

Jump from Parallel to Sequential Proofs: Multiplicatives*

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Abstract. We introduce a new class of multiplicative proof nets, *J-proof nets*, which are a typed version of Faggian and Maurel's multiplicative L-nets. In J-proof nets, we can characterize nets with different degrees of sequentiality, by gradual insertion of sequentiality constraints. As a byproduct, we obtain a simple proof of the sequentialisation theorem.

Introduction

Proof nets have been introduced by Girard [10] as an abstract representation of linear logic proofs; this representation has two main interests: to provide a tool for studying normalization, and to give a canonical representation of proofs.

In proof nets, information about the order in which the rules are performed is reduced to a minimum, only two kinds of information about sequentiality being kept: the one corresponding to the subformula trees and the one providing the axiom links.

To retrieve a sequent calculus derivation from a proof net, we need to recover more information about sequentiality. A sequentialization procedure gives instructions on how to introduce this sequentiality. Such a procedure usually relies on splitting lemmas, which are proved introducing the notion of empire.

In [11], Girard, as part of the correctness criterion for proof nets with quantifiers, introduces a more direct way to represent sequentiality constraints in a proof net, by using *jumps*: a jump is an untyped edge between two nodes (rules) a, b , which expresses a *dependency* relation: a precedes b (bottom-up) in the sequentialisation. Recently, the idea of using jumps as a way to represent sequentiality information has been developed by Faggian and Maurel ([8]) in the abstract context of the L-nets, a parallel variant of Ludics designs.

Here we define a representation of proofs where objects with different degree of parallelism live together, in the spirit of [6, 5]; for this purpose we introduce a new class of multiplicative proof nets, *J-proof nets*, that can be considered as a typed, concrete, version of multiplicative L-nets.

* This work has been partially supported by MIUR national project FOLLIA and a grant *Ville de Paris, 2005-2006*.

We prove that by gradual insertion of jumps in a J-proof net, one can move in a continuum from J-proof nets of minimal sequentiality to J-proof nets of maximal sequentiality. The former are proof nets in the usual sense, the latter directly correspond to sequent calculus proofs.

In this way, we realize, for the multiplicative fragment of Linear Logic, a proposal put forward by Girard.

Moreover, our technique results into a very simple proof of the sequentialisation theorem. Our main technical result is the *Arborisation Lemma*, which provides the way to add jumps to a J-proof net up to a maximum.

1 Focalization, MLL and HS

In the first part of the paper we consider the multiplicative fragment of the hypersequentialised calculus HS [12, 13, 9], which is a focussing version of Multiplicative Linear Logic (MLL), as we explain below.

We define proof nets for HS, and then introduce J-proof nets. The strong geometrical properties of HS will allow us to uncover a simple sequentialisation technique for the calculus. We will then be able to apply the same technique to MLL.

1.1 Focalization and MLL

It has been proved by Andreoli [2] that the sequent calculus of Linear Logic enjoys a property called *focalization*: a proof π of a sequent $\vdash T$ can be transformed into a proof π^{foc} of the same sequent which satisfies a specific discipline, called *focussing discipline*, which we describe below.

Here we stress that π^{foc} is obtained from π solely by permutation of the rules. As a consequence, if we restrict our attention to MLL, there is no difference in the proof net of π and π^{foc} . In fact, we have that:

1. π and π^{foc} are equivalent modulo permutation of the rules;
2. the proof net respectively associated to π and π^{foc} is the same;
3. an MLL proof net has always a focussing proof among its possible sequentialisations.

(2.) is an immediate consequence of (1.), while (3.) is actually the easier way to prove focalization for MLL (as first observed by Andreoli and Maieli in [1]). We revise this below.

Focalization relies on a distinction of Linear Logic connectives into two families, which are as follows.

Positive connectives: $\otimes, \oplus, 1, 0$.

Negative connectives: $\wp, \&, \perp, \top$.

From now on, we only consider the multiplicative fragment of Linear logic; the formulas are hence as follows

$$F ::= A \mid A^\perp \mid F \wp F \mid F \otimes F$$

where A, A^\perp are atoms.

To understand focalization, it helps to think of MLL proof nets rather than sequent calculus proofs.

