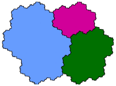
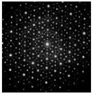
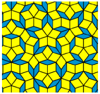
Topological properties of generalized Rauzy fractals: which novel issues?

Anne Siegel (CNRS, Rennes, France)

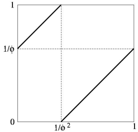
Repetitivity ?

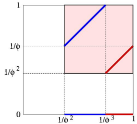
Topology: A compact set is self-similar if it can be decomposed into shrinked copies of itself.

Physics: A quasi-crystal is a distribution of atoms that is fixed by a decomposition process.

Dynamics: A map is self-induced if it is conjugate to its first return map over a well-chosen subset.





Number theory An algebraic number is somehow repetitive... $\phi^2 = \phi + 1$

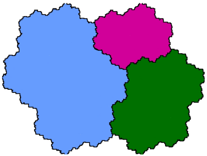
Combinatorics A symbolic dynamical system is substitutive if is it fixed by a letter-to-letter morphism.

$$\underbrace{12}_{1}\;\underbrace{1}_{2}\;\underbrace{12}_{1}\;\underbrace{12}_{1}\;\underbrace{1}_{2}\;\underbrace{12}_{1}\;\underbrace{12}_{1}\;\underbrace{12}_{1}\;\underbrace{1}_{2}\;\underbrace{12}_{1}\;\underbrace{12}_{1}\;\underbrace{1}_{2}\;\underbrace{12}_{1}\;\underbrace{1}_{2}\;\underbrace{12}_{1}\;\underbrace{12}_{1}\;\underbrace{1}_{2}\;....$$

A general scheme

- A repetitive object yields a replacement rule : combinatorial substitution.
- A combinatorial substitution yields a self-similar set.

Rauzy fractals *(Rauzy'81)*, Central tile *(Thurston'89)*



- **The properties of the initial mathematical object might appear in the fractal set.**

Examples of application of the scheme

Physics: Explicit cut and project schemes *(Gazeau&al'95, Berthé&S.'05)*

Dynamics:

- Tiling spaces with pure discrete spectrum *(Solomyak'98, Barge&Kwapisz&Diamond'06–10)&Bourdon-Berth/'e?Jolivet-S.'15&Barge'15*
- Invariant for tiling spaces? *(Barge'09, Sadun'10,Barge&Kellendonk'14)*
- Build Markov partitions for toral automorphisms *(Praggastis'98, Ito&Rao'06)*

Combinatorics/Algebra: Transversal dynamics for free group automorphisms *(Hillion&al'06–10)*

Number theory

- Best simultaneous rational approximation for vectors *(Rauzy'81, Messaoudi&Hubert'01)*
- Expansions in non-integer basis *(Akiyama&al'02–10,Steiner et al.'10-14)*
- Proofs for irrationality *(Adamczeswki&Frougny&S.&Steiner'10)*

Main purpose of the talk

... *Self-similar mathematical objects are related to topological properties of fractals* ...

Limitations of this scheme when considering generalized Rauzy fractals ?

How do we concretely investigate fractal topology

Investigating fractal topology ?

(Step 0) Symbolic dynamics and representation

(Step 1) Check that the representation is "relevant"

(Step 2) Understand the self-induction process

(Step 3) Build a finite graph to describe the self-induction process

(Step 4) Connect targeted applications to fractal topology

(Step 5) Study topological properties

(Step 0) Symbolic dynamics

Find a good representation of the mathematical process

Self-inducted process

→ Relevant partition of the space

→ Combinatorial substitution

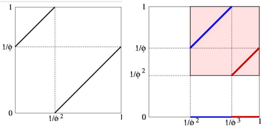
→ Symbolic (or tiling flow) dynamical system

