Non-stationary Markov Partitions and Brun Continued Fractions

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Admont, FAN days

Markov partitions of toral automorphisms

- We consider a unimodular square matrix of size n with integer entries. It defines an automorphism of the torus $\mathbb{R}^n/\mathbb{Z}^n$.
- An automorphism of the torus is hyperbolic if it has no eigenvalue of modulus 1.
- Markov partitions provide symbolic representations as shifts of finite type.
- Any hyperbolic automorphism of the torus admits a Markov partition [Sinaĭ'68, Bowen'70]
- The boundaries of the sets in a Markov partition for hyperbolic automorphisms of the 3-torus cannot be smooth [Bowen'78, Cawley'91]

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- The boundaries of the sets in a Markov partition for hyperbolic automorphisms of the 3-torus cannot be smooth [Bowen'78, Cawley'91]
- We want to generalize this situation to the nonstationary case for Brun algorithm following the formalism of [Arnoux-Fisher'05]

Topological partition

Let (X, T) be a dynamical system with T invertible

A topological partition of X is a finite collection $(X_i)_{i\in\mathcal{A}}$ of disjoint open sets whose closure covers X

$$X = \bigcup_{i\in\mathcal{A}} \overline{X_i}$$

The bilateral symbolic dynamical system associated with a topological partition is the set Ω_X endowed with the shift map S

$$\Omega_X = \{(\omega_n)_{n\in\mathbb{Z}} \in \{1, \ldots, d\}^{\mathbb{Z}};\ \exists x \in X,\ \forall n \in \mathbb{Z},\ T^n(x) \in X_{\omega_n}\}$$

A partition is generating if

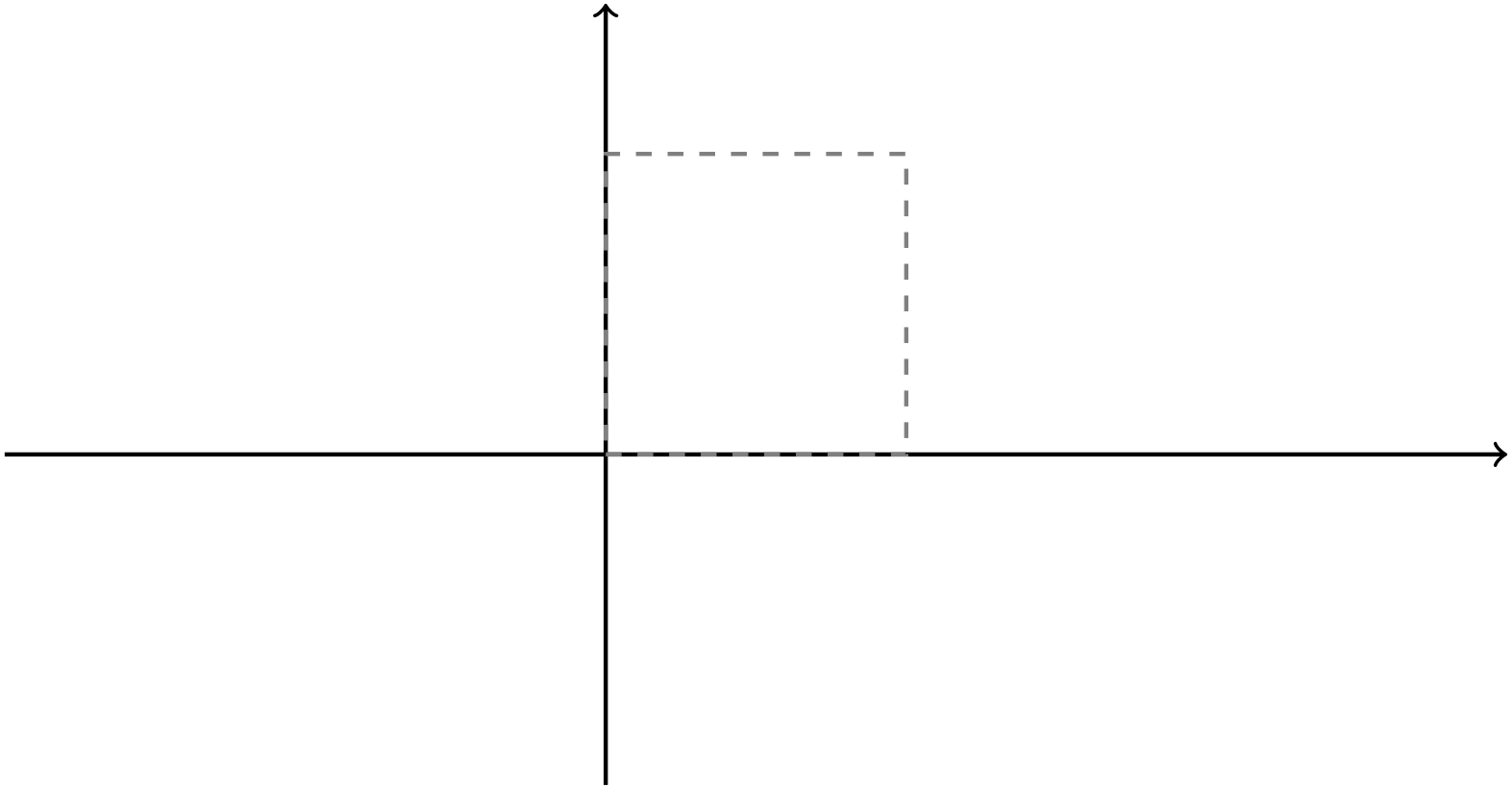
$$\bigcap_{-\infty}^{+\infty} \overline{T^{-k}(X_{\omega_k})}$$

is reduced to a point for $\omega \in \Omega_X$

Markov partition A generating topological partition $(X_1, \ldots, X_d)$ of X is a Markov partition of X if the bilateral symbolic dynamical system (Ω_X, S) is a shift of finite type

A toral automorphism

$$T : \begin{cases} [0,1]^2 & \to & [0,1]^2 \\ \binom{x}{y} & \mapsto & \binom{x}{y} = \binom{x+y}{x} \pmod 1 \end{cases}$$



