Non-stationary Markov partitions for Pisot cocycles

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The ubiquitous example:

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$
$$\sigma(1) = 12$$

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$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$
$$\sigma^2(1) = 1213$$

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$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$
$$\sigma^3(1) = 1213121$$

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$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$
$$\sigma^4(1) = 1213121121312$$

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$$\sigma^5(1) = 1213121121312121312112 13$$

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$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

$$\sigma^{\infty}(1) = 1213121121312121312112 13 \cdots \in \{1,2,3\}^{\mathbb{N}}$$

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$$M_\sigma = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right), \quad f(x) = x^3 - x^2 - x - 1$$

$\beta > 1$ Pisot root of $f(x) : |\beta'| < 1,\ \forall\, \beta'$ Galois conjugate of β

σ is an irreducible unimodular **Pisot** substitution.

- S = set of unimodular substitutions on $\mathcal{A} = \{1, \ldots, d\}$.
- $\boldsymbol{\sigma} = \cdots \sigma_{-2}\sigma_{-1}.\sigma_0\sigma_1 \cdots \in S^{\mathbb{Z}}$.
- $\mathcal{L}_{\boldsymbol{\sigma}}^{(m)}$ language of $\boldsymbol{\sigma}$: set of factors of $\sigma_{[m,n)}(i)$ for $i \in \mathcal{A}$, $n \in \mathbb{N}$.
- $\Sigma : (w_n)_{n\in\mathbb{Z}} \mapsto (w_{n+1})_{n\in\mathbb{Z}}$ shift.

$\boldsymbol{\sigma}$-shift

Shift space $(X_{\boldsymbol{\sigma}}, \Sigma)$ where $X_{\boldsymbol{\sigma}}$ is the set of bi-infinite words ω such that each factor is in $\mathcal{L}_{\boldsymbol{\sigma}} = \mathcal{L}_{\boldsymbol{\sigma}}^{(0)}$.
$\rightarrow$ usually entropy zero, pure point spectrum?

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Renormalization

(D, Σ, ν), $D \subseteq S^{\mathbb{Z}}$ sofic shift.

$$\boldsymbol{\sigma} = \cdots \sigma_{-2}\sigma_{-1}.\sigma_0\sigma_1 \cdots \in D$$

$\rightarrow$ hyperbolic system, action of toral automorphism

Sturmian word

A word $u \in \{0,1\}^{\mathbb{N}}$ is Sturmian if equivalently

1. its complexity function satisfies $p_u(n) = n + 1$.
2. it is a non eventually periodic 1-balanced word.
3. it is a natural coding of an irrational rotation $R_\alpha : \mathbb{T} \to \mathbb{T}$, $x \mapsto x + \alpha \bmod 1$.

Natural coding: $u = \mathcal{P}(x, R_\alpha)$ for $\mathcal{P} = \{[0, 1-\alpha), [1-\alpha, 1)\}$.

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Sturmian words are intimately related to continued fractions.

$$\sigma_0 : 0 \mapsto 0, 1 \mapsto 10, \qquad \sigma_1 : 0 \mapsto 01, 1 \mapsto 1$$
$$M_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \qquad M_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$$

- Derivation: $u = \sigma_{i_1} \cdots \sigma_{i_n}(u_n)$, where u_n is again Sturmian.

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- Write $i_1 i_2 \cdots = 0^{a_1} 1^{a_2} 0^{a_3} \cdots$ and set $\alpha = [a_1, a_2, a_3, \ldots]$.

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- Rauzy induction: renormalization procedure of R_α on $[-1, \alpha)$. Induced rotation $\leftrightarrow$ Gauss map $\alpha \mapsto \{1/\alpha\}$.

Random dynamical system (Arnold)

- (D, Σ, ν), $D \subset S^{\mathbb{Z}}$ sofic shift.

$$\boldsymbol{\sigma} = \cdots \sigma_{-2}\sigma_{-1}.\sigma_0\sigma_1 \cdots$$

- $A : D \to \mathsf{GL}(d, \mathbb{Z})$, $\boldsymbol{\sigma} \mapsto A(\boldsymbol{\sigma}) = M_0^{-1}$.

