METRICAL VERSIONS

OF THE TWO–DISTANCES THEOREM

Pseudo-randomness of a random Kronecker sequence

An instance of a dynamical analysis

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Outline of the talk

- The Kronecker sequence $n \mapsto n\alpha \mod 1$
  - The two–three distances Theorem.
  - Its relation with Continued Fraction Expansions
- Pseudo-randomness of a random truncated Kronecker sequence
  - Five parameters related to measures of pseudo-randomness.
  - Three types of truncation.
  - Four probabilistic models.
- Statement of some of the (sixty) results.
- A short description of the methods

A – The truncated Kronecker sequences.

Generalities on truncated sequence.

Consider a sequence $\mathcal{X} := \{n \mapsto x_n, n \geq 0\} \subset [0, 1]$

With a truncation integer T,

it defines a truncated sequence $\mathcal{X}_{\langle T \rangle} := \{n \mapsto x_n, n < T\}$

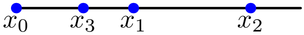
We consider – the ordered sequence $\{y_i, \quad i \in [0, T[\}$ on the unit torus.

– the distances δ_i's between two consecutive points

An instance for the Kronecker sequence

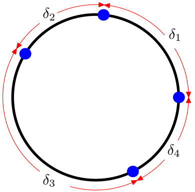
$$\mathcal{X} := \mathcal{K}(\alpha) := \{x_n := \{n\alpha\}, n \geq 0\}, \qquad \alpha = 7/17, \qquad T = 4$$

$$x_0 = 0, x_1 = 7/17,\ x_2 = 14/17,\ x_3 = 4/17$$

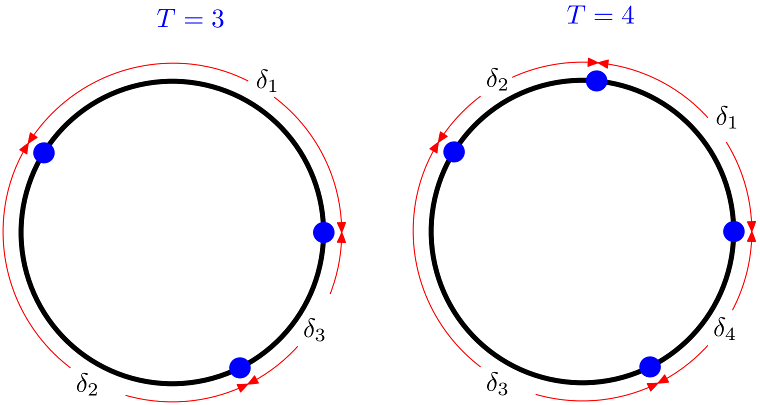


$$y_0 = 0,\ y_1 = 4/17,\ y_2 = 7/17,\ y_3 = 14/17$$

$$\delta = \frac{1}{17}(4, 3, 7, 3)$$

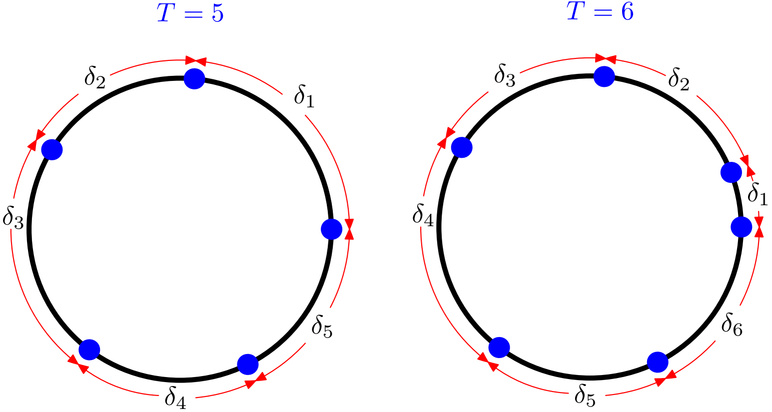


Example of the Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$ for $\alpha = 7/17$.



$$\delta = \frac{1}{17}(7, 7, 3) \qquad \delta = \frac{1}{17}(4, 3, 7, 3)$$

Example of the Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$ for $\alpha = 7/17$.



$$\delta = \frac{1}{17}(4, 3, 4, 3, 3) \qquad \delta = \frac{1}{17}(1, 3, 3, 4, 3, 3)$$

Main facts on the behaviour of the δ_i's for the truncated Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$

The behaviour of the sequence δ_i depends on
the Continued Fraction Expansion of the real number α.

