Dynamics of a family of piecewise linear maps: combinatorics and entropy.

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June 2014

Work (in progress) with

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Sefano Marmi (SNS)

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Sefano Marmi (SNS) Alessandro Profeti (SNS)

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Sefano Marmi (SNS) Alessandro Profeti (SNS)
Henk Bruin (Wien)

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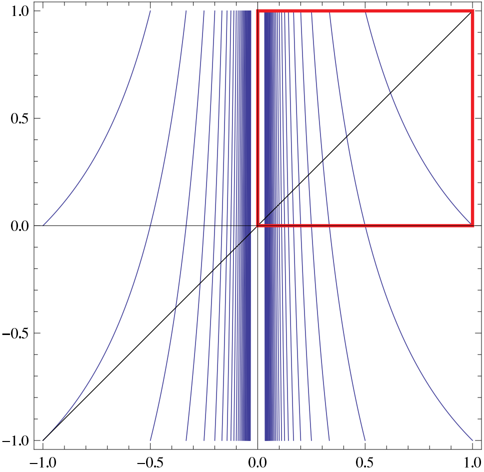

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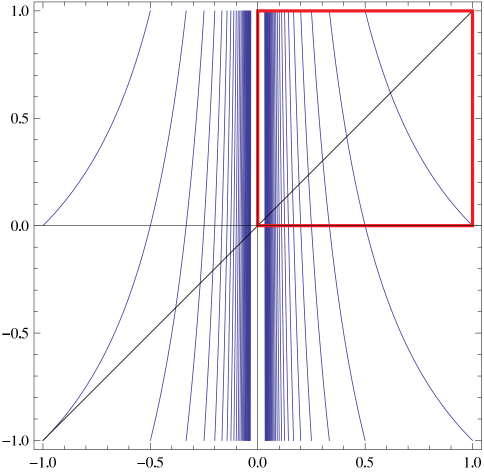

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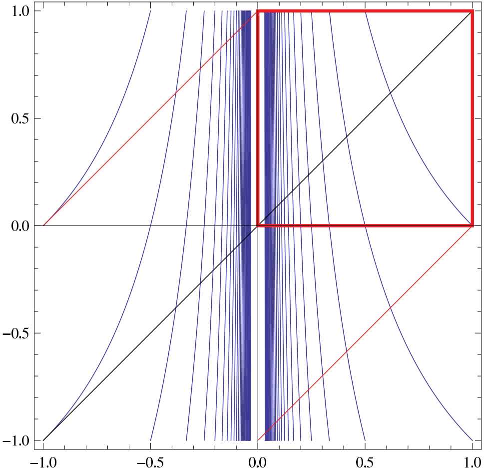

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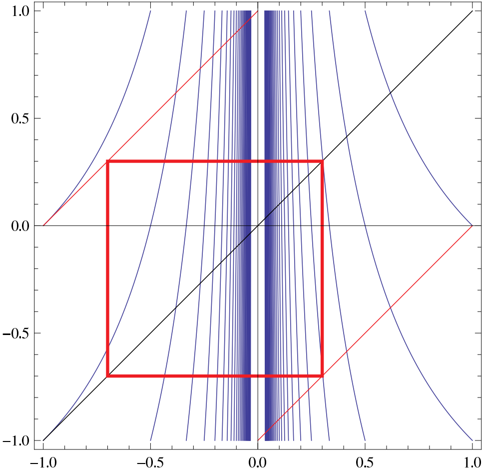

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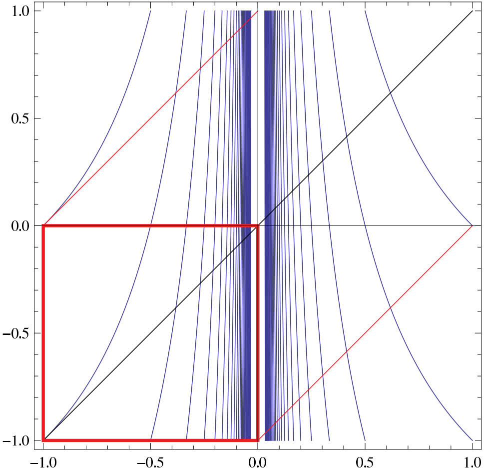

