Diophantine approximation of the orbit of 1 in the dynamical system of beta-expansions

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Outline

1. Diophantine approximation of the orbits of 1 under beta-transformations

2. The lengths of the cylinders in β-expansion

3. Distribution of regular cylinders in parameter space

Diophantine approximation of the orbits of 1 under beta-transformations

Backgrounds

- Poincaré Recurrence Theorem
  Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system (probability space) and $B \subset X$ with positive measure. Then

$$\mu\{x \in B : T^n x \in B \text{ infinitely often (i.o.)}\} = \mu(B).$$

- Birkhoff ergodic theorem
  Assume that μ is ergodic, then

$$\mu\{x \in X : T^n x \in B \text{ i.o.}\} = 1.$$

- shrinking target problem (Hill and Velani, 1995)
  Let $\{B_n\}_{n\geq 1}$ be a sequence of measurable sets with $\mu(B_n)$ decreasing to 0 as $n \to \infty$. Consider the metric properties of the following set

$$\{x \in X : T^n x \in B_n \text{ i.o.}\} = \limsup_{n\to\infty} T^{-n} B_n$$

Backgrounds

- well-approximable set
  Let d be a metric on X consistent with the probability space $(X, \mathcal{B}, \mu)$. Given a sequence of balls $B(y_0, r_n)$ with center $y_0 \in X$ and shrinking radius $\{r_n\}$, the set

$$F(y_0, \{r_n\}) := \{x \in X : d(T^n x, y_0) < r_n \text{ i.o.}\}$$

  is called the well-approximable set.

- dynamical Borel-Cantelli Lemma
  $\sum_{n=1}^{\infty} \mu(B(y_0, r_n)) < \infty \Rightarrow \mu(F(y_0, \{r_n\})) = 0$

  $\sum_{n=1}^{\infty} \mu(B(y_0, r_n)) = \infty +$ some condition $\Rightarrow \mu(F(y_0, \{r_n\})) = 1$

  (Kuraweil (1955), Philipp (1967), Kleinbock and Margulis (1999), Chernov and Kleinbock (2001), Kim (2007), Tseng (2008) etc)

Backgrounds

- well-approximable set : Hausdorff dimension of the set $F(y_0, \{r_n\})$ for the case $\sum_{n=1}^{\infty} \mu(B(y_0, r_n)) < \infty$

  (Hill and Velani (1995, 1997, 1999), Urbański (2002), Shen and Wang (2013), Bugeaud and Wang (2014), Li, Wang, Wu and Xu (2014) etc)

- inhomogeneous Diophantine approximation
  Let $S_\alpha : x \mapsto x + \alpha$ be the irrational rotation map on the circle with $\alpha \notin \mathbb{Q}$. The classic inhomogeneous Diophantine approximation can be written as

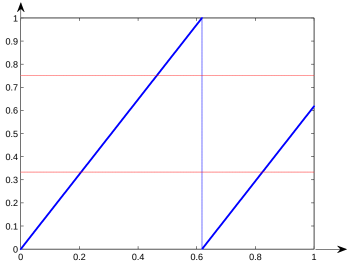
$$\left\{\alpha \in \mathbb{Q}^c : \|S_\alpha^n 0 - y_0\| < r_n, \text{ i.o. } n \in \mathbb{N}\right\}.$$

beta-transformations (greedy)

- $\beta > 1$
- β-transformation $T_\beta : [0,1] \to [0,1]$

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

  where $\lfloor \beta x \rfloor$ denotes the integer part of βx.
- Example : $\beta = \frac{1+\sqrt{5}}{2}$



- the orbit of 1 under T_β is crucial

Main problem

- well-approximable set in papameter space
  Fix a point $x_0 \in [0,1]$ and a given sequence of integers $\{\ell_n\}_{n\geq 1}$.

$$E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \big\{\beta > 1 : |T_\beta^n 1 - x_0| < \beta^{-\ell_n},\ \text{i.o.}\big\}$$

