Rotational beta expansion and self-similar tilings

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Recall **beta expansion** :

$$T(x) = \beta x - \lfloor \beta x \rfloor.$$

It is a generalization of binary and decimal expansion.

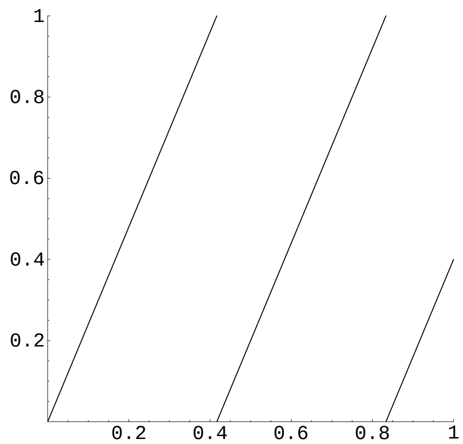


Figure 1: Beta expansion

It belongs to both ergodic theory and number theory.

- ACIM is unique and equivalent to the Lebesgue measure.

- Its density was made explicit

$$h(x) = \sum_{x < T^n(1)} \frac{1}{\beta^n}$$

- Symbolic property is well studied.

- Number theoretical results can be derived.

The orbit $T^n(1)$ produced so called **expansion of one** which is an infinite sequence

$$d_\beta(1) = c_1 c_2 c_3 \ldots$$

of letters in $\{0, 1, \ldots, \lfloor \beta \rfloor\}$ satisfying:

$$1 = \frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \ldots$$

where $T^n(1) = \sum_{i=1}^{\infty} c_{n+i-1} \beta^i$. This is used to prove the explicit shape of h.

Symbolic property of the beta expansion

If the orbit of discontinuity $(T^n(1))_{n=1,2,\dots}$ is finite, the system is sofic. If β is a Pisot number, then the system is sofic, which is equivalent to say that $d_\beta(1)$ is eventually periodic. If $d_\beta(1)$ is purely periodic, then the associated symbolic system is SFT. A lot of open questions remain, see Blanchard [5].

Number theoretical property by its dynamics: return time, shrinking targets problems, orbits of 1 (J. Wu, B. Wang, Wuhan group).

Under Pisot condition, a good natural extension characterizes periodic orbits: Ito-Rao [8], Berthé-Siegel [4].

Ito-Sadahiro [9] introduced the **negative beta expansion**

$$T : x \mapsto -\beta x - \lfloor -\beta x + \beta/(1+\beta) \rfloor$$

acting on $[-\beta/(1+\beta), 1/(1+\beta))$.

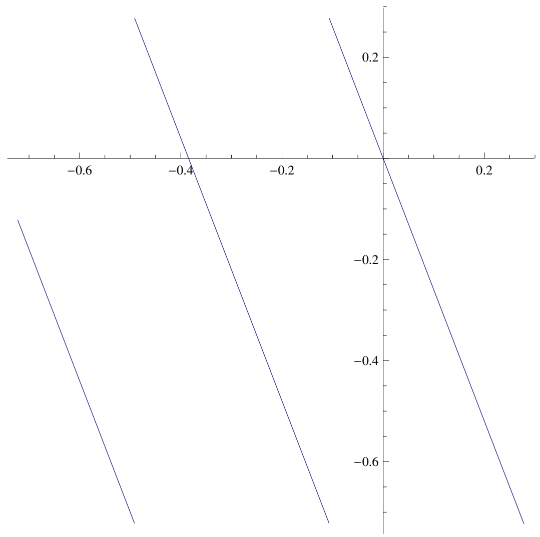


Figure 2: Negative Beta expansion for $\beta = 2.6$

The ACIM of T is unique. Its density is given by:

$$\sum_{x > T^n(-\beta/(1+\beta))} \frac{1}{(-\beta)^n}.$$

This expression is probably not intuitive. Liao-Steiner [12] proved that its ACIM is equivalent to the Lebesgue measure if and only if $\beta \geq (1+\sqrt{5})/2$. Symbolic dynamical study is parallel to the original beta expansion.

Rotational beta expansion

Let $1 < \beta \in \mathbb{R}$ and M be an element of the orthogonal group $O(m, \mathbb{R})$. Let $\mathcal{L}$ be a lattice of $\mathbb{R}^m$. Fix a fundamental domain $\mathcal{X}$ of $\mathcal{L}$. Then

$$\mathbb{R}^m = \bigcup_{d \in \mathcal{L}} (\mathcal{X} + d)$$

is a disjoint partition of $\mathbb{C}$. Define a map $T : \mathcal{X} \to \mathcal{X}$ by $T(z) = \beta M(z) - d$ where $d = d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta M(z) \in \mathcal{X} + d$.