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- Renormalization cocycle:

$$F : D \times \mathbb{T}^d \to D \times \mathbb{T}^d, \quad (\boldsymbol{\sigma}, w) \mapsto (\Sigma\boldsymbol{\sigma}, M_0^{-1} w)$$

and $F^n(\boldsymbol{\sigma}, w) = (\Sigma^n \boldsymbol{\sigma}, A^n(\boldsymbol{\sigma}) w)$ for $n \in \mathbb{Z}$, where

$$A^n(\boldsymbol{\sigma}) = \begin{cases} A(\Sigma^{n-1}\boldsymbol{\sigma}) \cdots A(\Sigma\boldsymbol{\sigma})A(\boldsymbol{\sigma}) = (M_0 \cdots M_{n-1})^{-1}, & \text{if } n > 0, \\ \mathrm{Id}, & \text{if } n = 0, \\ A(\Sigma^n\boldsymbol{\sigma})^{-1} \cdots A(\Sigma^{-1}\boldsymbol{\sigma})^{-1} = M_n \cdots M_{-1}, & \text{if } n < 0. \end{cases}$$

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- Cocycle fibers: $\{\boldsymbol{\sigma}\} \times \mathbb{T}^d$
- Orbits $\rightarrow$ Mapping families (Arnoux-Fisher 05)

$$\cdots \xrightarrow{M_{-2}^{-1}} \mathbb{T}^d_{-1} \xrightarrow{M_{-1}^{-1}} \mathbb{T}^d_0 \xrightarrow{M_0^{-1}} \mathbb{T}^d_1 \xrightarrow{M_1^{-1}} \cdots$$

$\mathbf{T} = \coprod_{n\in\mathbb{Z}} \mathbb{T}^d_n$
$f_{\boldsymbol{\sigma}} : \mathbf{T} \rightarrow \mathbf{T}$, $f_{\boldsymbol{\sigma}}(x) = M_n^{-1}(x)$ for $x \in \mathbb{T}^d_n$.
$(\mathbf{T}, f_{\boldsymbol{\sigma}})$ = mapping family

- Nature of the mapping family, e.g. hyperbolicity?
- Tool: Oseledets' multiplicative ergodic theorem

For ν-a.e. sequence $\boldsymbol{\sigma} \in D$ we have the splitting

$$\mathbb{R}^d = E_1(\boldsymbol{\sigma}) \oplus \cdots \oplus E_p(\boldsymbol{\sigma}), \quad \dim(E_i(\boldsymbol{\sigma})) = d_i$$

- Splitting is invariant: $A^n(\boldsymbol{\sigma})E_i(\boldsymbol{\sigma}) = E_i(\Sigma^n\boldsymbol{\sigma})$, $n \in \mathbb{Z}$.
- Dynamical characterisation:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|A^n(\boldsymbol{\sigma})x\| = \theta_i \quad \Leftrightarrow \quad x \in E_i(\boldsymbol{\sigma}) \setminus \{\mathbf{0}\}$$

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Lyapunov exponents

$$\theta_1 + \cdots + \theta_k = \lim_{n\to\infty} \frac{1}{n} \log\|\wedge^k A^n(\boldsymbol{\sigma})\|, \quad k = 1, \ldots, d$$

Lyapunov spectrum: $\{(\theta_i, d_i) : i = 1, \ldots, p\}$.

Hyperbolic cocycle: $\theta_i \neq 0$, $\forall\, i$

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Pisot cocycle: $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_{d-1} > 0 > \theta_d$

The PRICE to pay

(D, Σ, ν) ergodic shift with the Pisot condition such that there exists a cylinder of positive measure in D corresponding to a substitution with positive incidence matrix.

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- (C) C-balancedness: $\exists\, C > 0$ such that for each $\ell \in \mathbb{N}$ the number $n = n(\ell)$ can be chosen such that $\mathcal{L}_{\boldsymbol{\sigma}}^{(n+\ell)}$ is C-balanced.

$$u, v \in \mathcal{L}_{\boldsymbol{\sigma}},\ |u| = |v|, \quad -C \leq |u|_j - |v|_j \leq C, \quad \forall\, j \in \mathcal{A}$$

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- (E) Generalized left eigenvector: $\exists\, \mathbf{v} \in \mathbb{R}^d_{\ge 0} \setminus \{\mathbf{0}\}$ such that $(\mathbf{T}, f_{\boldsymbol{\sigma}})$ is Anosov in the past.

