Finite β-expansions and bounded remainder sets

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Let $\beta > 1$ be a real number. The β-transformation is defined by

$$T_\beta : x \mapsto \beta x - \lfloor \beta x \rfloor \ ,$$

where $\lfloor x \rfloor$ is the largest integer not exceeding x. By iterating this map and taking $\epsilon_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$, we obtain the greedy expansion of x:

$$x = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots = 0.\epsilon_1\epsilon_2\epsilon_3\ldots \ .$$

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- $\mathrm{Fin}(\beta)$ denotes the set consisting of all finite β-expansions.

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$$a_1 > a_2 + \cdots + a_m$$

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- GOAL: Give a sufficient condition for (F)

Let $G = (G_n)_{n\geq 0}$ be an increasing sequence of positive integers with $G_0 = 1$. Then every positive integer can be expanded in the following way

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) G_k \ ,$$

where $\varepsilon_k(n) \in \{0, \ldots, \lceil G_{k+1}/G_k \rceil - 1\}$ and $\lceil x \rceil$ denotes the smallest integer not less than $x \in \mathbb{R}$.

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$\mathcal{K}_G$ the subset of sequences satisfying (1) and its elements are called G-admissible.

Let $(G_n)_{n\in\mathbb{N}}$ be generated by a finite linear recurrence of order $d+1$

$$G_{n+d+1} = a_0 G_{n+d} + a_1 G_{n+d-1} + \cdots + (a_d + 1) G_n, \quad n \geq 0 \ ,$$

with positive coefficients and initial values

$$G_0 = 1 \ , \qquad G_{n+1} = \sum_{k=0}^{n} a_{n-k} G_k + 1 \ .$$

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Hypothesis B (Grabner-Liardet-Tichy 1995)

There exists an integer $b > 0$ such that for all k and

$$N = \sum_{i=0}^{k} \epsilon_i(N) G_i + \sum_{j=k+b+2}^{\infty} \epsilon_j(N) G_j,$$

the addition of G_m to N, where $m \geq k+b+2$, does not change the first $k+1$ digits in the greedy representation i.e.

$$N + G_m = \sum_{i=0}^{k} \epsilon_i(N) G_i + \sum_{j=k+1}^{\infty} \epsilon_j(N + G_m) G_j.$$

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$$n = \sum_{j \geq 0} \epsilon_j(n) G_j$$

be the G-expansion of an integer n. We define the β-adic Monna map $\phi_\beta : \mathcal{K}_G \to \mathbb{R}^+$ as

$$\phi_\beta(n) = \phi_\beta \left(\sum_{j \geq 0} \epsilon_j(n) G_j \right) = \sum_{j \geq 0} \epsilon_j(n) \beta^{-j-1} \ ,$$

where β is the Perron root of the characteristic polynomial

$$x^d = a_0 x^{d-1} + \ldots + a_{d-1}$$

associated to the numeration system G.

Theorem

Hypothesis B $\Longrightarrow$ *(F)*

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$$y = \epsilon_{-n}\beta^{-n} + \cdots + \epsilon_0 + \epsilon_1 + \cdots + \epsilon_k\beta^k \ , \quad \epsilon_i \in \mathbb{Z}_+, \epsilon_{-n} \neq 0$$

be the minimal element in $\mathbb{Z}_+[\beta^{-1}]$ such that $y \notin \mathrm{Fin}(\beta)$. This implies

$$\begin{aligned} x = y - 1 &= \epsilon_{-n}\beta^{-n} + \cdots + (\epsilon_0 - 1) + \epsilon_1 + \cdots + \epsilon_k\beta^k \\ &= \delta_{-m}\beta^{-m} + \cdots + \delta_0 + \cdots + \delta_l\beta^l \in \mathrm{Fin}(\beta) \ . \end{aligned}$$

Take $N = \beta^m x$. Then $N = (\eta_0 \ldots \eta_m \ldots \eta_{m+l})$.

Wlog, we can assume that there exists $b > 0$ such that

$$N = (\eta_0 \dots \eta_k 0^{(b+1)} \eta_{k+b+2} \dots \eta_{m+l}) \ .$$

Then for Hypothesis B the addition by G_m does not affects the first k digits of N, but

$$N + G_m = \beta^m x + \beta^m = \beta^m ((\eta_0 + 1)\eta_1 \dots \eta_k 0^{(b+1)} \eta_{k+b+2} \dots \eta_{m+l}) \ ,$$

leading to a contradiction.

Theorem (Hofer-I.-Tichy)

Let $G^1, \ldots, G^s$ be numeration systems given by

$$G^1_{n+d} = b_1(G_{n+d-1} + \cdots + G_n), \quad n \geq d \ ,$$

$$G^2_{n+d} = b_2(G_{n+d-1} + \cdots + G_n), \quad n \geq d \ ,$$

$$\vdots$$

$$G^s_{n+d} = b_s(G_{n+d-1} + \cdots + G_n), \quad n \geq d \ ,$$

with pairwise coprime, positive integers b_i. Furthermore let $\frac{\beta_i^k}{\beta_j^l} \notin \mathbb{Q}$, for all $l, k \in \mathbb{N}$, where $\beta_1, \ldots, \beta_s$ denote the characteristic roots of the numerations systems. Then

$$((\mathcal{K}_{G^1}, \tau_1) \times \ldots \times (\mathcal{K}_{G^s}, \tau_s)) \ ,$$

is uniquely ergodic.

- The $\boldsymbol{\beta}$-adic Halton sequence is given as

$$(\phi_{\boldsymbol{\beta}}(n))_{n\in\mathbb{N}} = (\phi_{\beta_1}(n), \dots, \phi_{\beta_s}(n))_{n\in\mathbb{N}} \ ,$$

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- A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $[0,1)^s$ is u. d. mod 1 if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_I(\mathbf{x}_n) = \lambda_s(I)$$

for all s-dimensional intervals $I \subseteq [0,1)^s$.

- The β-adic Halton sequence is given as

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- A natural measure of the uniformity of a finite sequence $(\mathbf{x}_1, \ldots \mathbf{x}_N)$ is the discrepancy, defined by

$$D_N = D_N(\mathbf{x}_n) = D_N(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sup_{I\subset[0,1)^s} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_I(\mathbf{x}_n) - \lambda_s(I) \right| .$$

- W. M. Schmidt in 1974 showed that, for any sequence, the discrepancy can never remain bounded as $N \to \infty$.

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- Steiner in 2006 proved that if β is a Pisot number with irreducible β-polynomial, then $D(N, [0, y))$ is bounded (in N) for $y \in [0, 1)$ if and only if the β-expansion of y is finite or its tail is the same as that of the expansion of 1 with respect to β.

Theorem (I.-Steiner-Tichy)

The s-dimensional box anchored at the origin $I = \prod_{i=1}^{s}[0, y_i)$ *is a BRS for the* $\boldsymbol{\beta}$*-adic Halton sequence* $(\phi_{\boldsymbol{\beta}}(n))_{n\in\mathbb{N}}$ *if and only if every* y_i *is a* β*-adic rational.*

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Theorem

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure preserving transformation. Then, for any $f \in L^2(\mu)$,

$$\sup_N \left\| \left(\sum_{n=1}^{N} f \circ T^n \right) \right\|_2 < \infty \iff \exists g \in L^2(\mu) \ : \ f = g - g \circ T \in L^2(\mu) \ .$$