Given a point z in $\mathcal{X}$, we obtain an expansion

$$\begin{aligned} z &= \frac{M^{-1}(d_1)}{\beta} + \frac{M^{-1}(T(z))}{\beta} \\ &= \frac{M^{-1}(d_1)}{\beta} + \frac{M^{-2}(d_2)}{\beta^2} + \frac{M^{-2}(T^2(z))}{\beta^2} \\ &= \sum_{i=1}^{\infty} \frac{M^{-i}(d_i)}{\beta^i} \end{aligned}$$

with $d_i = d(T^{i-1}(z))$. In this case, we write $d_T(z) = d_1 d_2 \ldots$. We call T the **rotational beta transformation** and $d_T(z)$ the **expansion** of z with respect to T.

For $m = 2$, $\beta > 1$ and M is in $SO(2, \mathbb{R})$, the algorithm is naturally written in complex plane. Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| = 1$, $\xi, \eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1/\eta_2 \notin \mathbb{R}$. Then

$$\mathcal{X} = \{\xi + x\eta_1 + y\eta_2 \mid x \in [0, 1), y \in [0, 1)\}$$

is a fundamental domain of the lattice $\mathcal{L} = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$ in $\mathbb{C}$. We are interested in the transform $T(z) = \beta\zeta z - d$ and its expansion:

$$z = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i \zeta^i} \in \mathbb{C}.$$

where $d_i \in \mathcal{L}$.

Motivations

- Another example of explicit ACIM ?

- Systematic construction of self-similar tilings.

ACIM's are not unique !

Example 1. $\zeta = \sqrt{-1}, \beta = 1.039, \eta_1 = 2.92, \eta_2 = \exp(\pi\sqrt{-1}/3)$ and $\xi = 0$.