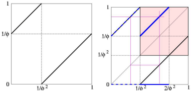
→ Representation as stairs

→ [Pisot assumption] **Rauzy fractal**

$x \in [0,1(\mapsto 2x - [2x] \in [0,1($

$$\phi^2 = \phi + 1$$



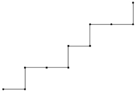


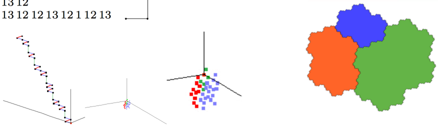
$(l)ong \mapsto (L)ong\,(S)hort$ $\quad (s)hort \mapsto (L)ong$

$\underbrace{12}_{Long}\ \underbrace{1}_{Short}\ \underbrace{12}_{Long}\ \underbrace{12}_{Long}\ \underbrace{1}_{Short}\ \underbrace{12}_{Long}\ \underbrace{1}_{Short}\ \underbrace{12}_{Long}$

$$1 \mapsto 12 \quad 2 \mapsto 1$$

1
12
12 13
12 13 12 1
12 13 12 1 12 13 12
12 13 12 1 12 13 12 12 13 12 1 12 13





(Step 1) Check wether the representation is "relevant"

What is a good representation? In the irreducible Pisot case, a domain exchange on the Rauzy fractal is a factor map of the shift-map on the substitutive system. *(Canterini-S.'01; Arnoux-Ito'01)*

Theorem *If σ is a substitution on d letters, of Pisot type, unimodular, and satisfying the coincidence condition, the symbolic dynamical system generated by σ is measure-theoretically conjugate to an exchange of domains in a self-similar compact subset of $\mathbb{R}^{d-1}$.*

Pisot conjecture: the substitutive dynamical system generated by a unit Pisot substitution has a pure discrete spectrum (it generates a self-similar aperiodic tiling of the space).

Sufficient conditions: (F), (W), (balanced pairs), (super coincidences)...

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(Many) approaches developped so far

Combinatorics	Coincidence	(*Dekking, Livshits*)	Sequences
	Balanced pairs	(Livshits, Sirvent&Solomyak)	
Arithmetics	Finiteness property	(*Solomyak*)	beta-substitutions
	Homoclinic condition	(*Vershik, Schmidt*)	homoclinic points
Tiling	Strong overlaps	(*Solomyak*)	Tiling space
Physics	Algebraic	(*Lee*)	Meyer sets
Geometry	Super-coincidence	(*Barge&Kwapisz, Ito&Rao*)	Strands
Topology	Boundary graph	(*Siegel, Thuswaldner*)	Fractals

Sufficient conditions: (F), (W), (balanced pairs), (super coincidences)...

(Step 2) Understand the self-induction process

Desubstitute any word in the substitutive langage

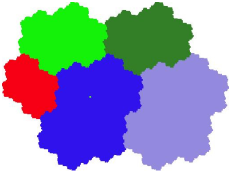
$$\sigma(1) = 12,\ \sigma(2) = 13,\ \sigma(3) = 1 \qquad W_1 = 12\,13\,12\,1\,1 = \sigma(\underbrace{1213}_{W_2})1 \in \mathcal{T}(2)$$

$$\underbrace{\pi\mathbf{P}(W)}_{\in\mathcal{T}(2)} = \mathbf{h}\,\underbrace{\pi\mathbf{P}(W_2)}_{\in\mathcal{T}(1)} + \pi(e_1)$$

Any prefix can be uniquely written as $W = \sigma(W_1)\underbrace{P}_{<\sigma(a)}$ such that $W_1 a$ is another prefix of the fixed point.