Let us partition the nodes which are respectively typed by \otimes and \wp into maximal trees of nodes with the same type (resp. positive and negative trees). We assume that there is at most one negative node which is conclusion of the proof net (otherwise, we put together all negative conclusions by making use of \wp).

Consider now sequentialization. That is, we associate a sequent calculus proof to a proof-net; to do this, essentially one has to “find a last rule”. The key result in proof-net theory is that this is always possible; a \otimes rule which can be removed from the proof net (and taken as the last rule in the sequent calculus derivation) is called a *splitting* \otimes . Let us now choose a specific sequentialization strategy, based on the notion of *hereditarily splitting* \otimes , whose existence was proved in ([7]).

- If R has a negative conclusion, we choose that conclusion as last rule of the sequent calculus proof, and remove it from the proof net. We persist until the whole negative tree has been removed.
- If R has only positive conclusions, we choose an *hereditarily splitting* \otimes . This means that we can choose a tree of \otimes , and persistently remove all the \otimes until the whole positive tree has been removed.

What we obtain is a sequent calculus derivation whose *bottom-up construction* satisfies the focussing discipline below.

Definition 1 (Focussing proofs). *A sequent calculus proof is called focussing if its bottom-up construction satisfies the following discipline:*

- *First keep decomposing a negative formula (if any) and its subformulas, until one get to atoms or positive subformulas;*
- *choose a positive formula, and keep decomposing it up to atoms or negative subformulas.*

1.2 Synthetic Connectives: HS.

Focalization implies that we can consider a maximal tree of connectives of the same polarity (positive or negative) as a single n-ary connective, called a *synthetic connective*, which can be introduced by a specific rule.

In [12] Girard has introduced a new calculus, HS, which uses focalization and synthetic connectives to force a “normal” form for MALL sequent calculus proofs.

HS introduces a polarization on the atoms. This constraint correspond to a hidden decomposition of the atoms, and does not introduce essential differences, while making the geometrical structure strong and clear. For this reason, we will first work with HS; in Section 7 we will then remove the polarization.

2 From Proof Nets to J-proof Nets

2.1 MHS sequent calculus

We indicate by MHS the multiplicative fragment of HS.

Formulas The formulas of MHS are as follows:

$$\begin{aligned} N &::= A^\perp \mid \wp(P, \dots, P) \\ P &::= A \mid \otimes(N, \dots, N) \end{aligned}$$

where A denotes a positive atom.

Rules. The rules for proving sequents are the following

$$\begin{aligned} &\frac{}{\vdash A, A^\perp} (Ax) && \frac{\vdash \Gamma, F \quad \vdash \Delta, F^\perp}{\vdash \Gamma, \Delta} (Cut) \\ &\frac{\vdash \Gamma_1, N_1 \dots \vdash \Gamma_n, N_n}{\vdash \Gamma_1, \dots, \Gamma_n, \otimes(N_1, \dots, N_n)} (+) && \frac{\vdash \Gamma, P_1, \dots, P_n}{\vdash \Gamma, \wp(P_1, \dots, P_n)} (-) \\ &\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta} (Mix) \end{aligned}$$

where all context Γ, Δ, \dots only contain P formulas.

Remark 1. The calculus admits a unary \wp (resp. \otimes) which is a negative (resp. positive) polarity inverter [13, 14] This polarity inverter is usually called a *negative (resp. positive) Shift* and denoted by \uparrow (resp. \downarrow).

Skeleton of a Sequent Calculus Proof. Let us give a first intuition of our approach. Consider a cut-free proof π , and type each rule application with its active formulas. Observe that if we forget everything but the types, we have a tree, where the nodes are MHS formulas, and the leaves have the form $\{A, A^\perp\}$. We can think of this tree as the skeleton of the sequent calculus derivation.

Here we do not push this intuition further, but it is possible to characterize the trees which correspond to sequent calculus derivations, in the spirit of [13], and extend the approach also to the cut rule.

2.2 MHS Proof-nets

In this section we define proof-nets for MHS.

Proof-nets provide a graph representation of proofs. Each node represents a rule of the sequent calculus, and it is only concerned with the active formulas.