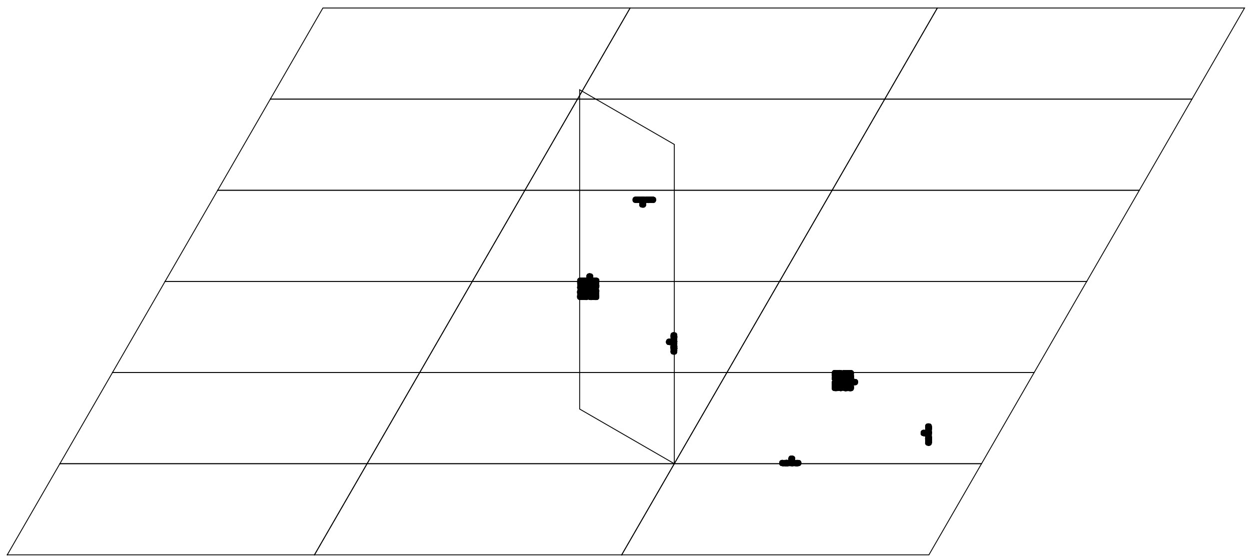
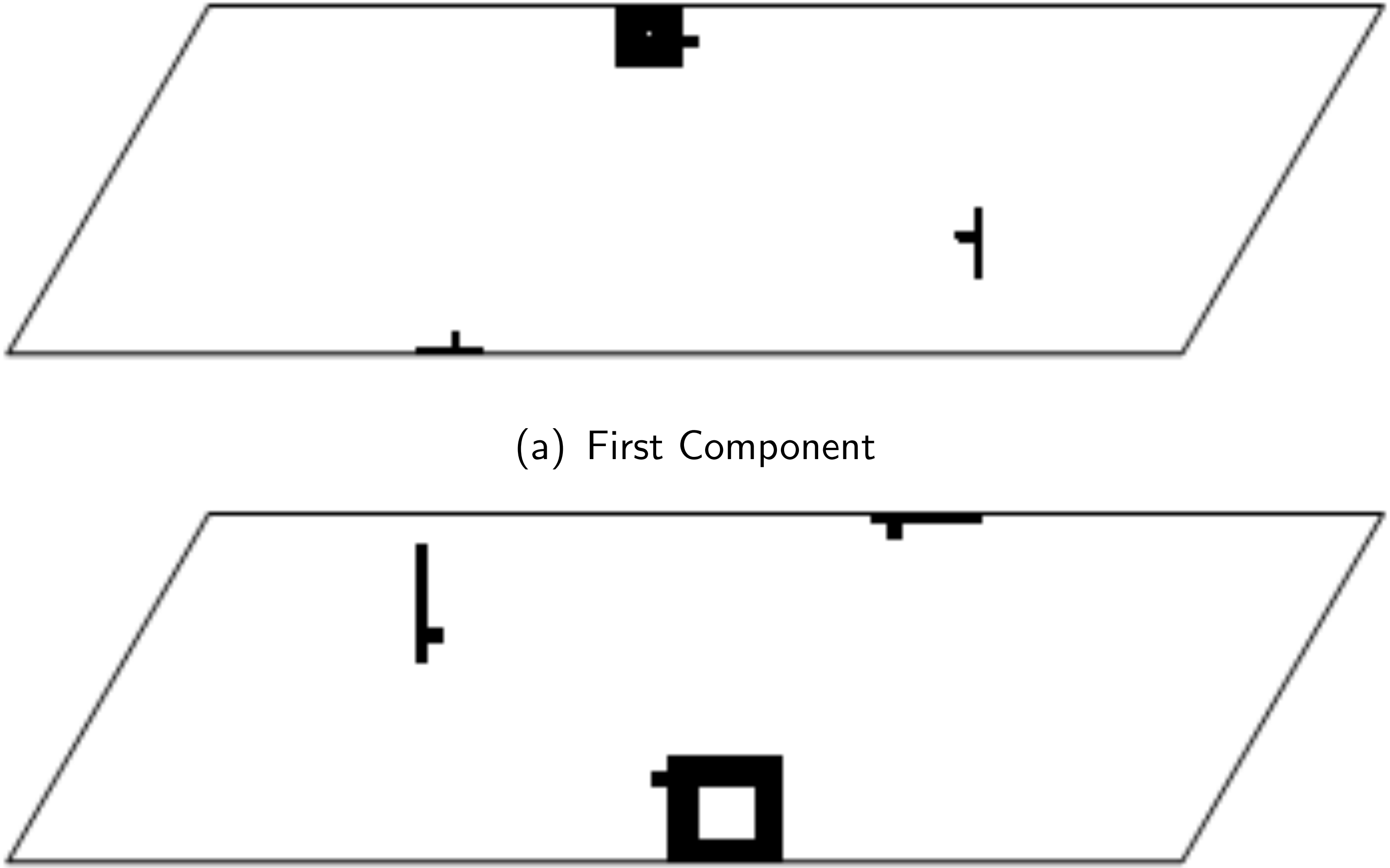