Stable and unstable spaces

For ν-almost every $\boldsymbol{\sigma} \in D$ the Oseldets' splitting is characterised by

$$\underbrace{E_1(\Sigma^n \boldsymbol{\sigma}) \oplus \cdots \oplus E_{d-1}(\Sigma^n \boldsymbol{\sigma}) = (\mathbf{v}^{(n)})^\perp}_{(d-1)\text{-dimensional}}, \quad \underbrace{E_d(\Sigma^n \boldsymbol{\sigma}) = \mathbb{R}\mathbf{u}^{(n)}}_{\text{one-dimensional}}$$

where

$$\mathbb{R}_+ \mathbf{u}^{(n)} = \bigcap_{k \geq n} M_{[n,k)} \mathbb{R}^d_+, \qquad \mathbb{R}_+ \mathbf{v}^{(n)} = \bigcap_{k \leq n} {}^t(M_{[k,n)}) \mathbb{R}^d_+$$

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- $\mathbf{u}^{(n)}$ and $\mathbf{v}^{(n)}$ exist by (P), (R) and (E).
- We have

$$\lim_{n \to -\infty} \| M_{[n,0)} \mathbf{x} \| = +\infty, \qquad \lim_{n \to \infty} (M_{[0,n)})^{-1} \mathbf{x} = \mathbf{0}, \qquad \forall\, \mathbf{x} \in \mathbb{R}\mathbf{u} \setminus \{\mathbf{0}\}$$

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Anosov property for both past and future!

Action of Σ^n on $\boldsymbol{\sigma}$ translates to

$$(\mathbf{u}, \mathbf{v}) \mapsto ((M_{[0,n)})^{-1}\mathbf{u}, {}^t(M_{[0,n)})\mathbf{v}) = (\mathbf{u}^{(n)}, \mathbf{v}^{(n)})$$

Two dual CF algorithms happening on these vectors.

$S = \{\beta_1, \beta_2, \beta_3\}$, where

$$\beta_1 : \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 23 \end{cases} \qquad \beta_2 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{cases} \qquad \beta_3 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}$$

with incidence matrices

$$M_{\beta_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{\beta_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad M_{\beta_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Brun algorithm

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Linear version defined on $B = \{(w_1, w_2, w_3) \in \mathbb{P}(\mathbb{R}^3_+) : w_1 < w_2 < w_3\}$

$$(w_1, w_2, w_3) \mapsto \mathrm{sort}(w_1, w_2, w_3 - w_2)$$

Let $B_i = M_{\beta_i} B \subset B$, $\mathbf{w}^{(0)} = (w_1^{(0)}, w_2^{(0)}, w_3^{(0)})$

$$\mathbf{w}^{(n-1)} \mapsto \mathbf{w}^{(n)} = M_{\beta_i}^{-1} \mathbf{w}^{(n-1)}, \quad \text{for } \mathbf{w}^{(n-1)} \in B_i$$

For $\boldsymbol{\sigma}$ satisfying PRICE define

$\mathcal{R}_{\mathbf{w}}^{(n)} = \bigcup_{i \in \mathcal{A}} \mathcal{R}_{\mathbf{w}}^{(n)}(i) \subset (\mathbf{w}^{(n)})^{\perp}$

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$\mathbf{l} : \mathcal{A}^* \to \mathbb{R}^d$, $i \mapsto \mathbf{e}_i$ abelianization.
$\omega^{(0)} = \sigma_{[0,n)}(\omega^{(n)})$ desubstitution.

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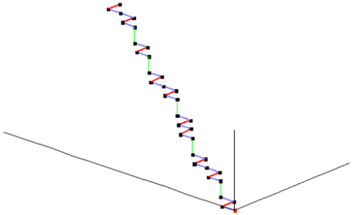
$S = \{\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1\}$.

M_σ-invariant decomposition: $\mathbb{R}^3 = E^u \oplus E^s \cong \mathbb{R} \oplus \mathbb{C}$.

Broken line (balanced): $\sigma^\infty(1) = 1213121121312121312112 13\cdots$.



