects, this makes it difficult in principle to extend the — very interesting — approach of ludics to a different setting, or build the interactive types on different computational objects.

Ludics, as introduced in [20], is constituted by several layers.

- At the *low level*, there is the definition of the *untyped computational structures* (strategies, there called *designs*) and their *dynamics* (interaction). Interaction allows the definition of *orthogonality.*
  - The computational objects satisfy certain remarkable properties, called *analytical theorems*, in particular **separation property**, the ludics analogue of Böhm theorem:

    two strategies $\mathcal{A}, \mathcal{B}$ are syntactically equal if and only if they are observationally equal (*i.e.*, for any counter-strategy $\mathcal{C}$, the strategies $\mathcal{A}, \mathcal{B}$ react in the same way to $\mathcal{C}$).
- At the *high level*, there is the definition of *interactive types*, which satisfy *internal completeness.*

By relying on less structure, we show that the high level architecture of ludics is somehow independent from the low level entities (strategies), and in fact could be built on other — more general — computational objects.

In particular, separation is a strong property. It is a great property, but it is not a common one to have. However, the fact that computational objects do not enjoy separation does not mean that it is not possible to build the "high level architecture" of ludics. In fact, we show (Section 5) that the proofs of internal and full completeness rely on much less structure, namely operational properties of the interaction.

We believe that discriminating between internal completeness and the properties which are specific to the objects is important both to improve understanding of the results, and to make it possible to build the same construction on different entities.

In particular, strategies with repetitions have weaker properties with respect to the the original — linear — ones. We show that it is still possible to have interactive types, internal completeness, and from this full completeness for polarized **MELL** (multiplicative-exponential-linear logic). The extension to full polarized linear logic **LLP** [26] is straightforward.

1.1.2. *Exponentials in ludics.* The treatment of exponentials has been the main open problem in ludics since [20]. Maurel [27] has been the first one to propose a solution (a summary of this solution can be found in [10, 21]). The focus of Maurel's work is to recover a form of separation when having repetitions; to this purpose, he develops a sophisticated setting, which is based on the use of *probabilistic strategies*: two probabilistic strategies "compose well" with a certain probability. This approach is however limited by its technical complexity; this is the main obstacle which stops Maurel from going further, and studying interpretation and full completeness issues.

In this work, we do not analyze the issue of separation, while we focus exactly into interactive types and internal completeness, and develop a fully complete interpretation from it.

Maurel also explores a simpler solution in order to introduce exponentials, but he does not pursue it further because of the failure of the separation property. Our work starts from an analysis of this simpler solution, and builds on it.

1.1.3. *Our approach.* In the literature, there are two standard branches of game semantics which have been extensively used to build denotational models of various fragments of linear logic. On the one hand, we have Abramsky-Jagadeesan-Malacaria style game semantics (AJM) [1] which is essentially inspired by Girard's geometry of interaction [17]. On the other hand, we have Hyland-Ong style game semantics (HO) [24], introducing *innocent strategies*. The main difference between those two game models is how the semantical structures corresponding to exponential modalities are built. In AJM, given a game A, $!A$ is treated as an infinite tensor product of A, where each copy of A receives a different labeling index. Two strategies in $!A$ which only differ by a different labeling of moves are identified. By contrast, in HO the notion of justification pointer substitutes that of index. The games A and $!A$ share the same arena. Informally, a strategy in $!A$ is a kind of "juxtaposition" of strategies of A such that by following the pointer structure, we can *unambiguously* decompose it as a set of strategies of A.

Girard's designs [20] are a linear form of HO innocent strategies [12]. Hence, the most natural solution to extend ludics to the exponentials is to consider as strategies, *standard HO innocent strategies* (on an universal arena). But in order to do so, there is a new kind of difficulty, which we deal with in this paper: we needs to have enough tests.

More precisely, as we illustrate in Section 6, we need *non-uniform* counter-strategies. We implement and concretely realize this idea of non-uniform (non-deterministic) tests by introducing a *non-deterministic sum* of strategies, which are based on work developed by Faggian and Piccolo [15]. More precise motivations and a sketch of the solution are detailed in Section 6.4.

1.2. **Plan of the paper.** In Section 2, we introduce the polarized fragment of linear logic **MELLS** for which we will show a fully complete model in Section 10.

In Section 3, we recall the basic notions of HO innocent game semantics, which we then use in Section 4 to present Girard's ludics.

In Section 5 we review the results of internal completeness for linear strategies and outline a direct proof of full completeness.

In Section 6, we provide the motivations and an informal description of non-uniform strategies, and in Section 7 we give the formal constructions.

In Section 8 we describe in detail the composition of non-uniform strategies.

In Section 9 we introduce semantical types for **MELLS**, and we extend internal completeness to non-linear strategies.

Full completeness is developed in Section 10.

2. Calculus

In this section, we introduce a calculus that we call **MELLS**, which is a variant of polarized **MELL** based on synthetic connectives. In Section 10, we prove that our model is fully complete for **MELLS**.

2.1. MELL and polarization.

Formulas of propositional multiplicative-exponential linear logic **MELL** [16] are finitely generated by the following grammar:

$$F ::= X \mid X^\perp \mid \mathbf{0} \mid \top \mid \mathbf{1} \mid \perp \mid F \otimes F \mid F \parr F \mid !F \mid ?F,$$

where $X, X^\perp$ are propositional variables (also called *atoms*). We give the sequent calculus in Figure 1. The notions of sequent, rule, derivation, etc for **MELL** are the standard ones.

$$\frac{}{\vdash X^\perp, X}\ \text{Ax} \qquad \frac{\vdash \Gamma, F \qquad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta}\ \text{Cut}$$

$$\frac{}{\vdash \mathbf{1}}\ \mathbf{1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \frac{}{\vdash \Gamma, \top}\ \top$$

$$\frac{\vdash \Gamma, F \qquad \vdash \Delta, G}{\vdash \Gamma, \Delta, F \otimes G}\ \otimes \qquad \frac{\vdash \Gamma, F, G}{\vdash \Gamma, F \parr G}\ \parr \qquad \frac{\vdash ?\Gamma, F}{\vdash ?\Gamma, !F}\ !$$

$$\frac{\vdash \Gamma, F}{\vdash \Gamma, ?F}\ ? \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ \text{C} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?F}\ \text{W}$$

Figure 1: **MELL**

Linear logic distinguishes formulas into:

- *linear formulas*: $\mathbf{0}, \mathbf{1}, \top, \perp, F \otimes F, F \parr F$;
- *exponential formulas*: $?F, !F$.

Linear formulas can only be used once, while the modalities $!, ?$ allow formulas to be repeated. The possibility of repeating formulas is expressed in the sequent calculus by the *contraction* rule on $?F$ formulas:

$$\frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ \text{C}$$

Dually, the modality ! allows proofs to be reused during cut-elimination procedure. In fact, we have that:

$$\frac{\dfrac{\dfrac{\vdots\, \pi}{\vdash ?F, ?F, \Gamma}}{\vdash ?F, \Gamma}\ \text{C} \qquad \dfrac{\vdots\, \rho}{\vdash !(F^\perp)}}{\vdash \Gamma}\ \text{Cut} \quad \rightsquigarrow \quad \frac{\dfrac{\dfrac{\vdots\, \pi}{\vdash ?F, ?F, \Gamma} \qquad \dfrac{\vdots\, \rho}{\vdash !(F^\perp)}}{\vdash ?F, \Gamma}\ \text{Cut} \qquad \dfrac{\vdots\, \rho}{\vdash !(F^\perp)}}{\vdash \Gamma}\ \text{Cut}$$

The proof ρ can be used several times, once for each duplication of $?F$.

Polarization. The connectives and constants of **MELL** can be split also according to their *polarity*, into two classes:

$$\textbf{Positive} \ : \ \mathbf{0}, \mathbf{1}, \otimes, ?, \qquad \textbf{Negative} \ : \ \top, \bot, \parr, !.$$

This distinction is motivated by properties of the connectives in proof construction [2, 19, 8], which we will briefly recall below. In particular, the rules which introduce negative connectives $\top, \bot, \parr, !$ are *reversible*: in the bottom-up reading, the rule is deterministic, *i.e.*, there is no choice. By contrast, a rule decomposing a positive connective involves a choice, *e.g.*, the spitting of the context in the $\otimes$ rule.

For the exponential modalities, the situation is a bit more complex[1].

There is not a well established notation for exponentials in a polarized setting. Following [31], we have chosen to write ! for the **negative** modality, and ? for the **positive** modality, because these symbols are more familiar. However, the reader should be aware that in a setting such as in [25, 8], the same connectives would be indicated by $\sharp$ (*negative modality*), and $\flat$ (*positive modality*). The contraction rule would be written as:

$$\frac{\vdash \flat P, \flat P, \Gamma}{\vdash \flat P, \Gamma}\,\mathrm{C}$$

2.2. **Synthetic connectives: MELLS.** We now introduce in detail the calculus **MELLS**. Formulas are here built by *synthetic connectives* [19, 8] *i.e.*, maximal clusters of connectives of the same polarity. The key ingredient that allows for the definition of synthetic connectives is *focalization* [2], a property which is based on the polarity of the formulas (*i.e.*, the polarity of the outermost connective). In [2], Andreoli demonstrates that if a sequent is provable in full linear logic, then it is provable with a proof which satisfies the following proof-search strategy (which is therefore complete).

In the bottom-up construction of a proof:

(1) If there is a negative formula, keep on decomposing it until we get to atoms or positive subformulas.
(2) If there are not negative formulas, choose a positive formula, and keep on decomposing it until we get to atoms or negative subformulas.

From the point of view of logic, focalization means that each cluster of formulas with the same polarities can be introduced by a single rule (with several premises), which allows for the definition of synthetic connectives. By using synthetic connectives, formulas are in a canonical form, where immediate subformulas have opposite polarity. This means that in a (cut-free) proof of **MELLS**, there is a *positive/negative alternation of rules*, which matches the standard Player (positive)/ Opponent (negative) alternation of moves in a strategy (see Section 3).

[1] In a polarized setting (such as [18, 25]), exponentials are often analyzed by decomposing them into:

$$!F =: \downarrow\sharp F, \qquad ?F =: \uparrow\flat F,$$

where $\sharp$ is negative, $\flat$ is positive, and $\downarrow$, $\uparrow$ are operators which change the polarity.

Formulas of **MELLS**. Formulas of **MELLS** split into positive P and negative N formulas, and they are finitely generated by the following grammar:

$$P := ?\mathbf{0} \mid ?X \mid ?(N \otimes \cdots \otimes N); \qquad N := !\top \mid !X^{\perp} \mid !(P \parr \cdots \parr P);$$

where X and $X^{\perp}$ are propositional variables.

We will use F as a variable for formulas and indicate the polarity also by writing F^{+} or F^{-}. To stress the immediate subformulas, we often write $F^{+}(N_1, \dots, N_n)$ and $F^{-}(P_1, \dots, P_n)$.

The involutive linear negation $^{\perp}$ is defined as usual:

$$(?\mathbf{0})^{\perp} := !\top; \qquad (?X)^{\perp} := !X^{\perp}; \qquad (?(N_1 \otimes \cdots \otimes N_n))^{\perp} = !({N_1}^{\perp} \parr \cdots \parr {N_n}^{\perp}).$$

A sequent of **MELLS** is a multi-set of formulas written $\vdash F_1, \dots, F_n$ such that it contains at most one negative formula.

Rules. For Γ multi-set of positive formulas, we have:

$$\frac{\vdash N_1, F^{+}, \Gamma \dots \vdash N_n, F^{+}, \Gamma}{\vdash F^{+}, \Gamma}\ \text{Pos} \qquad \frac{\vdash P_1, \dots, P_n, \Gamma}{\vdash F^{-}, \Gamma}\ \text{Neg}$$

$$\frac{}{\vdash ?X, !X^{\perp}, \Gamma}\ \text{Ax} \qquad \frac{}{\vdash !\top, \Gamma}\ !\top \qquad \frac{\vdash P, \Xi, \Gamma \qquad \vdash P^{\perp}, \Gamma}{\vdash \Xi, \Gamma}\ \text{Cut}$$

where Ξ is either empty or consisting of one negative formula.

Notice that usual linear logic structural rules (weakening, contraction, promotion and dereliction) are always *implicit* in our calculus.

Example 2.1. In standard **MELL** calculus, our positive rule

$$\frac{\vdash A, ?(A \otimes B), ?C \quad \vdash B, ?(A \otimes B), ?C}{\vdash ?(A \otimes B), ?C}\ Pos$$

would be decomposed as a dereliction and some contraction steps:

$$\cfrac{\cfrac{\cfrac{\vdash A, ?(A \otimes B), ?C \quad \vdash B, ?(A \otimes B), ?C}{\vdash A \otimes B, ?(A \otimes B), ?(A \otimes B), ?C, ?C}\ \otimes}{\vdash ?(A \otimes B), ?(A \otimes B), ?(A \otimes B), ?C, ?C}\ ?}{\cfrac{\vdots \text{ contractions } \vdots}{\vdash ?(A \otimes B), ?C}}$$

Proposition 2.2. *Cut-elimination property holds for* **MELLS**.

Remark 2.3 (Intuitionistic logic)**.** The calculus introduced above can be seen as a focalized version of the $\neg, \wedge$ fragment of the sequent calculus for intuitionistic logic **LJ** (see Appendix A).

3. HO innocent game semantics

An *innocent strategy* [24] can be described either in terms of all possible interactions for the player (*strategy as set of plays*), or in a more compact way, which provides only the minimal information for Player to move (*strategy as set of views*) [22, 9]. It is standard that the two presentations are equivalent: from a play one can extract the views, and from the views one can calculate the play.

In this paper we use the "strategy as set of views" description. Our presentation of innocent strategies adapts the presentation by Harmer [23] and Laurent [25].

Before introducing the formal notions, let us use an image. A strategy tells the player how to respond to a counter-player move. The dialog between two players — let us call them P (Player) and O (Opponent) — will produce an *interaction* (a play). The "universe of moves" which can be played is set by the *arena*. Each move belongs to only one of the players, hence there are P-moves and O-moves. For P, the moves which P plays are positive (active, output), while the moves played by O are negative (passive, input), to which P has to respond.

Polarities. Let Pol be the set of polarities, which here are positive (for Player) and negative (for Opponent); hence we have $Pol = \{+, -\}$. We use the symbol ϵ as a variable to range on polarities.

Arenas. An arena is given by *a directed acyclic graph*, d.a.g. for short, which describes a dependency relation between moves and *a polarity function*, which assigns a polarity to the moves.

Definition 3.1 (Arena)**.** An **arena** $(A, \vdash_A, \lambda_A)$ is given by:

- a **directed acyclic graph** $(A, \vdash_A)$ where:
  - A (elements of the d.a.g.) is the set of **moves**;
  - $\vdash_A$ (edges of the d.a.g.) is a well founded, binary **enabling relation** on A. If there is an edge from m to n, we write $m \vdash_A n$. We call **initial** each move m such that no other move enables it, and we write this as $\vdash_A n$.
- a **function** $\lambda_A : A \to Pol$ which labels each element with a polarity.

Enabling relation and polarity have to satisfy the following property of **alternation**:

if $n \vdash_A m$, they have opposite polarity.

If all the initial moves have the same polarity ϵ, we say that ϵ is the polarity of the arena. In this case we say that A is a **polarized arena** (of polarity ϵ) [25].

Strategies.

Definition 3.2 (Justified sequences)**.** Let A be an arena.

A **justified sequence** $s = s_0.s_1.\dots.s_n$ on A is a string $s \in A^*$ with pointers between the elements in the string which satisfies:

- *Justification.* For each non-initial move s_i of s, there is a unique pointer to an earlier occurrence of move s_j, called the justifier of s_i, such that $s_j \vdash_A s_i$.

The **polarity** of a move in a justified sequence is given by the arena. We sometimes put in evidence the polarity of a move x by writing x^+ or x^-.

Definition 3.3 (Views). A **view** s on A is a justified sequence on A which satisfies:

- *Alternation.* No two following moves have the same polarity.
- *View.* For each non-initial negative (Opponent) move s_i, its justifier is the immediate predecessor s_{i-1}.

In the following, we will use another formulation of view, originally suggested by Berardi. It is equivalent to Definition 3.3 here, but will allows for a generalization.

Proposition 3.4 (View). *The two following definitions are equivalent:*

(1) *Definition* 3.3.
(2) *A* **view** *s on A is a justified sequence where for each pair of consecutive actions s_i, s_{1+1} such that*

$$\lambda(s_i) = + \text{ or } \lambda(s_{i+1}) = -$$

we have that $s_i \vdash_A s_{i+1}$.

Definition 3.5 (Strategies). A **strategy** $\mathcal{D}$ on A, denoted by $\mathcal{D} : A$ is a prefix-closed set of views, such that:

- *Coherence.* If $s.m, s.n \in \mathcal{D}$ and $m \neq n$ then m, n are negative.
- *Maximality.* If $s.m$ is maximal in $\mathcal{D}$ (i.e. no other view extends it), then m is positive.

We call *positive* (resp. *negative*) a strategy on a positive (resp. negative) arena.

Composition of strategies. Composition of strategies as set of views has been studied in particular by Curien and Herbelin, who introduce the View-Abstract-Machine (VAM) [7, 10] by elaborating Coquand's Debates machine [6].

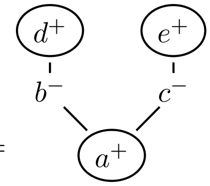
Notation. Emphasizing the tree structure of a strategy, it is often convenient to write a strategy whose first move is x as $\mathcal{D} = x.\mathcal{D}'$.

More precisely, if $\mathcal{D}$ is a positive strategy, we write it as $\mathcal{D} = a.\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$, instead of $\{a.s : s \in \mathcal{E}_i, 1 \leq i \leq n\}$, where $\mathcal{E}_i$ are negative strategies; conversely, if $\mathcal{E}$ is a negative strategy of root a, we write $\mathcal{E} = a.\mathcal{D}$ instead of $\{a.s : s \in \mathcal{D}\}$, where $\mathcal{D}$ is a positive strategy.

To better grasp the intuitions, we will draw strategies as trees whose nodes are labeled by moves. Nodes which are labeled by positive moves are circled.

Example 3.6. The positive strategy $\mathcal{D} = a^+.\{b^-.d^+, c^-.e^+\}$ can be represented by the following tree:

$\mathcal{D} =$



4. Ludics

In this and next section we give a compact but complete presentation of ludics [20], introducing all definitions and technical results which are relevant to our approach, including internal completeness and full completeness. Our choice here is to give a presentation which fits into the language of game semantics, and then restrict our attention to "linear strategies", and more specifically the ludics setting.

Let us first stress again the peculiarity of ludics in the panorama of game semantics. In game semantics, one defines constructions on arenas which correspond to the interpretation of types. A strategy is always "typed", in the sense that it is a strategy on a specific arena: first we have the "semantical type" (the arena), and then the strategy on that arena. When strategies are opportunely typed, they interact (compose) well.

In the approach of ludics, there is only one arena (up to renaming): the universal arena of all possible moves. Strategies are "untyped", in the sense that all strategies are defined on the universal arena. Strategies then interact with each other, and the interaction can *terminate well* (the two strategies "accept" each other) or not (*deadlock*).

Two opposite strategies $\mathcal{D}, \mathcal{E}$ whose interaction terminates well, are said **orthogonal**, written $\mathcal{D} \perp \mathcal{E}$.

Orthogonality allows us to define interactive types. A semantical type $\mathbf{G}$ is any set of strategies which react well to the same set of tests $\mathbf{E}$, which are themselves strategies (counter-strategies), that is $\mathbf{G} = \mathbf{E}^{\perp}$.

Daimon. The program of ludics was to overcome the distinction between syntax (the formal system) on one side and semantics (its interpretation) on the other side. Rather then having two separate worlds, proofs are interpreted via proofs. To determine and test properties, a *proof* of A should be tested with *proofs* of $A^{\perp}$. Ludics provides a setting in which proofs of A interact with proofs of $A^{\perp}$; to this end, it generalizes the notion of proof.

A proof should be thought in the sense of "proof-search" or "proof-construction": we start from the conclusion, and guess a last rule, then the rule above. What if we cannot apply any rule? A new rule is introduced, called **daimon**:

$$\frac{}{\vdash \Gamma}\ \dagger$$

Such a rule allow us to assume any conclusion, or said in other words, it allows to close any open branch in the proof-search tree of a sequent.

In the semantics, the daimon is a special action which acts as a *termination signal*.

4.1. Strategies on a universal arena.

Strategies on a universal arena. Strategies communicate on names. We can think of names as process algebras channels, which can be used to send outputs (if positive) or to receive inputs (if negative). Each strategy $\mathcal{D}$ has an *interface*, which provides the names on which $\mathcal{D}$ can communicate with the rest of the world, and the use (input/output) of each name.

A **name** (called *locus* in [20]) is a string of natural numbers. We use the variables $\xi, \sigma, \alpha, \ldots$ to range over names.Two names are *disjoint* if neither is a prefix of the other.

An **interface** Γ (called *base* in [20]) is a finite set of pairwise disjoint names, together with a polarity for each name, such that at most one name is negative. If a name ξ has polarity ϵ, we write $\xi^\epsilon \in \Gamma$. An interface Γ is *negative* if it contains a negative name, *positive* otherwise. In particular, the empty interface is positive.