Definition 2 (Typed structure). We call typed structure a directed acyclic graph where:

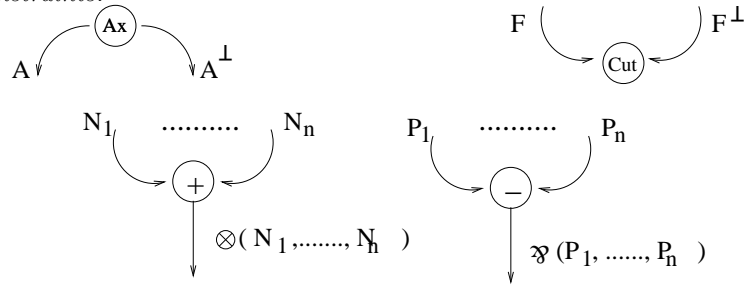
- the edges are possibly typed with MHS formulas

- the nodes (also called links) are typed either with a MHS formula or with a pair of atoms $\{A, A^\perp\}$.

Given a link, the incoming edges are called the premises of the link, and the outgoing edges are called conclusions of the link. We call positive (resp. negative) a link of type $\otimes(N_1, \dots, N_n)$ (resp. $\wp(P_1, \dots, P_n)$).

We admit pending edges. An edge which has no target is called a conclusion of the structure, and its source is called a terminal link.

Definition 3 (Proof structure). A proof structure is a typed structure where the nodes are typed as the conclusions, and the typing satisfies the following constraints:



Moreover, we ask that there is at most one negative terminal link.

Definition 4 (Switching path and cycle). Given a proof-structure, a switching path is an unoriented path which never uses two premises of the same negative link.

A switching cycle is a switching path which is a cycle.

Definition 5 (Proof-nets). A proof structure R is called a proof-net if it has no switching cycles.

Proposition 1. Given a sequent calculus proof π of MHS, we can associate to it a proof net π^* .

Proof. We proceed in the standard way.

Definition 6 (Sequentialization). A proof structure R is sequentialisable iff there exists a proof π of MHS (that we call a sequentialisation of R) s.t. $\pi^* = R$

2.3 J-proof Nets

We enrich proof-nets with jumps, which will allow us to graduate sequentiality.

Definition 7 (J-proof structure).

A J-proof structure (jumped proof structure) is a proof structure added with untyped edges called jumps, which connect a positive to a negative link (the orientation is from positive to negative).

Definition 8 (Switching path and cycle). Given a J-proof structure, a switching path is an unoriented path which never uses two premises of the same negative link (a jump is also a premise of its target); a switching cycle is a switching path which is a cycle.

Definition 9 (J-Proof nets). A J-proof structure R is called a J-proof-net if it has no switching cycles.

In Section A we sketch normalization of J-proof nets.

A proof-net is a special case of J-proof net. In the next section we will show that a sequent calculus proof (or rather its skeleton) can also be seen as a special case of J-proof net. This will allow us to define a new technique of sequentialization.

Note. From now on, we only consider J-proof structures without cut links. The cut can be smoothly dealt with essentially by identifying the cut node of premises F, F^\perp with the node of which the positive formula is conclusion.

2.4 Partial Order Associated to a J-proof net

Since a J-proof net R is a d.a.g., we associate to R in the standard way a strict partial order \prec_R on the typed nodes.

We recall that we can represent a strict partial order as a d.a.g., where we have an edge $a \leftarrow b$ whenever $a <_1 b$ (i.e. $a < b$, and there is no c such that $a < c$ and $c < b$.) Conversely (the transitive closure of) a d.a.g. G induces a strict partial order \prec_G on the nodes of G .

We call **skeleton** of a directed graph G , denoted $Sk(G)$, the minimal graph whose transitive closure is the same as that of G . An edge $a \leftarrow b$ is transitive if there is no node c such that $a \leftarrow c$ and $c \leftarrow b$.

With a slight abuse, we often identify \prec_G and the skeleton of G .

Given a J-proof net R , we call *minimal* (resp. *maximal*) a link c of R which is minimal in \prec_R , i.e. there is no node b such that $b \leftarrow c$ (resp. $c \leftarrow b$). Notice that, because of jumps, a node can be terminal, without being minimal.