- Question :

$$\dim_H E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = ?$$

- (Persson and Schmeling, 2008)
  When $x_0 = 0$ and $\ell_n = \gamma n (\gamma > 0)$, then

$$\dim_H E(\{\gamma n\}_{n\geq 1}, 0) = \frac{1}{1+\gamma}.$$

Main result

Theorem

Let $x_0 \in [0,1]$ *and let* $\{\ell_n\}_{n\geq 1}$ *be a sequence of integers such that* $\ell_n \to \infty$ *as* $n \to \infty$. *Then*

$$\dim_H E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \frac{1}{1+\alpha}, \text{ where } \alpha = \liminf_{n\to\infty} \frac{\ell_n}{n}.$$

The lengths of the cylinders in β-expansion

β-expansion

- digit set

$$\mathcal{A} = \begin{cases} \{0, 1, \ldots, \beta - 1\} & \text{when } \beta \text{ is an integer} \\ \{0, 1, \ldots, \lfloor \beta \rfloor\} & \text{otherwise.} \end{cases}$$

- digit function

$$\varepsilon_1(\cdot, \beta) : [0, 1] \to \mathcal{A} \text{ as } x \mapsto \lfloor \beta x \rfloor$$

- $\varepsilon_n(x, \beta) := \varepsilon_1(T_\beta^{n-1} x, \beta)$
- β-expansion (Rényi, 1957)

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \cdots$$

- notation :

$$\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots, \varepsilon_n(x, \beta), \ldots)$$

admissible sequence

- admissible sequence/word

  $\Sigma_\beta = \{\omega \in \mathcal{A}^{\mathbb{N}} : \ \exists\, x \in [0,1) \text{ such that } \varepsilon(x,\beta) = \omega\}$

  $\Sigma_\beta^n = \{\omega \in \mathcal{A}^n : \ \exists\, x \in [0,1) \text{ such that } \varepsilon_i(x,\beta) = \omega_i \text{ for all } i = 1, \cdots, n\}$

- β is an integer

$$\Sigma_\beta = \mathcal{A}^{\mathbb{N}} \quad \text{(except countable points)}$$

- Example : $\beta_0 = \frac{\sqrt{5}+1}{2}$

$$\Sigma_{\beta_0} = \{\omega \in \{0,1\}^{\mathbb{N}} : \text{ the word } 11 \text{ dosen't appear in } \omega\}$$

- number of admissible words of length n

$$\beta^n \le \sharp\Sigma_\beta^n \le \frac{\beta^{n+1}}{\beta - 1}$$

admissible sequence

- the infinite expansion of the number 1

$$\varepsilon^*(1,\beta) = \begin{cases} \varepsilon(1,\beta) & \text{if there are infinite many } \varepsilon_n(1,\beta) \neq 0 \text{ in } \varepsilon(1,\beta) \\ \big(\varepsilon_1(1,\beta),\cdots,(\varepsilon_n(1,\beta)-1)\big)^\infty & \text{otherwise, where } \varepsilon_n(1,\beta) \text{ is the last non-zero element in } \varepsilon(1,\beta). \end{cases}$$

Theorem (Parry, 1960)

Let $\beta > 1$ be a real number and $\varepsilon^*(1,\beta)$ the infinite expansion of the number 1. Then $\omega \in \Sigma_\beta$ if and only if

$$\sigma^k(\omega) \prec \varepsilon^*(1,\beta) \quad \textit{for all} \quad k \geq 0,$$

where $\prec$ means the lexicographical order.

self-admissible sequence

Corollary (Parry, 1960)

w is the β-expansion of 1 for some $\beta \Longleftrightarrow \sigma^k(w) \preceq w$ for all $k \geq 0$

- self-admissible sequence

$$\sigma^k(w) \preceq w \quad \text{for all} \quad k \geq 0$$

- cylinder of order n ($(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \Sigma_\beta^n$)

$$I_n(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = \{x \in [0,1) : \varepsilon_k(x) = \varepsilon_k, 1 \leq k \leq n\}$$

- full cylinder

$$\big|I_n(w_1, \cdots, w_n)\big| = \beta^{-n}$$

a kind of classification of $\beta > 1$

- $t_n(\beta) := \max\{\, k \geq 0 : \varepsilon^*_{n+1}(1,\beta) = \cdots = \varepsilon^*_{n+k}(1,\beta) = 0 \,\}$
- $t(\beta) = \limsup\limits_{n\to\infty} \frac{t_n(\beta)}{n}$
- A kind of classification of $\beta > 1$:

$$A_0 = \Big\{ \beta > 1 : \{t_n(\beta)\} \text{ is bounded } \Big\};$$
$$A_1 = \Big\{ \beta > 1 : \{t_n(\beta)\} \text{ is unbounded and } \ t(\beta) = 0 \Big\};$$
$$A_2 = \Big\{ \beta > 1 : t(\beta) > 0 \Big\}.$$