The Two–Three Distances Theorem proves that the sequence of δ_i's takes only TWO or THREE distinct values.

–The number of distinct values (two or three)
– The values themselves only depend on
– the *CFE* of the real α
– the position of T with respect to the *CFE*.

The Continued Fraction Expansion of α.

$$\alpha = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_p + \cfrac{1}{\ddots}}}}}$$

uses the family of linear fractional transformations (*LFT* in shorthand)

$$h_{[m]} : x \mapsto \frac{1}{m+x}$$

and their composition

$$h_{[m_1]} \circ h_{[m_2]} \circ \ldots \circ h_{[m_p]} \circ \ldots$$

Continuants q_k and distances θ_k

When split at depth k, the CFE of the real x produces

the beginning rational, the middle digit, and the ending real,

$$\frac{p_k}{q_k} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_k}}}} \qquad x_k = \cfrac{1}{m_{k+1} + \cfrac{1}{m_{k+2} + \cfrac{1}{\ddots}}}$$

The beginning part defines the *LFT*

$$g_k := h_{[m_1]} \circ h_{[m_2]} \circ \ldots \circ h_{[m_k]},$$

with $\quad g_k(y) = \dfrac{p_{k-1}y + p_k}{q_{k-1}y + q_k},$

and the rational $\dfrac{p_k}{q_k} = g_k(0).$

The real x_k is defined by the equality

$$x = g_k(x_k), \qquad \text{or} \quad x_k = \frac{\theta_{k+1}(x)}{\theta_k(x)}$$

which involves the distance

$$\theta_k(x) := |q_{k-1}x - p_{k-1}|\ .$$

Two-distances phenomenon for the truncated sequence $\mathcal{K}_{\langle T\rangle}(\alpha)$.

Consider the partition of $\mathbb{N}$ by the intervals $[q_{k-1}+q_k, q_{k+1}+q_k[$ $(k \geq 1)$

Any (truncation) integer in the interval $[q_{k-1}+q_k, q_{k+1}+q_k[$ is written as

$$T = mq_k + q_{k-1} + r, \qquad m \in [1..m_{k+1}], \quad r \in [0..q_k - 1].$$

(Remind $q_{k+1} = m_{k+1}q_k + q_{k-1}$)

It gives rise to the two-distances phenomenon iff $r = 0$.

– Such truncation integers are called two-distances integers.

– They are of the form

$$T = mq_k + q_{k-1}, \qquad m \in [1..m_{k+1}].$$

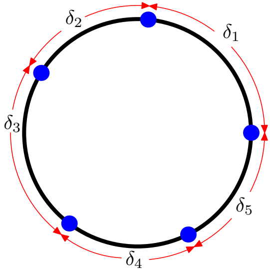
In the sequel, we only deal with this (particular) case.

B – Five parameters as measures of pseudo-randomness

Truncated Kronecker sequence $\mathcal{K}_{\langle T\rangle}(\alpha)$ with two distances.

I – First parameters of interest for $T = mq_k + q_{k-1}$

$$\alpha = \frac{7}{17},\ T = 5$$



$$\delta = \frac{1}{17}(4,3,4,3,3)$$

– The two distances. the small one and the large one

$$\widehat{\Gamma}_{\langle T\rangle} = \theta_{k+1}, \qquad \widetilde{\Gamma}_{\langle T\rangle} = \theta_k - (m-1)\theta_{k+1}$$

– The number of large distances : q_k

– The number of small distances :

$$T - q_k = (m-1)q_k + q_{k-1}$$

– The space covered by the large distance:

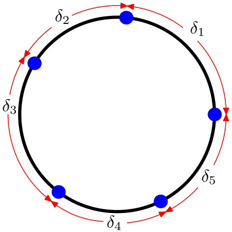
$$S_{\langle T\rangle} = q_k(\theta_k - (m-1)\theta_{k+1}).$$

Here : $S_{\langle 5\rangle} = \frac{12}{17}$

Truncated Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$ with two distances.