An **action** x is either the symbol $\dagger$ (called **daimon**) or a pair (ξ, I), where ξ is a name, and I is a finite subset of $\mathbb{N}$. Since in this paper we are not interested in interpreting the additives, from now on, we always assume that I is an initial segment of $\mathbb{N}$ *i.e.*, $I = \{1, 2, \dots, n\}$. We can think on the set I only just as an "arity provider".

Given an action (ξ, I) on the name ξ, the set I indicates the names $\{\xi i : i \in I\}$ which are generated from ξ by this action. The prefix relation (written $\xi \sqsubseteq \sigma$) induces a natural relation of dependency on names, which generates an arena.

Given an interface Γ, we call **initial actions** the action $\dagger$ and any action (ξ, I) such that $\xi \in \Gamma$.

Definition 4.1 (Universal arena on an interface)**.** Given an interface Γ, the **universal arena** $U(\Gamma)$ on Γ is the tuple $(U(\Gamma), \vdash, \lambda)$ where:

- The **set of moves** is the special action $\dagger$ together with the set of all actions of the form (ξ', I), for any $\xi \sqsubseteq \xi'$, $\xi \in \Gamma$ and I.
- The **polarity** of the initial actions (ξ, I) is the one indicated by the interface for ξ; the polarity of each other action is the one induced by alternation.
- The **enabling relation** is defined as follows:
  (1) $(\xi, I) \vdash (\xi i, J)$, for $i \in I$;
  (2) $x \vdash y$, for each x negative *initial* action, and y positive *initial* action.

Example 4.2. The universal arena $U(\xi^+)$ for the interface $\Gamma = \xi^+$ can be pictured as in Figure 2. The arrows denote enabling relation; the polarity of the actions is given as follows: actions lying on even (resp. odd) layers have positive (resp. negative) polarity.

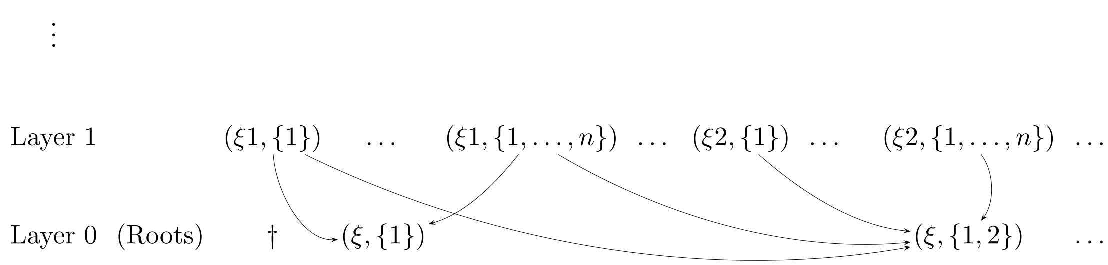


Figure 2: The universal arena $U(\xi^+)$

Definition 4.3 (Strategies on a universal arena (untyped strategies))**.** Let Γ be an interface. A **strategy** $\mathcal{D}$ **on** Γ**,** also written $\mathcal{D} : \Gamma$ is a strategy (in the sense of Definition 3.5) on the universal arena $U(\Gamma)$.

Examples 4.4 (Basic strategies: $\mathfrak{Dai}$, $\mathfrak{Fid}$, $\emptyset$)**.** Let us point out a few relevant strategies.

- There are two *positive* strategies which play a key role in ludics:
  $\mathfrak{Dai}$**:** the strategy which consists of only one action $\{\dagger\}$; it is called *daimon.*
  $\mathfrak{Fid}$**:** the *empty* strategy on a positive base; it is called *faith.*
  Observe that on the empty interface there are only two strategies: $\mathfrak{Dai}$ and $\mathfrak{Fid}$.
- We highlight also a simple example of negative strategy: the *empty* strategy on a negative base. We will denote this strategy simply by $\emptyset$.

Definition 4.5 (Totality)**.** We say that an untyped strategy is **total** when it is not $\mathfrak{Fid}$.

Definition 4.6 (Linearity)**.** Given a strategy $\mathcal{D} : \Gamma$, we say that an occurrence of action (ξ, I) in $\mathcal{D}$ is **linear** if the name ξ is only used by that occurrence of action. We say that $\mathcal{D} : \Gamma$ is **linear** if in $\mathcal{D}$ there is no repetition of actions on the same name.

Linear strategies are essentially the strategies introduced in [20] (there called *designs*). The linearity condition is there actually slightly more complex to take into account also the additive structures (additive duplication is allowed), but for our discussion it is enough to ask that in a strategy *each name is only used once.* Linearity has as consequence that that all pointers are trivial (each move has only one possible justifier and the prefix relation between names univocally tells us which is), and then can be forgotten.

4.2. **Dynamics.** The composition of untyped strategies can be described via the VAM machine (see Section 8). For the moment, we only describe normalization in the linear case (see [20, 11]). This case is simpler, but has all the key ingredients to follow most of the examples of this paper.

4.2.1. *Dynamics in the linear case.* We can compose two strategies $\mathcal{D}_1, \mathcal{D}_2$ when they have compatible interfaces, that is they have a common name, with opposite polarity. For example, $\mathcal{D}_1 : \sigma^+, \Gamma$ can communicate with $\mathcal{D}_2 : \sigma^-, \Delta$ through the name σ. The shared name σ, and all names hereditarily generated from σ, are said to be **internal**.

If $\mathcal{R} = \{\mathcal{D}_1, \ldots, \mathcal{D}_n\}$ is a finite set of strategies which have pairwise compatible interfaces, we call it a **cut-net**. A cut-net is **closed** if all names are internal. A typical example of closed net is when we compose strategies of opposite interface, like $\mathcal{D} : \xi^+$ and $\mathcal{E} : \xi^-$.

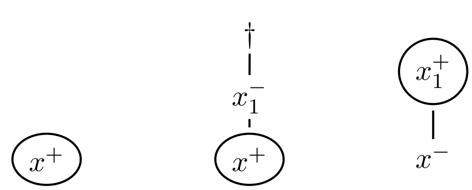
Given a cut-net $\mathcal{R}$, we denote by $[\![\mathcal{R}]\!]$ the result of the composition, also called **normal form**. Composition (also called normalization) follows the standard paradigm of *parallel composition (the interaction) plus hiding* of internal communication: $[\![\mathcal{R}]\!]$ is obtained from the result of the interaction by hiding all actions on internal names.

The most important case in ludics is the *closed one.* In this case, the normal form can be obtained very easily: one can just apply step by step the following rewriting rules:

$$\begin{array}{rcll} [\![x^+.\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}, x^-.\mathcal{D}, \mathcal{R}']\!] & \rightsquigarrow & [\![\mathcal{E}_1, \ldots, \mathcal{E}_n, \mathcal{D}, \mathcal{R}']\!]; & \text{(conversion)} \\ [\![\mathfrak{Dai}, \mathcal{R}']\!] & \rightsquigarrow & \mathfrak{Dai}; & \text{(termination)} \\ [\![\mathcal{R}]\!] & \rightsquigarrow & \mathfrak{Fid}; \qquad \text{otherwise.} & \text{(deadlock)} \end{array}$$

Since each action appears only once, the dynamics is extremely simple: we match actions of opposite polarity. Let us give an example of how interaction works.

Example 4.7. Let us consider the following small example of strategies (think $x = (\xi, I)$ and $x_1 = (\xi 1, K)$).

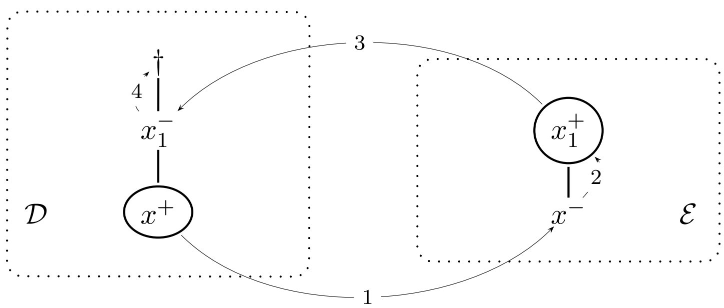


$$\mathcal{D}' := x^+ \quad \mathcal{D} := x^+.x_1^-.\dagger \quad \mathcal{E} := x^-.x_1^+$$

Let us have $\mathcal{D}$ interact with $\mathcal{E}$. $\mathcal{D}$ starts by playing the move x^+, $\mathcal{E}$ checks its answer to that move, which is x_1^+. If $\mathcal{D}$ receive input x_1, its answer is †, which terminates the interaction. Summing up, the interaction — the sequence of actions which matches — produces $x^+.x^-.x_1^+.x_1^-.\dagger$. If we hide the internal communication, we get †, *i.e.*, $[\![\mathcal{D},\mathcal{E}]\!] = \mathfrak{Dai}$.

If we have $\mathcal{E}$ interacting with $\mathcal{D}'$, we again match x^+ with x^-. Then $\mathcal{E}$ plays x_1, but $\mathcal{D}'$ has no considered the action x_1. Here we have a deadlock *i.e.*, $[\![\mathcal{D}',\mathcal{E}]\!] = \mathfrak{Fid}$.

4.2.2. *A notation to describe the interaction.* In the sequel, given two strategy $\mathcal{D}$,$\mathcal{E}$ we often describe their interaction in the following graphical way:



Here, we have taken $\mathcal{D}$ and $\mathcal{E}$ as in Example 4.7. We draw tagged arrows to denote the matching of actions (*e.g.*, x^+ matches x^- at step 1) and the (unique) positive action (the "answer") above a reached negative action (*e.g.*, , x_1^+ after x^-). The tags $1, 2, \ldots$ are only needed to record the chronological order in which actions are visited. Following the arrows with this order, we retrieve the sequence of actions $x^+.x^-.x_1^+.x_1^-.\dagger$ which correspond to the interaction of $\mathcal{D},\mathcal{E}$ of Example 4.7.

4.3. **Orthogonality and interactive types (behaviours).** The most important case of composition in ludics is the closed case, i.e. when all names are internal. We have already observed that there are only two possible strategies which have empty interface: $\mathfrak{Dai}$ and $\mathfrak{Fid}$. Hence, in the closed case, we only have two possible outcomes: either composition fails (deadlock), or it succeeds by reaching the action †, which signals termination. In the latter case, we say that the strategies are *orthogonal*.

Definition 4.8 (Orthogonality)**.** Given two strategies on interfaces of opposite polarity $\mathcal{D} : \xi^+$ and $\mathcal{E} : \xi^-$, they are **orthogonal**, written $\mathcal{D} \perp \mathcal{E}$, if $[\![\mathcal{D},\mathcal{E}]\!] = \mathfrak{Dai}$.
Given a set $\mathbf{E}$ of a strategies on the same interface, its **orthogonal** is defined as $\mathbf{E}^\perp := \{\mathcal{D} : \mathcal{E} \perp \mathcal{D} \text{ for any } \mathcal{E} \in \mathbf{E}\}$.

The definition of orthogonality generalizes to strategies of arbitrary interface, which form a closed net, for example, $\mathcal{D} : \xi^-, \alpha^+, \beta^+$ and $\mathcal{E}_1 : \xi^+, \mathcal{E}_2 : \alpha^-, \mathcal{E}_3 : \beta^-$.

Let $\Gamma = \xi_1^{\epsilon_1}, \ldots, \xi_n^{\epsilon_n}$; if $\mathcal{D} : \Gamma$ we must have a a family of counter-strategies $\mathcal{E}_1 : \xi_1^{\overline{\epsilon}}, \ldots, \mathcal{E}_n : \xi_n^{\overline{\epsilon}}$. We define $\mathcal{D} \perp \{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ if $[\![\mathcal{D}, \mathcal{E}_1, \ldots, \mathcal{E}_n]\!] = \mathfrak{Dai}$.

Orthogonality means that at each step any positive action x^+ *finds* its negative dual action x^-, and the computation terminates, that is it meets a † action.

Example 4.9. In example 4.7, $\mathcal{D} \perp \mathcal{E}$, while $\mathcal{D}'$ and $\mathcal{E}$ are not orthogonal.

Orthogonality allows the players to agree (or not), without this being guaranteed in advance by the type: $\{\mathcal{D}\}^\perp$ is the set of the counter-strategies which are consensual with $\mathcal{D}$.

Definition 4.10. A **behaviour (or interactive type)** on the interface Γ is a set $\mathbf{G}$ of strategies $\mathcal{D} : \Gamma$ such that $\mathbf{G}^{\perp\perp} = \mathbf{G}$ (it is closed by bi-orthogonal).

We say that a behaviour $\mathbf{G}$ is positive or negative according to its interface.

In the sequel, $\mathbf{P}$ will always denote a positive behaviour, $\mathbf{N}$ a negative one. When is useful to emphasize that $\mathbf{A}$ is a set of strategies on the name ξ, we may annotate the name ξ as a subscript: $\mathbf{A}_\xi$.

Remark 4.11. Orthogonality satisfies the usual closure properties.

- If $\mathbf{E}, \mathbf{F}$ are sets of strategies on the same interface, $\mathbf{E} \subseteq \mathbf{F}$ implies $\mathbf{F}^\perp \subseteq \mathbf{E}^\perp$;
- $\mathbf{E}^\perp = \mathbf{E}^{\perp\perp\perp}$.

Remark 4.12. Observe that the strategy $\mathfrak{Fid}$ can never belong to a behaviour, as it has no orthogonal. Hence all strategies in a behaviour are necessarily total.

4.4. **Type constructors.** In this section, we give the constructions on types which interpret linear formulas.

Let $\mathcal{D}_1 : \xi 1^-, \dots, \mathcal{D}_n : \xi n^-$ be negative strategies. We obtain a new positive strategy on the interface ξ^+, denoted by $\mathcal{D}_1 \bullet \dots \bullet \mathcal{D}_n$, by adding to the union of the strategies the positive root $(\xi, I)^+$, *i.e.*,

$$\mathcal{D}_1 \bullet \dots \bullet \mathcal{D}_n := (\xi, I)^+.\{\mathcal{D}_1, \dots, \mathcal{D}_n\} \qquad (I = \{1, \dots, n\})$$

It is immediate to generalize the previous construction to strategies $\mathcal{D}_1 : \xi 1^-, \Gamma, \dots, \mathcal{D}_n : \xi n^-, \Gamma$ to obtain $\mathcal{D}_1 \bullet \dots \bullet \mathcal{D}_n : \xi^+, \Gamma$.

Observe that the root $(\xi, I)^+$ is *always linear*.

Conversely, given any strategy $\mathcal{D} : \xi^+$, such that the root is linear, we can write it as $\mathcal{D} = x.\{\mathcal{D}_1, \dots \mathcal{D}_n\}$. It is immediate to check that each subtree $\mathcal{D}_i$ is a negative strategy on ξi^-. Given a strategy $\mathcal{D}$ as just described, we will write $\mathcal{D}\restriction_i$ for the operation which returns $\mathcal{D}_i$.

Let $\mathbf{N}_1, \mathbf{N}_2$ be *negative* behaviours, respectively on $\xi 1^-$ and $\xi 2^-$. We denote by $\mathbf{N}_1 \bullet \mathbf{N}_2$ the set $\{\mathcal{D}_1 \bullet \mathcal{D}_2 : \mathcal{D}_1 \in \mathbf{N}_1, \mathcal{D}_2 \in \mathbf{N}_2\} \cup \{\mathfrak{Dai}\}$. We define:

$$\begin{array}{lcll} \mathbf{N}_1 \otimes \mathbf{N}_2 & := & (\mathbf{N}_1 \bullet \mathbf{N}_2)^{\perp\perp} & \text{positive behaviour on } \xi^+; \\ \mathbf{N}_1{}^\perp \parr \mathbf{N}_2{}^\perp & := & (\mathbf{N}_1 \bullet \mathbf{N}_2)^{\perp} & \text{negative behaviour on } \xi^-. \end{array}$$

The interpretation $\mathbf{G}$ of a formula G will be a behaviour, *i.e.*, a set of strategies closed by biorthogonal: $\mathcal{D} \in \mathbf{G}$ *if and only if* $\mathcal{D} \perp \mathcal{E}$*, for each* $\mathcal{E} \in \mathbf{G}^\perp$. The interpretation of a sequent $\vdash G_1, \dots, G_n$ naturally follows the same pattern.

Definition 4.13 (Sequent of behaviours)**.** Let $\Gamma = \xi_1, \dots, \xi_n$ be an interface, and $\boldsymbol{\Gamma} = \mathbf{G}_{\xi_1}, \dots, \mathbf{G}_{\xi_n}$ a sequence of behaviours.

We define a new behaviour on Γ, which we call **sequent of behaviours** and denote by $\vdash \boldsymbol{\Gamma}$, as the set of strategies $\mathcal{D} : \Gamma$ which satisfy:

$$\mathcal{D}\bot\{\mathcal{E}_1,\dots,\mathcal{E}_n\}, \text{ for all } \mathcal{E}_1 \in \mathbf{G_1}^{\bot},\dots,\mathcal{E}_n \in \mathbf{G_n}^{\bot}$$

It is clear that a sequent of behaviours is itself a behaviour, i.e, a set of strategies closed by orthogonal. Observe that $\vdash \mathbf{P} = \mathbf{P}$ and $\vdash \mathbf{N} = \mathbf{N}$.

We will use the following property, which is immediate by associativity (Theorem 8.8).

Proposition 4.14. $\mathcal{D} \in \vdash \mathbf{\Gamma}, \mathbf{G}$ *if and only if for each* $\mathcal{E} \in \mathbf{G}^{\bot}$ $[\![\mathcal{D},\mathcal{E}]\!] \in \vdash \mathbf{\Gamma}$.

5. Ludics in the linear case: internal and full completeness

In this section we restrict our attention to linear strategies. In this case, the dynamics is quite simple and the reader should easily grasp the proofs. We introduce the notion of internal completeness and give a direct proof of internal completeness, as well as full completeness, without relying on separation.

In [20], the set of strategies which interpret **MALL** formulas satisfies a remarkable closure property, called *internal completeness*: the set $\mathbf{S}$ of strategies produced by the construction is (essentially) equal to its biorthogonal ($\mathbf{S} = \mathbf{S}^{\bot\bot}$). Since the biorthogonal does not introduce new objects, we have a complete description of all strategies in the behaviour.

The best example is the interpretation $\mathbf{A}_1 \otimes \mathbf{A}_2 := (\mathbf{A}_1 \bullet \mathbf{A}_2)^{\bot\bot}$ of a tensor formula. One proves that $(\mathbf{A}_1 \bullet \mathbf{A}_2) = (\mathbf{A}_1 \bullet \mathbf{A}_2)^{\bot\bot}$, *i.e.*, we do not add new objects when closing by biorthogonal: our description is already complete.

From this, full completeness follows. In fact, because of internal completeness, if $\mathcal{D} \in \mathbf{A}_1 \otimes \mathbf{A}_2$ we know we can decompose it as $\mathcal{D}_1 \bullet \mathcal{D}_2$, with $\mathcal{D}_1 \in \mathbf{A}_1$ and $\mathcal{D}_2 \in \mathbf{A}_2$. This corresponds to writing the rule:

$$\dfrac{\dfrac{\vdots}{\vdash A_1} \quad \dfrac{\vdots}{\vdash A_2}}{\vdash A_1 \otimes A_2}\ A_1 \otimes A_2$$

i.e., if each $\mathcal{D}_i$ corresponds to a proof of A_i, $\mathcal{D}$ corresponds to a proof of $A_1 \otimes A_2$.

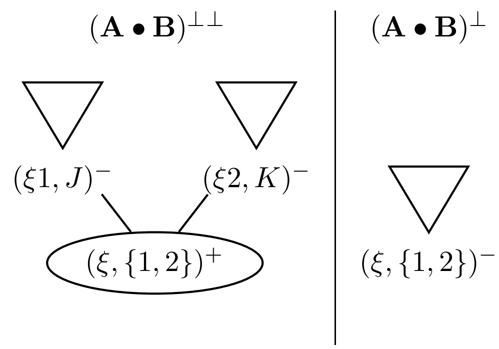
All along this section, we assume the following:

- $\mathbf{A}, \mathbf{B}$ are negative behaviours respectively on $\xi 1^-$ and $\xi 2^-$;
- all strategies are linear (even though what we really need is only linearity of the root);
- composition (normalization) is linear composition.

Let us consider $\mathbf{A} \bullet \mathbf{B}$. By construction, each strategy in $\mathbf{A} \bullet \mathbf{B}$ is on ξ^+ and has $x^+ = (\xi, \{1,2\})^+$ as root.

How is $(\mathbf{A} \bullet \mathbf{B})^{\bot}$? By definition of linear normalization, each strategy has as root the action $x^- = (\xi, \{1,2\})^-$ (otherwise, normalization would fail immediately). In particular we have the strategy $x^-.\dagger$.

How is $(\mathbf{A} \bullet \mathbf{B})^{\bot\bot}$? All strategies have a positive root, which, to normalize against $(\mathbf{A} \bullet \mathbf{B})^{\bot}$, must be either $\dagger$, or x^+. Hence, we know that a strategy $\mathcal{D} \in \mathbf{A} \otimes \mathbf{B}$ has the form $x^+.\{\mathcal{D}_1, \mathcal{D}_2\}$, where $\mathcal{D}_1 : \xi 1$ and $\mathcal{D}_2 : \xi 2$. The following picture represents this.