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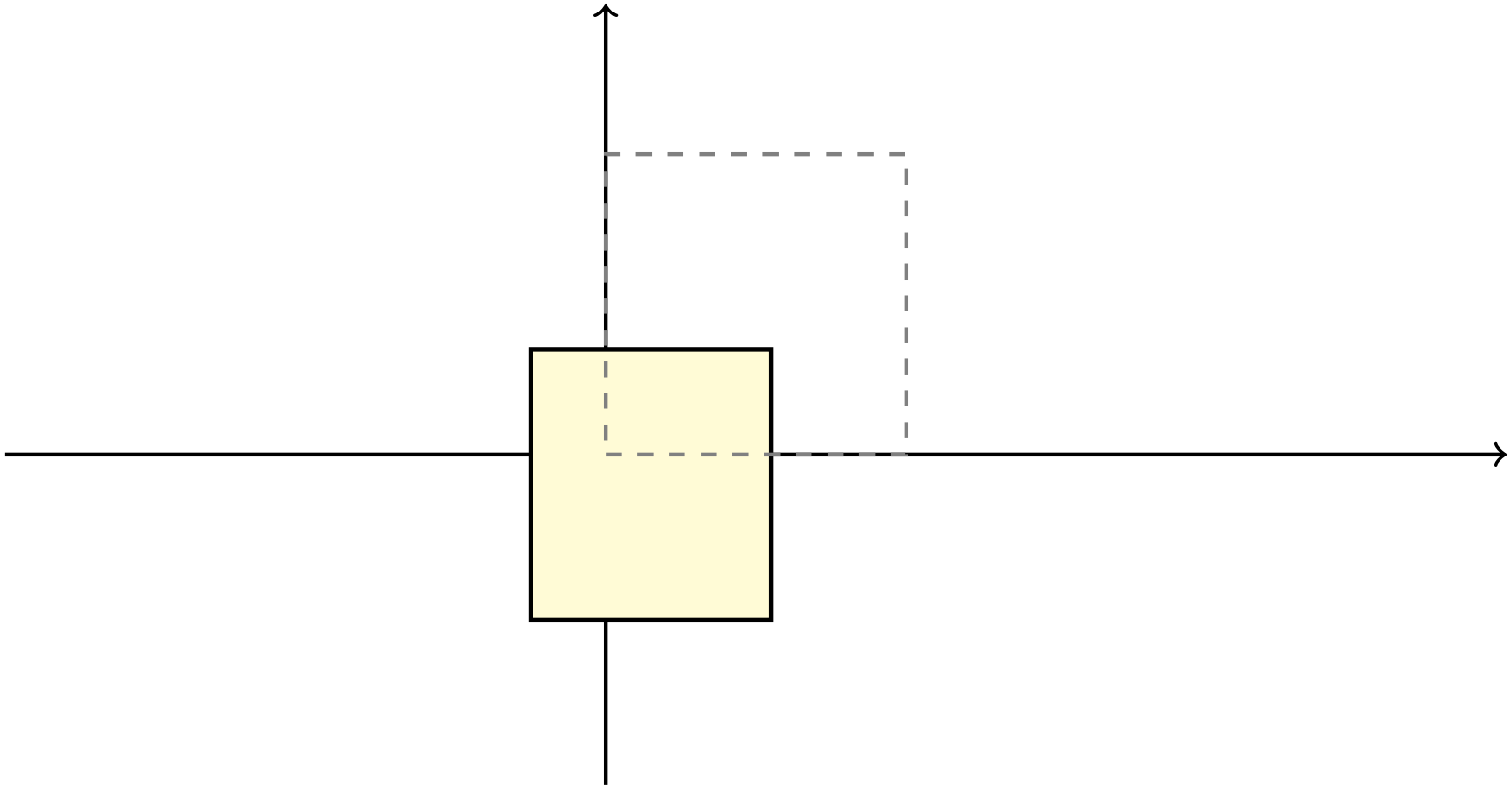

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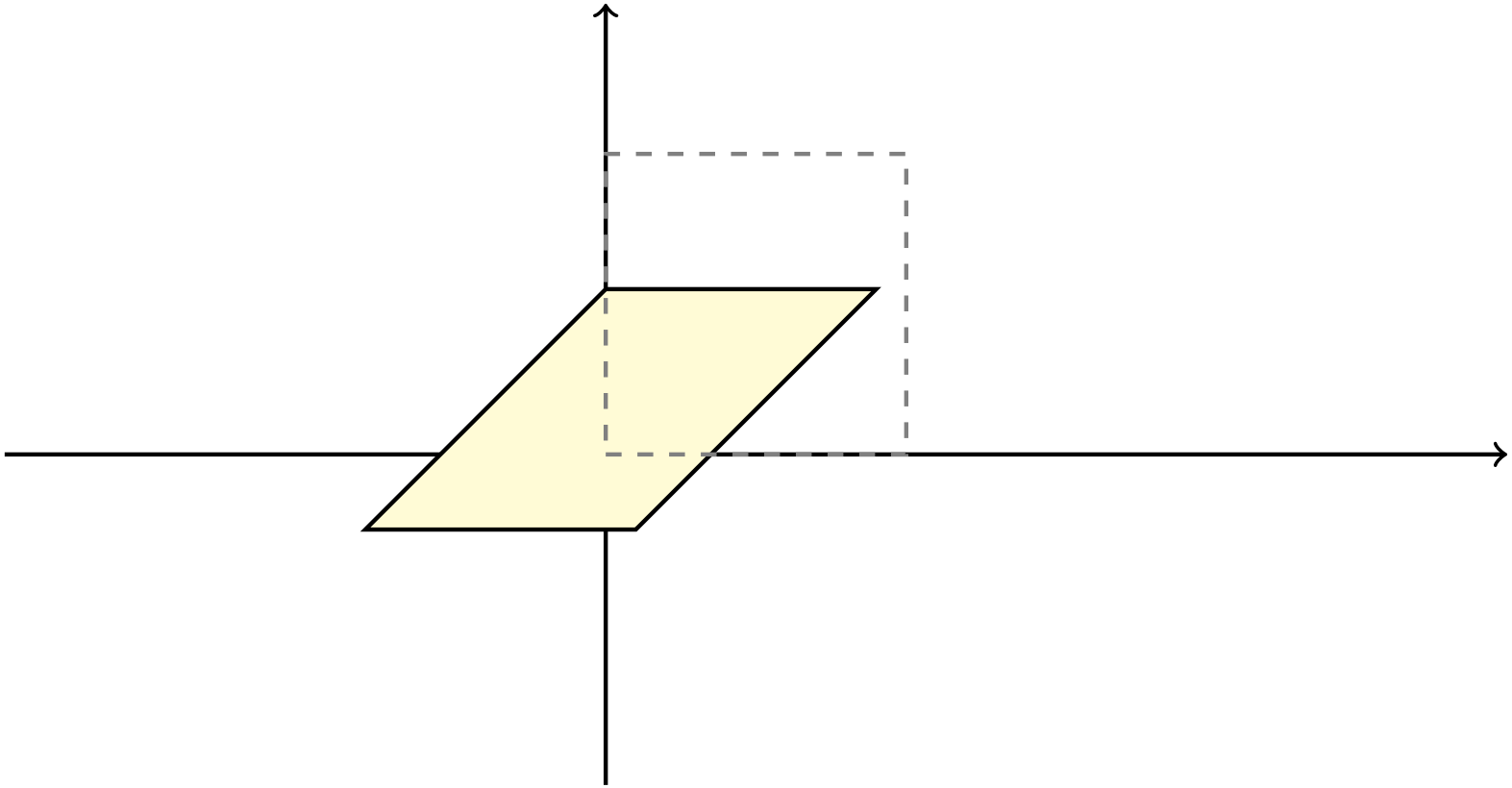