II - Discrepancy

$\alpha = \dfrac{7}{17}, T = 5$



$\delta = \dfrac{1}{17}(4, 3, 4, 3, 3)$

For a general sequence $\mathcal{X}$, the discrepancy $D_{\langle T \rangle}(\mathcal{X})$ compares the ordered sequence y_j to the sequence j/T.

$$D_{\langle T \rangle}(\mathcal{X}) = \sup_{j \in [1,T]} \left(\frac{j}{T} - y_j \right) + \sup_{j \in [1,T]} \left(y_j - \frac{j-1}{T} \right)$$

For the Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$:

$$\Delta_{\langle T \rangle}(\alpha) = T \cdot D_{\langle T \rangle}(\alpha) = 1 + (mq_k + q_{k-1} - 1)(\theta_k - m\theta_{k+1})$$

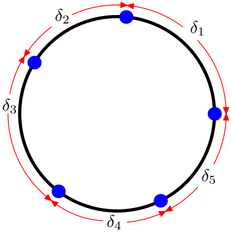
$$\Delta_{\langle T \rangle}(\alpha) \sim 1 + (mq_k + q_{k-1})(\theta_k - m\theta_{k+1}).$$

Here: $\quad \Delta_{\langle 5 \rangle} = \dfrac{21}{17}$

Truncated Kronecker sequence $\mathcal{K}_{\langle T\rangle}(\alpha)$ with two distances.

II - Discrepancy

$\alpha = \frac{7}{17}, T = 5$



$\delta = \frac{1}{17}(4,3,4,3,3)$

For a general sequence $\mathcal{X}$, the discrepancy $D_{\langle T\rangle}(\mathcal{X})$ compares the ordered sequence y_j to the sequence j/T.

$$D_{\langle T\rangle}(\mathcal{X}) = \sup_{j\in[1,T]}\left(\frac{j}{T} - y_j\right) + \sup_{j\in[1,T]}\left(y_j - \frac{j-1}{T}\right)$$

What is known on $\Delta_{\langle T\rangle}(\mathcal{X}) = T D_{\langle T\rangle}(\mathcal{X})$?

There exist C, C', such that for any $\mathcal{X}$:

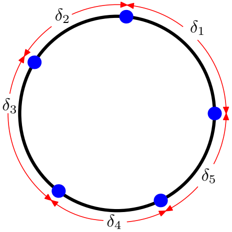
- one has $\Delta_{\langle T\rangle}(\mathcal{X}) \geq C$ for any T.
- there is T such that $\Delta_{\langle T\rangle}(\mathcal{X}) \geq C' \log T$

We are interested in a better understanding of this logarithmic behaviour in the case of a random Kronecker sequence.

Truncated Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$ with two distances.

III - Arnold measure

$$\alpha = \frac{7}{17}, T = 5$$



$$\delta = \frac{1}{17}(4, 3, 4, 3, 3)$$

For a general sequence $\mathcal{X}_{\langle T \rangle}$, the Arnold measure $A_{\langle T \rangle}(\mathcal{X})$ is the mean of the squares of the normalized $\hat{\delta}_i = T\delta_i$

$$A_{\langle T \rangle} = \frac{1}{T} \sum_{i=1}^{T} \left(\frac{\delta_i}{\frac{1}{T}} \right)^2 = T \sum_{i=1}^{T} {\delta_i}^2$$

For the Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$:

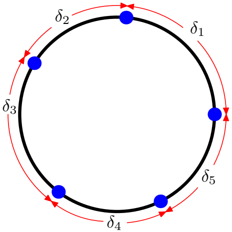
$$A_{\langle T \rangle}(\alpha) = (mq_k + q_{k-1}) \cdot \left[((m-1)q_k + q_{k-1})\theta_{k+1}^2 + q_k(\theta_k - (m-1)\theta_{k+1})^2 \right]$$

Here: $A_{\langle 5 \rangle} = \dfrac{295}{289}$

Truncated Kronecker sequence $\mathcal{K}_{\langle T \rangle}(\alpha)$ with two distances.

III - Arnold measure

$$\alpha = \frac{7}{17}, T = 5$$



$$\delta = \frac{1}{17}(4,3,4,3,3)$$

For a general sequence $\mathcal{X}_{\langle T \rangle}$, the Arnold measure $A_{\langle T \rangle}(\mathcal{X})$ is the mean of the squares of the normalized $\hat{\delta}_i = T\delta_i$

$$A_{\langle T \rangle} = \frac{1}{T}\sum_{i=1}^{T}\left(\frac{\delta_i}{\frac{1}{T}}\right)^2 = T\sum_{i=1}^{T}{\delta_i}^2$$

Arnold's proposal : The precise value of $A_{\langle T \rangle}(\mathcal{X})$ is a measure for the pseudo-randomness of $\mathcal{X}_{\langle T \rangle}$.

