Shortest paths on Sierpiński graphs and distances on the Sierpiński gasket

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Overview

- Definitions. Introducing the problem
- Sierpiński graphs in the Euclidean plane. Graph distances. Shortest paths.
- The geodesic distance on the Sierpiński gasket

Introducing the problem. Definitions

Let $P_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $P_2 = (0,0)$, and $P_3 = (1,0)$. For $i = 1,2,3$ let

$$\phi_i(x) = \frac{1}{2}(x - P_i) + P_i.$$

The invariant set of the set of contractions $\{\phi_1, \phi_2, \phi_3\}$,

$$\mathcal{G} = \bigcup_{i=1}^{3} \phi_i\left(\mathcal{G}\right)$$

is a self similar fractal called *the Sierpiński gasket*.

Sierpiński graphs

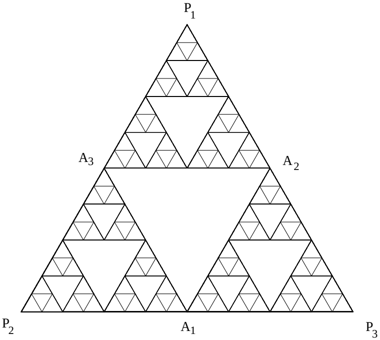
A sequence of graphs $\{G_n\}_{n\geq 0}$, $G_n = (V(G_n), E(G_n))$ related to $\mathcal{G}$ is defined as follows. Let $V(G_0) = \{P_1,\ P_2,\ P_3\}$, and $E(G_0) = \{\{P_1, P_2\},\{P_2, P_3\},\{P_1, P_3\}\}$. For $n \geq 1$, the *n-th Sierpiński graph* $G_n = (V(G_n), E(G_n))$ is defined by

$$V(G_n) = \bigcup_{1\leq i_1, i_2, \ldots, i_n \leq 3} \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n} (V(G_0)),$$

and $E(G_n) = \{\{\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n}(P_k), \phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_n}(P_l)\} | 1 \leq i_1, \ldots, i_n \leq 3,\ 1 \leq k, l \leq 3\}$.

Notation: d_n is the graph distance in G_n, for $n \geq 0$.

Example: G_4



Remark: The closure of the set $\cup_{n\geq 0} V(G_n)$ with respect to the Euclidean topology is the Sierpiński gasket.

(e.g., Yamaguti, Hata, Kigami (1997), Tichy, Grabner(1998),. . .)

The geodesic metric on the Sierpiński gasket

Let $x, y \in \mathcal{G}$ and, for all $n \geq 0$, let $\Delta_n(x), \Delta_n(y) \in \mathcal{T}_n$ be two elementary triangles of level n such that $x \in \Delta_n(x)$ and $y \in \Delta_n(y)$. For all $n \geq 0$ let x_n and y_n be the left lower vertices of $\Delta_n(x)$ and $\Delta_n(y)$, respectively. Thus $x_n, y_n \in V(G_n)$. The geodesic distance between x and y is defined as

$$d_{geod}(x, y) = \lim_{n \to \infty} 2^{-n} \cdot d_n(x_n, y_n).$$

- this distance also occurs at Barlow and Perkins (1988), Grabner and Tichy (1998), Strichartz (1999)

Already existing results

- Hinz, Schief (1990): for any $x \in \mathcal{G}$ there is a rectifiable curve in $\mathcal{G}$ with length ≤ 1 joining x and the vertex P_i, i=1,2,3. Application: definition of a geodesic metric on $\mathcal{G}$
- Hinz, Schief (1990): the average distance between 2 points on the gasket is $\frac{466}{885}$.
- Hinz (1989, 1992) - connection to Tower of Hanoi (graph) with 3 pegs.
- Grabner, Tichy (1998) - equidistribution and Brownian motion on $\mathcal{G}$
- Band, Mubarak (2004) - distribution of Euclidean and geodesic distances on $\mathcal{G}$

The problem: distances in Sierpiński graphs

Let $n \geq 1$, $x, y \in V(G_n)$, $x \neq y$, and m be an integer, $1 \leq m < n$ with the property that x and y lie in distinct elementary triangles of level m that contain a common vertex $z \in V(G_m)$. In order to construct a path of minimal length in G_n with respect to the graph distance d_n, one has to decide whether such a path passes through z or through $z_1, z_2 \in V(G_m)$, where z, z_1, and z_2 are the midpoints of other two sides of the triangle of level $m - 1$ that contains x and y.