We call predecessor of a node c , a node which immediately precedes c . Otherwise, we speak of hereditary predecessor. Similarly for the successor.

A strict order r on a set A is *arborescent* when each element has a unique predecessor.

If the order \prec_R associated to a J-proof net R is arborescent, the skeleton of R is a forest.

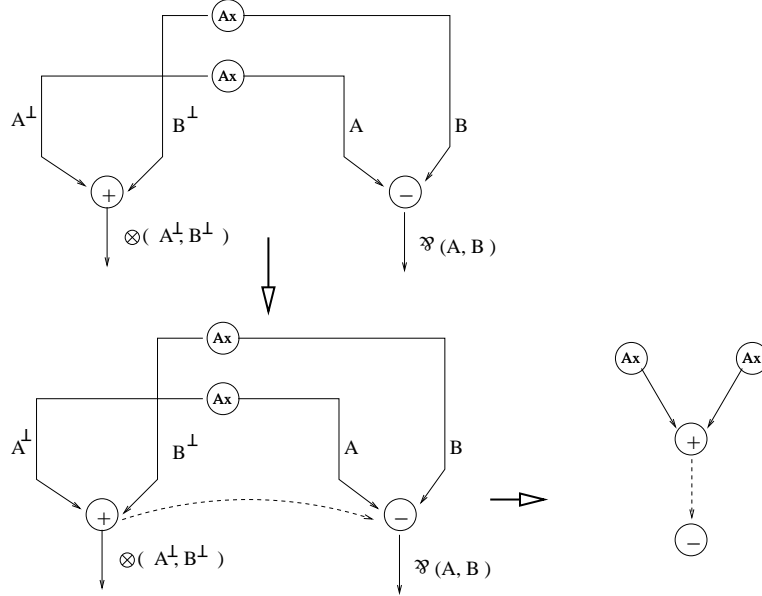
Finally, we observe that

Remark 2. Let R be a J-proof net.

- $Sk(R)$ is obtained from R by removing the edges which are transitive.
- Only an edge which goes from positive to negative can be transitive.

3 J-proof nets and Sequent Calculus

In the next section we will induce a sequentialisation of a proof net by adding jumps. Let us start with an example. Consider the proof-net below. We add a jump from the positive to the negative link, and consider the order induced on the links. We obtain a tree, and such a tree is the skeleton of a sequent calculus proof.



To sequentialize a J-proof net, we will then consider the order associated to a proof net as a directed acyclic graph, and add to it enough jumps, to make the order arborescent, and hence proof-like (Lemma 1).

Let us show that if the order on the nodes of a J-proof net is arborescent, it corresponds to a sequent calculus derivation. (A proof of this is given, in a more general setting and with full details, in [5].)

Proposition 2. *Let R be a J-proof net such that \prec_R is arborescent.*

(i) *We can associate to R a proof π^R in the sequent calculus MHS, possibly making use of the Mix Rule.*

(ii) *Moreover, if the order has a minimum and each negative link has a unique successor (i.e. if the skeleton is a tree which only branches on positive nodes) then π^R does not use the Mix rule.*

Proof. The proof is by induction on the number of links. For brevity, we show directly (ii).

1. $n = 1$: The only link of R is an Axiom link of conclusions A, A^\perp , to which we associate $\frac{}{\vdash A, A^\perp}$;
2. $n > 1$: we reason by cases, depending on the type of the minimal link c of R .

- $\wp(P_1 \dots P_n)$: let $\prec_{R'}$ be the order obtained by erasing c . By induction we associate a proof π' to $\prec_{R'}$. π^{\prec_R} is $\frac{\pi'}{\vdash \Gamma, \wp_{i \in \{1 \dots n\}}(P_i)}$, whose last rule is a $-$ rule on P_1, \dots, P_n (P_1, \dots, P_n are conclusions of π' by construction);
- $\otimes(N_1, \dots, N_n)$: let $\prec_{R_1}, \dots, \prec_{R_n}$ be the n orders obtained by erasing c . By induction we associate a proof π'_i to each \prec_{R_i} ; π^{\prec_R} is $\frac{\pi'_1 \dots \pi'_n}{\vdash \Gamma_1, \dots, \Gamma_n, \otimes_{i \in \{1 \dots n\}} N_i}$ whose last rule is a $+$ rule on N_1, \dots, N_n (by construction, N_1, \dots, N_n are respectively among the conclusions of π'_1, \dots, π'_n).