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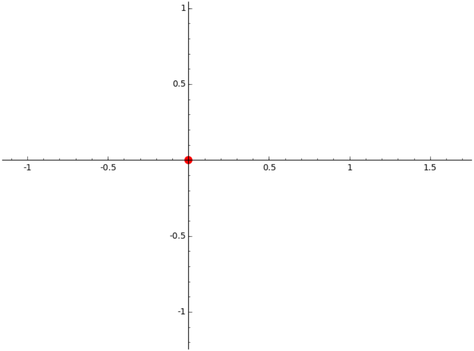

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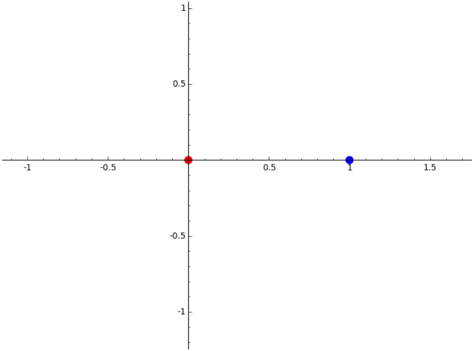

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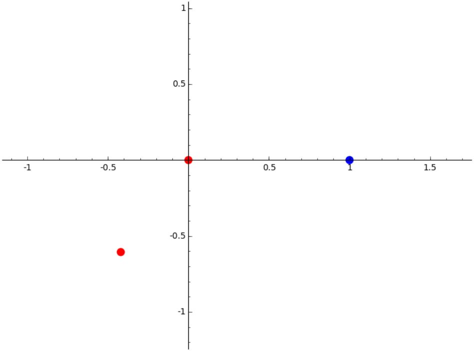

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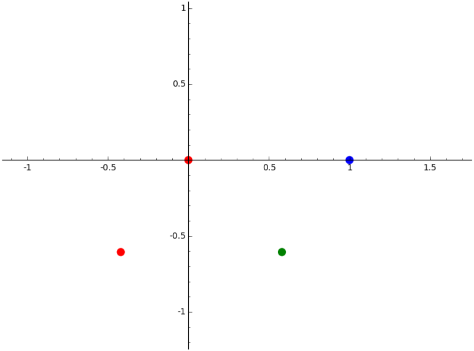

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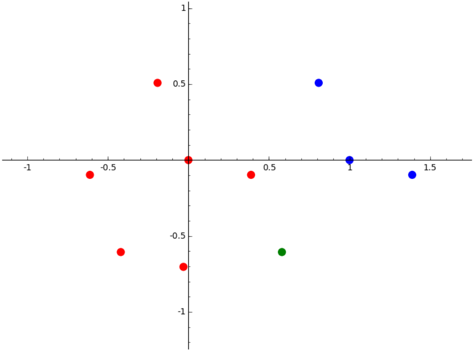

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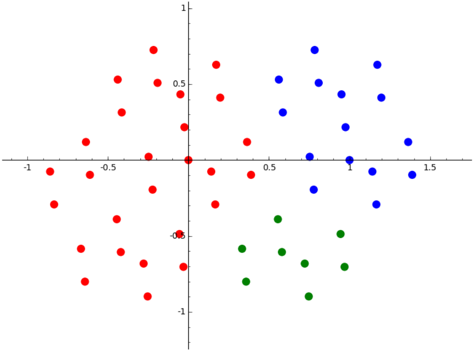

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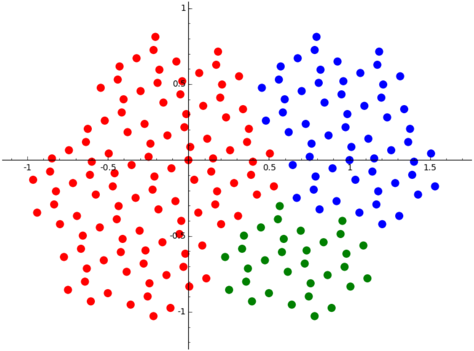

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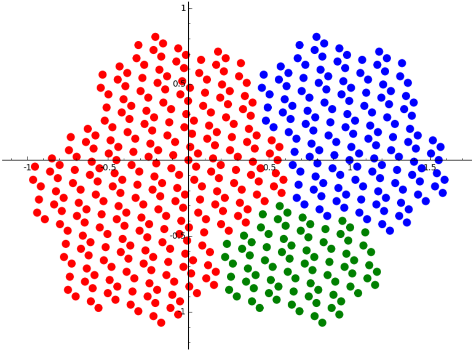