Theorem (Li and Wu, 2008)

(1) $\beta \in A_0 \Longleftrightarrow C\beta^{-n} \leq |I_n(x)| \leq \beta^{-n}$ for any $x \in [0,1]$ and $n \geq 1$, where C is a constant.
(2) $\beta \in A_0 \cup A_1 \Longleftrightarrow \lim\limits_{n\to\infty} -\frac{\log |I_n(x)|}{n} = \log \beta$ for any $x \in [0,1]$.

Distribution of regular cylinders in parameter space

cylinders in parameter space

- Recall :
  a word $w = (\varepsilon_1, \cdots, \varepsilon_n)$ is called self-admissible if $\sigma^i w \preceq w$ for all $1 \le i < n$, that is,

$$\sigma^i(\varepsilon_1, \cdots, \varepsilon_n) \preceq \varepsilon_1, \cdots, \varepsilon_n.$$

Definition

Let $(\varepsilon_1, \cdots, \varepsilon_n)$ be self-admissible. A cylinder in the parameter space is defined as

$$I_n^P(\varepsilon_1, \cdots, \varepsilon_n) = \Big\{\beta > 1 : \varepsilon_1(1, \beta) = \varepsilon_1, \cdots, \varepsilon_n(1, \beta) = \varepsilon_n\Big\},$$

i.e., the collection of β for which the β-expansion of 1 begins with $\varepsilon_1, \cdots, \varepsilon_n$.

cylinders in parameter space

- (Schmeling, 1997)
  The cylinder $I_n^P(\varepsilon_1, \cdots, \varepsilon_n)$ is a half-open interval $[\beta_0, \beta_1)$. The left endpoint β_0 is given as the only solution in $(1, \infty)$ to the equation
  $$1 = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n}.$$
  The right endpoint β_1 is given as the limit of the solutions $\{\beta_N\}_{N \geq 1}$ in $(1, \infty)$ to the equations
  $$1 = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n} + \frac{\varepsilon_{n+1}}{\beta^{n+1}} + \cdots + \frac{\varepsilon_N}{\beta^N},$$
  where $(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1}, \ldots, \varepsilon_N)$ is the maximal self-admissible word beginning with $\varepsilon_1, \cdots, \varepsilon_n$ in the lexicographical order. Moreover,
  $$\left| I_n^P(\varepsilon_1, \ldots, \varepsilon_n) \right| \leq \beta_1^{-n}.$$
- Remark : If the left endpoint of $I_n^P(\varepsilon_1, \cdots, \varepsilon_n)$ is 1, then the cylinder will be an open interval. For example, $I_2^P(1, 0) = (1, \frac{1+\sqrt{5}}{2})$.

maximal self-admissible sequence

Definition

Let $w = (\varepsilon_1, \cdots, \varepsilon_n)$ be a word of length n. The recurrence time $\tau(w)$ of w is defined as

$$\tau(w) := \inf\left\{k \geq 1 : \sigma^k(\varepsilon_1, \cdots, \varepsilon_n) = \varepsilon_1, \cdots, \varepsilon_{n-k}\right\}.$$

If such an integer k does not exist, then $\tau(w)$ is defined to be n and w is said to be of full recurrence time.