4 Sequentialization

Definition 10 (Saturated J-proof net). *A J-proof net R is saturated if for every negative link n and for every positive link p , adding a jump between n and p creates a switching cycle or doesn't increase the order \prec_R . Given a J-proof net R , a saturation R^J of R is a saturated J-proof net obtained from R by adding jumps.*

Our sequentialisation argument is as follows:

- If the order \prec_R associated to a J-proof net R is arborescent, we can associate to R a proof π^R in the sequent calculus.
- The order associated to a saturated J-proof net is arborescent.
- Any J-proof net can be saturated.

Lemma 1 (Arborisation). *Let R be a J-proof net. If R is saturated then \prec_R is arborescent. Any J-proof net can be saturated.*

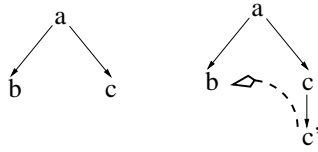
Proof. We prove that if \prec_R is not arborescent, then there exists a negative link c and a positive link b s.t. adding a jump between b and c doesn't create switching cycles and makes the order increase.

If \prec_R is not arborescent, then in \prec_R there exists a link a with two immediate predecessors b and c (they are incomparable). Observe that b and c are immediately below a in $Sk(R)$ and also in R .

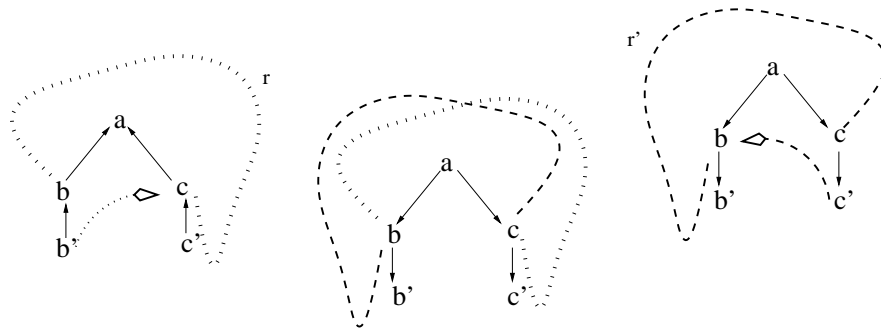
If a is an Axiom link, then necessarily b and c are respectively a positive link and a negative link; we draw a jump between b and c , this doesn't create a cycle and the order increases.

Otherwise, a is a positive link, and b and c are two negative links; we distinguish two cases:

1. either b or c is terminal in R . Let assume that b is terminal; then c cannot be terminal (by definition of jumped proof structure), and there is a positive link c' which immediately precedes c . If we add a jump between b and c' , this doesn't create cycles and the order increases.

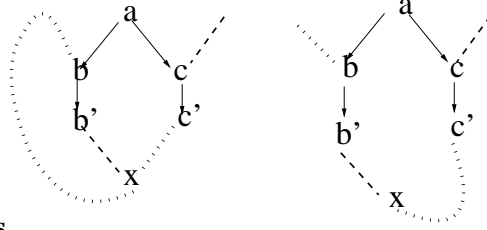


2. Neither b or c are terminal in R . Each of them has an immediate positive predecessor, respectively b' and c' . Suppose that adding a jump from b' to c creates a cycle: we show that adding a jump from c' to b cannot create a cycle. If adding to R the jump $b' \rightarrow c$ creates a cycle, that means that there is in R a switching path $r = \langle c, c' \dots b \rangle$; if adding the jump $c' \rightarrow b$ creates a cycle then there is a switching path $r' = \langle b, b' \dots c \rangle$. Assume that r and r' are disjoint: we exhibit a switching cycle in $R \langle c, c' \dots b, b' \dots c \rangle$ by concatenation of r and r' . This contradicts the fact that R is a proof net.