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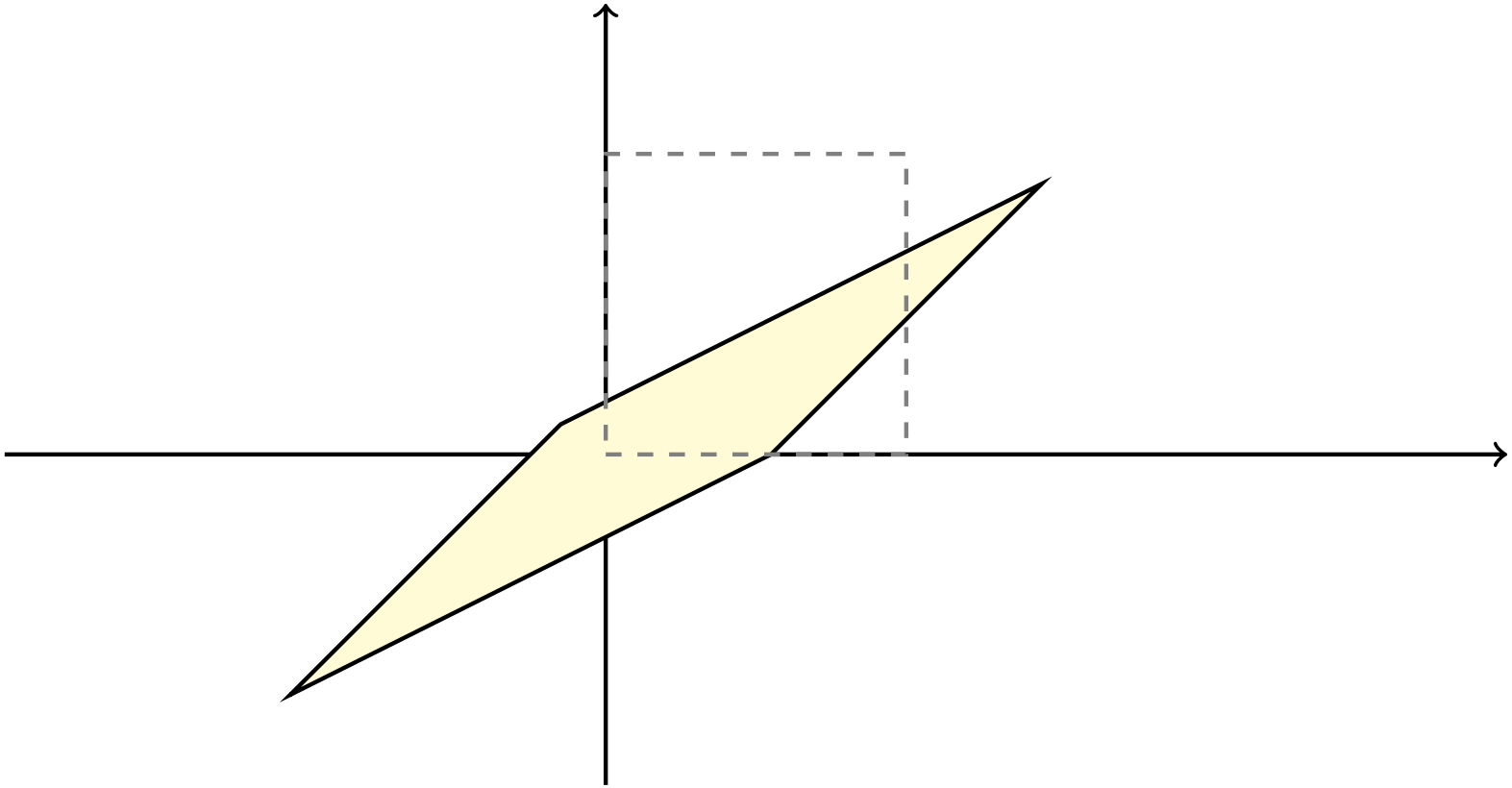

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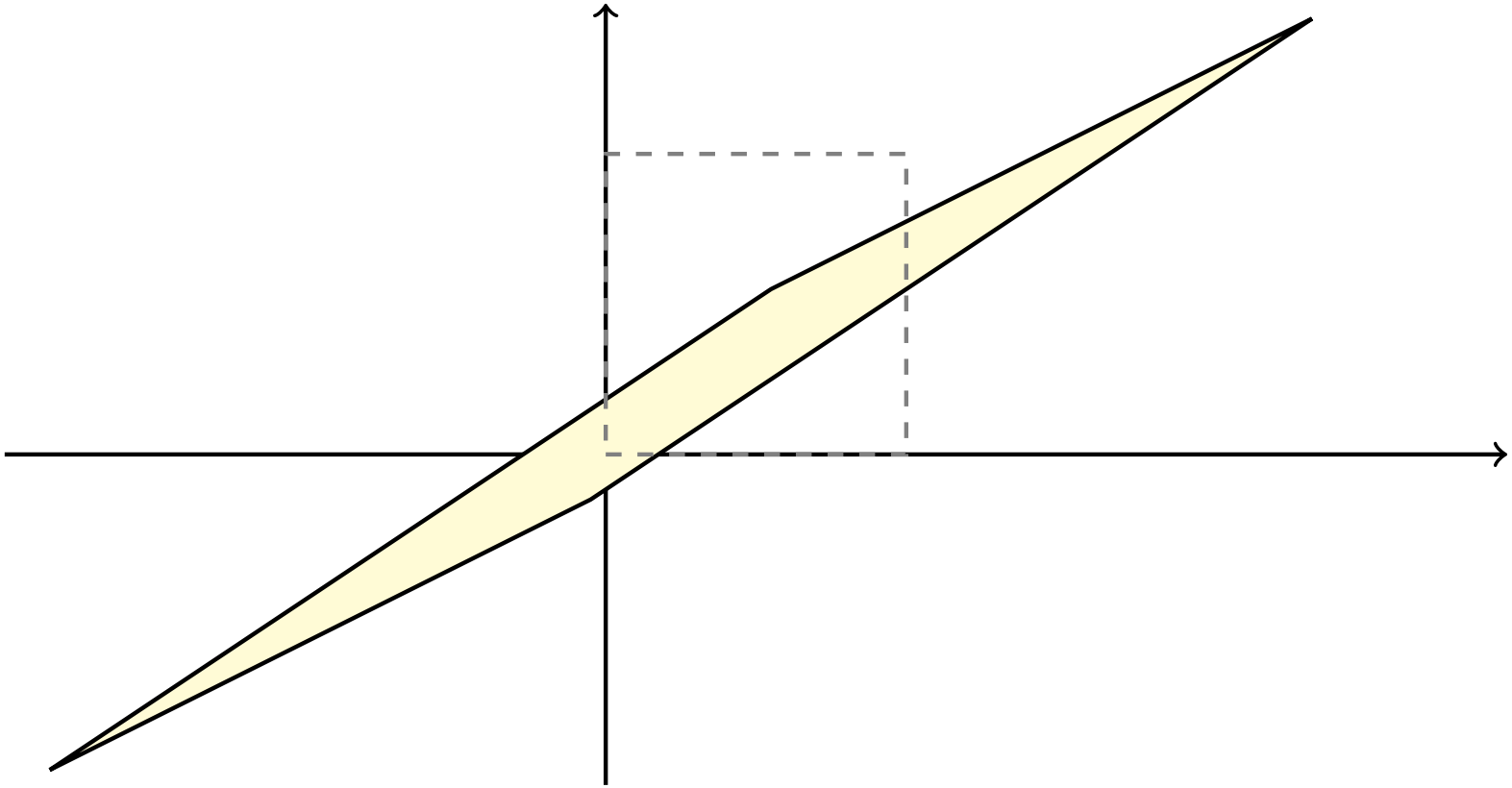