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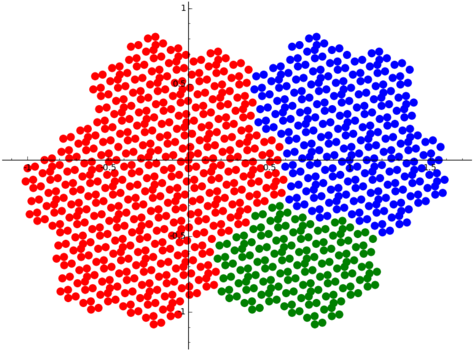

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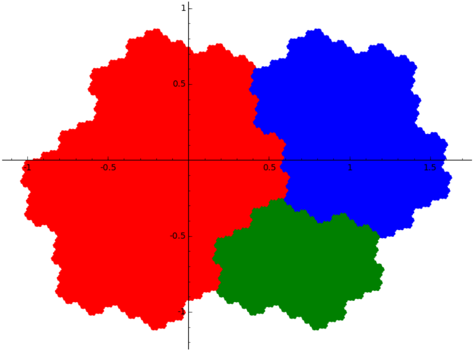

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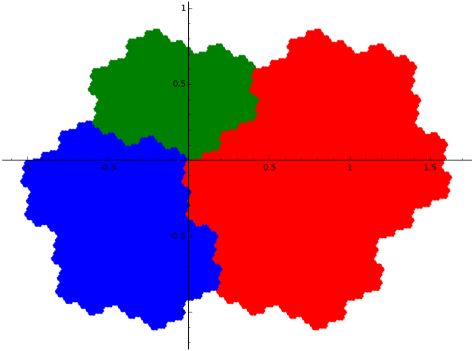

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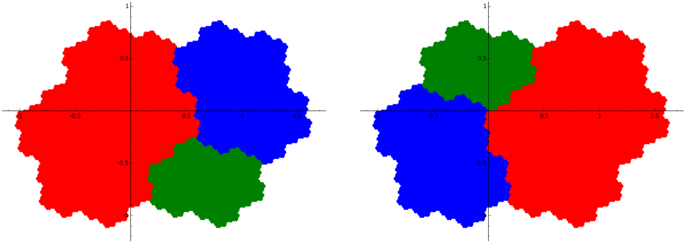

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Results of [Berthé-Steiner-Thuswaldner 14]:

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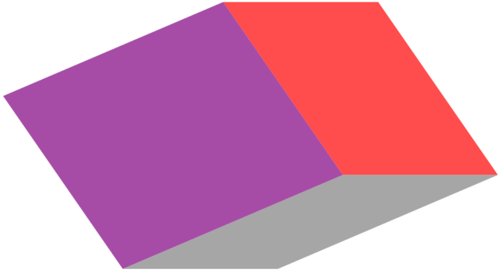
Dual action on $(d-1)$-dimensional faces:

$$\begin{aligned} E_1^*(\sigma) : [\mathbf{x}, 1] &\mapsto [M_\sigma^{-1}\mathbf{x}, 1] \cup [M_\sigma^{-1}\mathbf{x}, 2] \cup [M_\sigma^{-1}\mathbf{x}, 3] \\ [\mathbf{x}, 2] &\mapsto [M_\sigma^{-1}(\mathbf{x} + \pi(\mathbf{e}_1)), 1] \\ [\mathbf{x}, 3] &\mapsto [M_\sigma^{-1}(\mathbf{x} + \pi(\mathbf{e}_1)), 2] \end{aligned}$$



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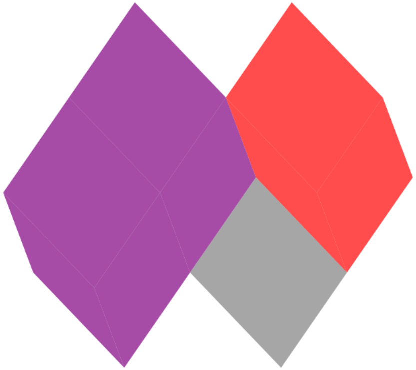