5.1. **Internal completeness.** We now prove that if $\mathcal{D} \in (\mathbf{A} \otimes \mathbf{B})$ then $\mathcal{D}_1 \in \mathbf{A}$ and $\mathcal{D}_2 \in \mathbf{B}$, which means that $(\mathbf{A} \bullet \mathbf{B})$ was already *complete*, *i.e.*, closed by biorthogonal.

Proposition 5.1 (Internal completeness of tensor)**.** *Let* $\mathbf{A}, \mathbf{B}$ *be negative behaviours, respectively on* $\xi 1^-$ *and* $\xi 2^-$*. We have that* $\mathbf{A} \otimes \mathbf{B} = \mathbf{A} \bullet \mathbf{B}$*.*

Proof. With the assumptions and the notation we have discussed above, we prove that $\mathcal{D}_1 \in \mathbf{A}$ and $\mathcal{D}_2 \in \mathbf{B}$.

(i) Given any $\mathcal{E} : \xi 1^+ \in \mathbf{A}^\perp$, we obtain the strategy $\mathcal{E}' : \xi^- = x^-.\mathcal{E}$ by adding the root x^-. For a generic $\mathcal{D} : \xi^+$ of the form $x^+.\{\mathcal{D}_1, \mathcal{D}_2\}$, we have that
$$[\![x^-.\mathcal{E}, \mathcal{D}]\!] = [\![\mathcal{E}, \mathcal{D}_1]\!] \tag{5.1}$$
by definition of normalization, and by the fact that since in $\mathcal{E}$ there are only names generated by $\xi 1$, $\mathcal{E} : \xi 1^+$ only interact with the subtree $\mathcal{D}_1 : \xi 1^-$.

(ii) $\mathcal{E}' \in (\mathbf{A} \bullet \mathbf{B})^\perp$, because by using equation (5.1) we deduce that $\mathcal{E}' \perp \mathcal{D}$, for any $\mathcal{D} \in (\mathbf{A} \bullet \mathbf{B})$.

(iii) Given any $\mathcal{D} \in \mathbf{A} \otimes \mathbf{B}$, it must be $\mathcal{D} \perp \mathcal{E}$ for each $\mathcal{E} \in (\mathbf{A} \bullet \mathbf{B})^\perp$. Hence in particular, for each $\mathcal{E} \in \mathbf{A}^\perp$, we have $\mathcal{D} \perp \mathcal{E}'$ ($\mathcal{E}'$ defined as above), and hence, again because of equation (5.1), $\mathcal{D}_1 \perp \mathcal{E}$. This says that $\mathcal{D}_1 \in (\mathbf{A}^\perp)^\perp = \mathbf{A}$. □

Remark 5.2 (Important)**.** Observe that here we only use two properties of the strategies: the dynamics (normalization), and the fact that *the root is the only action on the name* ξ (to say that occurrences of $\xi 1$ only appear inside $\mathcal{D}_1$).

Proposition 5.3 (Internal completeness of par)**.** *Let* $\mathbf{A}^\perp, \mathbf{B}^\perp$ *be positive behaviours respectively on* $\xi 1^+$ *and* $\xi 2^+$ *and* $x = (\xi, \{1,2\})$*. We have:*
$$x^-.\mathcal{E} \in \mathbf{A}^\perp \parr \mathbf{B}^\perp \;\Leftrightarrow\; \mathcal{E} \in \vdash \mathbf{A}^\perp, \mathbf{B}^\perp. \tag{5.2}$$

Proof. A strategy $x^-.\mathcal{E}$ belongs to $\mathbf{A}^\perp \parr \mathbf{B}^\perp$ if and only if for any $x^+.\{\mathcal{D}_1, \mathcal{D}_2\} \in \mathbf{A} \bullet \mathbf{B}$, we have that $x^-.\mathcal{E} \perp x^+.\{\mathcal{D}_1, \mathcal{D}_2\}$. By definition of normalization, $[\![x^-.\mathcal{E}, x^+.\{\mathcal{D}_1, \mathcal{D}_2\}]\!] = [\![\mathcal{E}, \mathcal{D}_1, \mathcal{D}_2]\!]$, and from this and the definition of sequent of behaviours (Definition 4.13) the claim immediately follows. □

5.2. **Full Completeness.** Full completeness for multiplicative-linear logic **MLL** follows from what we have seen in this section, by using the proof of internal completeness of tensor and par and Proposition 4.14. Again, while the proof in [20] relies on separation (unicity of the adjoint),we can give a simple and direct proof only using the properties of the dynamics. In this section, we only give the outline of the proof. We will prove the result in full detail in a setting which also includes exponentials in Section 10.5.

5.2.1. *Interpretation.* Let us denote by **MLLS** the multiplicative fragment of **MELLS**. Chosen an arbitrary name ξ, the interpretation $\langle F\rangle_\xi$ of a formula F of **MLLS** is a behaviour, which is defined by structural induction on F as follows:

$$\begin{array}{lllllll} \langle \mathbf{0}\rangle_\xi & := & \{\mathfrak{Dai}\}; & \quad & \langle N_1 \otimes \ldots \otimes N_n\rangle_\xi & := & \langle N_1\rangle_{\xi 1} \otimes \ldots \otimes \langle N_n\rangle_{\xi n}; \\ \langle \top\rangle_\xi & := & \{\mathfrak{Dai}\}^\perp; & & \langle P_1 \parr \ldots \parr P_n\rangle_\xi & := & \langle P_1\rangle_{\xi 1} \parr \ldots \parr \langle P_n\rangle_{\xi n}. \end{array}$$

A sequent $\vdash F_1, \ldots, F_n$ is interpreted by a sequent of behaviours $\vdash \mathbf{F}_1, \ldots, \mathbf{F}_n$ on a given interface $\xi_1, \ldots, \xi_n$.

The interpretation of a proof is a strategy, which satisfy some *winning condition*, *i.e.*, it is daimon-free and material (a notion which we can overlook for the moment; we will discuss it in Section 10— the reader can have a good intuition by reading the Example 10.1).

We have the following theorems.

Theorem 5.4 (Interpretation). *Let π be a proof of a sequent $\vdash \Gamma$ in* **MLLS**. *There exists a winning strategy $\mathcal{D} \in \vdash \mathbf{\Gamma}$ such that $\mathcal{D}$ is interpretation of π.*

Theorem 5.5 (Full Completeness). *If $\mathcal{D}$ is a winning strategy in a sequent of behaviours $\vdash \mathbf{\Gamma}$ then $\mathcal{D}$ is the interpretation of a cut-free proof π of the sequent $\vdash \Gamma$ in* **MLLS**.

5.2.2. *Outline of the proof of full completeness.* Let $\vdash \mathbf{\Delta}$ be the interpretation of the sequent $\vdash \Delta$, and $\mathcal{D} \in \vdash \mathbf{\Delta}$ a winning strategy. Our purpose is to associate to $\mathcal{D}$ a derivation $\mathcal{D}^\star$ of $\vdash \Delta$ in **MLLS** by progressively decomposing $\mathcal{D}$, *i.e.*, inductively writing "the last rule". To be able to use internal completeness, which is defined on behaviours (and not on sequents of behaviours), we will use — back and forth — the definition of sequent of behaviours and in particular Proposition 4.14.

The formula on which the last rule is applied is indicated by the name of the root action. For example, let us assume that the root of $\mathcal{D}$ is (ξ, I); then if $\mathcal{D} \in \vdash \mathbf{F}_\xi, \mathbf{G}_\sigma$, the behaviour which corresponds to the last rule is the one on ξ, *i.e.*, $\mathbf{F}_\xi$.

The proof is by induction.

The base case is immediate; if $\mathcal{D}$ is empty, it must be $\mathcal{D} \in \mathfrak{Dai}^\perp$, i.e. $\mathcal{D} \in \langle \top\rangle_\xi$, and we associate to $\mathcal{D}$ the $\top$-rule of **MLLS**.

Let $\mathcal{D} = x^+.\{\mathcal{D}_1, \ldots, \mathcal{D}_n\}$ be a *positive* winning strategy which belongs to $\vdash \mathbf{F}_\xi, \mathbf{G}_\alpha$, where $\mathbf{F}_\xi$ and $\mathbf{G}_\alpha$ are the interpretation of formulas F and G respectively. Let us assume $x = (\xi, \{1, 2\})$, and $\mathbf{F}_\xi = \mathbf{N}_1 \otimes \mathbf{N}_2$. We have $\mathcal{D} = x.\{\mathcal{D}_1, \mathcal{D}_2\}$. By Proposition 4.14, for any $\mathcal{E} \in {\mathbf{G}_\alpha}^\perp$, we have:

(1) $[\![\mathcal{D},\mathcal{E}]\!] \in \mathbf{F}_\xi$ and the root of $[\![\mathcal{D},\mathcal{E}]\!]$ is still x^+. This allows us to use internal completeness.
(2) By internal completeness of tensor (5.1), we have that $\mathcal{C} = [\![\mathcal{D},\mathcal{E}]\!]$ can be written as $\mathcal{C}_1 \bullet \mathcal{C}_2$, for some $\mathcal{C}_1 \in \vdash \mathbf{N}_1$ and $\mathcal{C}_2 \in \vdash \mathbf{N}_2$.
(3) The address α (which is the interface of $\mathcal{E}$) will appear either in $\mathcal{D}_1$ or in $\mathcal{D}_2$, not in both. Let assume it only appears in $\mathcal{D}_1$. By definition of normalization, it is immediate that:

$$[\![x^+.\{\mathcal{D}_1,\mathcal{D}_2\},\mathcal{E}]\!] = x^+.\{[\![\mathcal{D}_1,\mathcal{E}]\!],\mathcal{D}_2,\} = [\![\mathcal{D}_1,\mathcal{E}]\!] \bullet \mathcal{D}_2.$$

From this, we conclude that $[\![\mathcal{D}_1,\mathcal{E}]\!] \in \vdash \mathbf{N}_1$ and $\mathcal{D}_2 \in \vdash \mathbf{N}_2$.

By applying Proposition 4.14 again, we have that $\mathcal{D}_1 \in \vdash \mathbf{N}_1, \mathbf{G}_\alpha$ and then we can write the derivation:

$$\dfrac{\dfrac{\vdots\ \mathcal{D}_1^\star}{\vdash N_1, G} \quad \dfrac{\vdots\ \mathcal{D}_2^\star}{\vdash N_2}}{\vdash N_1 \otimes N_2, G}\ \text{Pos}$$

The negative case is an immediate application of the negative case of internal completeness (and again Proposition 4.14).

6. Ludics with repetitions: what, how, why

In the previous section, we assumed linearity of the strategies to prove internal completeness. From now on, we go back to the general definition of strategy (on an universal arena) as in Section 4, without any hypothesis of linearity. This means that strategies now allow repeated actions.

In this section, we mainly discuss the difficulties in extending the approach of ludics to this setting, and introduce our solution, which will be technically developed in Section 7.

First, let us introduce some operations which we will use to deal with repeated actions and describe the composition.

6.1. **Copies and renaming.**

Renaming. Given a strategy $\mathcal{E} : \xi$ of arbitrary polarity, let us indicate by $\mathcal{E}[\sigma/\xi]$ the strategy obtained from $\mathcal{E}$ by renaming, in all occurrences of action, the prefix ξ into σ, *i.e.*, each name $\xi.\alpha$ becomes $\sigma.\alpha$. Obviously, if $\mathcal{E} : \xi$, then $\mathcal{E}[\sigma/\xi] : \sigma$.

Renaming of the root. Given a positive strategy $\mathcal{D} : \xi^+$, let us indicate by $\underline{\sigma}(\mathcal{D})$ the strategy obtained by renaming the prefix ξ into σ in the root, and in all actions which are hereditarily justified by the root. If $\mathcal{D} : \xi^+$, we obtain a new strategy $\underline{\sigma}(\mathcal{D}) : \sigma^+, \xi^+$. We picture this in Figure 3, where we indicate an action on ξ simply with the name ξ.

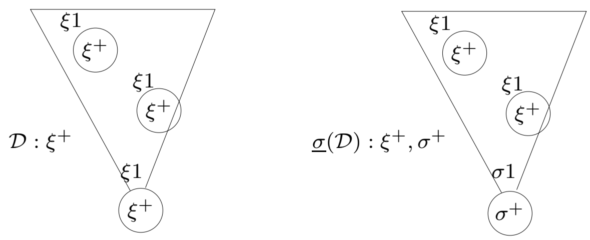


Figure 3: Renaming of the root

Copies of a behaviour. We remind that to emphasize that $\mathbf{A}$ is a set of strategies on interface ξ, we annotate the name ξ as a subscript: $\mathbf{A}_\xi$. If $\mathbf{A}_\xi$ is a set of strategies on the name ξ, we write $\mathbf{A}_\sigma$ for $\{\mathcal{D}[\sigma/\xi]$ *such that* $\mathcal{D} \in \mathbf{A}_\xi\}$. $\mathbf{A}_\sigma$ is a copy of $\mathbf{A}_\xi$: they are *equal up to renaming.*

6.2. **Composition (normalization).** In a strategy, actions can now be repeated. Composition of strategies as sets of views can be described via the VAM abstract machines introduced in [10]. We describe composition in details in Section 8.

We now give an example of composition of strategies using the graphical notation introduce before.

However, what we will really need is only that composition has a fundamental property, expressed by the following equation:

$$[\![\mathcal{D}, \mathcal{E}]\!] = [\![\underline{\sigma}(\mathcal{D}), \mathcal{E}, \mathcal{E}[\sigma/\xi]]\!] \tag{6.1}$$

This property will also hold for strategies with neutral actions we introduce later. The proof for the general case is given in Section 8 (Proposition 8.9).

From Equation (6.1), we have in particular:

Corollary 6.1. *$\mathcal{D} \perp \mathcal{E}$ if and only if $\underline{\sigma}(\mathcal{D}) \perp \{\mathcal{E}, \mathcal{E}[\sigma/\xi]\}$.*

Let us see how Equation (6.1) works by giving a description of the composition.

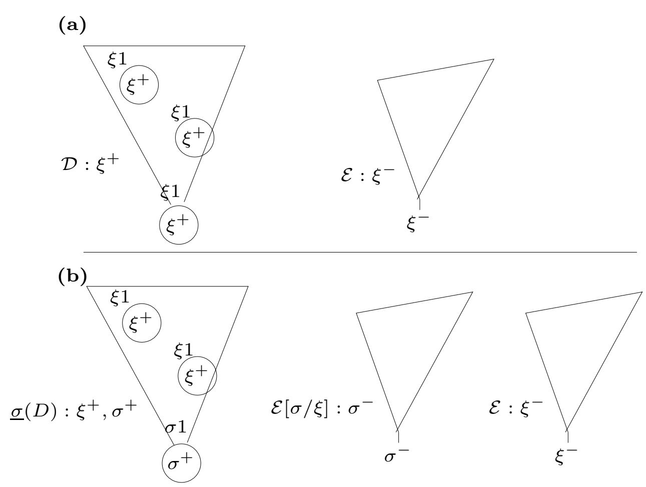


Figure 4: Composition (with repeated actions)

Let $\mathcal{D} : \xi^+$ and $\mathcal{E} : \xi^-$ be two strategies, which we represent in Figure 4 (a) (again, we indicate an action x on ξ simply with the name ξ). The idea behind the abstract machine in [10] is that, when the two strategies $\mathcal{D}$ and $\mathcal{E}$ interact, every time $\mathcal{D}$ plays an action x on ξ, a copy of $\mathcal{E}$ is created; *i.e.*, composition works as if we had a copy of $\mathcal{E}$ for each occurrence of x in $\mathcal{D}$. It is rather intuitive that the result of normalization is the same if we make this explicit, by renaming one occurrence of x (namely the root), and making an explicit copy of $\mathcal{E}$, as illustrated in Figure 4 (b).

Example 6.2. Let us consider the strategies $\mathcal{D}$ and $\mathcal{E}$ in Figure 5, where we indicate an action x on ξ simply with the name ξ. Observe that we explicitly need to draw a pointer from $\xi 11^+$ to the right occurrence of $\xi 1^-$ (the lowermost one in our case) which justifies it. The interaction is the sequence given by following the arrows and the normal form is $\mathfrak{Dai}$.

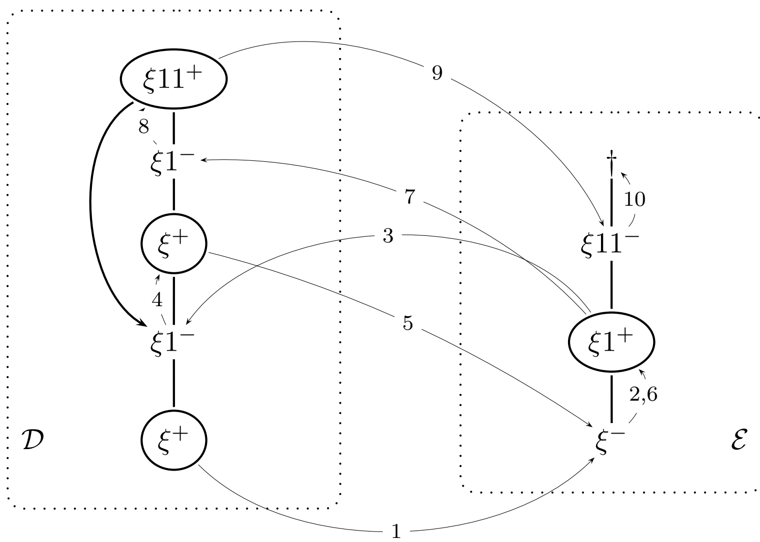


Figure 5: Example of interaction with repetitions

Example 6.3. We now check for $\mathcal{D}, \mathcal{E}$ in Example 6.2 that $[\![\mathcal{D}, \mathcal{E}]\!] = [\![\underline{\sigma}(\mathcal{D}), \mathcal{E}, \mathcal{E}[\sigma/\xi]]\!]$ as pictured in Figure 6. Since $\underline{\sigma}(\mathcal{D})$ is linear in this example, we no longer need to make pointers explicit.

6.3. **What are the difficulties.** We are ready to discuss which are the difficulties in extending the approach of ludics to a setting where strategies are non linear.

Problem 1: Separation. The first problem when strategies have repetitions is with separation. Let us give a simple example of why separation fails if we allow repetitions.

Example 6.4 ([27])**.** Let $\mathcal{D}_1, \mathcal{D}_2 : \xi^+$ and $\mathcal{E} : \xi^-$ be strategies as in Figure 7, where $x = (\xi, I)$, $y = (\xi i, J)$. We cannot find a strategy orthogonal to $\mathcal{D}_1$ but not orthogonal to $\mathcal{D}_2$. For example, the interaction between $\mathcal{D}_1$ and $\mathcal{E}$ is the same of $\mathcal{D}_2$ and $\mathcal{E}$ and in both cases the normal form is $\mathfrak{Dai}$.

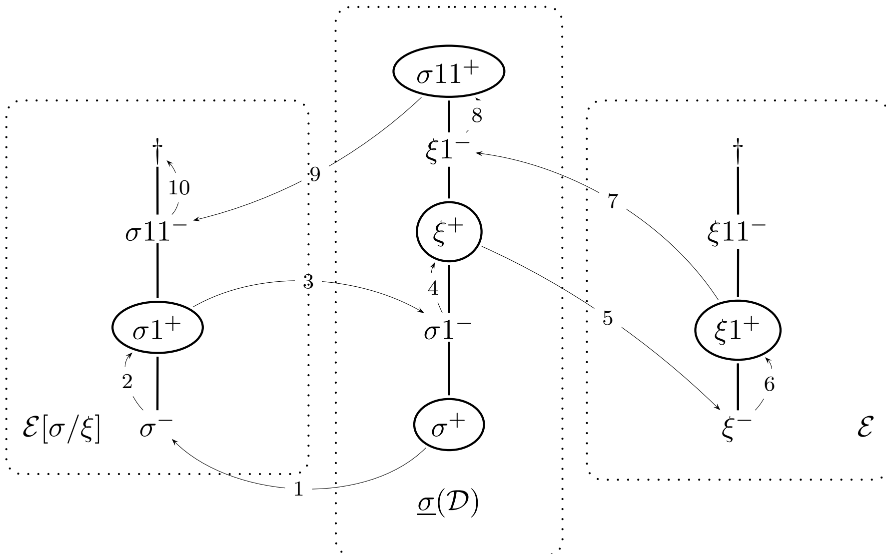


Figure 6: Example of interaction with copies

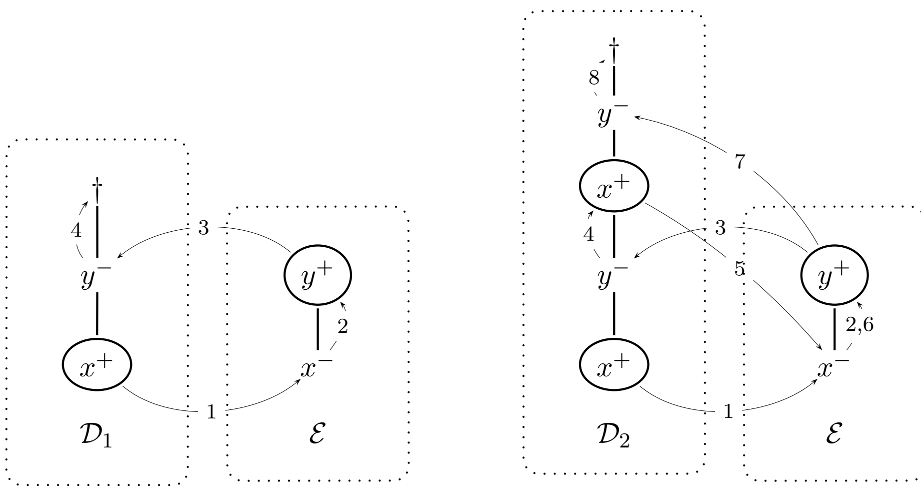


Figure 7: Non-separation

In this work, we ignore separation all together. As we discussed in Section 5, even if separation is an important property, we don't need it in order to have interactive types and internal completeness. In future work, it may be possible to refine our setting using Maurel techniques.