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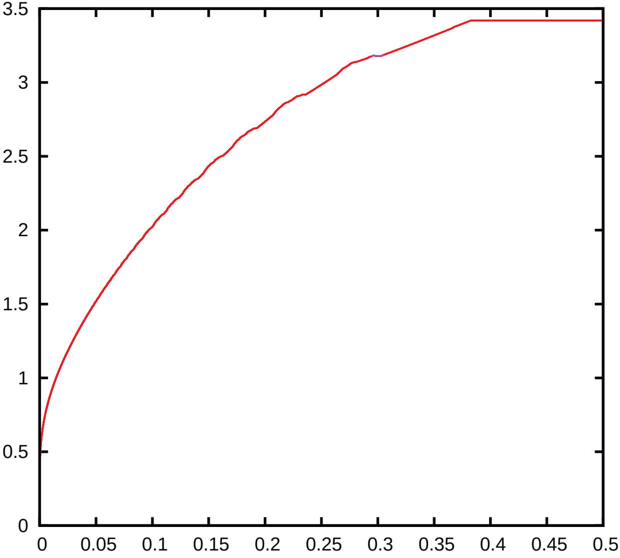

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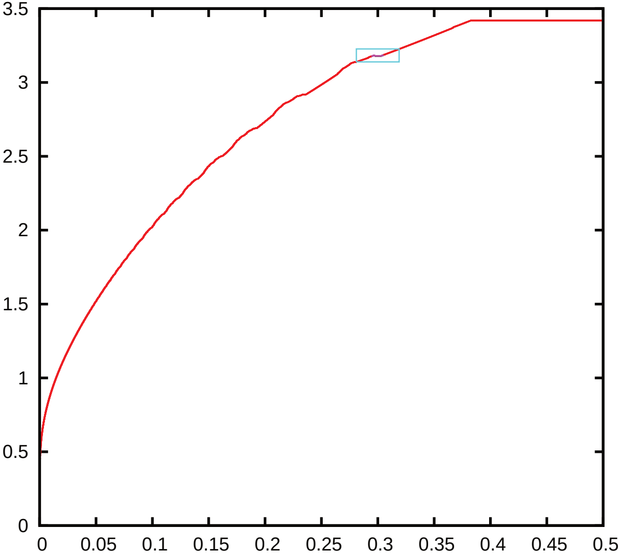

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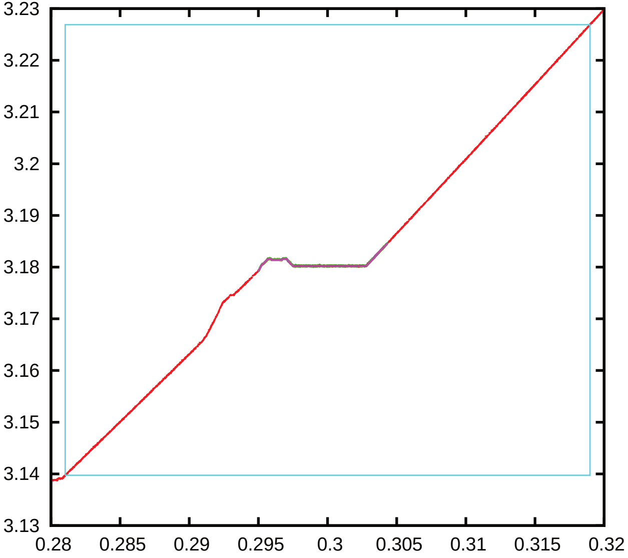

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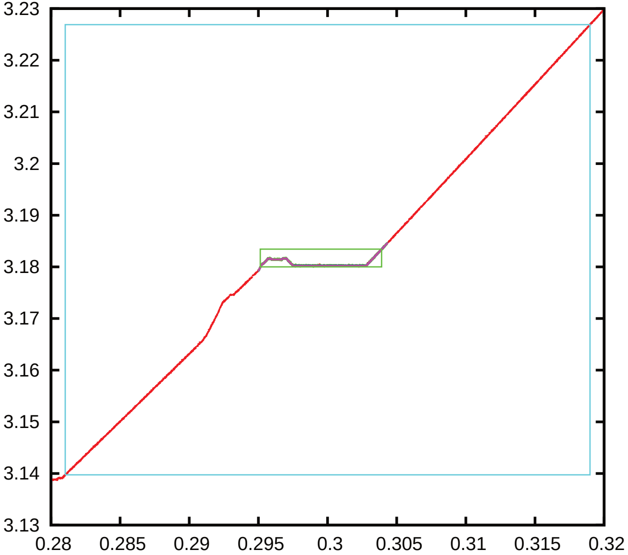