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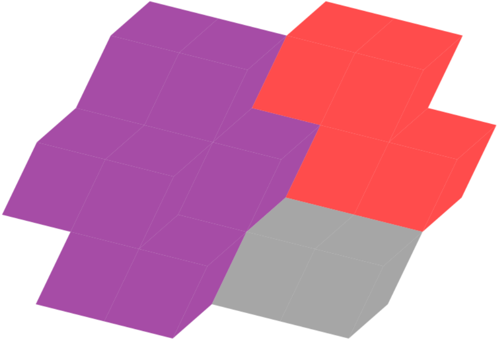

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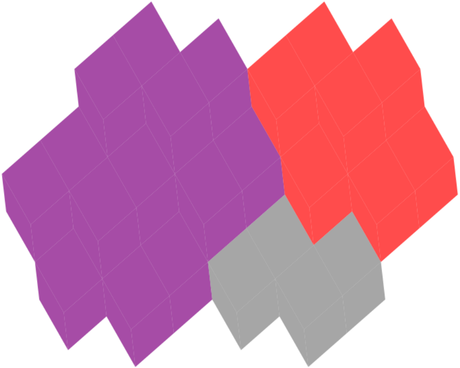

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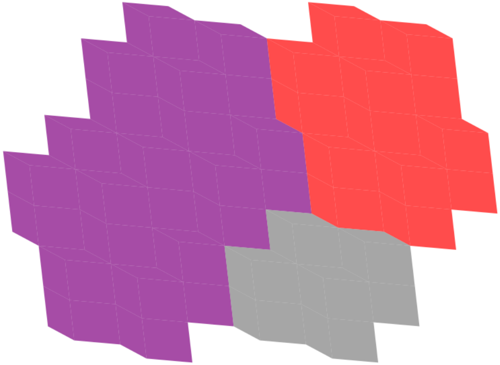

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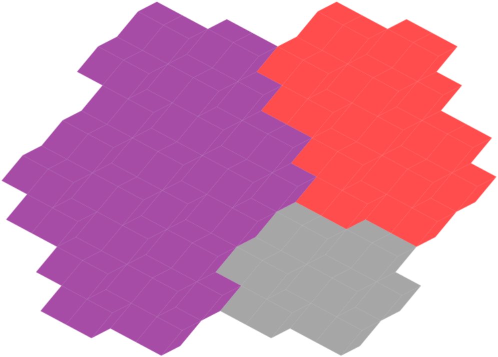

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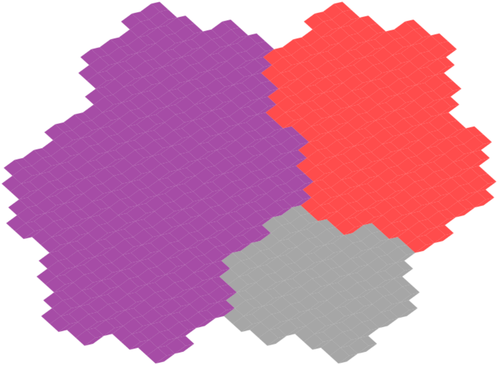

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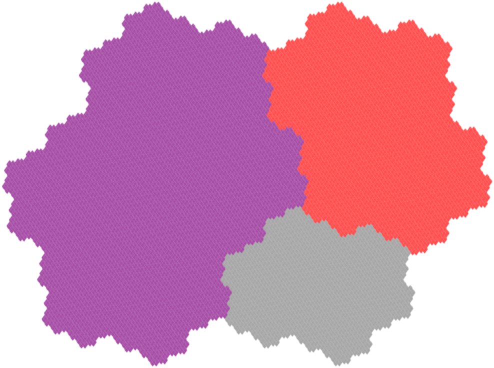

$$\mathcal{R}(i) = \lim_{n\to\infty} \pi(M_\sigma^n E_1^*(\sigma)^n([\mathbf{0}, i]))$$

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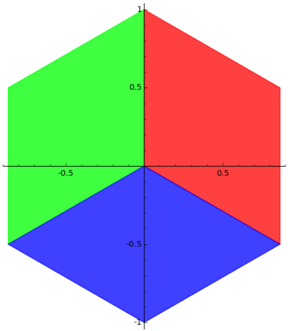