Problem 2: Enough tests (counter-strategies). The second problem — which we believe being the deeper one— has to do with having enough tests, *i.e.*, enough counter-strategies. As in [20], we have defined an interactive type to be any set of strategies closed by biorthogonal. Assume we have defined how to interpret formulas, and in particular $?A$ and $!A^{\perp}$.

We would like to associate to each "good" strategy in the interpretation of a formula, for example a behaviour that we indicate with $?\mathbf{A}$, a syntactical proof of $?A$ (full completeness).

If $\mathcal{D} : \xi^+ \in ?\mathbf{A}$, we would like to transform it into a strategy $\mathcal{D}' \in \vdash ?\mathbf{A}_\xi, ?\mathbf{A}_\sigma$ (where distinct names indicate distinct copies). This corresponds to the contraction rule (in its upwards reading).

The natural idea is to use the same technique as in [1], and to rename the root, and all the actions which are hereditarily justified by it. We have already illustrated this operation in Section 6.1 (Figure 3). From $\mathcal{D} : \xi^+$, we obtain a new strategy $\mathcal{D}' : \xi^+, \sigma^+$, where $\mathcal{D}' = \underline{\sigma}(\mathcal{D})$.
We would like to prove that:

$$(*)\ \mathcal{D} \in \vdash ?\mathbf{A}_\xi \quad \Rightarrow \quad (**)\ \underline{\sigma}(\mathcal{D}) \in \vdash ?\mathbf{A}_\xi, ?\mathbf{A}_\sigma.$$

To have $(**)$, we need (see Definition 4.13) to know that $\underline{\sigma}(\mathcal{D}) \perp \{\mathcal{E}, \mathcal{F}\}$ for each $\mathcal{E} \in (?\mathbf{A}_\xi)^\perp$ and each $\mathcal{F} \in (?\mathbf{A}_\sigma)^\perp$. Since $(?\mathbf{A}_\sigma)^\perp$ is a copy (renamed in σ) of $(?\mathbf{A}_\xi)^\perp$, we can also write this condition as

$$\underline{\sigma}(\mathcal{D}) \perp \{\mathcal{E}, \mathcal{F}[\sigma/\xi]\}, \tag{6.2}$$

where both $\mathcal{F}$ and $\mathcal{E}$ vary in $(?\mathbf{A}_\xi)^\perp$.

However, from Equation (6.1) we *only* have that $\underline{\sigma}(\mathcal{D}) \perp \{\mathcal{E}, \mathcal{E}[\sigma/\xi]\}$: two copies of *the same* (up to renaming) strategy $\mathcal{E}$. This fact can be rephrased by saying that in our "HO setting", strategies in $!\mathbf{C}$ are *uniform:* every time we find a repeated action in of "type" $?(\mathbf{C}^\perp)$, Opponent $!\mathbf{C}$ reacts in the same way.

6.4. **A solution: non-uniform tests.** The need for having enough tests appears similar to the one which has led Girard to the introduction of the *daimon rule*: in ludics, one typically opposes to an abstract "proof of A" an abstract "counter-proof of A". To have enough tests (that is, to have both proofs of A and proofs of $A^\perp$) there is a new rule which allow us to justify any premise.

Similarly here, when we oppose to a proof of $?\mathbf{A}$ a proof of $!\mathbf{A}^\perp$ $(= (?\mathbf{A})^\perp)$, we need enough counter-strategies. We are led to enlarge the universe of tests by introducing *non-uniform counter-strategies*. This is extremely natural to realize in an AJM setting [1, 3], where a strategy of type $!\mathbf{C}$ is a sort of infinite tensor of strategies on $\mathbf{C}$, each one with its *index of copy*. To have HO non-uniform counter-strategies, we introduce a non-deterministic sum of strategies. Let us illustrate the idea, which we will formalize in the next section.

Non-uniform counter-strategies. The idea is to allow a *"non-deterministic sum"* of *negative* strategies $\mathcal{E}, \mathcal{F}$. Let us, for now, *informally* write the sum of $\mathcal{E}$ and $\mathcal{F}$ this way:

$$\tau.\mathcal{E} + \tau.\mathcal{F}$$

- During the composition with other strategies, we might have to use several time this strategy, hence "entering" it several times. Every time is presented with this choice, normalization will non-deterministically chooses one of the two possible continuations. The choice could be different at each repetition.
- To define *orthogonality*, we set:
  $\mathcal{D} \perp (\tau.\mathcal{E} + \tau.\mathcal{F})$ *if and only if* $[\![\mathcal{D}, \tau.\mathcal{E} + \tau.\mathcal{F}]\!] = \mathfrak{Dai}$ *for each possible choice among the* τ*'s.*

It is immediate that:

$$\mathcal{D}\perp(\tau.\mathcal{E}+\tau.\mathcal{F}) \;\Rightarrow\; \mathcal{D}\perp\mathcal{E} \text{ and } \mathcal{D}\perp\mathcal{F}. \tag{6.3}$$

As we will see, if $\mathcal{E}\in\mathbf{G}$ and $\mathcal{F}\in\mathbf{G}$ for $\mathbf{G}$ *interpreting a formula* of **MELLS**, we have that $(\tau.\mathcal{E}+\tau.\mathcal{F})\in\mathbf{G}$, and vice-versa. Hence:

- if $\mathcal{D}\in ?\mathbf{A}$, for each $\mathcal{E},\mathcal{F}\in(?\mathbf{A})^{\perp}$ we have $\mathcal{D}\perp(\tau.\mathcal{E}+\tau.\mathcal{F})$.
- By using Equation (6.1) we have that $\underline{\sigma}(\mathcal{D})\perp\{(\tau.\mathcal{E}+\tau.\mathcal{F}),(\tau.\mathcal{E}+\tau.\mathcal{F})[\sigma/\xi]\}$.
- By using Equation (6.3), we deduce that $\underline{\sigma}(\mathcal{D})\perp\{\mathcal{E},(\mathcal{F}[\sigma/\xi])\}$, as we wanted.

Linearity of the root. Observe that by construction, in $\underline{\sigma}(\mathcal{D})$ the action at the root is positive and it is the only action on the name σ. We can hence apply the same argument we have already given in Section 5.1 for the internal completeness of tensor.

As a consequence, if $A=A_1\otimes A_2$, given $\mathcal{D}\in\vdash ?\mathbf{A}_\xi$, we have that $\underline{\sigma}(\mathcal{D})$ actually belongs to $\vdash\mathbf{A}_\sigma, ?\mathbf{A}_\xi$, and can be decomposed in strategies $\underline{\sigma}(\mathcal{D})_i\in\vdash\mathbf{A}_i, ?\mathbf{A}_\xi$, where $\mathbf{A}_i$ is consisting of strategies on interface σi^-.

This allows us to associate to $\mathcal{D}\in\vdash ?\mathbf{A},\mathbf{\Gamma}$ a proof which essentially has this form:

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash A_1, ?(A_1\otimes A_2),\Gamma} \quad \dfrac{\vdots}{\vdash A_2, ?(A_1\otimes A_2),\Gamma}}{\vdash A_1\otimes A_2, ?(A_1\otimes A_2),\Gamma}\ \otimes + \mathit{contraction}}{\vdash ?(A_1\otimes A_2),\Gamma}\ \mathit{dereliction} + \mathit{contraction}$$

7. Ludics with repetitions: non-uniform strategies

In this section we technically implement the ideas which we have presented in Section 6.4. In particular, we revise the definition of arena and strategy so to accommodate actions which correspond to the τ actions we have informally introduced. To this purpose, we consider a third polarity: the neutral one. In this section use notions which have been developed to bridge between ludics and concurrency in [15]; we refer the reader to that paper to understand the motivations behind the definitions.

Polarities and actions. We extend the set of polarities with a *neutral polarity*, hence we have now three possibilities: positive $(+)$, negative $(-)$ and neutral $(\mp)$ *i.e.*, $Pol=\{+,-,\mp\}$.

We extend the set of actions with a set $T=\{\tau_i : i\in\mathbb{N}\}$ of *indexed τ actions*, whose polarity is defined to be neutral.

We denote by T also the neutral arena: the set of moves is T, the enabling relation is empty, and the polarity is neutral.

Non-uniform strategies. Let A be a (positive or negative) arena. It is immediate to extend the definition of arena to $A \cup T$, by extending the polarity function λ_A to a function on $\lambda : A \cup T \to \{+, -, \mp\}$. We set $\lambda(x) = \lambda_A(x)$ if $x \in A$; $\lambda(x) = \mp$ if $x \in T$.

Definition 7.1 (Justified sequence and non-uniform view)**.**

- A **justified sequence** s on $A \cup T$ is a justified sequence in the sense of Definition 3.5.
- A **non-uniform view** s on A is a justified sequence on $A \cup T$, where: for each pair of consecutive actions s_i, s_{1+1} such that $\lambda(s_i) = +$ or $\lambda(s_{i+1}) = -$ we have that $s_i \vdash_A s_{i+1}$.

This above condition deserves some comments (*cf.* also Definition 3.3 and Proposition 3.4). If s is a sequence on $A \cup T$, the condition implies that immediate predecessors (resp. successors) of neutral actions are either negative or neutral (resp. either positive, or neutral). To understand the intuitions concerning this choice, it helps to think of a τ action as the result of matching a positive and a negative action. One should think of a neutral action as it were *an ordered pair consisting of a positive action followed by a negative action.* If s is a sequence on A (and hence we have the standard alternation between positive and negative actions), we already observed that the condition is equivalent to the standard one (Proposition 3.4).

Definition 7.2 (Non-uniform strategy)**.** A **non-uniform strategy** (n.u. strategy, for short) $\mathcal{D}$ on A is a prefix-closed set of non-uniform views on $A \cup T$, such that:

- *Coherence.* If $s.m, s.n \in \mathcal{D}$ and $m \neq n$ then m, n are either both negative or both neutral actions.
- *Maximality.* If $s.m$ is maximal in $\mathcal{D}$ (*i.e.*, no other view extends it), then m is positive or neutral.

We will call **deterministic** a n.u. strategy which has no τ actions.

As we have done for Definition 4.3, given a base Γ, a **n.u. strategy** $\mathcal{D}$ **on** Γ, written $\mathcal{D} : \Gamma$, is a n.u. strategy on the universal arena $U(\Gamma)$.

In Example 7.6, we show some n.u. strategies.

Remark 7.3. The index associated to a τ action is a technical choice, but it is irrelevant for the semantics; as in [32], we identify strategy which only differ for the index associated to an occurrence of τ action.

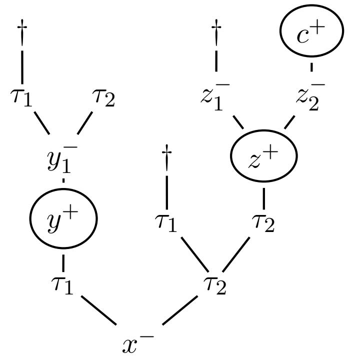
7.1. **Slices.** The following definitions are also introduced and motivated in [15].

Definition 7.4 (τ-cell)**.** Given a n.u. strategy $\mathcal{D}$, a τ**-cell** is a maximal set of occurrences of τ actions, which have the same immediate predecessor.

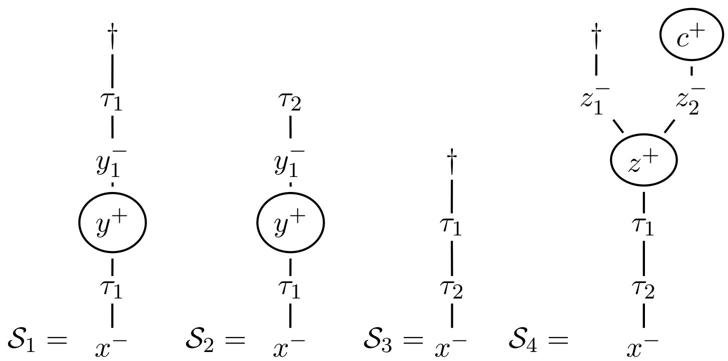
A τ-cell is in many respects similar to an additive &-rule [20, 11]; hence, similarly to the additive case, we can define a notion of slice.

Definition 7.5 (τ-slice)**.** Let $\mathcal{S}$ and $\mathcal{D}$ be n.u. strategies. We say that $\mathcal{S} \subseteq \mathcal{D}$ is a τ**-slice of** $\mathcal{D}$ if each τ-cell in $\mathcal{S}$ is *unary.* Given a τ-slice $\mathcal{S}$ of $\mathcal{D}$, $\mathcal{S}^{\approx}$ is obtained from $\mathcal{S}$ by hiding all τ actions; we also write $\mathcal{S} \approx \mathcal{S}^{\approx}$.

Example 7.6. For example, let us consider the following n.u. strategy:

$\mathcal{D} =$ 

It has four τ-slices:



Remark 7.7. $\mathcal{S}$ and $\mathcal{S}^{\approx}$ are easily checked to be **weekly bisimilar** (see [15] for details). The neutral actions are silent actions which are invisible from the point of view of the environment, and in fact irrelevant *w.r.t.* observational equivalence.

7.2. **Sum of strategies.** We will use n.u. strategies to capture the idea of "non-uniform" tests. As anticipated in Section 6.4, a n.u. strategies can be seen as a non-deterministic sum of "standard" strategies.

Definition 7.8 (τ-sum)**.** If $\{\mathcal{D}_i : \Gamma\}_{i\in S}$ ($S \subseteq \mathbb{N}$) is a family of positive n.u. strategies, we define their **sum**:

$$\sum_{i\in S}^{\tau} \mathcal{D}_i := \bigcup_{i\in S} \{\tau_i.\mathcal{D}_i\}.$$

When S is a finite set, say $\{1,\ldots,k\}$, we write $\mathcal{D}_1 +^{\tau} \ldots +^{\tau} \mathcal{D}_k$.

If $\{x^-.\mathcal{E}_i : \Gamma\}_{i\in S}$ ($S \subseteq \mathbb{N}$) is a family of negative n.u. strategies which have the same root x^-, we define their **sum**:

$$\sum_{i\in S}^{\tau} x^-.\mathcal{E}_i := x^-.\sum_{i\in S}^{\tau} \mathcal{E}_i$$

In the finite case, we write also $x^-.\mathcal{E}_1 +^{\tau} \ldots +^{\tau} x^-.\mathcal{E}_k$.

8. Normalization and orthogonality

In this section we define the composition of strategies and we show some properties of composition we need in the sequel. More specifically, we straightforwardly generalize Curien and Herbelin's View Abstract Machine [7, 10] (VAM) in order to compose strategies with neutral actions. We have already introduced the intuitions and the basic definitions concerning normalization in Section 4.2.1.

Two strategies $\mathcal{D}, \mathcal{E}$ have **compatible interfaces** if $\mathcal{D} : \Gamma, \xi^-$ and $\mathcal{E} : \Delta, \xi^+$, where ξ is the unique shared name, $\Gamma \cap \Delta = \emptyset$, and the set Γ, Δ of names which are not shared, forms an interface.

We will compose an arbitrary finite number of strategies at the same time.

Definition 8.1 (Cut-net [20]). A **cut-net** $\mathcal{R}$ is a non empty finite set $\mathcal{R} = \{\mathcal{D}_1, \dots, \mathcal{D}_n\}$ of strategies on *pairwise compatible interfaces* $\Gamma_1, \dots, \Gamma_n$, such that the graph whose nodes are given by the interfaces, and whose edges are given by linking the shared names (*e.g.*, we set a link from ξ^+, Γ to ξ^-, Δ) is *connected* and *acyclic*.

The *interface of a cut-net* $\mathcal{R}$ is the union of all interfaces $\bigcup_{1 \leq i \leq n} \Gamma_i$ where we delete all names which are shared.

A cut-net is **closed** if the interface is empty.

An action in $\mathcal{R}$ is said *internal* if it is hereditarily justified by an action (ξ, I) where ξ is a shared name, *visible* otherwise (observe that neutral actions *are always visible*).

Given a cut-net $\mathcal{R}$, let us impose a partial order $<$ on the interfaces $\Gamma_1, \dots, \Gamma_n$: the relation $\Gamma_i < \Gamma_j$ holds if there is a shared name ξ, such that $\xi^+ \in \Gamma_i$ and $\xi^- \in \Gamma_j$. By the definition of cut-net, it is easy to see that this order is tree-like. The *main strategy* of $\mathcal{R}$ is the strategy on the interface which is minimal *w.r.t.* this order.

8.1. The abstract machine.

We now introduce the machine VAM in two steps, which correspond to the standard paradigm for computing the composition of strategies "*parallel composition plus hiding*":

(1) Given a cut-net $\mathcal{R}$ we calculate its *interaction* $I(\mathcal{R})$ (Definition 8.4). This is a set of justified sequences of actions, which we call *plays*.
(2) From the interaction $I(\mathcal{R})$, we obtain the *compound strategy*, also called *normal form* $[\![\mathcal{R}]\!]$ (Definition 8.6) by hiding the internal communication (*i.e.*, the internal actions).

Since we allow for the repetition of actions, there might be several occurrence of the same action a in a strategy. However, distinct occurrences are distinguished by being the last element of distinct views; this holds because in our setting neutral actions are equipped with an index.

In order to define the interaction of a cut-net, we first need the following notion.

Definition 8.2 (View extraction). Let $s = x_1 \dots x_n$ be a justified sequence of actions. We define the **view** of s denoted by $\ulcorner s \urcorner$ as follows:

- $\ulcorner s \urcorner := s$ if s is empty;
- $\ulcorner s.x \urcorner := \ulcorner s \urcorner.x$ if x is positive of neutral;
- $\ulcorner s.x \urcorner := \ulcorner q \urcorner.x$ if x is negative, $s = q.r.x$ and x points to the the last action of q. If x is initial (*i.e.*, it does not point to any previous action), we set $\ulcorner s.x \urcorner = x$.

In words, we trace back from the end of s: (i) following the pointers of negative actions of s and erasing all actions under such pointers, (ii) bypassing positive and neutral actions, (iii) stopping the process when we reach an initial negative action.

Hence, given a justified sequence of actions $s = x_1 \dots x_n$ we obtain a subsequence $\ulcorner s \urcorner = x_{k_1} \dots x_{k_m}$, where $1 \leq k_1 < \dots < k_m \leq n$. Implicitly, we intend the operation $\ulcorner\ \urcorner$ to be pointer preserving: if x_i points to x_j in s and $k_s = i$, $k_r = j$, then x_i points to x_j in $\ulcorner s \urcorner$.

Example 8.3. Given $s = x_1.x_2 \dots x_9$ and $t = y_1.y_2 \dots y_8$ as follows:

$$s \ = \ \tau_1 \ a^+ \ a^- \ \tau_1 \ b^+ \ b_0^- \ \tau_1 \ a_0^+ \quad a_0^- \qquad\qquad t \ = \ \tau_1 \ a^+ \ a^- \ \tau_1 \ b^+ \ b_0^- \ \tau_1 \ a_0^+$$

we get $\ulcorner s \urcorner = x_1.x_2.x_9$ and $\ulcorner t \urcorner = y_3.y_4.y_5.y_6.y_7.y_8$:

$$\ulcorner s \urcorner \ = \ \tau_1 \ a^+ \ a_0^- \qquad\qquad \ulcorner t \urcorner \ = \ a^- \ \tau_1 \ b^+ \ b_0^- \ \tau_1 \ a_0^+$$

Before giving the definition, let us informally explain how the abstract machine calculates the interaction of a cut-net $\mathcal{R}$. The machine visits actions of strategies of $\mathcal{R}$ and collects the sequences of visited actions, proceeding as follows:

- We start on the roots of the main strategy of a cut net $\mathcal{R}$.
- If we visit a visible action a occurring in some $\mathcal{D} \in \mathcal{R}$, we continue to explore the current strategy $\mathcal{D}$. The process eventually branches when a is a branching node of $\mathcal{D}$.
- If we visit an internal action a^+ occurring in $\mathcal{D}$ we match it with its opposite a^- occurring in $\mathcal{E} \in \mathcal{R}$, then we continue to collect actions in $\mathcal{E}$ (this is a *jump* of the machine). Since there could be several occurrences of a^- in $\mathcal{E}$, we use $\ulcorner\ \urcorner$ to determinate the correct occurrence of action to which we have to move.
- We may eventually stop when either we reach a maximal action or an internal action which has no match.

We now give the formal definition of interaction.

Definition 8.4 (Interaction)**.** The **interaction** $I(\mathcal{R})$ of a cut-net $\mathcal{R} = \{\mathcal{D}_1, \dots, \mathcal{D}_n\}$ is a set of justified sequences of actions p called **plays** defined as follows:

Start: If a is a root of the main strategy of $\mathcal{R}$, then $a \in I(\mathcal{R})$. If the main strategy of $\mathcal{R}$ is empty, we set $I(\mathcal{R}) := \emptyset$.