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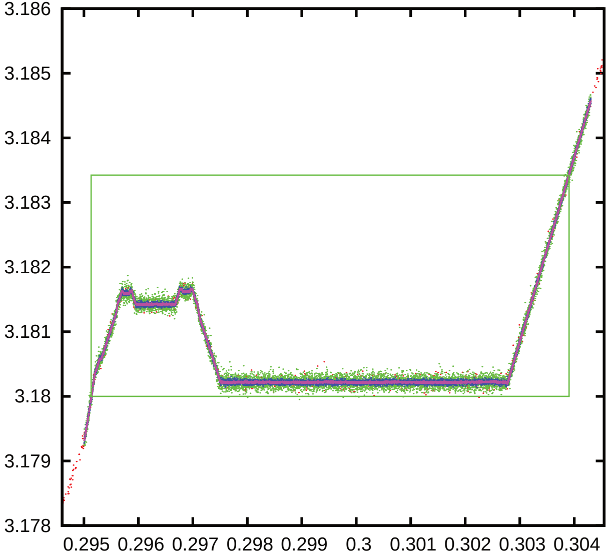

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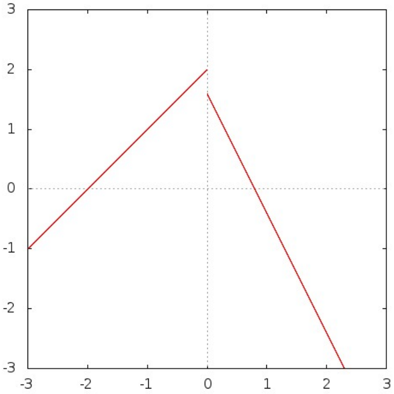

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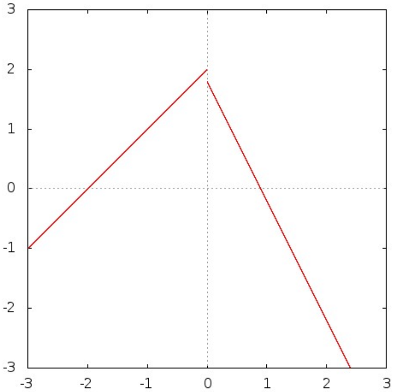

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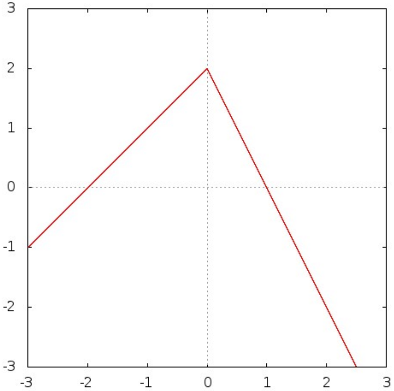

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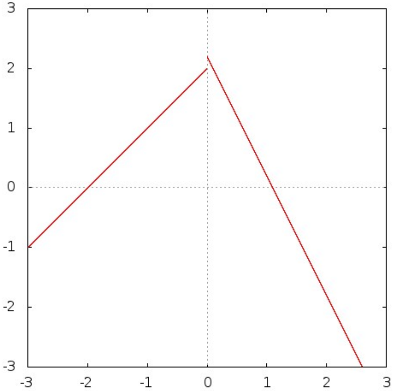

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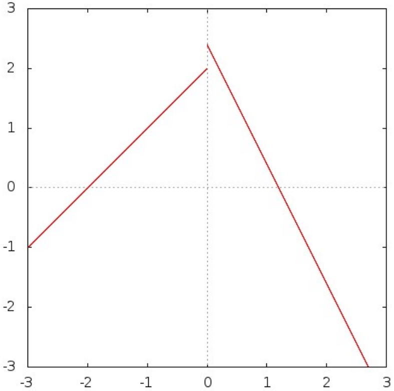

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The entropy can be computed using Rohlin formula

$$h(T_\beta) = \int \log|T'_\beta(x)| d\mu_\beta(x) = (\log s)\mu_\beta([0, +\infty]).$$

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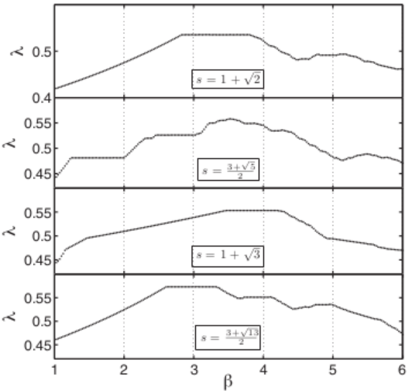

Figure 4. Lyapunov exponents as a function of β for the case of s taking the value of different quadratic Pisot numbers.