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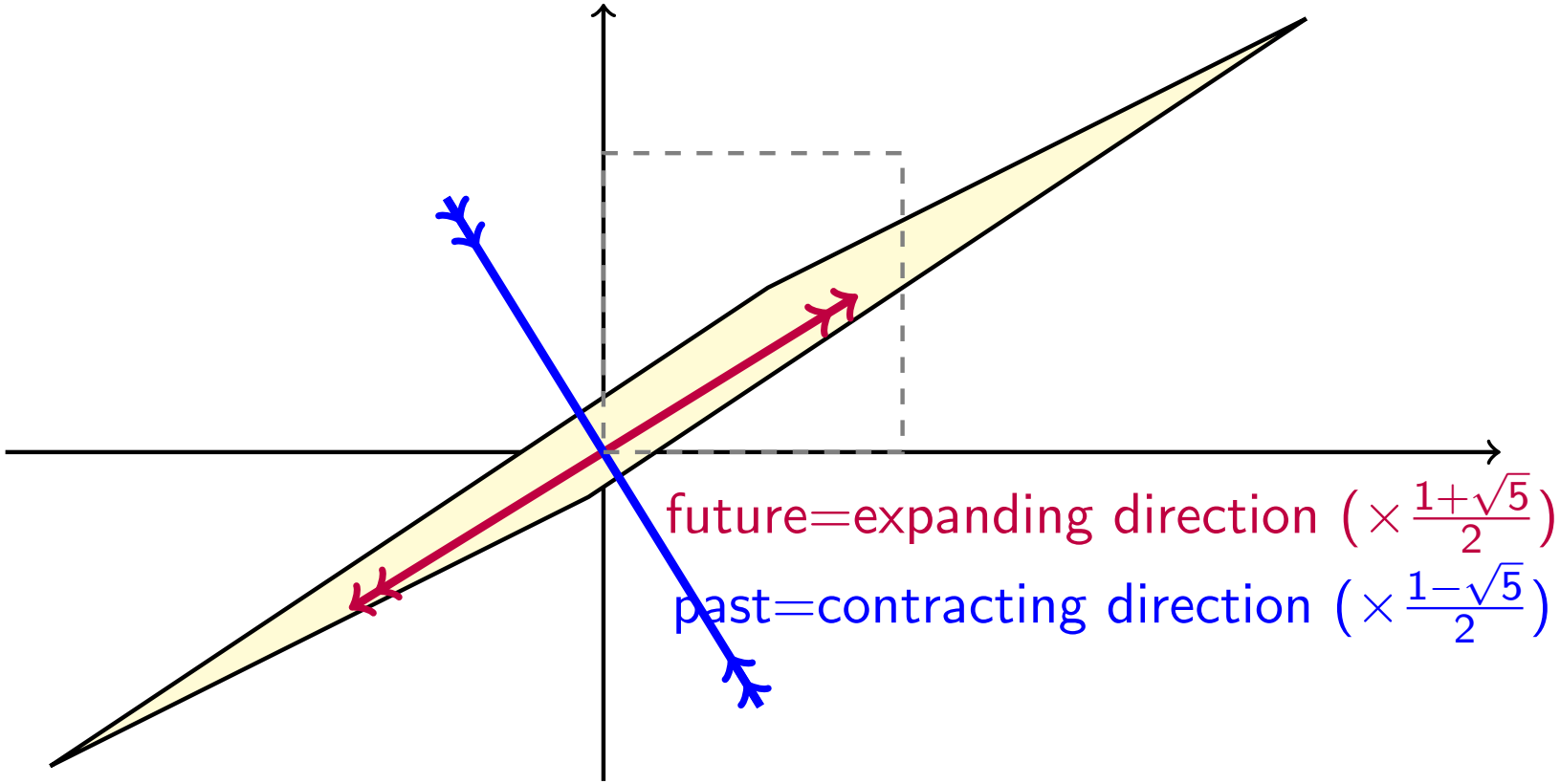
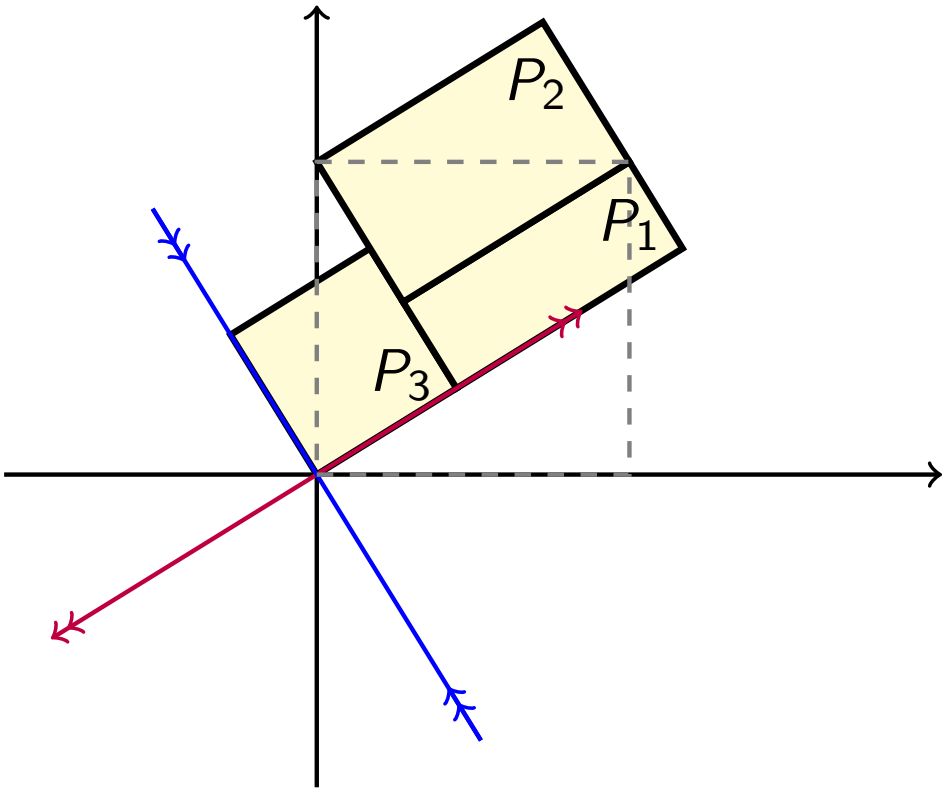
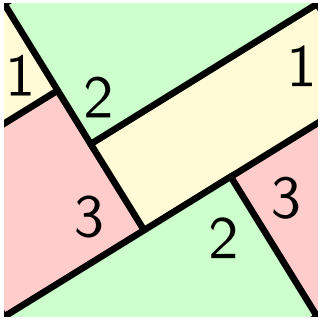