Continuation: If $p = x_1 \dots x_n \in I(\mathcal{R})$ and x_n is either a visible action or a internal negative action, then the interaction "continues" in the (unique) strategy $\mathcal{D}_i \in \mathcal{R}$ such that $\ulcorner p \urcorner \in \mathcal{D}_i$. So, for any action a which extends $\ulcorner p \urcorner$ in $\mathcal{D}_i$ *i.e.*, such that $\ulcorner p \urcorner.a \in \mathcal{D}_i$, we set $p.a \in I(\mathcal{R})$ where the pointer for a is given by the equation $\ulcorner p.a \urcorner = \ulcorner p \urcorner.a$.

Jump: If $p = x_1 \dots x_n \in I(\mathcal{R})$ and $x_n = a^+$ is a internal positive action, then we consider the sequence $p.a^-$ obtained by adding the action a^- to p together with a pointer from a^- to x_{i-1} in case x_n points to x_i in p. If there is $\mathcal{D}_i \in \mathcal{R}$ such that $\ulcorner p.a^- \urcorner \in \mathcal{D}_i$, we set $p.a^- \in I(\mathcal{R})$, otherwise (and we say that a^+ has not match) the play p is maximal, not extensible in $I(\mathcal{R})$.

Remark 8.5. In the case "continuation" above, the equation $\ulcorner p.a \urcorner = \ulcorner p \urcorner.a$ summarizes the following conditions:

- if a is negative, then a must point to x_n;
- if a has no pointer in $\mathcal{D}_i$ (*i.e.*, a is either neutral or visible positive action whose name is on the interface of $\mathcal{D}_i$ or daimon) then a does not point to any action in p;
- if a is positive and points to an occurrence of negative action b in $\ulcorner p \urcorner$ in $\mathcal{D}_i$, then a points at the unique occurrence of b occurring at the position x_i of p such that $\ulcorner p.a \urcorner = \ulcorner p \urcorner.a$ (notice that a different pointer, say from a to an occurrence of b at some different position x_j, gives a sequence which is not a view of $\mathcal{D}_i$).

Definition 8.6 (Normal form)**.** The **normal form** (or equivalently, the **composite**) of a cut net $\mathcal{R}$, denoted by $[\![\mathcal{R}]\!]$, is obtained by hiding the occurrences of internal actions from the plays of $I(\mathcal{R})$. Precisely:

(1) *Hiding.* We consider each sequence $p \in I(\mathcal{R})$ and we delete the internal actions. What remains is the subsequence of visible actions of p, written $\mathsf{hide}(p)$, with the obvious inherited pointer structure.
(2) *Garbage collection.* We erase from $\{\mathsf{hide}(p) : p \in I(\mathcal{R})\}$ the empty sequence and any maximal sequence ending with negative actions.

The normal form $[\![\mathcal{R}]\!]$ is then defined as the set of justified sequences obtained after hiding and garbage collection.

We have the following properties.

Proposition 8.7. *The normal form of a cut-net $\mathcal{R}$ is a strategy (on the interface of $\mathcal{R}$).*

Theorem 8.8 (Associativity)**.** *Let $\mathcal{D}, \mathcal{E}, F$ be strategies. We have:*

$$[\![\mathcal{D}, \mathcal{E}, \mathcal{F}]\!] = [\![[\![\mathcal{D}, \mathcal{E}]\!], \mathcal{F}]\!]$$

The proof is given in Appendix B.

The main result of composition we use is given by the following proposition.

Proposition 8.9 (Copies)**.** *Let $\mathcal{D} : \xi^+$ and $\mathcal{E} : \xi^-$ be strategies such that $\mathcal{D}$ has a positive action x as root. We have:*

$$[\![\mathcal{D}, \mathcal{E}]\!] = [\![\underline{\sigma}(\mathcal{D}), \mathcal{E}, \mathcal{E}[\sigma/\xi]]\!] \tag{8.1}$$

Proof. We define the following operation f which takes as input a play $p \in I(\{\mathcal{D}, \mathcal{E}\})$ and returns a justified sequence of actions $f(p)$ as follows.

- $f(p) = p$ is p is empty.
- Let $p = x_1 \dots x_n \in I(\{\mathcal{D}, \mathcal{E}\})$. We set:
  - $f(p) = f(x_1 \dots x_{n-1}).(\sigma.\alpha, I)^\epsilon$ if $x_n = (\xi.\alpha, I)^\epsilon$ and $(\xi.\alpha, I)^\epsilon$ hereditarily points to either $x_1 (= x^+)$ or $x_2 (= x^-)$.
  - $f(p) = f(x_1 \dots x_{n-1}).x_n$ otherwise.

  In both cases above, the pointer for x_n is given as follows: if x_n points to x_i in p then for $f(p) = y_1 \dots y_n$, y_n points to y_i in $f(p)$ (notice that by definition, f preserves the length of the sequences and the polarity of the actions).

It is not hard to prove that f provides a bijection from $I(\{\mathcal{D}, \mathcal{E}\})$ to $I(\{\underline{\sigma}(\mathcal{D}), \mathcal{E}, \sigma(\mathcal{E})\})$ and moreover, $\mathsf{hide}(p) = \mathsf{hide}(f(p))$. From this, we conclude $[\![\mathcal{D}, \mathcal{E}]\!] = [\![\underline{\sigma}(\mathcal{D}), \mathcal{E}, \mathcal{E}[\sigma/\xi]]\!]$. ☐

8.2. **Orthogonality.** We revise Definition 4.8.

Definition 8.10 (Orthogonality)**.** Let $\mathcal{D} : \xi^+$ and $\mathcal{E} : \xi^-$ be n.u. strategies. We define orthogonality as follows:

$$\mathcal{D}\perp\mathcal{E} \text{ if for each } \tau\text{-slice } \mathcal{S} \text{ of } [\![\mathcal{D},\mathcal{E}]\!],\ \mathcal{S}^{\approx} = \mathfrak{Dai}.$$

The definition immediately generalizes to strategies on an arbitrary interface. Let $\Gamma = \xi_1^{\epsilon_1}, \dots, \xi_n^{\epsilon_n}$; if $\mathcal{D} : \Gamma$ we must have a a family of counter-strategies $\mathcal{E}_1 : \xi_1^{\overline{\epsilon}}, \dots, \mathcal{E}_n : \xi_n^{\overline{\epsilon}}$, and we define in the straightforward way $\mathcal{D}\perp\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$.

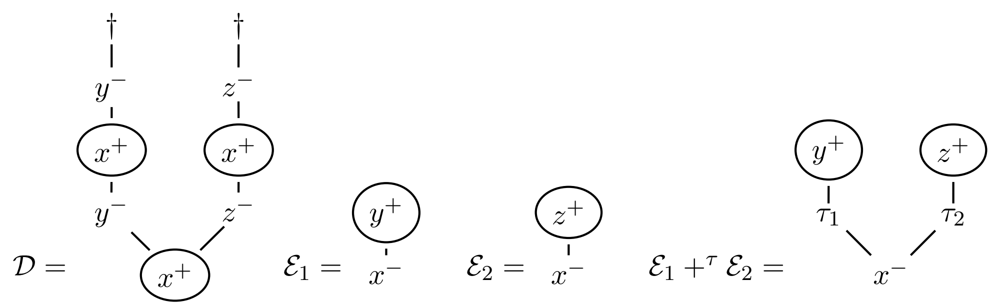
The following corollary is an immediate consequence of the definition of orthogonality:

Corollary 8.11. *Let $\mathcal{E} : \xi^-$ be a negative strategy such that $\mathcal{E} = \sum_{i\in S}^{\tau} \mathcal{E}_i$ and $\mathcal{D} : \xi^+$ a positive strategy. We have that:*

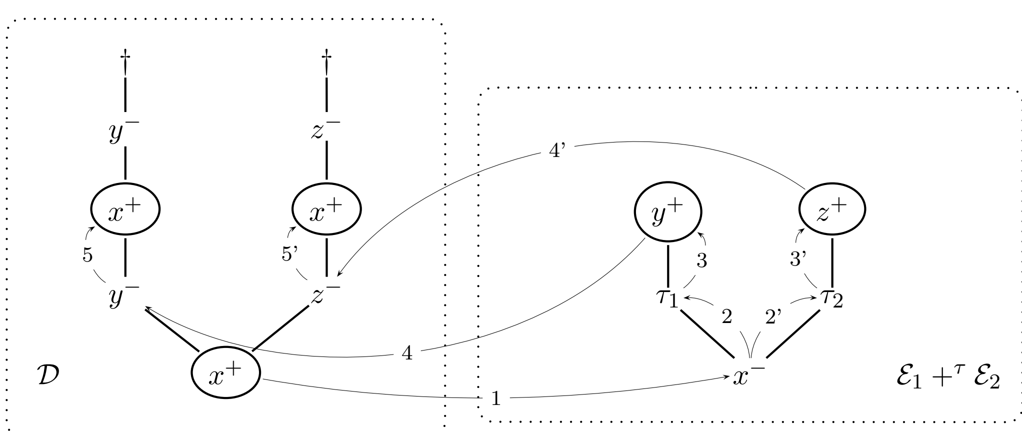
$$\mathcal{D}\perp\mathcal{E} \ \Rightarrow\ \mathcal{D}\perp\mathcal{E}_i, \quad \textit{for any } i \in S.$$

The converse does not hold in general. We now give a concrete example, which is also useful to better describe composition via our graphical notation.

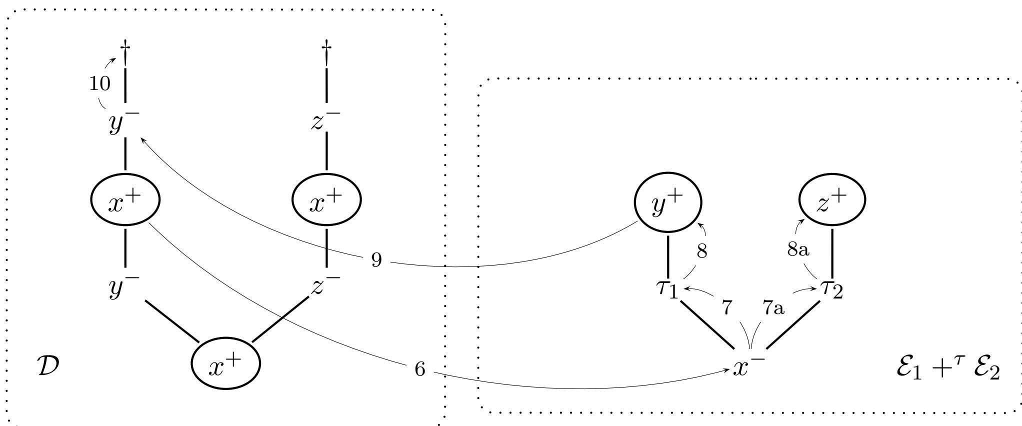
Example 8.12. Let us consider the following strategies.



If we compose $\mathcal{D}$ with $\mathcal{E}_i$, it is rather clear that we always reach $\dagger$, hence $\mathcal{D}\perp\mathcal{E}_1$ and $\mathcal{D}\perp\mathcal{E}_2$. On the other hand, if we compose $\mathcal{D}$ with $\mathcal{E}_1 +^\tau \mathcal{E}_2$, we have the interaction as (partially) described below.



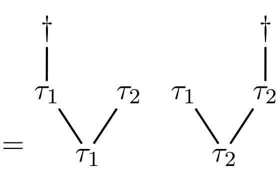
After the steps tagged by 5 and $5'$ the interaction "re-enters" in $\mathcal{E}_1 +^\tau \mathcal{E}_2$. The steps which follow 5 are described below (for the steps which follow $5'$ the situation is symmetric).



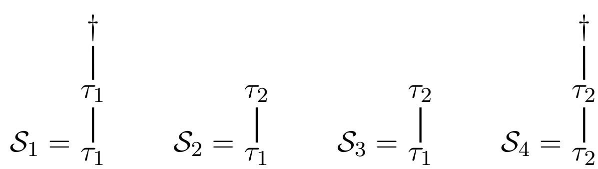
Notice that after the step tagged by $8a$ we have a *deadlock*: the action z^+ should match an action z^- above the *last visited* occurrence of x^+ (the leftmost one), but there is no such an action (we only have y^-).

The result of composition is:

$$[\![\mathcal{D}, \mathcal{E}_1 +^\tau \mathcal{E}_2]\!] =$$



which has four τ-slices:



That is:

$$\mathcal{S}_1^{\approx} = \mathfrak{Dai} \quad \mathcal{S}_2^{\approx} = \mathfrak{Fid} \quad \mathcal{S}_3^{\approx} = \mathfrak{Fid} \quad \mathcal{S}_4^{\approx} = \mathfrak{Dai}$$

From this we conclude that $\mathcal{D} \not\perp \mathcal{E}_1 +^\tau \mathcal{E}_2$.

We will use the following properties of normalization.

Lemma 8.13. *Let $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ be strategies on the same interface $\Gamma = \xi 1^+, \ldots, \xi n^+$. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ be a family of negative strategies such that $\mathcal{E}_i : \xi i^-$ (with $1 \leq i \leq n$).*

We have the following:

(a) $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\} \perp \mathcal{C}$ *if and only if* $x^+.\{\mathcal{E}_1, \ldots, \mathcal{E}_n\} \perp x^-.\mathcal{C}$*, where* $x = (\xi, \{1, \ldots, n\})$*;*

(b) $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\} \perp \mathcal{C}_1 +^\tau \mathcal{C}_2$ *if and only if* $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\} \perp \mathcal{C}_1$ *and* $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\} \perp \mathcal{C}_2$*.*

9. Ludics with repetitions: internal completeness

In this section we give constructions for behaviours, which correspond to the construction of **MELLS** formulas, and prove that they enjoy internal completeness.

As defined in Section 4.3, a behaviour $\mathbf{G}$ is a set of strategies closed by biorthogonal $\mathbf{G} = \mathbf{G}^{\perp\perp}$. The definition of sequent of behaviours remain the same; similarly Proposition 4.14 still hold:

Proposition 9.1. *$\mathcal{D} \in \vdash \mathbf{\Gamma}, \mathbf{G}$ if and only if for each $\mathcal{E} \in \mathbf{G}^{\perp}$, $[\![\mathcal{D}, \mathcal{E}]\!] \in \vdash \mathbf{\Gamma}$.*

9.1. **Constant types.** We define the positive (resp. negative) constant behaviour on ξ as follows:

$$?\mathbf{0} := \{\mathfrak{Dai}\}^{\perp\perp} \text{ on interface } \xi^{+}; \qquad !\top := \{\mathfrak{Dai}\}^{\perp} \text{ on interface } \xi^{-}.$$

We have that $?\mathbf{0}$ contains a unique deterministic strategy, which is $\mathfrak{Dai}$. On the other side, $!\top$ contains all negative strategies which has interface ξ, including the empty one.

9.2. **Compound types.** In this section, we use the same constructions on strategies and operations on sets of strategies as in Section 4.4.

Let us fix negative behaviours $\mathbf{N}_{\xi 1}, \ldots, \mathbf{N}_{\xi n}$ on the interfaces $\xi 1, \ldots, \xi n$ respectively. We define a new positive (resp. negative) behaviour on ξ as follows:

$$\mathbf{F}^{+}(\mathbf{N}_{\xi 1}, \ldots, \mathbf{N}_{\xi n}) := (\mathbf{N}_{\xi 1} \bullet \cdots \bullet \mathbf{N}_{\xi n})^{\perp\perp}; \qquad \mathbf{F}^{-}({\mathbf{N}_{\xi 1}}^{\perp}, \ldots, \mathbf{N}_{n}^{\perp}) := (\mathbf{N}_{\xi 1} \bullet \cdots \bullet \mathbf{N}_{\xi n})^{\perp}.$$

Up to the end of Section 9 we fix the following notation:

- we denote by x the action $(\xi, \{1, \ldots, n\})$;
- $\mathbf{N}_{\xi} = (\mathbf{N}_{\xi 1} \bullet \cdots \bullet \mathbf{N}_{\xi n})^{\perp}$ and $\mathbf{P}_{\xi} = (\mathbf{N}_{\xi 1} \bullet \cdots \bullet \mathbf{N}_{\xi n})^{\perp\perp}$.

Remark 9.2. It is important to observe that, by construction, all strategies in $\mathbf{N}_{\xi 1} \bullet \cdots \bullet \mathbf{N}_{\xi n}$ have as root $x = (\xi, \{1, \ldots, n\})$, which is *linear*. The repetitions of occurrences of x are obtained via the closure by biorthogonality, and hence only belong to $(\mathbf{N}_{\xi 1} \bullet \cdots \bullet \mathbf{N}_{\xi n})^{\perp\perp}$

Proposition 9.3 (Internal completeness of $\mathbf{F}^{-}$)**.**

$$(1)\ x^{-}.\mathcal{D} \in \mathbf{F}^{-}(\mathbf{P}_1, \ldots, \mathbf{P}_n) \ \Leftrightarrow\ (2)\ \mathcal{D} \in \vdash \mathbf{P}_1, \ldots, \mathbf{P}_n.$$

Proof. The proof follows immediately from the definitions. Expanding them, we obtain the two following properties, which are equivalent by using Lemma 8.13(a):

(1) $x^{-}.\mathcal{D} \perp x^{+}.\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$, for any $\mathcal{E}_1 \in {\mathbf{P}_1}^{\perp}, \ldots, \mathcal{E}_n \in {\mathbf{P}_n}^{\perp}$;
(2) $\mathcal{D} \perp \{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$, for any $\mathcal{E}_1 \in {\mathbf{P}_1}^{\perp}, \ldots, \mathcal{E}_n \in {\mathbf{P}_n}^{\perp}$. □

We will use the following technical properties.

Lemma 9.4.

(1) *Let us denote by* $[\mathbf{N}_\xi]$ *the set of all* $\mathcal{F} \in \mathbf{N}_\xi$ *such that* $\mathcal{F}$ *has a* unique *root. We have that* $[\mathbf{N}_\xi]^\perp = \mathbf{N}_\xi{}^\perp$ *i.e.,*
$$\mathbf{P}_\xi = [\mathbf{N}_\xi]^\perp.$$
(2) *Let* $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{N}_\xi$. *Assume* $\mathcal{F}_1 = x^-.\mathcal{D}_1, \mathcal{F}_2 = x^-.\mathcal{D}_2$. *We have* $x^-.(\mathcal{D}_1 +^\tau \mathcal{D}_2) \in \mathbf{N}_\xi$.
(3) *If* $\mathcal{D} \in \mathbf{P}_\xi$, *then for each* τ*-slice* $\mathcal{S}$ *of* $\mathcal{D}$ *(see Definition 7.5), the root of* $\mathcal{S}^\approx$ *is either* x^+ *or* $\dagger$.

Proof.

(1) It is immediate by definition of normalization; see Remark 9.2.
(2) By Internal completeness 9.3, for each i, $x^-.\mathcal{D}_i \in \vdash \mathbf{P}_1, \ldots, \mathbf{P}_n$. By Lemma 8.13 $\mathcal{D}_1 +^\tau \mathcal{D}_2 \in \vdash \mathbf{P}_1, \ldots, \mathbf{P}_n$. Using 9.3 again, we conclude $\mathcal{D}_1 +^\tau \mathcal{D}_2 \in \mathbf{N}$.
(3) It follows immediately from the fact that $P_\xi = [\mathbf{N}_\xi]^\perp$ (point (1) above). □

Observe that the property at the point (2) above does not hold in general, for arbitrary behaviours (*cf.* Example 8.12).

Lemma 9.5. *Let* $\mathcal{D} \in \mathbf{P}_\xi = (\mathbf{N}_{\xi 1} \bullet \cdots \bullet \mathbf{N}_{\xi n})^{\perp\perp}$, *such that* $\mathcal{D} = x.\mathcal{D}'$. *We have that* $\underline{\sigma}(\mathcal{D}) \in \vdash \mathbf{P}_\xi, \mathbf{P}_\sigma$. *Moreover, the new root on* σ *is linear.*

Proof. By Lemma 9.4(1), $\mathbf{P}_\xi = [\mathbf{N}_\xi]^\perp$. Moreover, for all pairs $\mathcal{E}, \mathcal{F} \in [\mathbf{N}_\xi]$, $\mathcal{D} \perp \mathcal{E}$ and $\mathcal{D} \perp \mathcal{F}$. By Lemma 9.4(2), we have that $\mathcal{D} \perp\ \mathcal{E} +^\tau \mathcal{F}$. Using Proposition 8.9, we have that $\underline{\sigma}(\mathcal{D}) \perp \{\mathcal{E} +^\tau \mathcal{F}, (\mathcal{E} +^\tau \mathcal{F})[\sigma/\xi]\}$ and by Corollary 8.11 we have that $\underline{\sigma}(\mathcal{D}) \perp \{\mathcal{E}, \mathcal{F}[\sigma/\xi]\}$, that is $\underline{\sigma}(\mathcal{D}) \in \vdash \mathbf{P}_\xi, \mathbf{P}_\sigma$. □

Proposition 9.6 (Internal completeness of $\mathbf{F}^+$)**.** *Let* $x.\mathcal{D} \in \mathbf{P}_\xi = \mathbf{F}^+(\mathbf{N}_{\xi 1}, \ldots, \mathbf{N}_{\xi n})$. *Then* $\underline{\sigma}(\mathcal{D}) = \mathcal{D}'_1 \bullet \cdots \bullet \mathcal{D}'_n$ *where each* $\mathcal{D}'_i \in \vdash \mathbf{N}_{\sigma i}, \mathbf{P}_\xi$.