δ_i's nearly equal	$A_{\langle T \rangle} \sim 1$.
all the δ_i's small, except one	$A_{\langle T \rangle} \sim T$.
T large, δ_i's i.i.d on the circle of length 1 with $\sum_{i=1}^{T}\delta_i = 1$	$s_*(T) = \frac{2T}{T+1}$ $\lim_{T\to\infty} A_{\langle T \rangle} = 2.$

C – Main principles for our study.

Summary : Five parameters of interest

relative to a truncation integer $T = mq_k + q_{k-1}$ with $m \in [1..m_{k+1}]$

Distances $\Gamma_{\langle T\rangle}$, Covered space $S_{\langle T\rangle}$, Discrepancy $\Delta_{\langle T\rangle}$, Arnold measure $A_{\langle T\rangle}$

$\widehat{\Gamma}_{\langle T\rangle}$	θ_{k+1}
$\widetilde{\Gamma}_{\langle T\rangle}$	$\theta_k - (m-1)\theta_{k+1}$
$S_{\langle T\rangle}$	$q_k(\theta_k - (m-1)\theta_{k+1})$
$\Delta_{\langle T\rangle}$	$1 + (mq_k + q_{k-1})(\theta_k - m\theta_{k+1})$
$A_{\langle T\rangle}$	$(mq_k + q_{k-1})\left[((m-1)q_k + q_{k-1})\theta_{k+1}^2 + q_k(\theta_k - (m-1)\theta_{k+1})^2\right]$

These parameters are linear combinations of elementary costs, of the form

$$R_k = m^e\, q_{k-1}^a\, q_k^b\, \theta_k^c\, \theta_{k+1}^d$$

with $m \in [1..m_{k+1}]$ and a, b, c, d, e some positive integers.

What about the mean value $\mathbb{E}[R_k]$?

Not so easy a priori since R_k is a product of correlated variables.

We are interested by the expectation of these parameters

– in suitable probabilistic models, to be defined;

– for suitable integers $m \in [1..m_{k+1}]$

Some objectives of the work.

Most of the existing works

- deal with a fixed real number α
- study the pseudo–randomness of the truncated sequence $\mathcal{K}_{\langle T\rangle}(\alpha)$

  mainly with the discrepancy measure.
- ask the question:

  For a given α, for which T, the discrepancy is maximal? minimal?

We ask (and answer) the same kind of questions. However, we deal with

- various subsets $A \subset [0,1]$ of real numbers α
- various families $\mathcal{T}$ of truncations T
- various parameters X for pseudo-randomness

  (not only the discrepancy)

And we study the asymptotics of the mean value,

$$\mathbb{E}_A[X_{\langle T\rangle}] \qquad \text{for } T \in \mathcal{T}, T \to \infty.$$

For which triples $(A, X, \mathcal{T})$ is there a logarithmic behaviour ?

Probabilistic study
of five parameters used as measures for quasi-randomness
of the truncated Kronecker sequence

Position for the truncation
How to choose $m \in [1..m_{k+1}]$?
How to choose the truncation $T = mq_k + q_{k-1}$?

Three types of parameters
Unbalanced, Balanced – extremal or non extremal–

Four probabilistic models
Real model versus rational model
Unconstrained versus Constrained

I (a)– Position of truncation integers $T = mq_k + q_{k-1}$ with $m \in [1..m_{k+1}]$.

The parameter m plays an important rôle.

- It may vary in the whole interval $[1..m_{k+1}]$,
- the quotient m_{k+1} has an infinite mean value.

We focus on the value of m with respect to m_{k+1}.

To a real $\mu \in [0, 1]$, called the position:

- We associate the integer $m = \lfloor 1 + \mu(m_{k+1} - 1) \rceil$
- this defines the truncation sequence at position μ,
$$T = T_k^{\langle \mu \rangle} = \lfloor 1 + \mu(m_{k+1} - 1) \rceil q_k + q_{k-1}$$
- For each parameter X of interest, this defines the sequence at position μ
$$X_k^{\langle \mu \rangle} := X_{\langle T \rangle} \text{ when } T = T_k^{\langle \mu \rangle}.$$

We are interested in the probabilistic behaviour of such a sequence.

We recover the two boundary cases:

$$m = 1 \text{ for } \mu = 0 \text{ and } m = m_{k+1} \text{ for } \mu = 1$$

In these cases, the quotient m_{k+1} does not appear in the expression of $X_{\langle T \rangle}$

(Up to a translation on index k, $m = m_{k+1}$ plays the same rôle as $m = 0$.)