Decompose the fractal following the desubstitution process

$$\mathcal{T}(1) = \mathbf{h}\mathcal{T}(1) \cup \mathbf{h}\mathcal{T}(2) \cup \mathbf{h}\mathcal{T}(3)$$
$$\mathcal{T}(2) = \mathbf{h}\mathcal{T}(1) + \pi\mathbf{P}(1)$$
$$\mathcal{T}(3) = \mathbf{h}\mathcal{T}(2) + \pi\mathbf{P}(1)$$



Depict self-similarity by a Graph-Iterated Function System

$$\mathcal{T}(a) = \bigcup_{b\in\mathcal{A},\ \sigma(b)=pas} \mathbf{h}(\mathcal{T}(b)) + \pi(\mathbf{l}(p))$$

(Step 3) Build a finite graph to describe the self-induction process

Describe intersections between tiles

- Subtiles in the decomposition have the shape $\mathbf{h}(\mathcal{T}(b) + \mathbf{e})$
- Assume that two subtiles intersect. Then there exists a translation vector $\mathbf{e}$ s.t.

$$\mathcal{I} = \mathcal{T}(a) \cap +(\mathcal{T}(b) + \mathbf{e}) \neq \emptyset$$

- Each tile can be decomposed as well

$$\mathcal{T}(a) = \bigcup_{\sigma(a_1)=p_1 a s_1} \mathbf{h}(\mathcal{T}(a_1) + \pi \mathbf{l}(p_1)). \qquad \mathcal{T}(b) = \bigcup_{\sigma(b_1)=p_2 b s_2} \mathbf{h}(\mathcal{T}(b_1) + \pi \mathbf{l}(p_2)).$$

- The intersection $\mathcal{I}$ can also be decomposed...

$$\mathcal{T}(a) \cap +(\mathcal{T}(b) + \mathbf{e}) = \bigcup \mathbf{h}\left(\mathcal{T}(a_1) \cap (\mathcal{T}(b_1) + \mathbf{e}_{(a_1,b_1)})\right) + \mathbf{f}_{(a_1,b_1)} \neq \emptyset$$

- **Magic point : the terms in the decomposition have the same structure and one of them is nonempty : relaunch the process !**

Graph construction

$$(a, b, \mathbf{e}) \to (a_1, b_1, \mathbf{e}_{(a_1,b_1)})$$

$$\text{edge label} = \mathbf{f}_{(a_1,b_1)}$$

Second magic point : Pisot implies that the graph is finite

(Step 3-bis) Build MANY finite graphs to describe the self-induction process

Construction

- Recode the place of tiles in expanded dynamics
- Repetitivity + Pisot assumption $\Longrightarrow$ finite graph
- Additional properties $\Longrightarrow$ reduce the graph

Applications in several frameworks

- cut and project schemes algebraic condition *(Lee'08)*,
- Tiling spaces strong overlap condition *(Solomyak'02, Akiyama&Lee'10)*,
- Hull of substitutive space Geometric coincidence condition *(Barge&Kwaplisz'06)*
- Discrete geometry Super coincidence condition *(Ito&Rao'06)*
- Topology/GIFS Boundary graph condition *(S.'04, Thuswaldner'06)*
- Number theory (W) property *(Akiyama'04)*

Strong connections between all conditions *(Berthé&S.&Thuswaldner'10)*

(Step 4) Connect targeted applications to fractal topology

Number theory

- Which β have a maximal set of finite expansions? 0 inner point
- How rational purely periodic expansions are spread over $[0, 1]$? Intersections of line and fractal boundary
- Which vectors have good rational approximations? larger ellipse contained in a fractal
- Relation between the norm of β and the expansion of 1? Connectivity

Dynamics

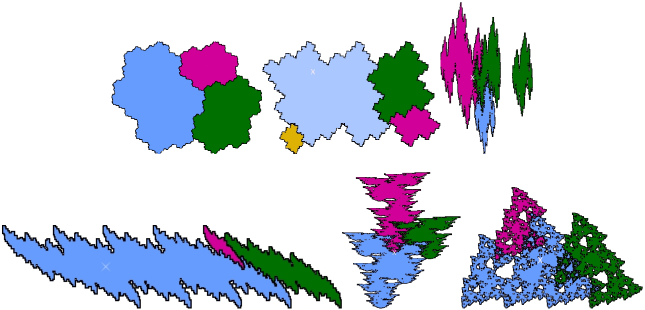
- Find Markov partition for toral automorphisms with a nice shape? Connectivity
- Compute invariants for tiling spaces ? Cutting points
- Compute cohomologies ? Intersection between tiles