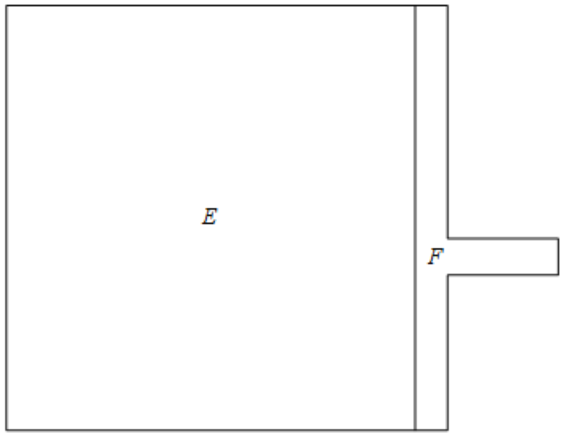
Figure 3: Non ergodic case



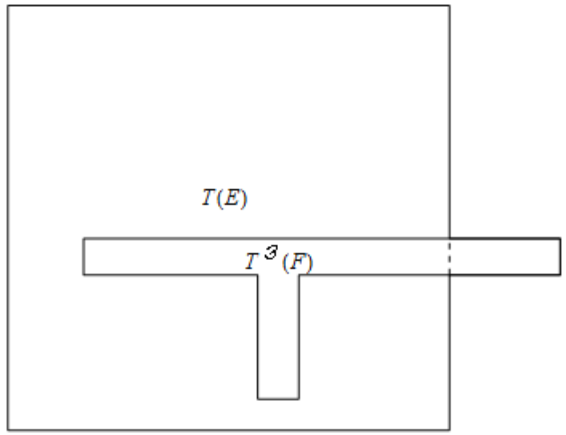
(a) First Component

(b) Second Component

Figure 4: Non unique ACIM



(a) E and F



(b) Confirmation of the set equation

The same situation happens when β and η_1 satisfy

$$\frac{\sqrt{3}}{2}\beta + 1 + \frac{\sqrt{3}}{\beta} - \frac{\sqrt{3}}{2\beta^3} \leq \eta_1 \leq \frac{1}{2} + \frac{\sqrt{3}}{\beta} + \frac{\sqrt{3}}{2\beta^3}$$

while other parameters are fixed.

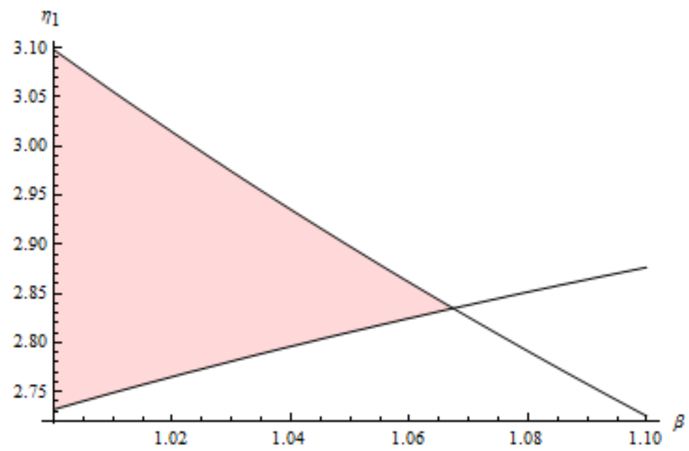


Figure 5: Non ergodic parameters

In this case, we have to study the Perron Frobenius operator:

$$P(h) = \sum_{y \in T^{-1}(x)} \frac{h(y)}{\mathrm{Jac}(T, y)}$$

acting on $L^1(\mathbb{R}^m, \mathbb{R})$. Then T is a very special case of piecewise expanding maps, studied by Keller, Gora-Boyarsky, Tsujii, Buzzi [10, 11, 7, 13, 14, 6, 15]. The main difficulty arises from the set of discontinuities. It becomes much more complicated than those in 1-dim.

We have to find a definition of total variation in higher dimension. The best one is found by Keller and used by Saussol [13]. Take a ball B and let

$$\operatorname{osc}(f, B) = \operatorname{esssup}_{x \in B} f(x) - \operatorname{essinf}_{x \in B} f(x),$$

the **oscillation** around B. Fix an $\varepsilon_0 > 0$ and put

$$\operatorname{Var}(f) = \sup_{0 < \varepsilon \leq \varepsilon_0} \frac{1}{\varepsilon} \int \operatorname{osc}(f, B(x, \varepsilon)) dx.$$

Then $\operatorname{Var}(f)$ is an analogy of the total variation and and the subspace $V = \{f \in L^1 \mid \operatorname{Var}(f) + \|f\| < \infty\}$ becomes relatively compact in $\mathcal{L}^1$.

Under some natural assumption on the piecewise expanding map, we can prove a Lasota-Yorke type inequality:

$$\mathrm{Var}(P^n(f)) < \eta \mathrm{Var}(f) + D\|f\|$$

with some $n \in \mathbb{N}$ and $0 < \eta < 1$. Iterating this inequality, from an infinite sequence

$$\frac{1}{N}\sum_{i=1}^{N} P^i(f), \quad N = 1, 2, \ldots$$

we can select a converging subsequence. This lead us to the unique limit, which satisfies $P(h) = h$.