This explains why the two cases $\mu = 0$ and $\mu = 1$ may be very particular.

I (b) –Expression of the five parameters at boundary positions $\mu = 0$ and $\mu = 1$.

	$m = 1 \; [\; \mu = 0 \;]$	$m = m_{k+1} \; [\; \mu = 1 \;]$
$\widehat{\Gamma}_{\langle T \rangle}$	θ_{k+1}	θ_{k+1}
$\widetilde{\Gamma}_{\langle T \rangle}$	θ_k	$\theta_k + \theta_{k+1}$
$S_{\langle T \rangle}$	$q_k \theta_k$	$q_k(\theta_k + \theta_{k+1})$
$\Delta_{\langle T \rangle}$	$1 + (q_k + q_{k-1})(\theta_k - \theta_{k+1})$	$1 + q_k \theta_{k+1}$
$A_{\langle T \rangle}$	$(q_k + q_{k-1}) \left[q_{k-1} \theta_{k+1}^2 + q_k \theta_k^2 \right]$	$q_k \left[(q_k - q_{k-1}) \theta_k^2 + q_{k-1} (\theta_k + \theta_{k+1})^2 \right]$

II(a) – Classification of costs $R_k = m^e\, q_{k-1}^a\, q_k^b\, \theta_k^c\, \theta_{k+1}^d$

A first easy study : $\mathbb{E}[\log R_k]$

when m is any integer in $[1..m_{k+1}]$ and α uniformy chosen in $[0, 1]$

Well known estimates involve the entropy $\mathcal{E} = \pi^2/(6 \log 2)$:

$$\mathbb{E}[\log q_k] = \frac{k\mathcal{E}}{2} + O(1) \qquad \mathbb{E}[\log \theta_k] = -\frac{k\mathcal{E}}{2} + O(1), \qquad \mathbb{E}[\log m_{k+1}] = \Theta(1).$$

Then, two main cases depending on the sum $(a + b)$ wrt the sum $(c + d)$

Unbalanced	$(a + b) \neq (c + d)$	$\mathbb{E}[\log R_k] \sim k(\mathcal{E}/2) \cdot [(a + b) - (c + d)]$
Balanced	$(a + b) = (c + d)$	$\mathbb{E}[\log R_k] = O(1)$

The two distances are unbalanced.

The other three costs are balanced with a balance $f = a + b = c + d$

- equal to $f = 1$, for the covered space and the discrepancy
- equal to $f = 2$ for the Arnold measure

II (b) – Balanced cost $R_k = m^e q_{k-1}^a q_k^b \theta_k^c \theta_{k+1}^d$ at position $\mu \in]0,1[$.

– Balanced cost $a + b = c + d = f$

– At position $\mu \in]0,1[$, one has $m^e \approx \mu^e m_{k+1}^e$.

– Replace q_{k-1} by $q_{k-1} = (1 - \theta_k q_k)/\theta_{k+1}$. Remark $\theta_{k+1}/\theta_k \approx 1/m_{k+1}$.

$$\Longrightarrow \qquad R_k \approx \mu^e (\theta_k q_k)^{f-a} (1 - \theta_k q_k)^a \quad {m_{k+1}}^{e-(d-a)}$$

Important rôle played by the exponent of m_{k+1} equal to $e - (d - a)$.

We prove : $\quad \mathbb{E}[R_k] = \infty \Longleftrightarrow (a = 0 \;\text{ and }\; e = 1 + d)$

In this case, the cost is called "extremal".

Extremal part of the three balanced parameters $m \approx \mu m_{k+1}, \quad \mu \in]0,1[$

$S_{\langle T\rangle}$	0
$\Delta_{\langle T\rangle}$	$m\, q_k \theta_k - m^2\, q_k \theta_{k+1}$
$A_{\langle T\rangle}$	$m\, q_k^2\, \theta_k^2 - 2\, m^2\, q_k^2\, \theta_k\, \theta_{k+1} + m^3 q_k^2\, \theta_{k+1}^2$

III(a) –Probabilistic models: Real model versus Rational model

For each parameter X, for μ fixed in $[0,1]$, the sequence at position μ is

$$X_k^{\langle\mu\rangle} := X_{\langle T\rangle} \text{ when } T = T_k^{\langle\mu\rangle} = \lfloor 1+\mu(m_{k+1}-1)\rceil q_k + q_{k-1}$$

This defines a sequence of random variables, which depend on α,

α may be a random real – or a random rational

It is "natural" to compare the two cases.