Combinatorics

- Compute matching rules to generate self-similar tilings? Simple connectivity

(Step 5) Study topological properties

- exhibit topological properties [connectivity, disklikeness...] within graphs *(S.&Thuswaldner'10)*
- Understand connectivity with local rules *(Berthé, Bourdon, Jolivet, S.'14)*



Remark Topological properties of central tiles are far from homogeneous...

Limitations

- Conditions are mostly algorithmics
- Few general properties (except for two and three dimensional cases).

Why is it so difficult to check topological properties?

Checking topological properties of GIFS is mostly undecidable ! *(Jolivet & Kari, MFCS, 2014)*

$$X_q = \bigcup_{r \in Q} \bigcup_{e \in E_{q,r}} f_e(X_r),$$

Theorem 4.2. *The following problem is undecidable. Instance: a d-dimensional affine GIFS $\mathcal{G}$ specified by maps with rational coefficients, and a state q of $\mathcal{G}$. Question: does X_q have empty interior? This problem remains undecidable if we restrict to 2-dimensional GIFS with 3 states.*

Actually Rauzy fractals are rather nice in terms of topological properties !!!!

- 0 inner point, computing boundary's haussdorf dimension, connectivity ... are decidable!

What about "generalized" Rauzy fractals?

Composition of substitutions according to continued-fraction algorithms?

- We need general results on families of substitutions

Tiles obtained by Shift Radix Systems?

- No more IFS but rather a family of IFS

Tiles obtained by S-adic systems ?

- Maybe not that bad?

Composition of substitutions within a family

Issue : What are the common properties of Rauzy fractals of

- Arnoux-Rauzy substitutions?
- Jacobi-Perron substitutions?
- beta-substitutions?

Novel class of question to address: which substitutions are Pisot within these families?

- Answer: all are Pisot, but it deserves specific studies *(Delecroix-Avila'13, ...)*

Global framework to study the family of fractals : local rules

(Ito&Otsuki, Furukado, Ei, Jolivet, Berthé, S....)

- **The study of local intersections between tiles is replaced by the exhaustive study of local rules placements**
- Study of connectivity, PDS...

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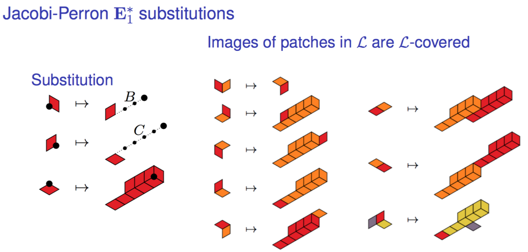

Composition of substitutions within a family: more insights

A magic recent result *(Barge'14 : arxiv)*

Theorem 3.12. *Suppose that ϕ is primitive, Pisot, injective on initial letters, and constant on final letters. Then $(\Omega_\phi, \mathbb{R})$ has pure discrete spectrum.*

- The theorem applies to the **reverse** of Arnoux-Rauzy substitutions, Jacobi-Perron substitutions and to all beta-substitutions (with additional combinatorial studies) *(Barge'15)*.
- The result is true on the tiling flow substitutive system
  $\rightarrow$ to be carefully applied when dealing with reducible substitutive dynamical systems.
- The theorem relies on a very smart understanding of proximalities *(Barge&Kellendonk'13)*
  $\rightarrow$ **What are the connection with other representations of proximalities: fractal trees...** *(Bressaud, Hillion...)* **?**

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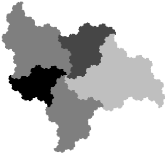
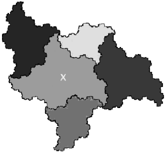
What does this condition mean in terms of fractal topology ?