A generating partition of $[0,1]^2$





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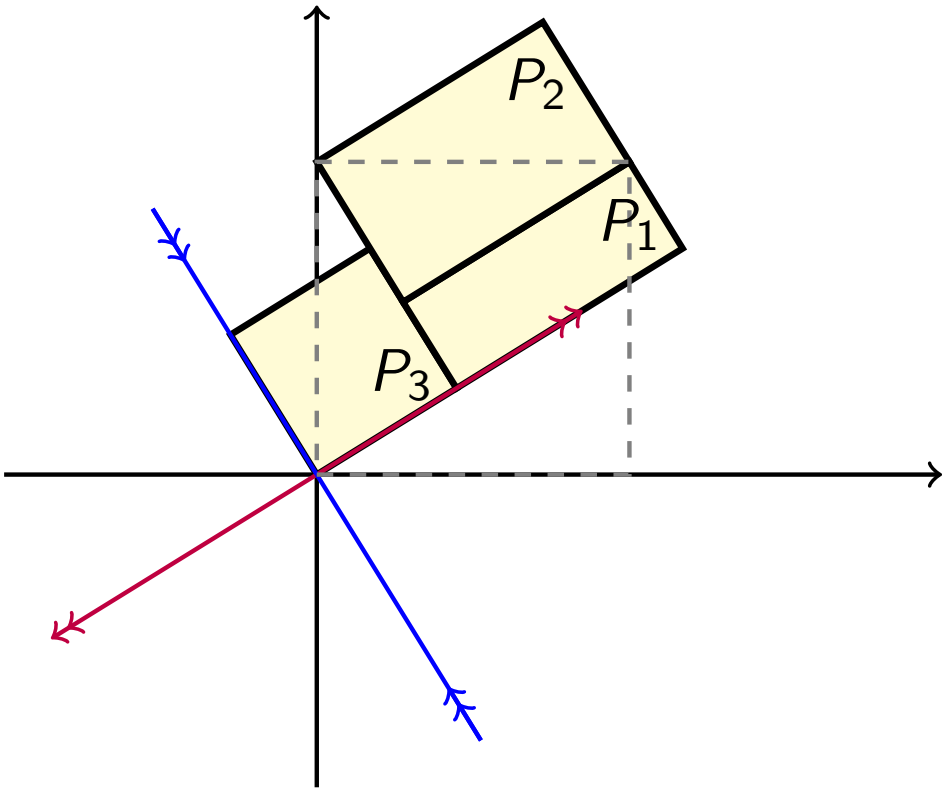
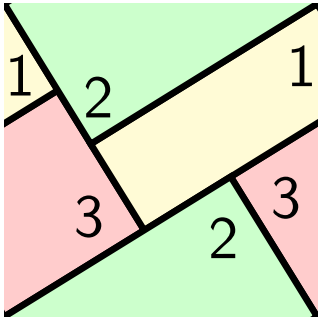
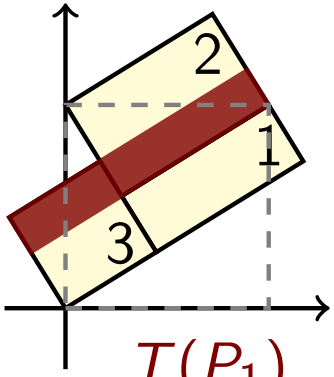
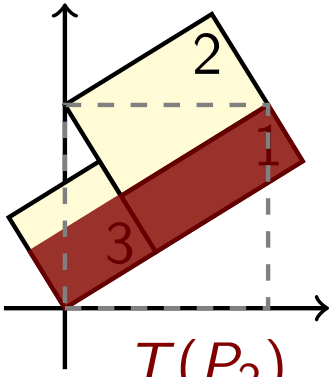






$T(P_1)$



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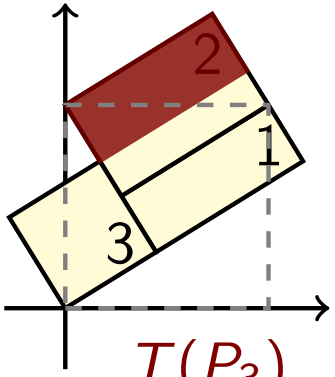


$T(P_3)$

Pisot substitution

Let σ be a Pisot irreducible substitution that has pure discrete spectrum (tiling)

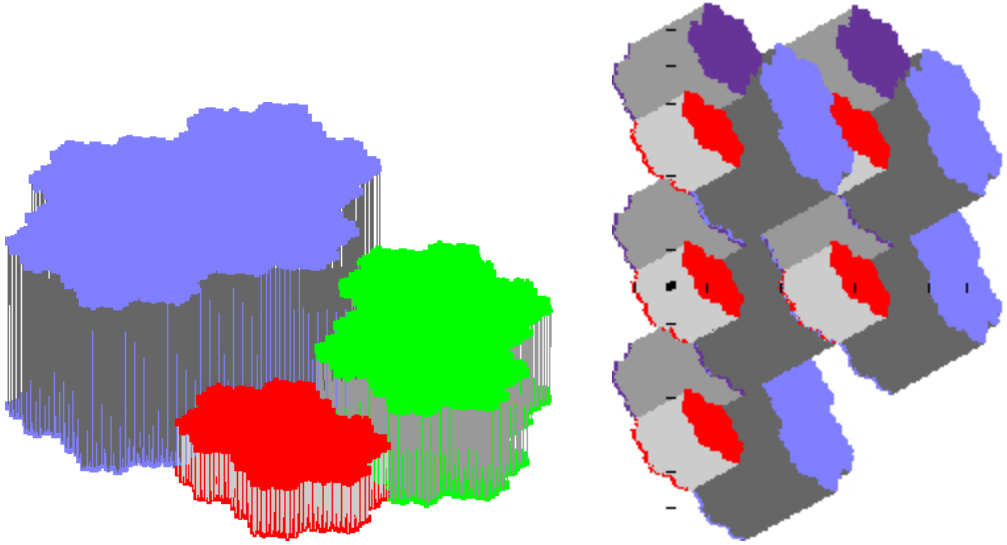
Pisot substitution σ is primitive (there exists a power of its incidence matrix which admits only positive entries) and its Perron–Frobenius eigenvalue (for its incidence matrix) is a Pisot number