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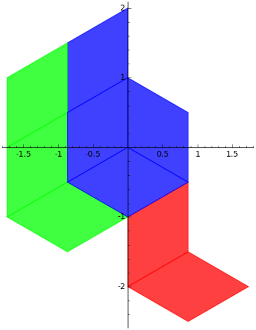

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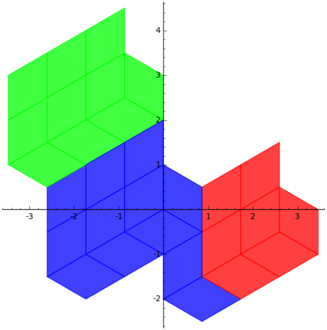

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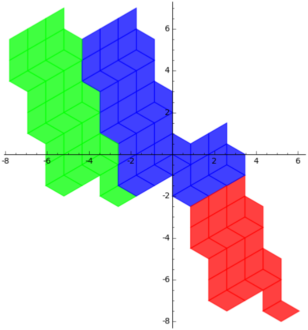

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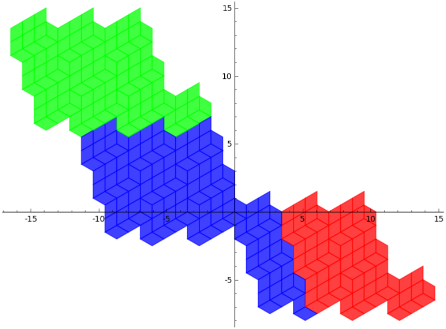

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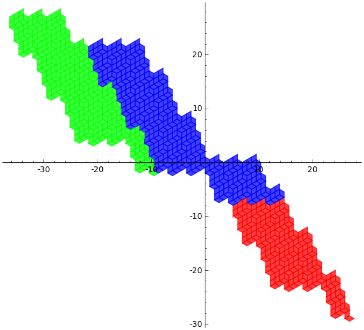

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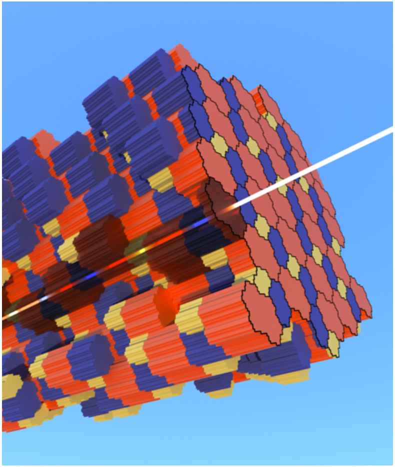

$\mathcal{P}_n = \{\mathrm{int}(\widehat{\mathcal{R}}^{(n)}(i)) \bmod \mathbb{Z}^d : i \in \mathcal{A}\}$ topological partition of $\mathbb{T}^d$.

Theorem [Arnoux, Berthé, M., Steiner, Thuswaldner]

$\mathcal{P}_n$ forms a non-stationary Markov partition for the mapping family $(\mathbf{T}, f_{\boldsymbol{\sigma}})$ associated with $\boldsymbol{\sigma}$.

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Theorem [Arnoux, Berthé, M., Steiner, Thuswaldner]

$\mathcal{P}_n$ forms a non-stationary Markov partition for the mapping family $(\mathbf{T}, f_{\boldsymbol{\sigma}})$ associated with $\boldsymbol{\sigma}$.

Proof.

- Each member of the partition has a pair of horizontal and vertical transverses.
- Actions of M_n^{-1} on $h_{n,i}(\mathbf{x})$ and of M_n on $v_{n,i}(\mathbf{x})$ are inflations.
- We are mod $\mathbb{Z}^d$ $\to$ cut-and-stack process
- Markov property: $v_{n,i}(M_n\mathbf{x}) \subset M_n v_{n+1,j}(\mathbf{x})$, $h_{n+1,j}(M_n^{-1}\mathbf{x}) \subset M_n^{-1} h_{n,i}(\mathbf{x})$ □

Subshifts of finite type

$\mathcal{P}_n$ is not generating in general: the "rectangles" $\mathrm{int}(\widehat{\mathcal{R}}^{(n)}(i))$ are not sufficiently small, the image of one of the could wrap around in the torus.