Theorem 3.12. *Suppose that ϕ is primitive, Pisot, injective on initial letters, and constant on final letters. Then $(\Omega_\phi, \mathbb{R})$ has pure discrete spectrum.*

(examples : $1 \to 21,\ 2 \to 31,\ 3 \to 1 \qquad 1 \to 21,\ 2 \to 31,\ 3 \to 41,\ 4 \to 51,\ 5 \to 1.$)





$$\mathcal{T}(a) = \bigcup_{b \in \mathcal{A},\ \sigma(b) = pas} \mathbf{h}(\mathcal{T}(b)) + \pi(\mathbf{l}(p))$$

Injective on letters?

- Each piece $\mathbf{h}\mathcal{T}(a)$ appears in the decomposition of a different piece of the fractal.
- $\mathbf{0}$ is at the intersection of all tiles $\mathcal{T}(a)$
- The preimages of the unit cube by $E_1(\sigma)^*$ are totally separated.

Constant on final letters?

- The reverse substitution starts with the same letter.
- Studying the reverse substitution is equivalent to applying the inverse shift map S^{-1}.
- When applying the domain exchange to the fractal tiles, one of the images of tiles contains a full copy of the fractal.

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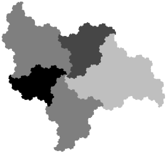
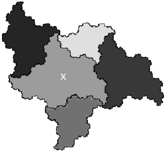




$$\mathcal{T}(a) = \bigcup_{b \in \mathcal{A},\ \sigma(b) = pas} \mathbf{h}(\mathcal{T}(b)) + \pi(\mathbf{l}(p))$$

Still to be understood...

- Why do these properties imply pure discrete spectrum ?
- What does it mean with respect to $E_1(\sigma)^*$?

Which combinatorial or numerical property/invariant are implied by Barge&Kellendonk's theorem ?

Shift Radix Systems

Larger family of numeration systems including canonical number system and beta numeration, with non unit cases

Definition 2.1 *(Shift radix system, finiteness property).* For $\mathbf{r} = (r_0, \ldots, r_{d-1}) \in \mathbb{R}^d$, $d \geqslant 1$, set

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d,$$
$$\mathbf{z} = (z_0, z_1, \ldots, z_{d-1})^t \mapsto (z_1, \ldots, z_{d-1}, -\lfloor \mathbf{r}\mathbf{z} \rfloor)^t,$$

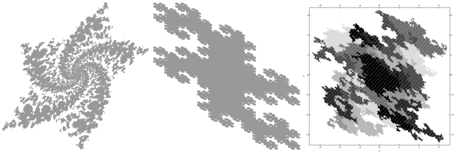
- Fractal shape but no GIFS: fixed point of an infinite tree (Bratelli diagram)

$$\mathcal{T}_{\mathbf{r}}(\mathbf{x}) = \bigcup_{\mathbf{y} \in \tau_{\mathbf{r}}^{-1}(\mathbf{x})} M_{\mathbf{r}} \mathcal{T}_{\mathbf{r}}(\mathbf{y}).$$

- Theorem: Finiteness property implies that the interiors are disjoint.

  Proof: roughly, the number of tiles containing a point is a.e. constant.

(Berthé, Siegel, Steiner, Surer, Thuswaldner'11)



- **Main problem: No graph description of the self-similar decomposition...**

S-adic systems

From Berthé, Steiner, Thuswaldner)

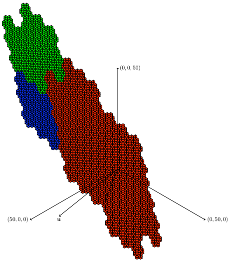
Definition An infinite word ω is said *S-adic* if there exist

- a finite set of substitutions $\mathcal{S}$
- an infinite sequence of substitutions $(\sigma_n)_{n\geq 1}$ with values in $\mathcal{S}$

such that

$$\omega = \lim_{n\to+\infty} \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n(0)$$