Pisot substitution

Let σ be a Pisot irreducible substitution that has pure discrete spectrum (tiling)

Theorem The Rauzy fractals provide basis of Markov partitions for Pisot unimodular irreducible substitutions under the tiling assumption

[Praggastis, beta-numeration, Ito-Rao, Siegel, substitutions]



Purely periodic β-expansions

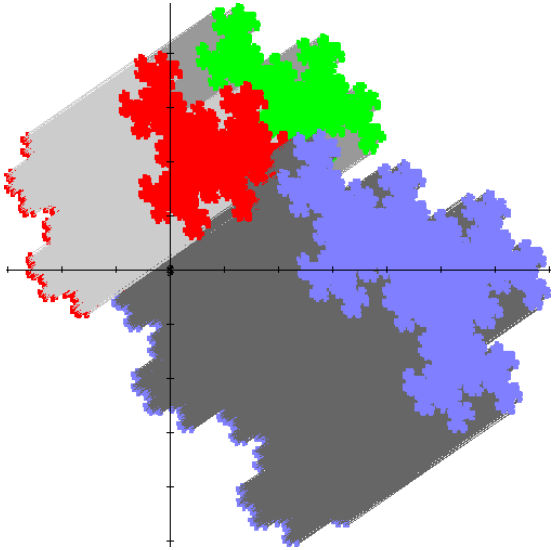
Theorem [K. Schmidt, A. Bertrand]

If β is a Pisot number, then x has an eventually periodic expansion iff $x \in \mathbb{Q}(\beta)$

Theorem [S. Ito, Y. Sano, R. Hui, V.B., A. Siegel]

If β is a Pisot number, then x has a purely periodic expansion iff $(x, x') \in \widetilde{\mathcal{R}_\beta}$

Natural extension for the beta-numeration



Theorem [Rauzy'82]

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

(X_σ, S) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$

We want a two-sided and S-adic version of Rauzy fractals in order to get non-stationary Markov partitions

S-adic expansions and non-stationary dynamics

Definition An infinite word ω is said S-adic if there exist

- a finite set of substitutions $\mathcal{S}$
- an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$

such that

$$\omega = \lim_{n\to+\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

The terminology comes from Vershik adic transformations
Bratteli diagrams

S stands for substitution, adic for the inverse limit
powers of the same substitution= partial quotients

Markov partitions Two-sided version of S-adic systems
cf. [Two-sided Markov compacta and suspension flows, Bufetov]

Dynamically

- One has the shift acting on zero entropy S-adic systems
- One has a renomalization cocycle given by the incidence matrices of the substitutions (inverse of the matrices of the Brun algorithm)
- We apply Oseledets theorem to get a splitting of the spaces to define stable and unstable spaces

Dynamically

- One has the shift acting on zero entropy S-adic systems
  Dictionary

$$\sigma^{\infty}(a) \quad \rightsquigarrow \quad \cdots \sigma_{-2}\sigma_{-1}.\sigma_0\sigma_1\sigma_2 \cdots$$

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Dictionary

$$\begin{tikzcd} \Omega_X \arrow[r, "S"] \arrow[d] & \Omega_X \arrow[d] \\ \mathbb{T}^d \arrow[r, "M_\sigma^{-1}"'] & \mathbb{T}^d \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \Omega^{(-1)} \arrow[r, "S"] \arrow[d, "\cdots"'] & \Omega^{(0)} \arrow[r, "S"] \arrow[d] & \Omega^{(1)} \arrow[r, "S"] \arrow[d] & \Omega^{(2)} \arrow[d, "\cdots"] \\ \mathbb{T}^d_{-1} \arrow[r, "f_{-1}"] & \mathbb{T}^d_0 \arrow[r, "f_0"] & \mathbb{T}^d_1 \arrow[r, "f_1"] & \mathbb{T}^d_2 \end{tikzcd}$$

One-sided case

- We apply a multidimensional continued fraction algorithm to the line in $\mathbb{R}^3$ directed by a given vector $\mathbf{u} = (u_1, u_2, u_3)$
- We then associate with the matrices produced by the algorithm substitutions, with these substitutions having the matrices produced by the continued fraction algorithm as incidence matrices

$$\mathbf{u} = \mathbf{u}_0 \xleftarrow{M_1} \mathbf{u}_1 \xleftarrow{M_2} \mathbf{u}_2 \xleftarrow{M_3} \cdots \xleftarrow{M_k} \mathbf{u}_k$$

$$w = w_0 \xleftarrow{\sigma_1} w_1 \xleftarrow{\sigma_2} w_2 \xleftarrow{\sigma_3} \cdots \xleftarrow{\sigma_k} w_k \in \{1, 2, 3\}$$

$$\mathbf{u} = M_1 \cdots M_k \mathbf{u}_k$$

S-adic Rauzy fractals

We associate with every translation acting on $\mathbb{T}^d$ (i.e., with any line in $\mathbb{R}^d$)

- an S-adic sequence

$$\omega = \lim_{n \to +\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

- such that X_ω is isomorphic to a Kronecker map
- with finite symbolic discrepancy
- provided by a multidimensional continued fraction algorithm (e.g. Brun algorithm)

S-adic Rauzy fractals

We associate with almost every translation acting on $\mathbb{T}^d$ (i.e., with any line in $\mathbb{R}^d$)

- an S-adic sequence

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- such that X_ω is isomorphic to a Kronecker map
- with finite symbolic discrepancy
- provided by a multidimensional continued fraction algorithm (e.g. Brun algorithm)

Brun and Jacobi-Perron algorithms are "Pisot"
a.e. exponential convergence [Broise-Guivarc'h]

S-adic Pisot dynamics

Theorem [B.-Steiner-Thuswaldner]

- For almost every $(\alpha, \beta) \in [0,1]^2$, the S-adic system provided by the Brun multidimensional continued fraction algorithm applied to (α, β) is measurably conjugate to the translation by (α, β) on the torus $\mathbb{T}^2$
- For almost every Arnoux-Rauzy word, the associated S-adic system has pure discrete spectrum