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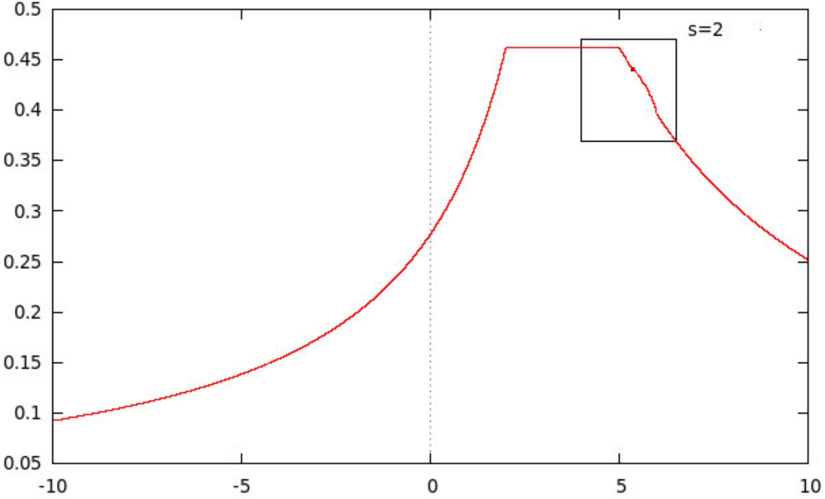

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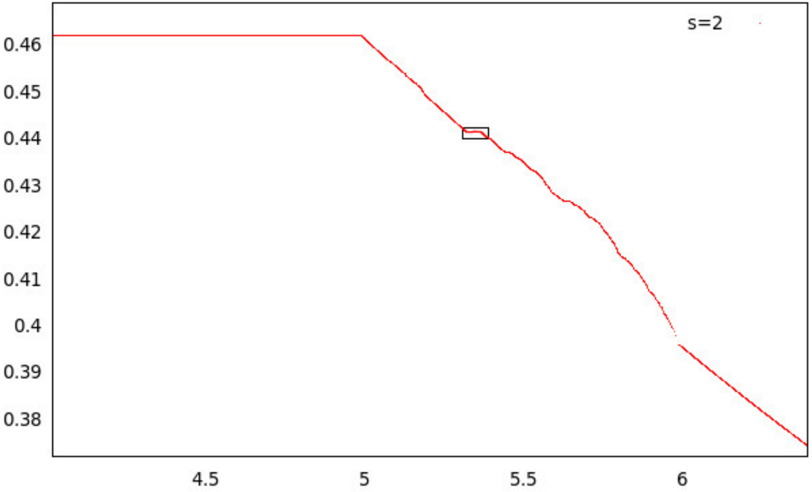

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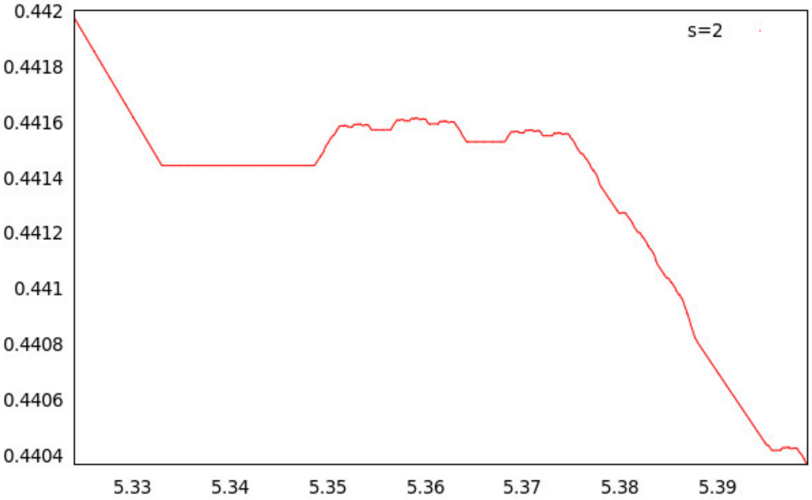

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When this condition holds, we can compute both the invariant density (which is locally constant) and the entropy.