We know that there exists an ACIM μ whose support contains a ball of positive density. This implies the number of ergodic components is finite and bounded by

$$\frac{1}{\pi}\left(\frac{D}{1-\eta}\right)^2.$$

However, the bound is not practically good since η is usually close to 1. For a bounded set $A \in \mathbb{R}^m$, define the **width** $w(A)$ of A as the minimum distances of two pararell hyperplanes sandwiches A. The **covering radius** of a relatively dense subset $P \in \mathbb{R}^m$ is

$$r(P) := \sup_{x \in \mathbb{R}^m} \inf_{y \in P} \|x - y\|$$

Theorem 2 ([1]). Assume that $2r(\mathcal{L}) < \beta w(\mathcal{X})$. If $\beta > m + 1$ then there is a unique ACIM of T equivalent to the m-dim Lebesgue measure.

We use the result by Bang[3], which solved Tarski's plank problem. For $m = 2$ we get more precise results.

Let $\theta(\mathcal{X}) \in (0, \pi)$ be the angle between η_1 and η_2.

$$B_1 = \begin{cases} 2 & \text{if } \frac{1}{2} < \tan\left(\frac{\theta(\mathcal{X})}{2}\right) < 2 \\ 1 + \frac{2}{1+\sin\left(\frac{\theta(\mathcal{X})}{2}\right)} & \text{if } \sin(\theta(\mathcal{X})) < \sqrt{5} - 2 \\ \frac{3}{2} + \frac{1}{16}\cot^2\left(\frac{\theta(\mathcal{X})}{2}\right) + \tan^2\left(\frac{\theta(\mathcal{X})}{2}\right) & \text{otherwise} \end{cases}$$

and

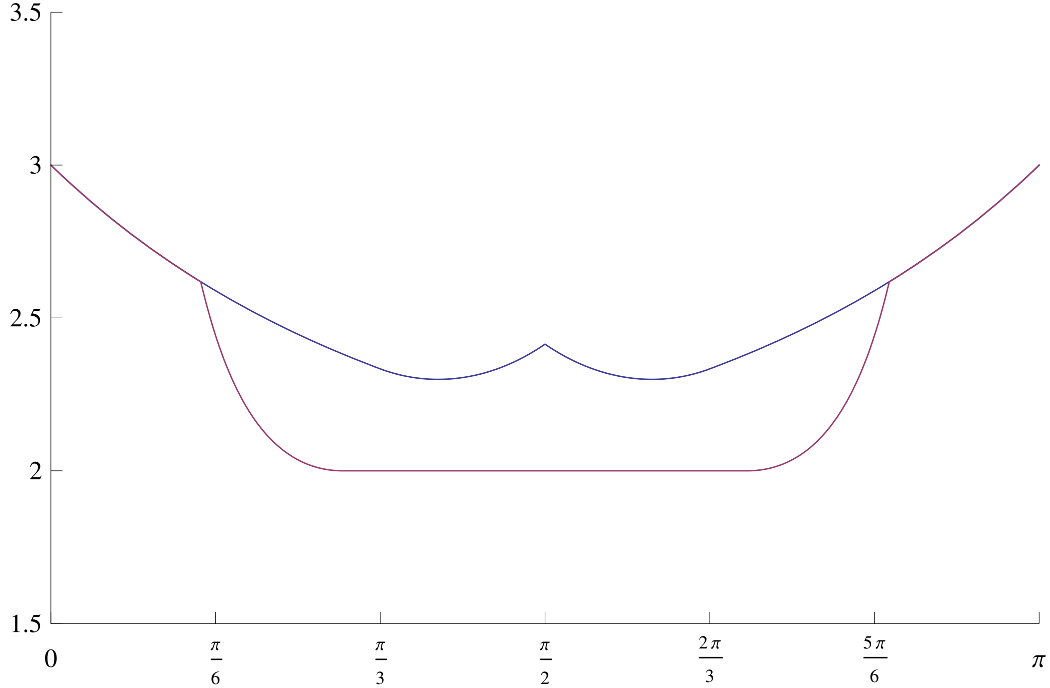
$$B_2 := \begin{cases} \frac{|\cos(\theta(\mathcal{X}))|+1}{2(|\cos(\theta(\mathcal{X}))|+\sin(\theta(\mathcal{X}))-1)} & \text{if } \frac{\pi}{3} < \theta(\mathcal{X}) < \frac{2\pi}{3} \\ 1 + \frac{2}{1+\sin\left(\frac{\theta(\mathcal{X})}{2}\right)} & \text{otherwise.} \end{cases}$$

Theorem 3. If $\beta > B_1$ then $(\mathcal{X}, T)$ has a unique ACIM μ. Moreover, if $\beta > B_2$ then μ is equivalent to the 2-dimensional Lebesgue measure restricted to $\mathcal{X}$.