Proof Based on

- "adic IFS" (Iterated Function System)
- Theorem [Avila-Delecroix]
  - The Arnoux-Rauzy S-adic system is Pisot
- Theorem [Avila-Hubert-Skripchenko]
  - A measure of maximal entropy for the Rauzy gasket
- Finite products of Brun/Arnoux-Rauzy substitutions have pure discrete spectrum [B.-Bourdon-Jolivet-Siegel]

Random dynamical systems and linear cocyles

- Let $(X, \mathcal{B}, \mu)$ be a probability space, T an invertible transformation on $(X, \mathcal{B}, \mu)$ (base transformation)
- Let $A\colon X \to GL(d, \mathbb{R})$
- Linear cocycle

$$(T, A) : X \times \mathbb{R}^d \to X \times \mathbb{R}^d, \quad (x, v) \mapsto (Tx, A(x)v)$$

$$(T, A)^n = (T^n, A_n)$$

$$A_n(x) := A(T^{n-1}x) \cdots A(x) \quad n \geq 0$$

$$A_n(x) := A(T^{-n}x)^{-1} \cdots A(x)^{-1} \quad n < 0$$

Brun algorithm

Brun Start with three entries $0 \leq x_1 \leq x_2 \leq x_3$
We subtract the second largest and we reorder

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - x_2)$$

Linear version Start with $\mathbf{w}^{(0)} = (w_1^{(0)}, w_2^{(0)}, w_3^{(0)})$ with

$$0 \leq w_1^{(0)} \leq w_2^{(0)} \leq w_3^{(0)}$$

Define

$$\mathbf{w}^{(n)} = M_{i_n}^{-1} \mathbf{w}^{(n-1)}$$

where M_{i_n} is chosen among the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

according to $w_3^{(n-1)} - w_2^{(n-1)}$ compared to $w_1^{(n-1)}$ and $w_2^{(n-1)}$

Brun algorithm

Additive form

Let

$$\Delta_2 := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, 0 \le x_1 \le x_2 \le 1\}$$

$$T_{\mathrm{Brun}} : \Delta_2 \to \Delta_2$$

$$T_{\mathrm{Brun}}\big(w_1^{(n-1)}/w_3^{(n-1)}, w_2^{(n-1)}/w_3^{(n-1)}\big) = \big(w_1^{(n)}/w_3^{(n)}, w_2^{(n)}/w_3^{(n)}\big)$$

$$T_{\mathrm{Brun}} : (x_1, x_2) \mapsto \begin{cases} \left(\frac{x_1}{1-x_2}, \frac{x_2}{1-x_2}\right), & \text{for } x_2 \le \frac{1}{2} \\ \left(\frac{x_1}{x_2}, \frac{1-x_2}{x_2}\right), & \text{for } \frac{1}{2} \le x_2 \le 1 - x_1 \\ \left(\frac{1-x_2}{x_2}, \frac{x_1}{x_2}\right), & \text{for } 1 - x_1 \le x_2 \end{cases}$$

Brun substitutions

$$\beta_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases} \qquad \beta_2 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{cases} \qquad \beta_3 : \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 23 \end{cases}$$

Their incidence matrices coincide with the three matrices associated with Brun's algorithm

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Brun cocyle

S is the shift
D is the S-adic shift generated by Brun

Symbolic version

$$F_s : D \times \mathbb{T}^d \to D \times \mathbb{T}^d, \quad (\boldsymbol{\sigma}, w) \mapsto (S\boldsymbol{\sigma}, M_{\sigma_0}^{-1} w)$$

$$\boldsymbol{\sigma} = (\sigma_n) \in S^{\mathbb{Z}} \quad \lim_{n\to\infty} \sigma_{[-n,0)} . \sigma_{[0,n)} = \cdots \sigma_{-2}\sigma_{-1} . \sigma_0\sigma_1\sigma_2 \cdots$$

Arithmetic version

$$F_a : [0,1]^2 \times \mathbb{T}^d \to [0,1]^2 \times \mathbb{T}^d, \quad ((x_1, x_2), w) \mapsto (T_{\mathrm{Brun}}(x_1, x_2), M_{\mathrm{Brun}} w)$$

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Arithmetic version

$$F_a : [0,1]^2 \times \mathbb{T}^d \to [0,1]^2 \times \mathbb{T}^d, \quad ((x_1, x_2), w) \mapsto (T_{\mathrm{Brun}}(x_1, x_2), M_{\mathrm{Brun}} w)$$

$F_s^n(\boldsymbol{\sigma}, w) = (S^n\boldsymbol{\sigma}, A^n(\boldsymbol{\sigma})w)$ for $n \in \mathbb{Z}$, where

$$A_n(\boldsymbol{\sigma}) := A(S^{n-1}\boldsymbol{\sigma}) \cdots A(S\boldsymbol{\sigma})A(\boldsymbol{\sigma}) = M_{[0,n)}^{-1}, \quad \text{if } n > 0$$
$$A_n(\boldsymbol{\sigma}) := A(S^n\boldsymbol{\sigma})^{-1} \cdots A(S^{-1}\boldsymbol{\sigma})^{-1} = M_{[n,0)}, \quad \text{if } n < 0$$

Let $\mathcal{L}_{\boldsymbol{\sigma}}$ be the language associated with $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = (\sigma_n) \in S^{\mathbb{Z}}$$

The $\boldsymbol{\sigma}$-subshift with directive sequence $\boldsymbol{\sigma}$ is $(X_{\boldsymbol{\sigma}}, \Sigma)$, where $X_{\boldsymbol{\sigma}}$ denotes the set of infinite words ω such that each factor of ω is an element of $\mathcal{L}_{\boldsymbol{\sigma}}$

- Primitivity For each $k \in \mathbb{Z}$, $M_{[k,\ell)}$ is positive for some $\ell > k$
- Algebraically irreducible For each $k \in \mathbb{Z}$, the characteristic polynomial of $M_{[k,\ell)}$ is irreducible for all sufficiently large ℓ
- Balance A pair of words $u, v \in \mathcal{A}^*$ with $|u| = |v|$ is C-balanced if

  $$-C \leq |u|_j - |v|_j \leq C \quad \text{for all } j \in \mathcal{A}.$$

  A language $\mathcal{L}$ is C-balanced if each pair of words $u, v \in \mathcal{L}$ with $|u| = |v|$ is C-balanced.