Assume that r and r' are not disjoint. Let x be the first node (starting from b) where r and r' meet. Observe that x must be negative (otherwise there would be a cycle). Each path uses one of the premises, and the conclusion (hence the path meets also in the node below x). From the fact that x is the first point starting from b where r and r' meet it follows that: (i) r' enters in x from one of the premises, and exits from the conclusion; (ii) each of r and r' must use a different premise of x . Then we distinguish two cases:

- r enters x from one of the premises; we build a switching cycle taking the sub path $\langle b, \dots, x \rangle$ of r' and the sub path $\langle x, \dots, b \rangle$ of r .
- r enters x from the conclusion; then we build a switching cycle composing the sub path of $r \langle c, \dots, x \rangle$, the reversed sub path of $r' \langle x, \dots, b \rangle$ and the path $\langle b, a, c \rangle$.



5 Properties

In this section we deal with three standard results one usually has on proof nets. In 5.1 we get rid of the Mix rule, in 5.2 we give an immediate proof of the usual splitting Lemma, in 5.3 we prove that the sequentialization we have defined is correct w.r.t. Definition 6.

The novelty here is the argument. When adding jumps, we gradually transform the skeleton of a graph into a tree. We observe that some properties are invariant under the transformation we consider: adding jumps and removing transitive edges. Our argument is always reduced to simple observations on the final tree (the skeleton of R^J), and on the fact that each elementary graph transformation preserves some properties of the nodes.

5.1 Connectness

Lemma 2. (i) *Two nodes are connected in a d.a.g. G (i.e. there exists a sequence of connected edges between the two nodes) iff they are connected in the skeleton of G .*

(i) *If two nodes are connected in R , then they are connected in R^J .*

(iii) *If R is connected as a graph so are R^J and $Sk(R^J)$.*

Proof. Immediate, because adding edges, or deleting transitive edges, preserves connectness.

We now deal with a more peculiar notion of connectness, to get rid of the mix rule, as is standard in the theory of proof-nets.

Definition 11 (Correction graph). *Given a typed graph R , we call switching a function s which associates to every negative node of R one of its premises (again, jumps also are premises of their target); a correction graph $s(R)$ is the graph obtained by erasing for every negative node of R the premises not chosen by s .*

Definition 12 (s-connected). *A J -proof net R is s -connected if given a switching of R , its correction graph is connected.*

Remark 3. We only need to check a single switching. The condition that a proof structure has not switching cycles is equivalent to the condition that *all* correction graphs are acyclic.

A simple graph argument shows that assuming that all correction graphs are acyclic, if for a switching s the correction graph $s(R)$ is connected, then for all other switching s' $s'(R)$ is connected.

Proposition 3. *If R is s-connected, then its skeleton is a tree which only branches on positive nodes (i.e., each negative link has a unique successor).*

Proof. First we observe that:

- any switching of R is a switching of R^J , producing the same correction graph. Hence if R is s-connected, R^J is s-connected.
- Given a J-proof net G , any switching of its skeleton is also a switching of G , because the skeleton is obtained by erasing the edges which are transitive. A transitive edge can be premise only of a negative node.

As a consequence, any switching of $Sk(R^J)$ induce a correction graph which is connected. However, $Sk(R^J)$ is a tree, so we cannot erase any edge. Hence each negative link has a unique premise, and the graph has only one switching.

From Proposition 2, it follows that

Proposition 4. *If R is s-connected, and R^J a saturation, we can associate to it a proof π^{R^J} which does not use the Mix rule.*

5.2 Splitting

Observe that a minimal link of S is a root of its skeleton.

Definition 13 (Splitting). *Let R be a typed structure, c a positive link, and b_1, \dots, b_n the nodes which are immediately above c (the premises of c have the same type as b_1, \dots, b_n). We say that c is splitting for R if it is terminal, and removing c there is no more connection (i.e. no sequence of connected edges) between any two of the nodes b_i .*

Remark 4. Assume that R is a connected graph. It is immediate that if R is a J-proof net whose terminal links are all positive, the removal of c splits R into n disjoint connected components R_1, \dots, R_n , and each component is a J-proof net.