Movie(s)

Remark: when the fixed point is no more in the image of the left branch the support of the invariant measure gets disconnected
Show movie!

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Thus for $s > 1$ (fixed) we consider the following family of maps depending on $\gamma \in \mathbb{R}$

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$\beta := 2(1 + s)(1 - \gamma)$

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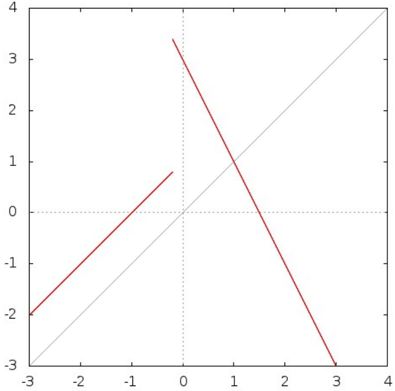

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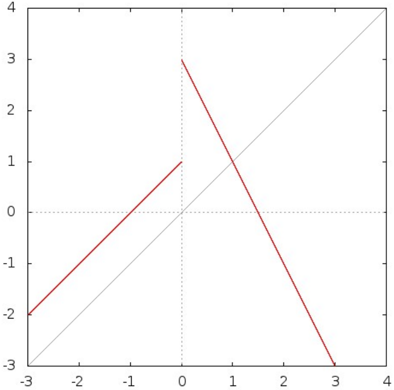

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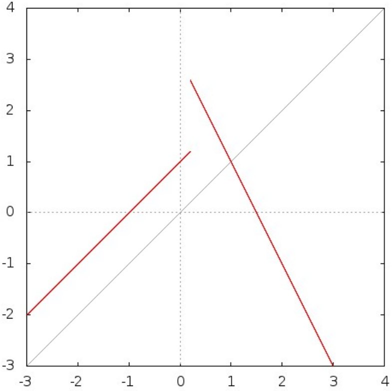

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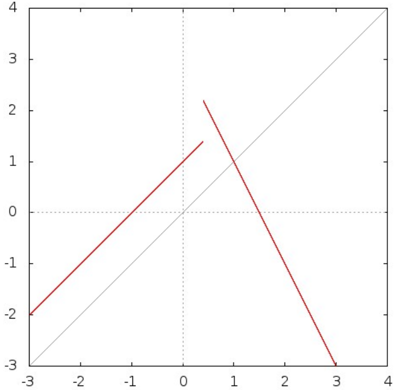

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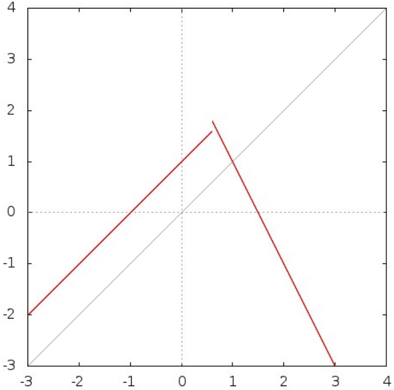

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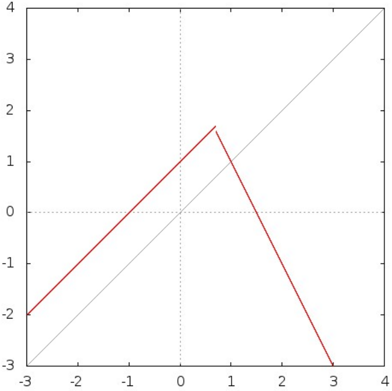

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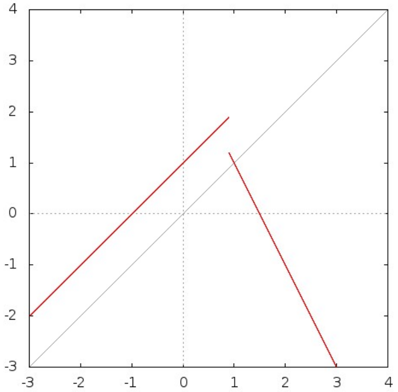

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Here $k^- = 3$, $k^+ = 2 + s^2$, and derivatives match as well, since the expanding branch has been used the same number of times for the upper and lower orbit.