The terminology comes from Vershik adic transformations (cf. Bratteli diagrams)



Generalized Rauzy fractal : increasing approximations of a family of discrete planes

Brattelli-like GIFS equation

Proposition 5.6. *Let $\boldsymbol{\sigma} = (\sigma_n)$ be a sequence of unimodular substitutions with generalized right eigenvector $\mathbf{u}$. Then for each $[\mathbf{x}, i] \in \mathbb{Z}^d \times \mathcal{A}$ and all $k < \ell$, we have the set equation*

$$\pi_{\mathbf{u},\mathbf{w}}^{(k)} \mathbf{x} + \mathcal{R}_{\mathbf{w}}^{(k)}(i) = \bigcup_{[\mathbf{y},j]\in E_1^*(\sigma_{[k,\ell)})[\mathbf{x},i]} M_{[k,\ell)} \left(\pi_{\mathbf{u},\mathbf{w}}^{(\ell)} \mathbf{y} + \mathcal{R}_{\mathbf{w}}^{(\ell)}(j)\right). \tag{5.3}$$

Still new classes of questions...

- What are the relations between the translation vectors involved in the Brattelli-IFS?
- What is the convergence speed of $M_k \times M_{k+1} \times \cdots \times M_l$?
- Is 0 an inner point?
- What is the Haussdorf dimension of the boundary?

S-adic systems: towards graphs?

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A graph could be defined...

- Nodes $[i, j, \mathbf{e}]$
- Edges $[i, j, \mathbf{e}] \to [i_1, j_1, \mathbf{e}_1]$ labeled by $[\mathbf{f}, M]$ iff $\mathcal{R}^{(k)}(i) \cap (\mathcal{R}^{(k)}(j) + \pi^{(k)}\mathbf{e})$ contains $\mathbf{f} + \mathbf{M}(\mathcal{R}^{(k)}(i_1) \cap (\mathcal{R}^{(k)}(j_1) + \pi^{(k)}\mathbf{e}_1))$ in its decomposition
- The graph could be pruned according to conditions on matrices M (rational langage).

Reminder Graph definition in the IFS case

$$\mathcal{I} = \mathcal{T}(a) \cap +(\mathcal{T}(b) + \mathbf{e}) \neq \emptyset$$

- Each tile can be decomposed as well

$$\mathcal{T}(a) = \bigcup_{\sigma(a_1) = p_1 a s_1} \mathbf{h}(\mathcal{T}(a_1) + \pi \mathbf{l}(p_1)). \qquad \mathcal{T}(b) = \bigcup_{\sigma(b_1) = p_2 b s_2} \mathbf{h}(\mathcal{T}(b_1) + \pi \mathbf{l}(p_2)).$$

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- **Magic point : the terms in the decomposition have the same structure and one of them is nonempty : relaunch the process !**

Graph construction

$$(a, b, \mathbf{e}) \to (a_1, b_1, \mathbf{e}_{(a_1,b_1)})$$

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S-adic systems: towards graphs

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- Nodes $[i, j, \mathbf{e}]$
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Although...

- Which condition on matrices will ensure that expansions given by the graph converge?
  $\to$ Discrepancy? Lyapounov exponents?
- Which condition will ensure that the graph is finite?
  $\to$ Appropriate choice of the projections $\pi^{(k)}$?

Conclusion

Take home message
When we have a good understanding of algebraic relations within a GIFS equation, we can prove interesting features of fractal tiles topology and apply it to nice mathematical issues.

Future goal ? **Describe the topology of families of generalized fractals with graphs !**

- Based on the combinatorics of substitution chainings (S-adic)
- Taking benefit of combinatorial properties that contained in Barge& Kelledonk theorem about proximalities?
- How does Barge's theorem applies in the S-adic framework? What is the corresponding tiling space?