This is an improvement of the result in the [2], in particular if θ is small.

One can confirm the inequality $B_1 \leq B_2$ in Figure 6.

Figure 6: Comparison of B_1 and B_2



Idea of the proof.

Starting from a r-covering of $\mathcal{X}$, inductively we create a finer one by looking the inverse images of T. Then we can show that if β is large, then for any $\epsilon > 0$ and any point $z \in \mathcal{X}$ that $\bigcup_{n=1}^{m} T^{-n}(z)$ is an ϵ-covering. Assuming two ergodic ACIM, this fact means two ergodic components have non negligible communication, which gives a contradiction.

Put $\mathcal{A} := \{d(z) \mid z \in \mathcal{X}\}$. Let $\mathcal{A}^{\mathbb{Z}}$ (resp. $\mathcal{A}^*$) be the set of all bi-infinite (resp. finite) words over $\mathcal{A}$. We say $w \in \mathcal{A}^*$ is admissible if w appears in the expansion $d_T(z)$ for some $z \in \mathcal{X} \setminus \bigcup_{n\in\mathbb{Z}} T^n(\partial(\mathcal{X}))$. Let

$$\mathcal{X}_T := \left\{ w = (w_i) \in \mathcal{A}^{\mathbb{Z}} \middle| \, w_i w_{i+1} ... w_j \text{ is admissible } \right\}.$$

The symbolic dynamical system associated to T is the topological dynamics $(\mathcal{X}_T, s)$ given by the shift operator $s((w_i)) = (w_{i+1})$. We say $(\mathcal{X}_T, s)$ (or simply, $(\mathcal{X}, T)$) is sofic if there is a finite directed graph G labeled by $\mathcal{A}$ such that for each $w \in \mathcal{X}_T$, there exists a bi-infinite path in G labeled w and vice versa.

Lemma 4. The system $(\mathcal{X}, T)$ is sofic if and only if $\bigcup_{n=1}^{\infty} T^n(\partial(\mathcal{X}))$ is a finite union of segments.

A problem on the definition of soficness

One may define **complete soficness** by considering all orbits in $\mathcal{X}$ instead of $\mathcal{X} \setminus \bigcup_{n\in\mathbb{Z}} T^n(\partial(\mathcal{X}))$. Then the results will be:

Lemma 5. The system $(\mathcal{X}, T)$ is completely sofic if and only if $(T^n(\partial(\mathcal{X})))_{n=1,2,\ldots}$ is eventually periodic as a sequence of sets.

We are not sure these definitions are the same.

So $(\mathcal{X}, T)$ to be sofic, ζ must be a root of unity. Assume that ζ is a q-th root of unity with $q > 2$ and $\xi, \eta_1, \eta_2 \in \mathbb{Q}(\zeta, \beta)$ with $\eta_1/\eta_2 \notin \mathbb{R}$.

Theorem 6. Let ζ be a q-th root of unity $(q > 2)$ and β be a Pisot number. Let $\eta_1, \eta_2, \xi \in \mathbb{Q}(\zeta, \beta)$. If $\cos(2\pi/q) \in \mathbb{Q}(\beta)$, then the system $(\mathcal{X}, T)$ is sofic.

Corollary 7. If ζ is a $3rd$, $4th$ or $6th$ root of unity, then the system $(\mathcal{X}, T)$ is sofic for any Pisot number β.

Corollary 8. For any positive integer q, there exists a Pisot number β which satisfies above conditions. Thus there is a self-similer tiling in $\mathbb{R}^2$ with inflation constant β and q-fold rotation action, whose all tiles are polygons.

On the other hand, we can give a family of non-sofic systems when $\zeta + \zeta^{-1} \notin \mathbb{Q}(\beta)$.

Theorem 9. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/5)$. If $\beta > 2.90332$ such that $\sqrt{5} \notin \mathbb{Q}(\beta)$, then $(\mathcal{X}, T)$ is not a sofic system.