For $\gamma \in]\frac{s}{s+1}, +\infty[$ matching holds as well:

lower	$\gamma \mapsto \gamma + 1 \overset{+}{\mapsto} 1 - s\gamma \mapsto \ldots \mapsto 1 + s - s\gamma$	$k^- = 2 + s$
upper	$\gamma \overset{+}{\mapsto} 1 + s - s\gamma$	$k^+ = 1$

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For all integer values s of the slope, matching is prevalent (has full measure).

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Being bold, one could also ask the following question:
what conditions on the slope s characterize prevalence of matching?

Matching $\Rightarrow$ monotonicity: NN formula.

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Moreover, if $\gamma = \frac{p}{s^q}$ with $p, q \in \mathbb{N}$ then both the upper and the lower rbit of γ end up in the fixed point, and matching holds.

On the other hand, if the upper (or lower) orbit of γ ends up in the fixed point then $\gamma = \frac{p}{s^q}$.

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Let $x \in [0, 1)$ and let $R(x)$ denote the first return of $Q_\gamma^k(x) on [0, 1)$. Then

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Of course R is not defined for $x = 0$ because in this case $Q_\gamma^k(x) = 1$ for all $k \geq 1$ (never returns).

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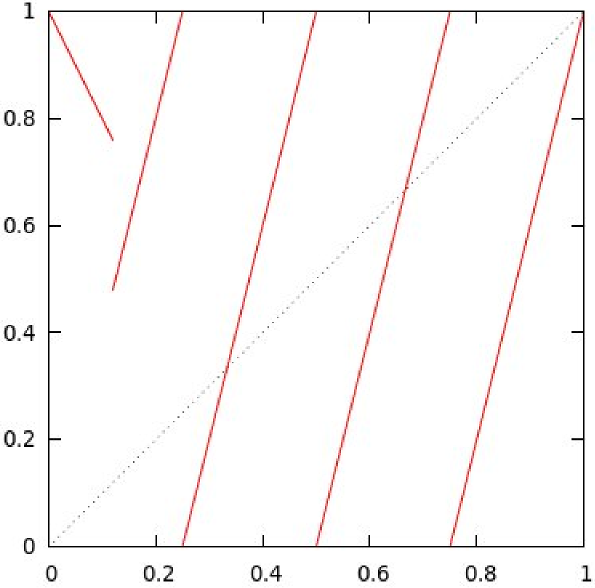

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It is easy to check that the upper and lower orbit of γ begin as follows:

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2. some lower and upper iterates both attain $g^k(\gamma)$ for some k (matching).

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Characterization of the bifurcation set

Theorem

The bifurcation set is

$$\mathcal{E} := \{\gamma \in [0,1] \ : \ g^k(\gamma) \geq \gamma \ \forall k \in \mathbb{N}\}. \qquad (4)$$

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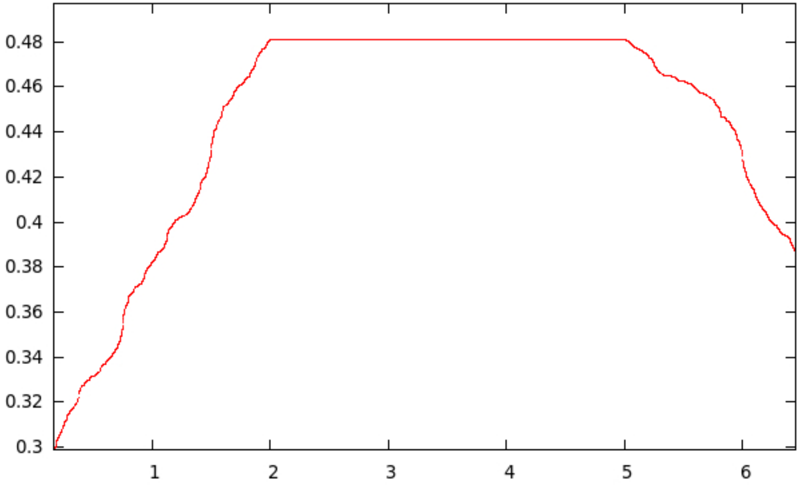
Theorem

For s integer, the bifurcation set $\mathcal{E}$ has zero lebesgue measure.

The end

Thank for your attention.

Topological entropy



Magic circle

Who has seen this guy?

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