Proof. By Lemma 9.5, we have that if $x.\mathcal{D} \in \mathbf{P}_\xi$, then $\underline{\sigma}(x).\mathcal{D} \in \vdash \mathbf{P}_\xi, \mathbf{P}_\sigma$. Moreover, the root is an action on the name σ, and it is the only occurrence of action on σ. By using the same argument as in Proposition 5.1, we have that $\underline{\sigma}(x).\mathcal{D} = \mathcal{D}'_1 \bullet \cdots \bullet \mathcal{D}'_n$ and $\mathcal{D}'_i \in \vdash \mathbf{N}_{\sigma i}, \mathbf{P}_\xi$. □

10. Ludics with repetitions: full completeness

We now show that our model is fully complete with respect to **MELLS** (Section 2). In this paper, we limit ourselves to the constant fragment of **MELLS**: ground, atomic formulas are therefore only $?\mathbf{0}$ and $!\top$. From now on, by **MELLS** we always mean its constant fragment.

As usual in game semantics (*e.g.*, [26, 28]), not all strategies are suitable to be interpretation of a proof. In general, strategies which are interpretation of a proof have to satisfy some *winning conditions* which describe a "good" strategy. Our winning strategies are those that are finite, deterministic, daimon-free and *material* (see below).

We recall that we say that a strategy is deterministic if it is free of neutral actions; we now introduce the notion of materiality.

10.1. **Materiality.** By definition of composition (Section 8.1), at each step, the interaction collects (*i.e.*, *visits*) some occurrences of action of the strategies which we are composing. Moreover, when visiting an occurrence of action a in some view $w.a$, all actions in w have already been visited. We also say that the view $w.a$ is *used* during the composition.

The idea of *materiality* is immediate to understand; let us consider the following example.

Example 10.1. Let $\mathcal{D}, \mathcal{E}, \mathcal{F}$ be the strategies in Figure 8.

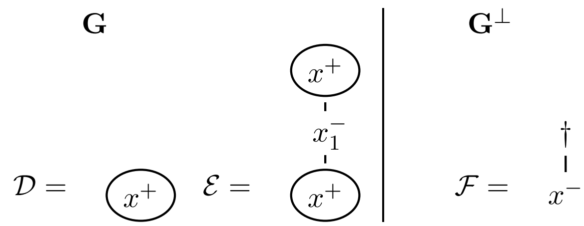


Figure 8: Materiality

Consider $\mathbf{G} = \{\mathcal{D}\}^{\perp\perp}$ and notice that $\mathcal{F} \in \mathbf{G}^{\perp}$. Observe also that $\mathcal{E} \in \mathbf{G}$, but the normalization between $\mathcal{E}$ and $\mathcal{F}$ uses only the first action x^+; the action x_1^- is never visited through the interaction between $\mathcal{E}$ and $\mathcal{F}$.

As the example shows us, normalization does not necessarily visit all the actions of a strategy. The notion of materiality exactly captures the significant part of a strategy from the point of view of a behaviour $\mathbf{G}$, that is the part that is really used to react to the tests (strategies of $\mathbf{G}^{\perp}$).

Definition 10.2 (Materiality)**.** Let $\mathcal{D}, \mathcal{E}$ be orthogonal strategies; we denote by $\mathcal{D}_{\mathcal{E}}$ the set of views of $\mathcal{D}$ which are used during the normalization against $\mathcal{E}$.

Let $\mathcal{D}$ be a strategy of $\mathbf{G}$. We define the **material part** of $\mathcal{D}$ in $\mathbf{G}$,

$$|\mathcal{D}|_{\mathbf{G}} = \{ \bigcup_{\mathcal{E} \in \mathbf{G}^{\perp}} \mathcal{D}_{\mathcal{E}} \}.$$

A strategy is said **material** in $\mathbf{G}$ if $\mathcal{D} = |\mathcal{D}|_{\mathbf{G}}$.

The content of this definition is made explicit by the points (2),(3) below.

Lemma 10.3.

(1) *$\mathcal{D}_{\mathcal{E}}$ and $|\mathcal{D}|_{\mathbf{G}}$ are strategies.*
(2) *$[\![\mathcal{D}_{\mathcal{E}}, \mathcal{E}]\!] = [\![\mathcal{D}, \mathcal{E}]\!]$.*
(3) *$|\mathcal{D}|_{\mathbf{G}} \perp \mathcal{E}$ for each $\mathcal{E} \in \mathbf{G}^{\perp}$.*

Proof.

(1) It is a consequence of the fact that $\mathcal{D}_{\mathcal{E}} \subseteq \mathcal{D}$ and $|\mathcal{D}|_{\mathbf{G}} \subseteq \mathcal{D}$, which guarantees most of the conditions, in particular coherence. Maximality is given by the definition of normalization (an action which is not matched, is not considered "visited").
(2) and (3) express exactly the content of the definition of materiality: $\mathcal{D}_{\mathcal{E}}$ is all what is used in $\mathcal{D}$ to react with $\mathcal{E}$. So, in particular,

$$[\![|\mathcal{D}|_{\mathbf{G}}, \mathcal{E}]\!] = [\![\mathcal{D}, \mathcal{E}]\!] \text{ for each } \mathcal{E} \in \mathbf{G}^{\perp}.$$

□

Obviously, from the previous lemma we have:

Corollary 10.4. *If* $\mathcal{D} \in \mathbf{G}$ *then* $|\mathcal{D}|_{\mathbf{G}} \in \mathbf{G}$.

Coming back to Example 10.1, we can see that $\mathcal{D}$ is material in $\mathbf{G}$ whereas $\mathcal{E}$ is not.

The notion of materiality naturally extends to sequent of behaviours (considering families of counter-strategies instead of single counter-strategies).

Example 10.5 (Materiality with constant types)**.** Let us fix a name ξ. The only material and deterministic strategy in the constant type $?\mathbf{0}$ is $\mathfrak{Dai}$. What about $!\top = \{\mathcal{D} : \mathcal{D}$ is on interface $\xi^-\}$? The unique material strategy which inhabits $!\top$ is the empty negative one $\emptyset$ (*cf.* Example 4.4). Indeed, no view of any other strategy $\mathcal{D} \in\, !\top$ can be visited by the interaction with $\mathfrak{Dai}$.

10.2. **Completeness theorems.** In Section 10.3, we describe the interpretation of a formula F of **MELLS** into a behaviour $\mathbf{F}$ and similarly the interpretation of syntactical sequents of **MELLS** into sequents of behaviours. Derivations of sequents in **MELLS** will be interpreted by *winning strategies*.

Definition 10.6 (Winning strategy)**.** A strategy $\mathcal{D} \in\, \vdash \mathbf{\Gamma}$ is said **winning** if it is finite, deterministic, daimon-free and material in $\vdash \mathbf{\Gamma}$.

In the sequel, finiteness, determinism, daimon-freeness and materiality are also called *winning conditions*.

Remark 10.7 (Finiteness condition)**.** We here assume *finiteness* among the winning conditions. However, recent work by Basaldella and Terui [5] shows an exciting property of interactive types: any material, deterministic and daimon free strategy in a behaviour which is interpretation of logical formula is finite. We are confident that this result is also valid our setting; we need to check this in detail, and for lack of time we postpone it in a subsequent work.

The rest of this article is then devoted to prove the following theorems.

Soundness: (Theorem 10.11) Let π be a derivation of a sequent $\vdash \Gamma$ in **MELLS**. There exists a winning strategy $\pi^\star \in\, \vdash \mathbf{\Gamma}$ such that $\pi^\star$ is interpretation of π, where $\vdash \mathbf{\Gamma}$ is the interpretation of $\vdash \Gamma$. Moreover, if π reduces to ρ by means of cut-elimination, then $\pi^\star = \rho^\star$.

Full Completeness: (Theorem 10.12) Let $\vdash \mathbf{\Gamma}$ be the interpretation of a sequent $\vdash \Gamma$, and let $\mathcal{D} \in\, \vdash \mathbf{\Gamma}$. If $\mathcal{D}$ is winning, then $\mathcal{D}$ is the interpretation of a cut-free derivation π of the sequent $\vdash \Gamma$ in **MELLS**.

10.3. **Interpretation of formulas, sequents and derivations.** The interpretation $\langle F \rangle_\xi$ of a formula F of **MELLS** is given by a behaviour $\mathbf{F}$ on a chosen name ξ by structural induction on F as follows:

$$\begin{array}{lllllll} \langle !\top \rangle_\xi & := & !\top_\xi; & \qquad & \langle F^+(N_1,\ldots,N_n) \rangle_\xi & := & \mathbf{F}^+(\langle N_1 \rangle_{\xi 1},\ldots,\langle N_n \rangle_{\xi n})_\xi; \\ \langle ?\mathbf{0} \rangle_\xi & := & ?\mathbf{0}_\xi; & & \langle F^-(P_1,\ldots,P_n) \rangle_\xi & := & \mathbf{F}^-(\langle P_1 \rangle_{\xi 1},\ldots,\langle P_n \rangle_{\xi n})_\xi. \end{array}$$

In the sequel, we indicate the behaviour $\langle F \rangle_\xi$ by $\mathbf{F}_\xi$ (by $\mathbf{F}(\xi)$ in case of multiple subscripts) or just by $\mathbf{F}$. We will always assume that a behaviour is an interpretation of a formula of **MELLS**. Precisely, we will only consider behaviours inductively defined as follows

$$\mathbf{P} ::= ?\mathbf{0} \mid \mathbf{F}^+(\mathbf{N}_1, \ldots, \mathbf{N}_n); \qquad \mathbf{N} ::= !\top \mid \mathbf{F}^-(\mathbf{P}_1, \ldots, \mathbf{P}_n);$$

using the types constructors introduced in Section 9.

A sequent $\vdash F_1, \ldots, F_n$ of **MELLS** is interpreted by the sequent of behaviours $\vdash \mathbf{F}_1(\xi_1), \ldots, \mathbf{F}_n(\xi_n)$ on a given interface $\xi_1, \ldots, \xi_n$.

In order to interpret derivations of **MELLS**, we first need the following lemma:

Lemma 10.8 (Contraction)**.**
(1) *If* $\mathcal{D} \in \vdash \mathbf{P}_\xi, \mathbf{P}_\sigma, \mathbf{\Gamma}$ *then* $\mathcal{D}[\xi/\sigma] \in \vdash \mathbf{P}_\xi, \mathbf{\Gamma}$.
(2) *Moreover, if* $\mathcal{D}$ *is winning in* $\vdash \mathbf{P}_\xi, \mathbf{P}_\sigma, \mathbf{\Gamma}$*, then* $\mathcal{D}[\xi/\sigma]$ *is winning in* $\vdash \mathbf{P}_\xi, \mathbf{\Gamma}$.

Proof.

1. Let $\mathbf{\Gamma} = \mathbf{F}_1, \ldots, \mathbf{F}_n$. By hypotheses, for arbitrary $\mathcal{E}_1 \in {\mathbf{F}_1}^\perp, \ldots, \mathcal{E}_k \in {\mathbf{F}_n}^\perp$ and $\mathcal{A}, \mathcal{B} \in \mathbf{P}_\xi^\perp$, we have $\mathcal{D} \perp \{\mathcal{A}, \mathcal{B}[\sigma/\xi], \mathcal{E}_1, \ldots, \mathcal{E}_n\}$. In particular, we also have $\mathcal{D} \perp \{\mathcal{A}, \mathcal{A}[\sigma/\xi], \mathcal{E}_1, \ldots, \mathcal{E}_n\}$ which implies $\mathcal{D}[\xi/\sigma] \perp \{\mathcal{A}, \mathcal{E}_1, \ldots, \mathcal{E}_n\}$, since the interactions of the cut-nets $\mathcal{R} = \{\mathcal{D}, \mathcal{A}, \mathcal{A}[\sigma/\xi], \mathcal{E}_1, \ldots, \mathcal{E}_n\}$ and $\mathcal{R}' = \{\mathcal{D}[\xi/\sigma], \mathcal{A}, \mathcal{E}_1, \ldots, \mathcal{E}_n\}$ are essentially the same ($p \in I(\mathcal{R})$ if and only if "$p[\xi/\sigma]$" $\in I(\mathcal{R}')$, with the obvious intuitive meaning for $p[\xi/\sigma]$). Hence, $\mathcal{D}[\xi/\sigma] \in \vdash \mathbf{P}_\xi, \mathbf{\Gamma}$.
2. For winning conditions, the only one which is not immediate to check is materiality. By Lemma 9.4(1), we can choose counter-strategies $\mathcal{A}, \mathcal{B} \in [{\mathbf{P}_\xi}^\perp]$. Materiality follows by observing that for any part $\mathcal{D}' \subseteq \mathcal{D}$ which can be visited by normalizing $\mathcal{D}$ with $\{\mathcal{A}, \mathcal{B}[\sigma/\xi], \mathcal{E}_1, \ldots, \mathcal{E}_n\}$ and $\{\mathcal{B}, \mathcal{A}[\sigma/\xi], \mathcal{E}_1, \ldots, \mathcal{E}_n\}$, the corresponding part $\mathcal{D}'[\xi/\sigma] \subseteq \mathcal{D}[\xi/\sigma]$ can be visited by using $\{\mathcal{A} +^\tau \mathcal{B}, \mathcal{E}_1, \ldots, \mathcal{E}_n\}$, since by Lemma 9.4(2), $\mathcal{A} +^\tau \mathcal{B} \in \mathbf{P}_\xi^\perp$. □

Proposition 10.9. *Let* π *be a derivation of a sequent* $\vdash \Gamma$ *in* **MELLS**. *There exists a strategy* $\pi^\star \in \vdash \mathbf{\Gamma}$ *such that* π^* *is interpretation of* π.

Proof. The proof is given by induction on the depth of π. We have four cases, one for each rule of **MELLS** without propositional variables.

!⊤-rule: $\dfrac{}{\vdash !\top, \Gamma}\,!\top$
We take $\pi^\star = \emptyset$ on the sequent of behaviour $\vdash !\top, \mathbf{\Gamma}$.

Pos-**rule:** $$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash N_1, F^+, \Gamma\end{array} \quad \ldots \quad \begin{array}{c}\vdots\,\pi_n\\ \vdash N_n, F^+, \Gamma\end{array}}{\vdash F^+, \Gamma}\,\text{Pos}$$
By inductive hypotheses, there are strategies $\pi_i^\star \in \vdash \mathbf{N}_i(\sigma i), \mathbf{F}^+(\xi), \mathbf{\Gamma}$ (for $1 \leq i \leq n$). We now take $(\sigma, \{1, \ldots, n\})^+.\{\pi_1^\star, \ldots, \pi_n^\star\}$ which is by construction in $\vdash \mathbf{F}_\sigma^+, \mathbf{F}_\xi^+, \mathbf{\Gamma}$. Finally, we consider $\pi^\star := (\sigma, \{1, \ldots, n\})^+.\{\pi_1^\star, \ldots, \pi_n^\star\}[\xi/\sigma]$. By Lemma 10.8 (1), $\pi^\star \in \vdash \mathbf{F}_\xi^+, \mathbf{\Gamma}$.

Neg-**rule:** $$\dfrac{\begin{array}{c}\vdots\,\zeta\\ \vdash P_1, \ldots, P_n, \Gamma\end{array}}{\vdash F^-, \Gamma}\,\text{Neg}$$
By inductive hypotheses, we have a strategy $\zeta^\star \in \vdash \mathbf{P}_1(\xi 1), \ldots, \mathbf{P}_n(\xi n), \mathbf{\Gamma}$. We take $\pi^\star := (\xi, \{1, \ldots, n\})^-.\zeta^\star$. By construction, $\pi^\star \in \vdash \mathbf{F}_\xi^-, \mathbf{\Gamma}$.

Cut-**rule:**
$$\dfrac{\dfrac{\vdots\, \pi_1}{\vdash P, \Xi, \Gamma} \qquad \dfrac{\vdots\, \pi_2}{\vdash P^{\perp}, \Gamma}}{\vdash \Xi, \Gamma}\,\text{Cut}$$

Let Γ be $P_1, \ldots, P_k$. By inductive hypotheses, we can consider two strategies $\pi_1^{\star} \in \vdash \mathbf{P}_{\xi}, \boldsymbol{\Xi}_{\varphi}, \boldsymbol{\Gamma}_{\vec{\alpha}}$, and $\pi_2^{\star} \in \vdash \mathbf{P}_{\xi}^{\perp}, \boldsymbol{\Gamma}_{\vec{\beta}}$, where $\boldsymbol{\Gamma}_{\vec{\alpha}} = \mathbf{P}_1(\alpha_1), \ldots, \mathbf{P}_k(\alpha_k)$ and $\boldsymbol{\Gamma}_{\vec{\beta}} = \mathbf{P}_1(\beta_1), \ldots, \mathbf{P}_k(\beta_k)$ with $\vec{\alpha} = \alpha_1, \ldots, \alpha_k$ and $\vec{\beta} = \beta_1, \ldots, \beta_k$ disjoint interfaces. By composing them, we get the strategy $[\![\pi_1^{\star}, \pi_2^{\star}]\!] \in \vdash \boldsymbol{\Xi}_{\varphi}, \boldsymbol{\Gamma}_{\vec{\alpha}}, \boldsymbol{\Gamma}_{\vec{\beta}}$. For $\vec{\gamma} = \gamma_1, \ldots, \gamma_k$, let $\varphi^-, \vec{\gamma}$ be the interface associated to the interpretation of the final sequent $\vdash \Xi, \Gamma$. We take $\pi^{\star} := [\![\pi_1^{\star}, \pi_2^{\star}]\!][\vec{\gamma}/\vec{\alpha}, \vec{\gamma}/\vec{\beta}]$, where $[\![\pi_1^{\star}, \pi_2^{\star}]\!][\vec{\gamma}/\vec{\alpha}, \vec{\gamma}/\vec{\beta}]$ is given by replacing any occurrence of name α_i and β_i in $[\![\pi_1^{\star}, \pi_2^{\star}]\!]$ by γ_i, for any $(1 \leq i \leq k)$. Applying several times Lemma 10.8(1), we have that $\pi^{\star} \in \vdash \boldsymbol{\Xi}_{\varphi}, \boldsymbol{\Gamma}_{\vec{\gamma}}$. □

10.4. **Soundness.** In order to prove Soundness, we first show the following lemma, which express that our interpretation is invariant under cut-elimination procedure of **MELLS**.

Lemma 10.10. *Let π be a derivation of a sequent $\vdash \Xi, \Gamma$ in* **MELLS** *ending with a cut-rule.*

$$\dfrac{\dfrac{\vdots\, \pi'}{\vdash P, \Xi, \Gamma} \qquad \dfrac{\vdots\, \pi''}{\vdash P^{\perp}, \Gamma}}{\vdash \Xi, \Gamma}\,\text{Cut}$$

If π reduces to ρ by a step of cut-elimination, then π, ρ are interpreted by the same strategy $\mathcal{D} \in \vdash \boldsymbol{\Xi}, \boldsymbol{\Gamma}$.

The proof is quite technical and lengthy; it can be found in Appendix C.

We now have all the ingredients for proving:

Theorem 10.11 (Soundness)**.** *Let π be a derivation of a sequent $\vdash \Gamma$ in* **MELLS**. *There exists a winning strategy $\pi^{\star} \in \vdash \boldsymbol{\Gamma}$ such that $\pi^{\star}$ is interpretation of π, where $\vdash \boldsymbol{\Gamma}$ is the interpretation of $\vdash \Gamma$. Moreover, if π reduces to ρ by means of cut-elimination, then $\pi^{\star} = \rho^{\star}$.*

Proof. By Proposition 10.9 and Lemma 10.10, it only remains to prove that those strategies given by Proposition 10.9 are winning in their sequents of behaviours.