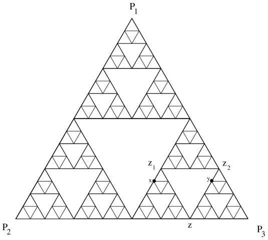


Figure: G_n for $n = 4$, $m = 2$

The problem: distances in Sierpiński graphs

Problem: decide (geometric criteria!), without actually constructing paths, and without comparing lengths of different paths, whether a shortest path that connects x and y in G_n passes through z or through z_1 and z_2

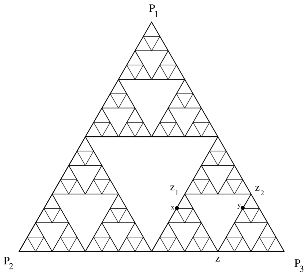
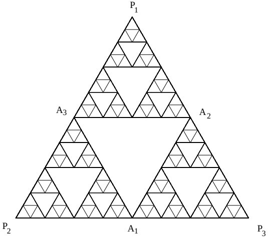


Figure: Here $n = 4$, $m = 2$ and the shortest path from x to y in G_4 does not pass through $z \in V(G_2)$.

Sierpiński graphs in the Euclidean plane. Graph distances

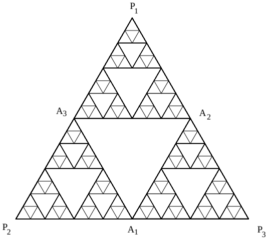
Notations

- $n \geq 0$, A, B, C points in the Euclidean plane, $\overline{ABC}$ the convex hull of A, B and C, $(\overline{ABC})_n = \overline{ABC} \cap V(G_n)$
- A_i the midpoint of the side of the triangle $\overline{P_1P_2P_3}$ opposite to P_i
- for A, B two points in the plane, $A \neq B$:
  $\sigma(A, B)$ the straight line that contains A and B,
- d_{eucl} the Euclidean distance



Sierpiński graphs in the Euclidean plane. Graph distances.

Main result: geometric criterion



Theorem

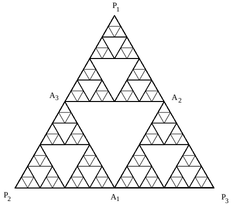
Let $x \in (\overline{A_1 A_3 P_2})_n$ *and* $y \in (\overline{A_2 A_1 P_3})_n$.

1. *If* $\frac{3}{2} + x_1 - \sqrt{3}x_2 \geq y_1 + \sqrt{3}y_2$ *then* $d_n(x, y) = d_n(x, A_1) + d_n(A_1, y)$ *and*
2. *otherwise* $d_n(x, y) = d_n(x, A_3) + d_n(A_3, A_2) + d_n(A_2, y)$.

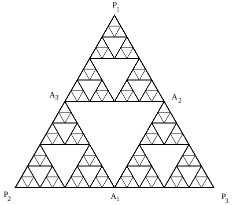
Sierpiński graphs in the Euclidean plane. Graph distances

Proposition

Let $n \geq 0$ and $x \in V(G_n)$. Then, for $i \in \{1, 2, 3\}$, we have $d_n(x, P_i) \leq 2^n$, where the equality holds if and only if x lies on the side of $\overline{P_1 P_2 P_3}$ opposite to P_i.

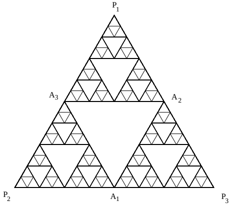


Sierpiński graphs in the Euclidean plane. Graph distances



Proof.
For $i = 1$, by induction on n. For $n = 0$, trivial.
Assume $d_{n-1}(x, P_1) \leq 2^{n-1}$, for all $x \in V(G_{n-1})$, with equality if and only if $x \in V(G_{n-1}) \cap \overline{P_2P_3}$. Let $x \in V(G_n)$.
First, we consider the case when $x \in (\overline{A_3A_2P_1})_n$. By the induction hypothesis applied to $(\overline{A_3A_2P_1})_n$ and x (since the subgraph of G_n induced by the vertex set $(\overline{A_3A_2P_1})_n$ is isomorphic to G_{n-1}), we have $d_n(x, P_1) \leq 2^{n-1} < 2^n$.