Solution: take the first subdivision of the rectangles $\widehat{\mathcal{P}}_n$ according to the set equation.

With $\widehat{\mathcal{P}}_n$ generating define the transition matrices

$$A_{ij}^{(n)} = \begin{cases} 1 & \text{if} \quad \mathrm{int}\big(\widehat{\mathcal{R}}^{(n+1)}(i)\big) \cap M_n^{-1}\mathrm{int}\big(\widehat{\mathcal{R}}^{(n)}(j)\big) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The map $\pi : (\Omega_A, \Sigma) \to (\mathbf{T}^d, f_\sigma)$ defined by $(x_n) \mapsto \bigcap_{n\in\mathbb{Z}} M_n^{-1}\widehat{\mathcal{R}}^{(n)}(x_n)$ is one-to-one except on the set of boundary pullbacks.

$$\begin{array}{ccccccc} \Omega_A^{-1} & \xrightarrow{\Sigma} & \Omega_A^{0} & \xrightarrow{\Sigma} & \Omega_A^{1} & \xrightarrow{\Sigma} & \Omega_A^{2} \\ \cdots\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow\cdots \\ \mathbb{T}^d_{-1} & \xrightarrow{M_{-1}^{-1}} & \mathbb{T}^d_0 & \xrightarrow{M_0^{-1}} & \mathbb{T}^d_1 & \xrightarrow{M_1^{-1}} & \mathbb{T}^d_2 \end{array}$$

- Oseledets' splitting:

$$E_1(\Sigma^n\boldsymbol{\sigma}) \oplus \cdots \oplus E_d(\Sigma^n\boldsymbol{\sigma}) = \mathbb{R}\mathbf{u}_1^{(n)} \oplus \cdots \oplus \mathbb{R}\mathbf{u}_d^{(n)}$$

  Renormalize $\|\mathbf{u}_1^{(n)}\| = 1$, $\langle \mathbf{u}_1^{(n)}, \mathbf{v}^{(n)} \rangle = 1$, where $(\mathbf{v}^{(n)})^\perp = \operatorname{span}(\mathbf{u}_2^{(n)}, \ldots, \mathbf{u}_d^{(n)})$.

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- Then

$$M_n^{-1} B_n \text{diag}(\lambda_1^{(n)}, \ldots, \lambda_d^{(n)}) = B_{n+1}$$

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- The B_n's are cross-sections of the Weyl chamber flow

$$\mathrm{diag}(e^{t_2 + \cdots + t_d}, e^{-t_2}, \ldots, e^{-t_d})$$

- $(S^{\mathbb{Z}}, \Sigma, \nu)$ satisfies the Pisot condition [Avila, Delecroix 15]
- $\mathcal{R} + \Lambda$ tiling [Berthé-Bourdon-Jolivet-Siegel 14] $\Rightarrow \widehat{\mathcal{R}} + \mathbb{Z}^d$ tiling. The Rauzy suspensions are fundamental domains of $\mathbb{T}^d$.
- For ν-a.e. every $\boldsymbol{\sigma} \in S^{\mathbb{Z}}$ words in $X_{\boldsymbol{\sigma}}$ are natural codings of a translation on $\mathbb{T}^2$.

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- Set equations for β_3:

$$\begin{aligned}
M_{\beta_3}^{-1}\mathcal{R}^{(0)}(1) &= \mathcal{R}^{(1)}(1),\\
M_{\beta_3}^{-1}\mathcal{R}^{(0)}(2) &= \mathcal{R}^{(1)}(2),\\
M_{\beta_3}^{-1}\mathcal{R}^{(0)}(3) &= \mathcal{R}^{(1)}(3) \cup (\mathcal{R}^{(1)}(2) + \pi^{(1)} M_{\beta_3}^{-1}\mathbf{e}_2),
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- From the cut-and-stack

$$(m_1, m_2, m_3) \mapsto (m_1, m_2, m_3 - m_2) \qquad \text{action of } M_{\beta_3}^{-1}$$
$$(h_1, h_2, h_3) \mapsto (h_1, h_2 + h_3, h_3) \qquad \text{action of } {}^t M_{\beta_3}$$

$\rightarrow$ Natural extension for Brun CF algorithm

Thanks for the attention !