Lemma 3 (Splitting lemma). *Let R be a J-proof net whose terminal nodes are all positive, and R^J a saturation; the minimal link c of R^J (i.e. the root of $Sk(R^J)$) is splitting for R .*

Proof. Observe that c is obviously splitting in the skeleton of R^J , because c is the root of a tree. Hence it is splitting in R^J , as a consequence of Lemma 2, (i). Similarly, c must be splitting in R , as a consequence of Lemma 2, (ii).

5.3 Sequentialisation Is Correct

Proposition 5. *Let R be a J-proof-net. For any saturation R^J of R , if $\pi = \pi^{R^J}$ then $(\pi)^* = R$.*

Proof. For brevity, we assume that R is s-connected. Hence, the skeleton of R^J is a tree. The proof is by induction on the number of links of R .

1. $n = 1$: then R consists of a single Axiom link, and π is the corresponding Axiom rule.
2. $n > 1$. We consider the minimal link k of R^J .
 Observe that the last rule of π is the rule which correspond to the root k . Let us call π_1, \dots, π_n the premises of the rule, and R_1^J, \dots, R_n^J the subnets obtained from R^J by removing k . By definition, each π_i is the proof associated to an R_i^J .
 - Assume k is positive. By the splitting lemma, k is splitting in R .
 R_1^J, \dots, R_n^J are obviously saturated (we have not erased any jump) so by induction hypothesis on R_1, \dots, R_n which are the n sub nets obtained by removing k from R , $(\pi^{R_i^J})^* = R_1, \dots, (\pi^{R_n^J})^* = R_n$; by composing all the $\pi^{R_i^J}$ with the rule corresponding to k , we get a proof which is equal to π^{R^J} and we find that $R = (\pi^{R^J})^*$.
 - Assume k is negative. Similarly, we remove k from R and apply induction to obtain the conclusion.

6 Partial Sequentialisation and Desequentialization

The approach we have presented is well suited for *partially* introducing or removing sequentiality, by adding (deleting) a number of jumps.

Actually, it would be straightforward to associate to a sequent calculus proof π a saturated J-proof net. In this way, to π we could associate either a maximal sequential or a maximal parallel J-proof net, on the lines of [6, 5].

Given a J-proof net R , let us indicate with $Jump(R)$ ($DeJump(R)$) a J-proof net resulting from (non deterministically) introducing (eliminating) a number of jumps in such a way that every time the order increases (decreases).

The following result apply to a J-proof net of *any* degree of sequentiality.

Theorem 1 (Partial sequentialisation/desequentialization). *Let R, R' be J-proof nets.*

If $R' = Jump(R)$ then there exists $DeJump(R')$ such that $DeJump(R') = R$.

If $R' = Dejump(R)$ then there exists $Jump(R')$ such that $Jump(R') = R$.

Proof. Immediate, since we can reverse any step...

7 MLL

Our sequentialisation proof can now be extended to MLL. It is straightforward to translate an MLL proof net into MHS, however, here we prefer a more direct approach (where the translation is implicit). We proceed in two steps, first by introducing a variant of Andreoli's focussing calculus based on synthetic connectives, and then working directly with MLL.

7.1 MHS^+

The polarization of HS makes the geometrical structure clean and clear. We now eliminate the polarization constraints, still keeping the calculus focussing.

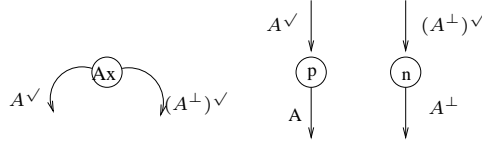
We call this calculus MHS^+ . The grammar of the formulas is the following:

$$\begin{aligned} N &::= A \mid A^\perp \mid \wp(P, \dots, P) \\ P &::= A \mid A^\perp \mid \otimes(N, \dots, N) \end{aligned}$$

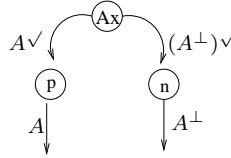
Remark 5. Observe that now we have *all the formulas* of MLL, modulo clustering/declustering into synthetic connectives. For example, $A\wp A^\perp$ is a formula of MHS^+ .