Real model. The real α is uniformly drawn in the unit interval $\mathcal{I}$.
We study the behaviour of the mean value $\mathbb{E}[X_k^{\langle\mu\rangle}]$ for $k\to\infty$,

Rational model. For a rational α of depth $P(\alpha)$, the index k satisfies $k \le P(\alpha)$.
The index k is chosen as an *admissible* function K of the depth P, i.e., $k = K(P)$

with $\beta^- P \le K(P) \le \beta^+ P$ for two constants $0 < \beta^- < \beta^+ < 1$.

– We consider the (finite) set Ω_N of rationals α with $\mathrm{den}(\alpha) \le N$,
endowed with the uniform probability,

– We choose and fix an admissible function K for the index

We study the behaviour of the mean value $\mathbb{E}_N[X_K^{\langle\mu\rangle}]$ for $N\to\infty$.

III(b) – Probabilistic models: Unconstrained model versus constrained model

There is a close connection between the two behaviours

- the truncated Kronecker sequence $\mathcal{K}_{\langle T\rangle}(\alpha)$
- the (boundness) of the sequence of digits (m_k) in the *CFE* of α

This is why we wish to deal with the "constrained" models,

where all the digits m_k are bounded by some constant M

and then let M tend to ∞ to obtain the unconstrained model.

Two main constrained cases.

- The real case:

  the Cantor set $\mathcal{I}^{[M]}$ of real numbers with digits $m_k \leq M$
- The rational case:

  the set $\Omega_N^{[M]}$ of rational numbers with $\mathrm{den} \leq N$ and digits $m_k \leq M$

Probabilistic study

of five parameters used as measures for quasi-randomness

of the truncated Kronecker sequence

Three types of positions for the truncation
Boundary cases $\mu = 0$ and $\mu = 1$ Generic case $\mu \in]0, 1[$

Three types of parameters		
Unbalanced	Balanced with a zero extremal part for a generic position $\mu \in]0, 1[$	Balanced with a non zero extremal part for a generic position $\mu \in]0, 1[$
Two distances	Covered Space	Discrepancy and Arnold Measure

Four probabilistic models
Real model versus rational model   Unconstrained versus Constrained

D –Some of our results.

Distances in the M–constrained model $[M \le \infty]$.

Real case: α is uniformy chosen in the set $\mathcal{I}^{[M]}$ of reals with $m_k \le M$.

The mean value of any distance [small or large] is exponentially decreasing,

$$\mathbb{E}^{[M]}[\Gamma_k] = \Theta({\gamma_M}^k) \qquad (k \to \infty),$$

The rate γ_M involves the dominant eigenvalue $\lambda_M(s)$ of the operator

$$\mathbf{H}_{M,s}[g](x) := \sum_{m \le M} \frac{1}{(m+x)^{2s}}\, g\left(\frac{1}{m+x}\right),$$

and the Hausdorff dimension σ_M of $\mathcal{I}^{[M]}$ with the relation $\lambda_M(\sigma_M) = 1$.

The rate γ_M equals $\lambda_M\left(\sigma_M + \frac{1}{2}\right)$.
When $M = \infty$, the rate equals $\lambda_\infty\left(\frac{3}{2}\right)$.

This value $\lambda_\infty\left(\frac{3}{2}\right) \sim 0.3964$ is "new" in Euclidean probabilistic analyses.

Constrained case $\to$ Unconstrained case :

$$\gamma_M = \gamma_\infty\left[1 + O\left(\frac{1}{M}\right)\right] \quad (M \to \infty)$$

Distances in the probabilistic M–constrained model $[M \le \infty]$.

Rational case: α is uniformy chosen in $\Omega_N^{[M]}$.

The index K is chosen as the δ–fraction of the depth

$K = K_{\langle \delta \rangle} = \lfloor \delta P \rfloor$, $\delta \in \mathbb{Q} \cap [0,1]$

The mean value of any distance [small or large] is exponentially decreasing,

$$\mathbb{E}_N^{[M]}[\Gamma_{K_{\langle\delta\rangle}}] = \Theta\left(N^{2[\sigma_M(\delta) - \sigma_M]}\right) \qquad [N \to \infty].$$

The exponent $\sigma_M(\delta)$ is the unique real solution of the equation

$$\lambda_M^{1-\delta}(\sigma)\, \lambda_M^{\delta}\left(\sigma + \frac{1}{2}\right) = 1 \qquad \text{with } \sigma_M(0) = \sigma_M, \quad \sigma_M(1) = \sigma_M - (1/2).$$

Constrained case $\to$ Unconstrained case :

$$(\forall \delta \in [0,1]) \quad \sigma_M(\delta) = \sigma_\infty(\delta)\left[1 + O\left(\frac{1}{M}\right)\right] \quad (M \to \infty)$$

Study of the balanced parameters – Covered space, Discrepancy, Arnold Measure–.

at a boundary position $\mu = 0$ [case -] and $\mu = 1$ [case $+$].