Theorem

Let $w = (\varepsilon_1, \ldots, \varepsilon_n)$ be self-admissible with $\tau(w) = k$. Then the periodic sequence

$$(\varepsilon_1, \cdots, \varepsilon_k)^\infty$$

is the maximal self-admissible sequence beginning with $\varepsilon_1, \cdots, \varepsilon_n$.

lengths of cylinders in parameter space

Theorem

Let $w = (\varepsilon_1, \cdots, \varepsilon_n)$ be self-admissible with $\tau(w) = k$. Let β_0 and β_1 be the left and right endpoints of $I_n^P(\varepsilon_1, \cdots, \varepsilon_n)$. Then we have

$$\left|I_n^P(\varepsilon_1, \cdots, \varepsilon_n)\right| \geq \begin{cases} C\beta_1^{-n}, & \text{when } k{=}n\,; \\ C\left(\frac{\varepsilon_{t+1}}{\beta_1^{n+1}} + \cdots + \frac{\varepsilon_k+1}{\beta_1^{(\ell+1)k}}\right), & \text{otherwise.} \end{cases}$$

where $C := (\beta_0 - 1)^2$ is a constant depending on β_0 ; the integers t and ℓ are given as $\ell k < n \leq (\ell+1)k$ and $t = n - \ell k$.

- regular cylinder
  When $(\varepsilon_1, \cdots, \varepsilon_n)$ is of full recurrence time, the length

$$C\beta_1^{-n} \leq |I_n^P(\varepsilon_1, \cdots, \varepsilon_n)| \leq \beta_1^{-n},$$

in this case, $I_n^P(\varepsilon_1, \cdots, \varepsilon_n)$ is called regular cylinder.

distribution of regular cylinders in parameter space

- Denote by C_n^P the collection of cylinders of order n in parameter space.

Corollary

Among any n consecutive cylinders in C_n^P, there is at least one with full recurrence time, hence with regular length.

- This corollary was established for the first time by Persson and Schmeling (2008).

Recall main result

Theorem

Let $x_0 \in [0,1]$ and let $\{\ell_n\}_{n\geq 1}$ be a sequence of integers such that $\ell_n \to \infty$ as $n \to \infty$. Then

$$\dim_H E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \frac{1}{1+\alpha}, \text{ where } \alpha = \liminf_{n\to\infty} \frac{\ell_n}{n}.$$

- The generality of $\{\ell_n\}_{n\geq 1}$ arises no extra difficulty compared with special $\{\ell_n\}_{n\geq 1}$.
- The difficulty comes from that $x_0 \neq 0$ has no uniform β-expansion for different β.
- When $x_0 \neq 1$, the set $E(\{\ell\}_{n\geq 1}, x_0)$ can be regarded as a type of shrinking target problem. While $x_0 = 1$, it becomes a type of recurrence properties.
- The notion of the recurrence time of a word in symbolic space is introduced to characterize the lengths and the distribution of cylinders in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$.

More general theorem

- the set $E\big(\{\ell_n\}_{n\geq 1}, x_0\big)$ concerns points in the parameter space $\{\,\beta > 1 : \beta \in \mathbb{R}\,\}$ for which the orbit $\{\,T_\beta^n 1 : n \geq 1\,\}$ is close to the same magnitude $x(\beta) = x_0$ for infinitely many moments in time.
- What can be said if the magnitude $x(\beta)$ is also allowed to vary continuously with $\beta > 1$?
- Let $x = x(\beta)$ be a function on $(1, +\infty)$, taking values on $[0, 1]$. The set $E\big(\{\ell_n\}_{n\geq 1}, x_0\big)$ changes to

$$\widetilde{E}\big(\{\ell_n\}_{n\geq 1}, x\big) = \Big\{\, \beta > 1 : |T_\beta^n 1 - x(\beta)| < \beta^{-\ell_n},\ \text{i.o.} \Big\}.$$

Theorem

Let $x = x(\beta) : (1, +\infty) \to [0, 1]$ be a Lipschtiz continuous function and $\{\ell_n\}_{n\geq 1}$ be a sequence of positive integers such that $\ell_n \to \infty$ as $n \to \infty$. Then

$$\dim_H \widetilde{E}\big(\{\ell_n\}_{n\geq 1}, x\big) = \frac{1}{1+\alpha}, \quad \textit{where } \alpha = \liminf_{n\to\infty} \frac{\ell_n}{n}.$$

Application : sizes of A_0, A_1, A_2

Theorem

(1) $\mathcal{L}(A_0) = 0$ *and* $\dim_H(A_0) = 1$ *(already known by Schmeling, 1997).*
(2) The set A_1 *is of full Lebesgue measure.*
(3) $\mathcal{L}(A_2) = 0$ *and* $\dim_H(A_2) = 1$.

Thanks for your attention !