The *sequent calculus rules* are (formally) the same as those of MHS. Observe however that now we consider negative atoms also as P-formulas. This means that the contexts Γ, Δ, \dots may also contain negative atoms. Moreover, a negative atom can appear in the premises of a negative rule, and a positive atom can appear in the premises of a positive rule.

Proof nets. We modify the Axiom link, by introducing a (formal) decomposition of the atoms. Any atom A can be decomposed into A^\vee , of opposite polarity (technically, the A^\vee has been introduced by Girard, and is used also by Laurent in [14]). Hence we have:



To the identity axiom we associate



where the n and p links, respectively negative and positive, can be considered as steps of decomposition of the atoms: we call these links *hidden*. They do not appear in the sequent calculus, but provide space for the jumps.

The definitions of the previous sections can be applied to MHS^+ , with this variant: when we associate an order $<_R$ to a J-proof net of MHS^+ , we ignore the hidden links. It is straightforward to check that the results of the previous sections (and in particular the Arborisation Lemma) still hold in this case.

Remark 6. MHS^+ is a variant of MLL^{Foc} : to a proof of MHS^+ corresponds a proof of MLL^{Foc} , and vice-versa.

Similarly, the proof nets closely correspond to focussing proof nets, as defined by Andreoli [3].

7.2 MLL

It is immediate now that we can

- transform an MLL proof net R into a MHS^+ proof net R' ;
- transform MHS^+ sequent calculus derivation π into an MLL sequent calculus derivation by “declustering” the rules. The sequent calculus derivation which we obtain is focussing.

Observe that this transformations can simply be “*virtual*”. To sequentialize an MLL proof net R , we expand the axiom links, and treat each maximal tree of \otimes as a $+$ link, and each maximal tree of \wp as a $-$ link. Using our procedure, we obtain again an arborescent order on $+$, $-$ and axiom links. Observe that we can substitute each step in Proposition 2 with an expanded version.

8 Conclusions and Future Work

J-proof nets provide a representation of proofs where objects with different degrees of parallelism live together; furthermore, by the use of jumps as sequentiality constraints, we can transform any proof net of MLL in a sequent calculus proof, which seems a very natural way to approach sequentialisation.

Jumps are related with the notion of *empire* of a formula in a proof net; we wish to investigate this relationship, in order to understand the differences between our proof of sequentialisation and the traditional ones.

Also, we would like to understand the relation with work by Banach [4], where the use of an order on the links of the proof net as a tool for sequentialization has a precedent.

As a future research direction, we hope to be able to extend this work to consider a larger fragments of linear logic; recent developments in the theory of L-nets [6, 5] seem to make plausible an extension to $MALL$.

Acknowledgements

The authors are indebted with Jean-Yves Girard, for having first suggested the idea of sequentialising with jumps. Many thanks also to Lorenzo Tortora de Falco and Olivier Laurent for useful and fruitful discussions.

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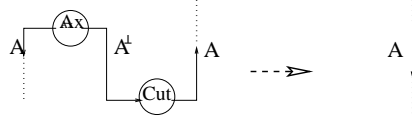
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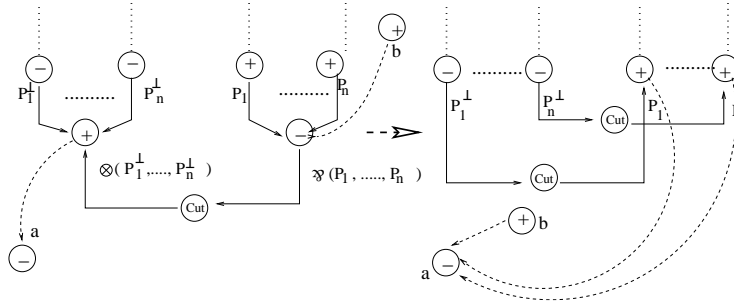
A Normalization of J-proof Nets

We can define cut elimination on J-proof nets in the same way as for L-nets:

– Ax



– cut $+/-$



The procedure is confluent and strong normalizing, and preserves correction; furthermore it preserves the order on the links (if a precedes b before the reduction, it still precedes b afterwards). Notice that cuts are oriented from negative to positive: actually, we modify the orientation of the edges of the cut link when we associate an order to a proof net, so to get the above good properties.