- $!\top$-**rule:** If π ends with the $!\top$-rule, then $\pi^{\star} = \emptyset$ is clearly a winning strategy in $\vdash !\top, \boldsymbol{\Gamma}$.
- Pos-**rule:** If π ends with a positive rule and the premises are interpreted by winning strategies, $\pi_i^{\star} \in \vdash \mathbf{N}_i(\sigma i), \mathbf{F}^+(\xi), \boldsymbol{\Gamma}$ (for $1 \leq i \leq n$), then the strategy $(\sigma, \{1, \ldots, n\})^+ . \{\pi_1^{\star}, \ldots, \pi_n^{\star}\}$ is winning in $\vdash \mathbf{F}_{\sigma}^+, \mathbf{F}_{\xi}^+, \boldsymbol{\Gamma}$. By Lemma 10.8(2), $\pi^{\star} = (\sigma.\{1, \ldots, n\})^+ . \{\pi_1^{\star}, \ldots, \pi_n^{\star}\}[\xi/\sigma]$ is winning in $\vdash \mathbf{F}_{\xi}^+, \boldsymbol{\Gamma}$.
- Neg-**rule:** If π ends with a negative rule and its premise is interpreted by a winning strategy $\zeta^{\star} \in \vdash \mathbf{P}_1(\xi 1), \ldots, \mathbf{P}_n(\xi n), \boldsymbol{\Gamma}$, then it is immediate to check that $\pi^{\star} = (\xi, \{1, \ldots, n\})^- . \zeta^{\star}$ is winning in $\vdash \mathbf{F}_{\xi}^-, \boldsymbol{\Gamma}$.
- Cut-**rule:** If π ends with a cut-rule, let $\pi_1^{\star} \in \vdash \mathbf{P}_{\xi}, \boldsymbol{\Xi}_{\varphi}, \boldsymbol{\Gamma}_{\vec{\alpha}}$ and $\pi_2^{\star} \in \vdash \mathbf{P}_{\xi}{}^{\perp}, \boldsymbol{\Gamma}_{\vec{\beta}}$ be the winning strategies which interpret the premises of the cut. We have to show that $[\![\pi_1^{\star}, \pi_2^{\star}]\!] \in \vdash \boldsymbol{\Xi}_{\varphi}, \boldsymbol{\Gamma}_{\vec{\alpha}}, \boldsymbol{\Gamma}_{\vec{\beta}}$ is winning. By Lemma 10.10, $[\![\pi_1^{\star}, \pi_2^{\star}]\!]$ is also the interpretation of a *cut-free* derivation ρ of $\vdash \Xi, \Gamma, \Gamma$ obtained by applying cut-elimination procedure to π. Hence, by the previous steps above, $[\![\pi_1^{\star}, \pi_2^{\star}]\!]$ is winning

in $\vdash \mathbf{\Xi}_\varphi, \mathbf{\Gamma}_{\vec{\alpha}}, \mathbf{\Gamma}_{\vec{\beta}}$. Finally, applying Lemma 10.8(2) several times, we have that $\pi^\star = [\![\pi_1^\star, \pi_2^\star]\!][\vec{\gamma}/\vec{\alpha}, \vec{\gamma}/\vec{\beta}]$ is winning in $\vdash \mathbf{\Xi}_\varphi, \mathbf{\Gamma}_{\vec{\gamma}}$. □

10.5. **Full completeness.**

Theorem 10.12 (Full Completeness)**.** *Let $\vdash \mathbf{\Gamma}$ be the interpretation of a sequent $\vdash \Gamma$, and let $\mathcal{D} \in \vdash \mathbf{\Gamma}$. If $\mathcal{D}$ is winning, then $\mathcal{D}$ is the interpretation of a cut-free derivation π of the sequent $\vdash \Gamma$ in* **MELLS***.*

Proof. Since our strategies are finite, we can reason by induction on the number of actions of $\mathcal{D}$.

Empty case. We already observed that in a behaviour, all strategies are total. Hence, if $\mathcal{D}$ is empty, it must be a negative strategy. We have $\mathcal{D} \in \vdash \mathbf{N}, \mathbf{P}_1, \ldots, \mathbf{P}_n$, and by definition of sequent of behaviours, for all $\mathcal{E}_i \in \mathbf{P}_i{}^\perp$ $(1 \leq i \leq n)$, $[\![\mathcal{D}, \mathcal{E}_1, \ldots, \mathcal{E}_n]\!] \in \mathbf{N}$. By definition of normalization, we have $[\![\mathcal{D}, \mathcal{E}_1, \ldots, \mathcal{E}_n]\!] = \emptyset$. We conclude that $\mathbf{N} = !\top$, and hence that $\mathcal{D} \in \vdash !\top, \mathbf{P}_1, \ldots, \mathbf{P}_n$. The empty strategy is therefore the interpretation of the $!\top$-rule.

Non empty case. Let $\vdash \mathbf{\Delta}$ be the interpretation of the sequent $\vdash \Delta$, and $\mathcal{D} \in \vdash \mathbf{\Delta}$ a winning strategy. Our purpose is to associate to $\mathcal{D}$ a derivation $\mathcal{D}^\star$ of $\vdash \Delta$ in **MELLS**, by progressively decomposing $\mathcal{D}$, *i.e.*, inductively writing "the last rule". To be able to use internal completeness, which is defined on behaviours (and not on sequents of behaviours), we will use — back and forth — the definition of sequent of behaviours and in particular Proposition 9.1.

The formula on which the last rule is applied is indicated by the name of the root action. Since $\mathcal{D}$ is non-empty, there is a minimal action, which is the root. Such a minimal action is unique, because $\mathcal{D}$ is deterministic and material. For example, let us assume that the root of $\mathcal{D}$ is (ξ, I); then if $\mathcal{D} \in \vdash \mathbf{F}_\xi, \mathbf{C}_\sigma$, the behaviour which corresponds to the last rule is the one on ξ, *i.e.*, $\mathbf{F}_\xi$.

Without loss of generality, in the following we will consider $\mathcal{D} \in \vdash \mathbf{F}, \mathbf{C}$; moreover, we assume $\mathbf{F}$ binary. Of course the argument straightforwardly generalizes to the cases $\mathcal{D} \in \vdash \mathbf{F}, \mathbf{\Gamma}$ and $\mathbf{F}$ n-ary.

Positive case. Let $\mathcal{D} = (\xi, \{1,2\})^+.\{\mathcal{D}_1, \mathcal{D}_2\}$ be a positive winning strategy which belongs to $\vdash \mathbf{F}^+_\xi, \mathbf{C}_\alpha$, where $\mathbf{F}^+ = \mathbf{F}^+(\mathbf{N}, \mathbf{M})$ and $\mathbf{C}$ are the interpretation of formulas $F^+(N, M)$ and C respectively. By Proposition 9.1, for any $\mathcal{E} \in \mathbf{C}^\perp$, we have:

(1) $[\![\mathcal{D}, \mathcal{E}]\!] \in \mathbf{F}^+_\xi$ and the root of $[\![\mathcal{D}, \mathcal{E}]\!]$ is still $(\xi, \{1,2\})^+$. This allows us to use internal completeness.
(2) By internal completeness of positive connectives (Proposition 9.6), we have that $\underline{\sigma}([\![\mathcal{D}, \mathcal{E}]\!])$ can be written as $\mathcal{C}_1 \bullet \mathcal{C}_2$, for some $\mathcal{C}_1 \in \vdash \mathbf{N}_{\sigma 1}, \mathbf{F}^+_\xi$ and $\mathcal{C}_2 \in \vdash \mathbf{M}_{\sigma 2}, \mathbf{F}^+_\xi$.
(3) By definition of normalization, it is immediate that:

$$\begin{aligned}\underline{\sigma}([\![\mathcal{D}, \mathcal{E}]\!]) = [\![\underline{\sigma}(\mathcal{D}), \mathcal{E}]\!] &= [\![(\sigma, \{1,2\})^+.\{\mathcal{D}'_1, \mathcal{D}'_2\}, \mathcal{E}]\!] \\ &= (\sigma, \{1,2\})^+.\{[\![\mathcal{D}'_1, \mathcal{E}]\!], [\![\mathcal{D}'_2, \mathcal{E}]\!]\} \\ &= [\![\mathcal{D}'_1, \mathcal{E}]\!] \bullet [\![\mathcal{D}'_2, \mathcal{E}]\!].\end{aligned}$$

From this, we conclude that $[\![\mathcal{D}'_1, \mathcal{E}]\!] \in \vdash \mathbf{N}_{\sigma 1}, \mathbf{F}^+_\xi$ and $[\![\mathcal{D}'_2, \mathcal{E}]\!] \in \vdash \mathbf{M}_{\sigma 2}, \mathbf{F}^+_\xi$.

By applying Proposition 9.1 again, we have that $\mathcal{D}'_1 \in \vdash \mathbf{N}_{\sigma 1}, \mathbf{F}^+_\xi, \mathbf{C}_\alpha$ and $\mathcal{D}'_2 \in \vdash \mathbf{M}_{\sigma 2}, \mathbf{F}^+_\xi, \mathbf{C}_\alpha$ and then we can write the derivation:

$$\dfrac{\dfrac{\vdots\ \mathcal{D}'^{\star}_1}{\vdash N_1, F^+(N,M), C} \quad \dfrac{\vdots\ \mathcal{D}'^{\star}_2}{\vdash N_2, F^+(N,M), C}}{\vdash F^+(N,M), C}\ \text{Pos}$$

It is immediate to check that winning conditions are preserved in both $\mathcal{D}'_1, \mathcal{D}'_2$ and the number of actions decreases. Hence, the inductive hypotheses applies.

Negative case. Let us now consider a negative winning strategy $\mathcal{D} \in \vdash \mathbf{F}^-_\xi, \mathbf{C}_\alpha$, where $\mathbf{F}^- = \mathbf{F}^-(\mathbf{P}, \mathbf{Q})$ and $\mathbf{C}$ are the interpretations of formulas $F^-(P,Q)$ and C respectively. All strategies in $\mathbf{F}^{-\perp}$ have essentially only one possible root action x (by Lemma 9.4(3)); x is hence the only root action which can be used by normalization. Since $\mathcal{D}$ is material, it follows that $\mathcal{D}$ can only have a single root action: $\mathcal{D}$ is of the form $x^-.\mathcal{D}'$. Let assume $x = (\xi, \{1,2\})$.

For any $\mathcal{E} \in \mathbf{C}^\perp$, we have

(1) $[\![\mathcal{D}, \mathcal{E}]\!] \in \mathbf{F}^-(\mathbf{P}, \mathbf{Q})$, and the root is still x^-. This allows us to use internal completeness.
(2) By internal completeness of negative connectives (Proposition 9.3), we conclude that $[\![\mathcal{D}, \mathcal{E}]\!]$ is of the form $x^-.\mathcal{D}''$ with $\mathcal{D}'' \in \vdash \mathbf{P}_{\xi 1}, \mathbf{Q}_{\xi 2}$.
(3) By the definition of normalization,

$$[\![\mathcal{D}, \mathcal{E}]\!] = [\![x^-.\mathcal{D}', \mathcal{E}]\!] = x^-.[\![\mathcal{D}', \mathcal{E}]\!].$$

From this, we have that $\mathcal{D}'' = [\![\mathcal{D}', \mathcal{E}]\!]$ and hence $[\![\mathcal{D}', \mathcal{E}]\!] \in \vdash \mathbf{P}_{\xi 1}, \mathbf{Q}_{\xi 2}$.

By applying Proposition 9.1 again, we have that $\mathcal{D}' \in \vdash \mathbf{P}_{\xi 1}, \mathbf{Q}_{\xi 2}, \mathbf{C}_\alpha$. Then, we can write the derivation:

$$\dfrac{\dfrac{\vdots\ \mathcal{D}'^{\star}}{\vdash P, Q, C}}{\vdash F^-(P,Q), C}\ \text{Neg}$$

It is immediate to check that winning conditions are preserved in $\mathcal{D}'$ and the number of actions decreases. Hence, the inductive hypotheses applies. □

11. Conclusion

In this work, we started by recalling the standard notion of HO strategy and we have shown how ludics strategies can be expressed in term HO strategies by giving an universal arena. We have revised the main results of the higher-level part of ludics (namely, *internal completeness*) giving direct proofs of them using basic properties of the dynamics only. We have motivated and introduced the notion of non-uniform strategy and shown that we still have internal completeness when strategies are non-linear and non-uniform. From this, we finally have shown a full completeness result with respect to the constant-only fragment of **MELLS**.

Related and future work.

Maurel's exponentials. Maurel [27] has built a sophisticated setting to recover a form of separation when having repetitions in ludics; however, the complexity of the setting prevent him from going furthen and studying interpretation and full completeness issues. In this paper, we ignore separation all together, and in fact we show that we don't need it in order to have interactive types and internal completeness. In future work, we hope it may be possible to refine our setting by using Maurel techniques.

Let us discuss this perspective. In Maurel's setting, strategies have a quantitative information carried by probabilistic values (*coefficients*). The values in the coefficients have a central role, and must satisfy a set of "quantitative conditions" inspired by measure theory. This is fundamentally different from our indexed neutral (τ) actions, as the specific natural number which is chosen as index for a τ action is irrelevant (in particular, all the indexes can be interchanged, and this does not affect orthogonality), and there are no condition attached. Our indexed τ have the same role as in [32]. However, in a way, we think that our use of τ actions could be seen as a simplification — or rather a quotient — on Maurel's coefficients; on this grounds, we hope it may be possible to refine our neutral actions by attaching probabilities to them, without loosing our high-level results.

AJM style exponentials for Ludics. A different solution that uses AJM style exponentials has been studied by the first of the two authors in [3]: essentially, the strategies which inhabit a semantical type $!\mathbf{A}$ are those of the form $(\mathcal{N}, 1) \cup (\mathcal{M}, 2) \cup \ldots$: an *indexed superimposition* of strategies $\mathcal{N}, \mathcal{M}, \ldots$ of $\mathbf{A}$.

However, the approach we use in this paper, which exploits similar ideas, is considerably simpler, and we hope more suitable for more applicative uses of ludics [14, 29, 30].

τ-actions and innocent strategies. The n.u. strategies we introduce in this paper rely on previous work developed by Faggian and Piccolo [15, 13, 14], to bridge between ludics and process calculus by means of a more general language, that of event structures. The notion of τ actions and τ-cell are there introduced and motivated, as well as the conditions to generalize the definition of innocent strategies.

While the setting by Faggian and Piccolo is linear, we show here that the generalized definition of innocence (with τ actions) extends also to the case with repetition.

A closely related work —where indexed τ actions are first introduced— is [32].

Computational ludics. By using the approach we present in this paper, Basaldella and Terui [5] have recently extended Terui's computational ludics [30] in order to accommodate exponentials.

Their paper is aimed to analyzing the traditional logical duality between proofs and models from the point of view of ludics and they get an alternative proof of full completeness based on a direct construction of a counter-model. Very interestingly, that work also enlighten an exciting property of the "*interactive types*". Unlike in standard HO game semantics, finiteness does not need to be requested as a condition for strategies to be winning; it is rather an outcome of the closure by orthogonality. In fact, Basaldella and Terui show that any material, deterministic strategy in a behaviour which is interpretation of logical formula is finite. We are confident that this result is also valid our setting, however we still need to check all the details.

Non-deterministic innocent strategies. They have been introduced by Harmer in [22], with the purpose of modeling non-determinism (PCF with erratic choice).

In this paper we introduce non-uniform strategies, which are realized by means of *non-deterministic sums*. However, the purpose of our *non-deterministic sums* is to implement *non uniformity* via "formal sums" of strategies, in order to provide enough tests to make possible the interactive approach of ludics. The different purpose is reflected in the composition, which is simpler in our setting, where is in fact reduced to deterministic composition. Our strategies could be seen as a "concrete" implementation of Harmer's solution, in a simplified setting. Harmer overcomes the problems with composition moving from naive non-deterministic strategies $S : A \to B$ to an "indirect" definition of strategies of the kind $S : A \times \mathbb{N} \to B$. We have instead 3 kinds of actions: Player (+), Opponent (−), and tau actions (τ). Tau actions carry an index $i \in \mathbb{N}$, and have a two-fold role: they guard the sum (as in [32]), and provide an "index of copy" (as in AJM game semantics, but here the index is unfold only when needed).

Polarized game semantics. We build on the variant of HO strategies introduced in [26]. Moreover, we are interested in connections with the resource modalities of games semantics introduced by Melliès and Tabareau in [28].

Abstract machines. Curien and Herbelin in [10] have studied composition of strategies as sets of views. In particular they have developed the View-Abstract-Machine (VAM) (see also [7]) which is the device we use in this paper.

References

[1] Abramsky, S., Jagadeesan, R., Malacaria, P.: Full Abstraction for PCF. Inf. Comput. 163 2 (2000) 409–470.
[2] Andreoli, J.-M.: Logic Programming with Focusing Proof in Linear Logic. J. Log. Comput. 2(3) (1992) 297–347.
[3] Basaldella, M.: On Exponentials in Ludics. PhD Thesis (2008) University of Siena.
[4] Basaldella, M., Faggian, C.: Ludics with repetitions (exponentials, interactive types and completeness). In: LiCS. (2009) 375–384.
[5] Basaldella, M., Terui, K.: On the meaning of logical completeness. In: TLCA. (2009) 50–64.
[6] Coquand, T.: A semantics of evidence for classical arithmetic. J. Symb. Log. 60(1) (1995) 325-337.
[7] Curien, P.-L.: Abstract Böhm trees. Mathematical Structures in Computer Science 8(6) (1998) 559–591.
[8] Curien, P.-L.: Introduction to Linear Logic and Ludics, part II. Advances of Mathematics (China) 35(1) (2006) 1–44.
[9] Curien, P.-L.: Notes on game semantics. (2006) Manuscript.
[10] Curien, P.-L., Herbelin, H.: Abstract machines for dialogue games. CoRRabs/0706.2544 (2007).
[11] Faggian, C.: Travelling on designs. In: CSL. (2002) 427–441.
[12] Faggian, C., Hyland, J.M.E.: Designs, disputes and strategies. In : CSL. (2002) 442–457.
[13] C. Faggian and M. Piccolo. A graph abstract machine describing event structure composition. In: GT - VC, Graph Transformation for Verification and Concurrency. ENTCS (2007) 21–36.
[14] Faggian, C., Piccolo, M.: Ludics is a model for the finitary linear pi-calculus. In: TLCA. (2007) 148–162.
[15] Faggian, C., Piccolo, M.: Partial Orders, Event Structures, and Linear Strategies. In: TLCA. (2009) 95–111.
[16] Girard, J.-Y.: Linear Logic. Theor. Comput. Sci. 50(1) (1987) 1–102.
[17] Girard, J.-Y.: Geometry of interaction I: Interpretation of System F. Logic Colloquium 88, In R. Ferro et al., (1989) 221–260.

[18] Girard, J.-Y.: On the meaning of logical rules I: syntax vs. semantics. Computational Logic (U. Berger and H. Schwichtenberg eds) Heidelberg Springer-Verlag (1999) 215 – 272.
[19] Girard, J.-Y.: On the meaning of logical rules II : multiplicatives and additives. Foundation of Secure Computation (2000) 183–212.
[20] Girard, J.-Y.: Locus solum: From the rules of logic to the logic of rules. Mathematical Structures in Computer Science 11 (2001) 301–506.
[21] Girard, J.-Y.: Le Point Aveugle, Cours de logique, Tome II: Vers l'imperfection. Visions des Sciences. Hermann (2007).
[22] Harmer, R. S.: Games and Full Abstraction for Nondeterministic Languages. PhD Thesis (1999) University of London.
[23] Harmer, R. S.: Innocent game semantics. (2006) Manuscript.
[24] Hyland, J.M.E., Ong, C.H.L.: On full abstraction for PCF: I, II, and III. Inf. Comput. 163 (2000) 285–408.
[25] Laurent, O.: Polarized Games. Ann. Pure Appl. Log. 130(1-3) (2004) 217–264.
[26] Laurent, O.: Syntax vs. semantics: A polarized approach. Theor. Comput. Sci. 343(1-2) (2005) 177–206.
[27] Maurel, F.: Un cadre quantitatif pour la Ludique. PhD Thesis (2004) Université Paris VII.
[28] Melliès, P.-A., Tabareau, N.: Resource modalities in game semantics. In: LiCS. (2007) 389–398.
[29] Saurin, A.: Towards Ludics Programming: Interactive Proof Search. In: ICLP. (2008) 253–268.
[30] Terui, K.: Computational ludics. (2008) To appear in Theor. Comput. Sci.
[31] Yoshida, N., Berger, M., Honda, K.: Strong Normalisation in the pi-Calculus. In: LiCS. (2001) 311–322.
[32] Varacca, D., Yoshida, N.: Typed Event Structures and the *pi*-Calculus: Extended Abstract. Electr. Notes Theor. Comput. Sci. 158 (2006) 373–397.

Appendix A. Intuitionistic logic calculus $\mathbf{LJ}_0$

MELLS is equivalent to the following calculus $\mathbf{LJ}_0$. Formulas of $\mathbf{LJ}_0$ are defined by the following grammar:

$$F := \ \bot \mid X \mid \neg(F \wedge \ldots \wedge F).$$

Rules of $\mathbf{LJ}_0$ are the following. Let F be $\neg(F_1 \wedge \ldots \wedge F_n)$:

$$\frac{}{\Gamma, X \vdash X} \qquad \frac{}{\Gamma, \bot \vdash} \qquad \frac{\Gamma, F_1, \ldots, F_n \vdash}{\Gamma \vdash F}$$

$$\frac{\Gamma, F \vdash F_1 \quad \ldots \quad \Gamma, F \vdash F_n}{\Gamma, F \vdash} \qquad \frac{\Gamma \vdash F \quad \Gamma, F \vdash \Xi}{\Gamma \vdash \Xi}$$

where Ξ is either empty or consisting of one formulas. $\mathbf{LJ}_0$ is a focalized version of the fragment $\neg, \wedge$ of intuitionistic logic. A translation $(\ )^*$ of formulas and sequent of **MELLS** in $\mathbf{LJ}_0$ can be given by induction as follows:

$$\begin{array}{rclrcl} ?\mathbf{0}^{\ *} & := & \bot; & !\top^{\ *} & := & \neg\bot; \\ F^+(N_1, \ldots, N_n)^{\ *} & := & \neg(N_1^* \wedge \ldots \wedge N_n^*); & F^-(P_1, \ldots, P_n)^{\ *} & := & \neg(P_1^* \wedge \ldots \wedge P_n^*); \\ \vdash P_1, \ldots, P_n^{\ *} & := & P_1^*, \ldots, P_n^* \vdash; & \vdash N, P_1, \ldots, P_n^{\ *} & := & P_1^*, \ldots, P_n^* \vdash N^*. \end{array}$$

Appendix B. Normalization of n.u. strategies : associativity

The aim of this section is to give a proof of associativity of composition [20, 8] in our setting with repetitions of actions and neutral actions:

Theorem B.1 (Associativity)**.** *Let $\mathcal{D}$ a strategy and $\mathcal{R}_1$ and $\mathcal{R}_2$ be cut-nets such that $\mathcal{D}$ and $\mathcal{R}_1 \cup \mathcal{R}_2$ form a cut net. We have:*

$$[\![\mathcal{D}, \mathcal{R}_1 \cup \mathcal{R}_2]\!] = [\![[\![\mathcal{D}, \mathcal{R}_1]\!], \mathcal{R}_2]\!].$$

For the sake of readability, here we will only consider the case $\mathcal{R}_1 = \{\mathcal{E}\}$ and $\mathcal{R}_2 = \{\mathcal{F}\}$ and hence the equation

$$[\![\mathcal{D}, \mathcal{E}, \mathcal{F}]\!] = [\![[\![\mathcal{D}, \mathcal{E}]\!], \mathcal{F}]\!]. \tag{B.1}$$

For *deterministic* strategies (with repetition of actions), associativity of composition is well-known result. One could either translate our "strategies as set of views" into " strategies as set of plays" and use standard game semantics arguments. Alternatively, one can also translate our deterministic strategies into abstract Böhm trees [7, 10] (which better fits our presentation of "strategies as set of views"), for which associativity of composition is proven [7]. A direct translation of deterministic strategies into of abstract Böhm trees can be found in [27].