Strong convergence Let $\boldsymbol{\sigma}$ be an algebraically irreducible sequence of substitutions with generalized right eigenvector $\mathbf{u}$ and balanced language $\mathcal{L}_{\boldsymbol{\sigma}}$

$$\bigcap_{n\in\mathbb{N}} M_{[0,n)}\mathbb{R}_+^d = \mathbb{R}_+\mathbf{u}$$

Then the coordinates of $\mathbf{u}$ are rationally independent

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Then the coordinates of $\mathbf{u}$ are rationally independent

Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a primitive, algebraically irreducible, and recurrent sequence of substitutions with balanced language $\mathcal{L}_{\boldsymbol{\sigma}}$. Then

$$\lim_{n\to\infty} \pi_{\mathbf{u},\mathbf{1}} M_{[0,n)}\, \mathbf{e}_i = \mathbf{0} \quad \text{for all } i \in \mathcal{A}.$$

Theorem Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{Z}}$ be a primitive, algebraically irreducible, and recurrent sequence of substitutions with balanced language $\mathcal{L}_{\boldsymbol{\sigma}}$. Then the mapping family $(\mathbf{T}, f)$ associated with $\boldsymbol{\sigma}$ is eventually Anosov

Theorem Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a primitive and algebraically irreducible sequence of unimodular substitutions over the finite alphabet $\mathcal{A}$. Assume that there is $C > 0$ such that for each $\ell \in \mathbb{N}$, there is $n \geq 1$ with $(\sigma_n, \ldots, \sigma_{n+\ell-1}) = (\sigma_0, \ldots, \sigma_{\ell-1})$ and the language $\mathcal{L}_{\boldsymbol{\sigma}}^{(n+\ell)}$ is C-balanced. If moreover one has tiling, then the partition $\widehat{\mathcal{R}}^{(n)}$ made of the suspensions of the Rauzy fractals form a Markov partition for the mapping family $(\mathbf{T}, f)$

These pieces are connected [B.-Bourdon-Jolivet-Siegel]

Non-stationary composition of toral automorphisms

Consider the sequence of toral homeomorphisms

$$\cdots \xrightarrow{f_{-2}} \mathbf{T}_{-1} \xrightarrow{f_{-1}} \mathbf{T}_0 \xrightarrow{f_0} \mathbf{T}_1 \xrightarrow{f_1} \cdots$$

- each $\mathbf{T}_i$ is a manifold homeomorphic to the d-dimensional torus by a given map $\varphi_i \colon \mathbf{T}_i \to \mathbb{R}^d/\mathbb{Z}^d$
- $f_i : \mathbf{T}_i \to \mathbf{T}_{i+1}$ is a map such that $\varphi_{i+1} \circ f_i \circ \varphi_i^{-1}$ is an automorphism of $\mathbb{R}^d/\mathbb{Z}^d$ given in the canonical coordinates by left multiplication by the inverse matrix M_i^{-1}

  Let $\mathbf{T}$ be the disjoint union of the $\mathbf{T}_i$, let $f : \mathbf{T} \to \mathbf{T}$ be the total map which equals f_i on the component $\mathbf{T}_i$. We call $(\mathbf{T}, f)$ the mapping family associated with $\boldsymbol{\sigma}$.

Mapping family

The mapping family associated with $\boldsymbol{\sigma}$ is eventually Anosov if there exist splittings $E_s^{(n)} \oplus E_u^{(n)}$ of $\mathbb{R}^d$ so that the following properties hold.

- f-**invariance** For all n, $f_n(E_s^{(n)}) = E_s^{(n+1)}$, $f_n(E_u^{(n)}) = E_u^{(n+1)}$.
- **Hyperbolicity** For some (and hence for all) $k \in \mathbb{Z}$

$$\lim_{n\to+\infty} \inf\{\|M_{[k,n)}^{-1}\mathbf{x}\|/\|\mathbf{x}\| : \mathbf{x} \in E_u^{(k)} \setminus \{\mathbf{0}\}\} = +\infty, \quad n > k,$$

$$\lim_{n\to+\infty} \sup\{\|M_{[k,n)}^{-1}\mathbf{x}\|/\|\mathbf{x}\| : \mathbf{x} \in E_s^{(k)} \setminus \{\mathbf{0}\}\} = 0, \quad n > k,$$

$$\lim_{n\to-\infty} \sup\{\|M_{[n,k)}\mathbf{x}\|/\|\mathbf{x}\| : \mathbf{x} \in E_u^{(k)} \setminus \{\mathbf{0}\}\} = 0, \quad n < k,$$

$$\lim_{n\to-\infty} \inf\{\|M_{[n,k)}\mathbf{x}\|/\|\mathbf{x}\| : \mathbf{x} \in E_s^{(k)} \setminus \{\mathbf{0}\}\} = +\infty, \quad n < k.$$

Theorem Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{Z}}$ be a primitive, algebraically irreducible, and recurrent sequence of substitutions with balanced language $\mathcal{L}_{\boldsymbol{\sigma}}$. Then the mapping family $(\mathbf{T}, f)$ associated with $\boldsymbol{\sigma}$ is eventually Anosov.

Proof Under hypotheses of primitivity and recurrence of the directive sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{Z}}$ we have the existence of two positive vectors $\mathbf{u}$ and $\mathbf{v}$ defined as

$$\bigcap_{n\in\mathbb{N}} M_{[0,n)}\mathbb{R}_+^d = \mathbb{R}_+\mathbf{u}, \qquad \bigcap_{m\in\mathbb{N}} {}^t(M_{[-m,0)})\mathbb{R}_+^d = \mathbb{R}_+\mathbf{v}.$$

Set

$$\begin{aligned} \mathbf{u}^{(n)} &= (M_{[0,n)})^{-1}\mathbf{u}, & \mathbf{v}^{(n)} &= {}^t(M_{[0,n)})\mathbf{v}, & &\text{for } n \geq 0 \\ \mathbf{u}^{(n)} &= M_{[n,0)}\mathbf{u}, & \mathbf{v}^{(n)} &= {}^t(M_{[n,0)})^{-1}\mathbf{v}, & &\text{for } n < 0. \end{aligned}$$

$$E_u^{(n)} := (\mathbf{v}^{(n)})^\perp, \quad E_s^{(n)} := \langle \mathbf{u}^{(n)} \rangle.$$

Note that $E_u^{(n)}$ has codimension 1 whereas $E_s^{(n)}$ has dimension 1.

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One checks that f-invariance holds by looking at the definitions of $\mathbf{u}^{(n)}$ and $\mathbf{v}^{(n)}$

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The hyperbolicity for $k = 0$ (this is sufficient to do it for $k = 0$) comes from the following limits

$$\lim_{n\to-\infty} \|M_{[n,0)}\mathbf{u}\| = +\infty, \quad \lim_{n\to\infty} M_{[0,n)}^{-1}\mathbf{u} = 0$$

$$\lim_{n\to-\infty} M_{[n,0)}\mathbf{x} = 0, \quad \lim_{n\to\infty} \|M_{[0,n)}^{-1}\mathbf{x}\| = +\infty, \quad \text{for all } \mathbf{x} \in \mathbf{v}^\perp \setminus \{\mathbf{0}\}.$$