For example, $\beta = 3, 4, 5 \ldots$ are not sofic in this setting.

Summary of our results

	unique	Lebesgue	density	sofic
Beta	Yes	Yes	Yes	β: Pisot
Negative	Yes	$\beta \geq \frac{1+\sqrt{5}}{2}$	Yes	β: Pisot
Rotation	$\beta > B_1$	$\beta > B_2$	?	Pisot & $\cos(2\pi/q) \in \mathbb{Q}(\beta)$

Open questions

- Improve the constants B_1 and B_2. They seem not optimal.
- Make explicit the density of ACIM. Possible in sofic cases.

Example 10. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/3)$. Put $\beta = 1 + \sqrt{2}$. We have 9 cylinders.

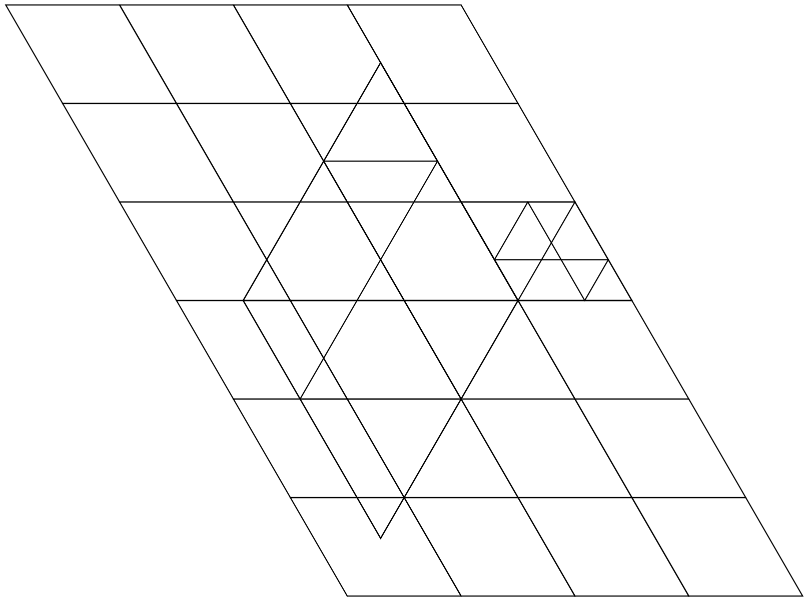


Figure 7: 3-fold sofic case

Example 11. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/5)$. Let $\beta = (1+\sqrt{5})/2$. There are 40 cylinders.

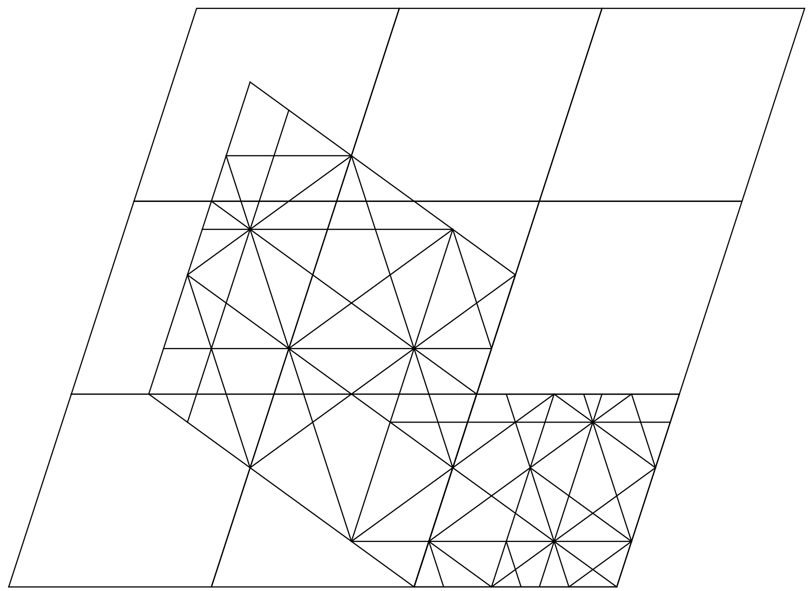


Figure 8: 5-fold sofic case

Example 12. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/7)$. Let $\beta = 1 + 2\cos(2\pi/7) \approx 2.24698$. From $r(\mathcal{L}) = 1/(2\cos(\pi/7))$, $w(\mathcal{X}) = \sin(2\pi/7)$ we have $\beta > B_1 \approx 2.00272$ and there is a unique ACIM by Theorem 3, but $\beta < B_2 \approx 2.41964$. From Theorem 6, we know that the corresponding dynamical system is sofic. Figure 9 shows the sofic dissection of $\mathcal{X}$ by 224 discontinuity segments. The number of cylinders is 3292 (!), computed by Euler's formula.