To prove associativity for strategies *with neutral action*, we define a translation of n.u. strategies into deterministic one.

To have an intuition, we first observe that by our definition of normalization, a neutral action τ_i *operationally* behaves as it were an ordered pair (b^+, b_i^-) consisting of a visible positive action followed by a visible negative action. Our idea is then to replace any neutral action τ_i occurring in a view of a strategy by a suitable pair of actions (b^+, b_i^-).

We recall (Definition 7.2) that a n.u. strategy $\mathcal{D}$ on an arena A is a set of views on the arena $A \cup T$. In order to formally "replace" a neutral action $\tau_i \in T$ occurring in a view $w \in \mathcal{D}$ by a pair of action $b^+.b_i^-$, we introduce the following arena B.

The (two-layered) arena $B = (B, \vdash_B, \lambda_B)$ is given as follows:

- The set of moves B consists of a move b together with a denumerable set of moves b_i *i.e.*, $B = \{b\} \cup \{b_i : i \in \mathbb{N}\}$;
- The enabling relation is given by $b \vdash_B b_i$ for any b_i;
- The polarity is given by $\lambda_B(b) = +$ and $\lambda_B(b_i) = -$ for any b_i.

We are now ready to define our translation.

Definition B.2 (Translation into deterministic strategies)**.** Let $\mathcal{D}$ be a non deterministic strategy on an arena A. Let $w = w'.a \in \mathcal{D}$ a view on $A \cup T$. We define the operation d from views on $A \cup T$ to views on $A \cup B$ as follows:

- $\mathsf{d}(w) := \mathsf{d}(w').a$ if a is either a positive or negative action;
- $\mathsf{d}(w) := \mathsf{d}(w').b^+.b_i^-$ if $a = \tau_i$ and w is not maximal in $\mathcal{D}$;
- $\mathsf{d}(w) := \mathsf{d}(w').b^+.b_i^-.b^+$ if $a = \tau_i$ and w is maximal in $\mathcal{D}$.

The pointer structure of $\mathsf{d}(w)$ is hereditarily given from w and in addition, any new action b_i^- points to the previous occurrence of initial action b^+. We define the *deterministic* strategy $\mathsf{d}(\mathcal{D})$ on A as $\mathsf{d}(\mathcal{D}) := \{\mathsf{d}(w) : w \in \mathcal{D}\}$.

To see the operation d more graphically, let us consider Figure 9. In the left side (a) we have pictured a piece of strategy in which a negative action a^- is followed by a τ-cell τ_1, τ_2. On the right part (b), we have pictured the same piece of strategy after the operation d.

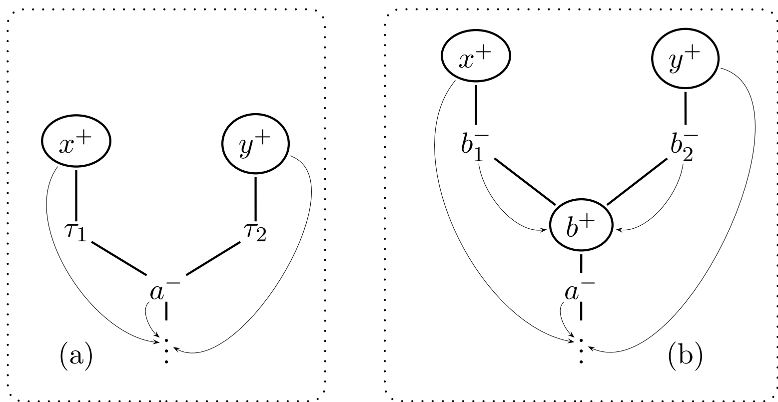


Figure 9: Translation into deterministic strategies

Given a strategy of the form $\mathsf{d}(\mathcal{D})$ on A, we define the inverse operation, which we call u which restores the neutral action that were in $\mathcal{D}$. More precisely, given a view $w \in \mathsf{d}(\mathcal{D})$ on $A \cup B$, we define:

- $\mathsf{u}(w) := \mathsf{u}(w').a$ if $w = w'.a$ and a is either a positive or negative action;
- $\mathsf{u}(w) := \mathsf{u}(w').\tau_i$ if $w = w'.b^+.b_i^-$ and w is not maximal in $\mathcal{D}$;
- $\mathsf{u}(w) := \mathsf{u}(w').\tau_i$ if $w = w'.b^+.b_i^-.b^+$ and w is maximal in $\mathcal{D}$.

It is immediate that $\mathsf{u}(\mathsf{d}(\mathcal{D})) = \mathcal{D}$.

Is important to notice that our translation works as well for strategy given on interfaces: given a strategy $\mathcal{D} : \Gamma$, we obtain a *deterministic* strategy $\mathsf{d}(\mathcal{D}) : \Gamma$. In particular, if $\mathcal{D}, \mathcal{E}$ are n.u. strategies that can be composed together, then also the deterministic strategies $\mathsf{d}(\mathcal{D}), \mathsf{d}(\mathcal{E})$ can be composed together.

The most important property of our translation is that d commutes with normalization:

$$\mathsf{d}([\![\mathcal{D}_1, \dots, \mathcal{D}_n]\!]) = [\![\mathsf{d}(\mathcal{D}_1), \dots, \mathsf{d}(\mathcal{D}_n)]\!] \tag{B.2}$$

In order to show associativity of composition for n.u. strategies (Equation B.1) we can now use associativity of composition of of deterministic strategies (noted by AD, below) together with Equation B.2. We have:

$$\begin{array}{rcll}
[\![\mathcal{D}, E, \mathcal{F}]\!] & = & \mathsf{u}(\mathsf{d}([\![\mathcal{D}, \mathcal{E}, \mathcal{F}]\!])) & \\
& = & \mathsf{u}([\![\mathsf{d}(\mathcal{D}), \mathsf{d}(\mathcal{E}), \mathsf{d}(\mathcal{F})]\!]) & \text{by Equation B.2} \\
& = & \mathsf{u}([\![[\![\mathsf{d}(\mathcal{D}), \mathsf{d}(\mathcal{E})]\!], \mathsf{d}(\mathcal{F})]\!]) & \text{by AD} \\
& = & \mathsf{u}([\![\mathsf{d}([\![\mathcal{D}, \mathcal{E}]\!]), \mathsf{d}(\mathcal{F})]\!]) & \text{by Equation B.2} \\
& = & \mathsf{u}(\mathsf{d}([\![[\![\mathcal{D}, \mathcal{E}]\!], \mathcal{F}]\!])) & \text{by Equation B.2} \\
& = & [\![[\![\mathcal{D}, \mathcal{E}]\!], \mathcal{F}]\!]. &
\end{array}$$

Appendix C. Proof of Lemma 10.10

To prove Lemma 10.10, we naturally generalize the renaming operator defined in Section 6.1. We now allow a simultaneous renamings in arbitrary interfaces.

Let $\Delta = \alpha_1, \ldots, \alpha_k$, $\Delta' = \beta_1, \ldots, \beta_k$ and Γ be non necessarily disjoint interfaces such that $\Delta \cup \Gamma$ and $\Delta' \cup \Gamma$ are interfaces. Let $\mathcal{D}$ be a strategy in $\Delta \cup \Gamma$. By $\mathcal{D}[\Delta'/\Delta]$ we denote the strategy on interface $\Delta' \cup \Gamma$ obtained from $\mathcal{D}$ be renaming, in all occurrences of action, the prefix α_i into β_i for any $1 \leq i \leq k$.

We use the following fact, which is a straightforward generalization of Proposition 8.9.

Let $\mathcal{D} : \xi^+, \Gamma$ and $\mathcal{E} : \xi^-, \Delta$ be strategies such that $\mathcal{D}$ has a positive action (ξ, I) as root. Let σ, Δ' be arbitrary fresh names. We have:

$$[\![\mathcal{D}, \mathcal{E}]\!] = [\![\underline{\sigma}(\mathcal{D}), \mathcal{E}, \mathcal{E}[\sigma/\xi, \Delta'/\Delta]]\!][\Delta/\Delta'] \tag{C.1}$$

where $\underline{\sigma}(\mathcal{D}) : \xi^+, \sigma^+, \Gamma$, $\mathcal{E} : \xi^-, \Delta$ and $\mathcal{E}[\sigma/\xi, \Delta'/\Delta] : \sigma^-, \Delta'$.

Observe that the renaming $[\Delta'/\Delta]$ in $\mathcal{E}[\sigma/\xi, \Delta'/\Delta]$ is needed to ensure that $\{\underline{\sigma}(\mathcal{D}), \mathcal{E}, \mathcal{E}[\sigma/\xi, \Delta'/\Delta]\}$ is a correct cut-net on interface Γ, Δ, Δ'. The latest one $[\Delta/\Delta']$ is needed to have the strategy on the r.h.s. of Equation C.1 on the same interface Γ, Δ of $\{\mathcal{D}, \mathcal{E}\}$.

For the use we make below, one can also assume that $\mathcal{D}$ and $\mathcal{E}$ above are deterministic strategies.

Proof of Lemma 10.10. We only consider the most interesting case of cut-elimination. For the sake of simplicity, we consider: $F^+ := F^+(N, M)$ and $(F^+)^\perp := F^-(P, Q)$ with $N = P^\perp$ and $M = Q^\perp$. We only analyze the situation in which F^+ is immediately decomposed in the left premise of the cut. The most general one, which involves some commutations of rules, can be easily derived from this simpler one. Moreover, for the sake of readability, we consider the case in which the final sequent consists of a single positive formula C. So, let π be the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{\vdash N, F^+, C} \quad \dfrac{\vdots\ \pi_2}{\vdash M, F^+, C}}{\vdash F^+, C} \qquad \dfrac{\dfrac{\vdots\ \zeta}{\vdash P, Q, C}}{\vdash F^-, C}}{\vdash C}\ \text{Cut}$$

We consider:

- $\mathcal{A}_1 \in \vdash \mathbf{N}_{\sigma 1}, \mathbf{F}^+_\xi, \mathbf{C}_\alpha$ (the interpretation of π_1);
- $\mathcal{A}_2 \in \vdash \mathbf{M}_{\sigma 2}, \mathbf{F}^+_\xi, \mathbf{C}_\alpha$ (the interpretation of π_2);
- $\mathcal{E} \in \vdash \mathbf{P}_{\xi 1}, \mathbf{Q}_{\xi 2}, \mathbf{C}_\beta$ (the interpretation of ζ).

By Proposition 10.9, we have that:

$$\dfrac{\dfrac{\mathcal{A}_1 \in \vdash \mathbf{N}_{\sigma 1}, \mathbf{F}^+_\xi, \mathbf{C}_\alpha \qquad \mathcal{A}_2 \in \vdash \mathbf{M}_{\sigma 2}, \mathbf{F}^+_\xi, \mathbf{C}_\alpha}{(\sigma, \{1,2\})^+.\{\mathcal{A}_1.\mathcal{A}_2\}[\xi/\sigma] \in \vdash \mathbf{F}^+_\xi, \mathbf{C}_\alpha} \qquad \dfrac{\mathcal{E} \in \vdash \mathbf{P}_{\xi 1}, \mathbf{Q}_{\xi 2}, \mathbf{C}_\beta}{(\xi, \{1,2\})^-.\mathcal{E} \in \vdash \mathbf{F}^-_\xi, \mathbf{C}_\beta}}{[\![(\sigma, \{1,2\})^+.\{\mathcal{A}_1, \mathcal{A}_2\}[\xi/\sigma], (\xi, \{1,2\})^-.\mathcal{E}]\!][\gamma/\alpha, \gamma/\beta] \in \vdash \mathbf{C}_\gamma}$$

Let us write $\mathcal{A}$ for the strategy $(\sigma,\{1,2\})^+.\{\mathcal{A}_1,\mathcal{A}_2\}$, $\mathcal{B}$ for $(\xi,\{1,2\})^-.\mathcal{E}$ and $\mathcal{D}$ for the interpretation of π respectively. By Equation C.1 and associativity, we have that:

$$\begin{array}{rcl}
\mathcal{D} & = & [\![\mathcal{A}[\xi/\sigma],\mathcal{B}]\!][\gamma/\alpha,\gamma/\beta] \\
& = & [\![\mathcal{A},\mathcal{B},\mathcal{B}[\sigma/\xi,\nu/\beta]]\!][\beta/\nu][\gamma/\alpha,\gamma/\beta] \quad (\nu \text{ fresh name}) \\
& = & [\![\mathcal{A},\mathcal{B},\mathcal{B}[\sigma/\xi,\nu/\beta]]\!][\gamma/\alpha,\gamma/\beta,\gamma/\nu] \\
& = & [\![[\![\mathcal{A},\mathcal{B}]\!],\mathcal{B}[\sigma/\xi,\nu/\beta]]\!][\gamma/\alpha,\gamma/\beta,\gamma/\nu] \\
& = & [\![[\![\ (\sigma,\{1,2\})^+.\{\mathcal{A}_1,\mathcal{A}_2\},\mathcal{B}]\!],\mathcal{B}[\sigma/\xi,\nu/\beta]]\!][\gamma/\alpha,\gamma/\beta,\gamma/\nu] \\
& = & [\![(\sigma,\{1,2\})^+.\{[\![\mathcal{A},\mathcal{B}]\!][\alpha/\beta],[\![\mathcal{A}_2,\mathcal{B}]\!][\beta/\alpha]\},\mathcal{B}[\sigma/\xi,\nu/\beta]]\!][\gamma/\alpha,\gamma/\beta,\gamma/\nu] \\
& = & [\![[\![\mathcal{A}_1,\mathcal{B}]\!][\alpha/\beta],[\![\mathcal{A}_2,\mathcal{B}]\!][\beta/\alpha],\mathcal{E}[\nu/\beta,\sigma1/\xi1,\sigma2/\xi2]]\!][\gamma/\alpha,\gamma/\beta,\gamma/\nu] \\
& = & [\![[\![[\![\mathcal{A}_1,\mathcal{B}]\!][\alpha/\beta],\mathcal{E}[\nu/\beta,\sigma1/\xi1,\sigma2/\xi2]]\!],[\![\mathcal{A}_2,\mathcal{B}]\!][\beta/\alpha]]\!][\gamma/\alpha,\gamma/\beta,\gamma/\nu] \\
& = & [\![[\![[\![\mathcal{A}_1,\mathcal{B}]\!][\alpha/\beta],\mathcal{E}[\nu/\beta,\sigma1/\xi1,\sigma2/\xi2]]\!][\alpha/\nu],[\![\mathcal{A}_2,\mathcal{B}]\!][\beta/\alpha]]\!][\gamma/\alpha,\gamma/\beta].
\end{array}$$

The derivation π reduces by cut-elimination to the following derivation ρ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\,\pi_1'}{\vdash N,F^+,[Q],C} \quad \dfrac{\dfrac{\vdots\,\zeta'}{\vdash P,Q,[Q],C}}{\vdash F^-,[Q],C}}{\vdash N,[Q],C}\,\text{Cut} \quad \dfrac{\vdots\,\zeta}{\vdash P,Q,C}}{\vdash Q,C}\,\text{Cut} \quad \dfrac{\dfrac{\vdots\,\pi_2}{\vdash M,F^+,C} \quad \dfrac{\dfrac{\vdots\,\zeta}{\vdash P,Q,C}}{\vdash F^-,C}}{\vdash M,C}\,\text{Cut}}{\vdash C}\,\text{Cut}$$

where π_1' are ζ' are respectively obtained by adding an occurrence of formula Q —that we point out with brackets [ ] — to every sequent of π_1 and ζ (implicit weakening). These occurrences are only needed to make the contexts of the cut-rule matching. Moreover, as they have been introduced, they do not play any active role (*i.e.*, they are never decomposed) in π_1' and ζ'. Hence, it is immediate to see that *even if typed by different sequents of behaviours*, the strategies interpreting π_1 and π_1' (resp. ζ and ζ') are the exactly the same.

For ρ, the interpretation of the sub-derivation ending with $\vdash N,[Q],C$ is given by:

$$\dfrac{\mathcal{A}_1 \in \vdash \mathbf{N}_{\sigma 1},\mathbf{F}^+_\xi,[\mathbf{Q}_\delta],\mathbf{C}_\alpha \qquad \dfrac{\mathcal{E} \in \vdash \mathbf{P}_{\xi 1},\mathbf{Q}_{\xi 2},[\mathbf{Q}_\varrho],\mathbf{C}_\beta}{\mathcal{B} \in \vdash \mathbf{F}^-_\xi,[\mathbf{Q}_\varrho],\mathbf{C}_\beta}}{[\![\mathcal{A}_1,\mathcal{B}]\!][\alpha/\beta,\delta/\varrho] \in \vdash \mathbf{N}_{\sigma 1},[\mathbf{Q}_\delta],\mathbf{C}_\alpha}$$

the sub-derivation ending with $\vdash Q,C$ is given by:

$$\dfrac{[\![\mathcal{A}_1,\mathcal{B}]\!][\alpha/\beta,\delta/\varrho] \in \vdash \mathbf{N}_{\sigma 1},[\mathbf{Q}_\delta],\mathbf{C}_\alpha \qquad \mathcal{E}[\nu/\beta,\sigma1/\xi1,\sigma2/\xi2] \in \vdash \mathbf{P}_{\sigma 1},\mathbf{Q}_{\sigma 2},\mathbf{C}_\nu}{[\![[\![\mathcal{A}_1,\mathcal{B}]\!][\alpha/\beta,\delta/\varrho],\mathcal{E}[\nu/\beta,\sigma1/\xi1,\sigma2/\xi2]]\!][\alpha/\nu,\sigma2/\delta] \in \vdash \mathbf{Q}_{\sigma 2},\mathbf{C}_\alpha}$$

and finally the sub-derivation ending with $\vdash M,C$ is so given:

$$\dfrac{\mathcal{A}_2 \in \vdash \mathbf{M}_{\sigma 2},\mathbf{F}^+_\xi,\mathbf{C}_\alpha \qquad \dfrac{\mathcal{E} \in \vdash \mathbf{P}_{\xi 1},\mathbf{Q}_{\xi 2},\mathbf{C}_\beta}{\mathcal{B} \in \vdash \mathbf{F}^-_\xi,\mathbf{C}_\beta}}{[\![\mathcal{A}_2,\mathcal{B}]\!][\beta/\alpha] \in \vdash \mathbf{M}_{\sigma 2},\mathbf{C}_\beta}$$

Summing up, the interpretation of ρ is given by the following $\mathcal{D}' \in \vdash \mathbf{C}_\gamma$:

$$\mathcal{D}' := [\![[\![[\![\mathcal{A}_1,\mathcal{B}]\!][\alpha/\beta,\delta/\varrho],\mathcal{E}[\sigma1/\xi1,\sigma2/\xi2,\nu/\beta]]\!][\alpha/\nu,\sigma2/\delta],[\![\mathcal{A}_2,\mathcal{B}]\!][\beta/\alpha]]\!][\gamma/\alpha,\gamma/\beta]$$

. Since by construction, no action on names δ and ρ occurs in $[\![\mathcal{A}_1, \mathcal{B}]\!]$, we have that $[\![\mathcal{A}_1, \mathcal{B}]\!][\alpha/\beta, \delta/\varrho] = [\![\mathcal{A}_1, \mathcal{B}]\!][\alpha/\beta]$. Hence,

$$\begin{array}{rcl} \mathcal{D}' & = & [\![[\![[\![\mathcal{A}_1, \mathcal{B}]\!][\alpha/\beta, \delta/\varrho], \mathcal{E}[\sigma 1/\xi 1, \sigma 2/\xi 2, \nu/\beta]]\!][\alpha/\nu, \sigma 2/\delta], [\![\mathcal{A}_2, \mathcal{B}]\!][\beta/\alpha]]\!][\gamma/\alpha, \gamma/\beta] \\ & = & [\![[\![[\![\mathcal{A}_1, \mathcal{B}]\!][\alpha/\beta], \mathcal{E}[\sigma 1/\xi 1, \sigma 2/\xi 2, \nu/\beta]]\!][\alpha/\nu], [\![\mathcal{A}_2, \mathcal{B}]\!][\beta/\alpha]]\!][\gamma/\alpha, \gamma/\beta] = \mathcal{D}. \end{array}$$

□