Sierpiński graphs in the Euclidean plane. Graph distances



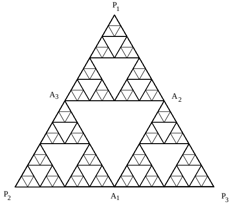
In the case $x \notin (\overline{A_3 A_2 P_1})_n$, let us assume, without loss of generality, that $x \in (\overline{A_1 A_3 P_2})_n$. First, we note that $d_n(x, A_3) \leq d_n(x, A_2)$, since $d_n(x, A_3) \leq 2^{n-1}$ by the induction hypothesis and $d_n(x, A_2) = \min\{d_n(x, A_3) + d_n(A_3, A_2), d_n(x, A_1) + d_n(A_1, A_2)\} \geq 2^{n-1}$. Thus, $d_n(x, P_1) = d_n(x, A_3) + d_n(A_3, P_1) = d_n(x, A_3) + 2^{n-1}$. By the induction hypothesis, we have $d_n(x, A_3) \leq 2^{n-1}$ with equality if and only if $x \in (\overline{A_1 A_3 P_2})_n$ is collinear with P_2 and A_1.

Sierpiński graphs in the Euclidean plane. Graph distances

Proposition

Let $n \geq 0$. For any $x \in V(G_n)$, we have

$$\sum_{i=1,2,3} d_n(x, P_i) = 2^{n+1}.$$

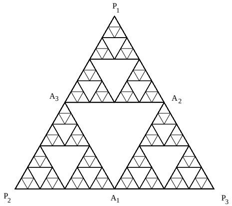


Proof by induction.

Sierpiński graphs in the Euclidean plane. Graph distances

For two integers $i, n \geq 0$, let

- $h_1^n(i) = \{x \in (\overline{A_1 A_3 P_2})_n \mid d_n(x, A_1) - d_n(x, A_3) = i\}$ and
- $h_2^n(i) = \{x \in (\overline{A_2 A_1 P_3})_n \mid d_n(x, A_1) - d_n(x, A_2) = 2^{n-1} - i\}$.



Proposition

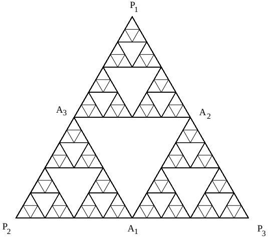
Let $x \in (\overline{A_1 A_3 P_2})_n$ and $y \in (\overline{A_2 A_1 P_3})_n$.

1. *There is one and only one i_1 such that $-2^{n-1} \leq i_1 \leq 2^{n-1}$ and $x \in h_1^n(i_1)$.*
2. *There is one and only one i_2 such that $0 \leq i_2 \leq 2^n$ and $y \in h_2^n(i_2)$.*

Sierpiński graphs in the Euclidean plane. Graph distances

Proposition

Let $x \in (\overline{A_1 A_3 P_2})_n$, $y \in (\overline{A_2 A_1 P_3})_n$, $x \in h_1^n(i_1)$, *and* $y \in h_2^n(i_2)$, *for some integers* i_1, i_2. *Then*
$d_n(x, A_1) + d_n(A_1, y) = d_n(x, A_3) + d_n(A_3, A_2) + d_n(A_2, y) + i_1 - i_2.$



Corollary

Let $x \in (\overline{A_1 A_3 P_2})_n$*,* $y \in (\overline{A_2 A_1 P_3})_n$*,* $x \in h_1^n(i_1)$*, and* $y \in h_2^n(i_2)$*, for some integers* i_1, i_2*. Then we have*

1. $d_n(x, A_1) + d_n(A_1, y) = d_n(x, A_3) + d_n(A_3, A_2) + d_n(A_2, y)$ *if and only if* $i_1 = i_2$*,*
2. $d_n(x, A_1) + d_n(A_1, y) > d_n(x, A_3) + d_n(A_3, A_2) + d_n(A_2, y)$ *if and only if* $i_1 > i_2$*,*
3. $d_n(x, A_1) + d_n(A_1, y) < d_n(x, A_3) + d_n(A_3, A_2) + d_n(A_2, y)$ *if and only if* $i_1 < i_2$*.*

Sierpiński graphs in the Euclidean plane. Geometric aspects

Let T_0 be the triangle whose set of vertices is $V(G_0)$.

Proposition

Let $P \in V(G_0)$ and σ be the straight line that contains the side of T_0 that lies opposite the vertex P and let $I_t^n(P) = \{x \in V(G_n) \mid d_n(x, P) = t\}$, where $t \in \mathbb{Z}$, $0 \leq t \leq 2^n$, and $n \geq 0$.