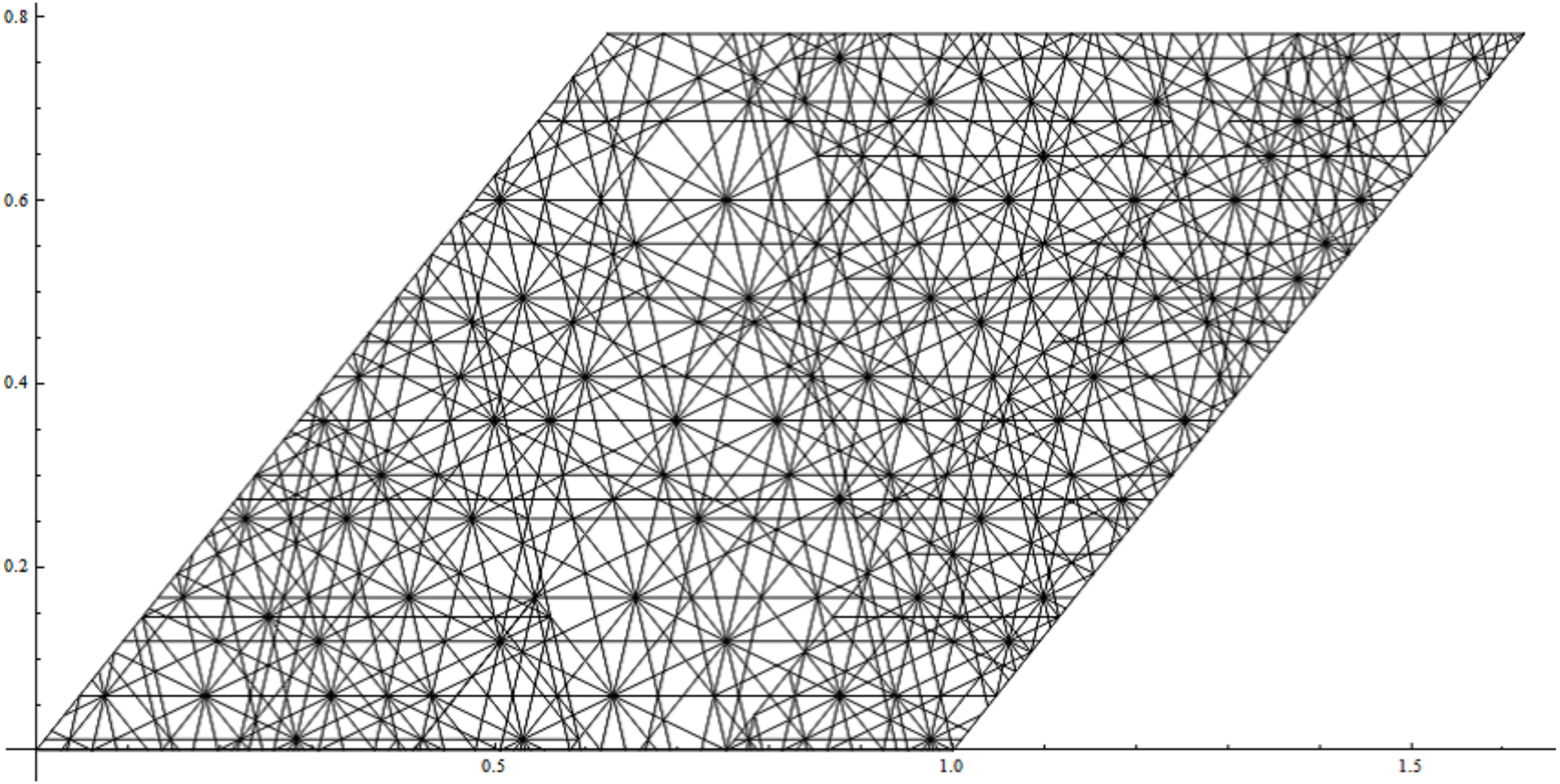


Figure 9: Sofic 7-fold rotation

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