Case of the random variables $X_k^\pm \in \{S_k^\pm, \Delta_k^\pm, A_k^\pm\}$

(i) For each X, the expectations $\mathbb{E}^{[M]}[X_k^\pm]$ have the same finite limit $\chi_M^\pm$ in the real case and the rational case.

(ii) Constrained case $\to$ Unconstrained case :

$$\chi_M^\pm = \chi_\infty^\pm \left[1 + O\left(\frac{1}{M}\right)\right] \quad (M \to \infty)$$

(iii) The values $\chi_\infty^\pm$ are explicit:

$$s^- = \frac{1}{2} + \frac{1}{4\log 2} \sim 0.861, \quad d^- = 1 + \frac{1}{2\log 2} \sim 1.721 \quad a^- = \frac{2}{3} + \frac{1}{3\log 2} \sim 1.147$$

$$s^+ = \frac{1}{2} \qquad d^+ = 1 + \frac{1}{4\log 2} \sim 1.360, \quad a^+ = \frac{2}{3} + \frac{1}{4\log 2} \sim 1.027.$$

Study of the balanced parameters – Discrepancy and Arnold Measure– at a generic position $\mu \in]0,1[$

Case of the random variables $X_k^{\langle\mu\rangle}$ with $X \in \{\Delta, A\}$

(i) The expectations $\mathbb{E}[X_k^{\langle\mu\rangle}]$ in the real unconstrained model are infinite.

(ii) In the rational unconstrained model, there is a logarithmic behaviour

$$\mathbb{E}_N[\Delta_K^{\langle\mu\rangle}] \sim \mu(1-\mu)\log_2 N, \quad \mathbb{E}_N[A_K^{\langle\mu\rangle}] \sim \mu(1-\mu)^2 \log_2 N, \quad (N \to \infty)$$

(iii) In the unconstrained models, the expectations have

the same finite limit in the real and the rational models

$$\begin{array}{lll} [\text{Real Case.}] & \text{for } k \to \infty: & \mathbb{E}^{[M]}[\Delta_k^{\langle\mu\rangle}] \to d_M^{\langle\mu\rangle}, \quad \mathbb{E}^{[M]}[A_k^{\langle\mu\rangle}] \to a_M^{\langle\mu\rangle} \\ [\text{Rational Case.}] & \text{for } N \to \infty: & \mathbb{E}_N^{[M]}[\Delta_K^{\langle\mu\rangle}] \to d_M^{\langle\mu\rangle}, \quad \mathbb{E}_N^{[M]}[A_K^{\langle\mu\rangle}] \to a_M^{\langle\mu\rangle}. \end{array}$$

(iv) Constrained $\to$ Unconstrained : a logarithmic behaviour

$$d_M^{\langle\mu\rangle} \sim \mu(1-\mu)\log_2 M, \qquad a_M^{\langle\mu\rangle} \sim \mu(1-\mu)^2 \log_2 M, \qquad (M \to \infty)$$

(v) Maximum for the mean discrepancy at $\mu = 1/2$.

Maximum for the mean Arnold measure at $\mu = 1/3$.

Conclusion.

We try to answer the question:

Is a random Kronecker sequence $\mathcal{K}_{\langle T\rangle}(\alpha)$ pseudo-random?

We describe a model with various possible choices

– Five parameters of pseudo-randomness,

– Families of truncations T

– Specific subsets for α.

where a precise answer can be provided.

We also study the random behaviour of general parameters

which are polynomials in q_k, θ_k, m_{k+1}

We limit ourselves to the two–distances framework.

In the three-distances framework,

– similar behaviours can be exhibited for four parameters,

– but, for the discrepancy,

we do not have a close formula as a polynomial function in θ_k, q_k, m_{k+1}.

D) Some hints on the methods.

Dynamical analysis method for the rational setting

The main tool for studying a cost $R\left(\frac{u}{v}\right)$ on each

$$\Omega_N = \{u/v : \ \gcd(u,v) = 1, \ 0 \le u \le v \le N\}$$

is the Dirichlet generating function of cost R on the set $\Omega = \bigcup_N \Omega_N$:

$$S_R(s) := \sum_{u/v \in \Omega} \frac{1}{v^{2s}} R\left(\frac{u}{v}\right) = \sum_k \frac{c_k}{k^{2s}}, \qquad \text{with} \qquad c_k := \sum_{\substack{u \le k \\ \gcd(u,k)=1}} R\left(\frac{u}{k}\right).$$

Then, $\qquad \mathbb{E}_N[R] := \dfrac{\sum_{k \le N} c_k}{\sum_{k \le N} a_k}$

where a_k is the coefficient of the series $S_{[R]}$ for $R = 1$.

Three main steps.

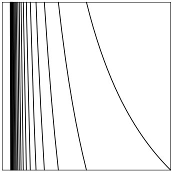
Step 1. Look for an alternative form of S_R with dynamical systems.

Step 2. Study singularities of S_R

Step 3. Transfer these informations on the asymptotics of $\mathbb{E}_N[R]$.

Euclidean Dynamical System and the continuous world

The Gauss map $T : [0,1] \to [0,1]$



$$T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor, \quad T(0) = 0.$$

$$\mathcal{H} := \{h : x \mapsto \frac{1}{m+x}, \quad m \geq 1\}$$

is the set of inverse branches of T

Density Transformer:

For a density f on $\mathcal{I}$, $\mathbf{H}[f]$ is the density on $\mathcal{I}$ after one iteration of shift T

$$\mathbf{H}[f](x) = \sum_{h \in \mathcal{H}} |h'(x)| f \circ h(x)$$

$$= \sum_{m \in \mathbb{N}} \frac{1}{(m+x)^2} f(\frac{1}{m+x}).$$

Transfer operator (Ruelle):

$$\mathbf{H}_s[f](x) = \sum_{h \in \mathcal{H}} |h'(x)|^s f \circ h(x).$$

The k-th iterate satisfies:

$$\mathbf{H}_s^k[f](x) = \sum_{h \in \mathcal{H}^k} |h'(x)|^s f \circ h(x)$$

Discrete world.
Generation of q_k, θ_k, m_{k+1} via transfer operators

Main fact. If $h : x \mapsto h(x) = \dfrac{ax+b}{cx+d}$, then $h'(x) = \dfrac{\det h}{(cx+d)^2}$

For coprime (u, v), if $\dfrac{u}{v} = h(0)$, then $\dfrac{1}{v^2} = |h'(0)|$.

Since continuants q_k, distances θ_k, digits m_{k+1} are denominators,

the operators $\mathbf{H}_s^k$, with some extensions, are able to *generate* continuants q_k, distances θ_k of depth k.

The transfer operators used as generating operators.

$$R_k = (\theta_k q_k)^{f-a}(1-\theta_k q_k)^a \quad {m_{k+1}}^e \left(\frac{\theta_{k+1}}{\theta_k}\right)^{d-a}$$

Name	Use	Definition of the component operator
$\mathbf{H}_{s+a}$	$(\theta_{k+1}/\theta_k)^{2a}$	$\lvert h'(x)\rvert^{s+a} \cdot g \circ h(x)$
$\mathbf{H}_{(s+b,-b)}$	$(\theta_k q_k)^b$	$\lvert h'(x)\rvert^{s+b} \lvert h'(y)\rvert^{-b} \cdot G(h(x), h(y))$
$\underline{\mathbf{H}}_{(s,c)}$	m_{k+1}^{-2c}	$\lvert h'(x)\rvert^{s} \, \lvert h'(0)\rvert^{-c} \cdot g \circ h(x)$

In the constrained models, we use constrained operators (where the sum is taken over $h_{[m]}$ with $m \leq M$)

Main differences with the usual approach in Dynamical Analysis

A Dirichlet series $S_R(s)$ is associated to each parameter R.

Each series is expressed as a sum of powers of previous operators $\mathbb{G}_s$

— The classical setting deals with the usual transfer operator.

The Dirichlet series involves the quasi-inverse of this operator

$$(I - \mathbf{H}_s)^{-1} = \sum_{p \geq 1} \mathbf{H}_s^p.$$

— Here, one deals with the red and blue transfer operators,

The Dirichlet series involve pseudo-quasi-inverses:

$$\sum_{p=1}^{\infty} \mathbf{H}_{s+a}^{p-K(p)} \circ \mathbf{H}_{(s+b,-b)}^{K(p)} \cdots$$