1. *The set $I_t^n(P)$ is contained in a straight line $\omega_t = \omega_t(P)$ that is parallel to σ.*
2. *The Euclidean distance between ω_t and σ is $\frac{\sqrt{3}}{2^{n+1}}(2^n - t)$.*

Sierpiński graphs in the Euclidean plane. Geometric aspects

Proof. W.l.o.g. assume $P = P_1$. For $t = 0$ the affirmation is trivial. Let $1 \le t \le 2^n$ be arbitrarily fixed and $\omega_t(P_1)$ be the straight line containing the points x_t and y_t, which are defined as follows. The vertex $x_t \in V(G_n)$ lies on the side $\overline{P_1 P_2}$ of T_0, such that $d_n(x_t, P_1) = t$, and the vertex $y_t \in V(G_n)$ lies on the side $\overline{P_1 P_3}$ of T_0, such that $d_n(x_t, P_1) = t$. Thus $\omega_t(P_1)$ is parallel to σ_1, the straight line containing P_2 and P_3.
We proceed in two steps. At the first step, we show (by induction on n) that for $x \in V(G_n) \cap \omega_t(P_1)$ we have $d_n(x, P_1) = t$, (i.e., $V(G_n) \cap \omega_t(P_1) \subseteq I_t^n(P_1)$). At the second step, we show that $d_n(x, P_1) \neq t$ for $x \in V(G_n) \setminus \omega_t(P_1)$.

Sierpiński graphs in the Euclidean plane. Geometric aspects

Proposition

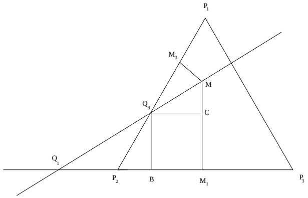
Let σ_k *be the straight line that contains the side of* T_0 *that lies opposite the vertex* P_k *for* $1 \leq k \leq 3$ *and let* $1 \leq i < j \leq 3$*. For all real numbers* a *with* $|a| \leq \frac{\sqrt{3}}{2}$*, the set* $D_{ij}(a) = \{x \in \mathbb{R}^2 \mid d_{eucl}(x, \sigma_i) - d_{eucl}(x, \sigma_j) = a\}$ *is contained in a straight line* $\gamma_{ij}(a)$*, where*

1. $\gamma_{12}(a) : x_1 + \sqrt{3}x_2 = 1 + \frac{2}{\sqrt{3}}a$,
2. $\gamma_{13}(a) : -x_1 + \sqrt{3}x_2 = \frac{2}{\sqrt{3}}a$, *and*
3. $\gamma_{23}(a) : x_1 = \frac{3-2\sqrt{3}a}{6}$,

where by x_1, x_2 *we denote the coordinates in the Euclidean plane.*

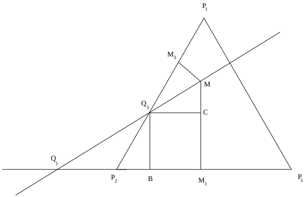
Sierpiński graphs in the Euclidean plane. Geometric aspects

Proof. (for $D_{13}(a)$)



$Q_3 = (\frac{\sqrt{3}}{3}a, a) \in D_{13}(a) \cap \sigma_3$ and thus $D_{13}(a) \neq \emptyset$. Let M be a point in the interior of the triangle T_0, $M \in D_{13}(a)$, i.e., $d_{eucl}(M, \sigma_1) - d_{eucl}(M, \sigma_3) = a$. Assume, w.l.o.g., that $a > 0$ (the case $a < 0$ can be solved analogously). Let M_1 and M_3 be the orthogonal projections of M on σ_1 and σ_3, resp., and B and C be the orthogonal projections of Q_3 on σ_1 and $\sigma(M, M_1)$. Then $d_{eucl}(M, M_1) - d_{eucl}(M, M_3) = a$ and $a = d_{eucl}(Q_3, \sigma_1) = d_{eucl}(Q_3, B) = d_{eucl}(C, M_1)$ $\quad$ ($\overline{BM_1CQ_3}$ rectangle).

Sierpiński graphs in the Euclidean plane. Geometric aspects



We obtain $d_{eucl}(M, C) = d_{eucl}(M, M_3)$ and herefrom, the angles $\angle CQ_3M$ and $\angle MQ_3M_3$ have 30°. Let Q_1 be the intersection point of σ_1 and $\sigma(Q_3, M)$. Then $\angle P_2Q_1Q_3$ has 30° and we infer that $Q_1 = (-\frac{2\sqrt{3}}{3}a, 0)$. Moreover, it follows that $\sigma(Q_1, Q_3) = \gamma_{13}(a)$. As M was arbitrarily chosen in $D_{13}(a)$ and $\sigma(M, Q_3) = \sigma(Q_1, Q_3)$, we conclude that $D_{13}(a) \subseteq \gamma_{13}(a)$.

Sierpiński graphs in the Euclidean plane. Geometric aspects

Proposition

Let $1 \leq i < j \leq 3$. For an integer k, with $-2^n \leq k \leq 2^n$, the points in $\{x \in V(G_n) \mid d_n(x, P_i) - d_n(x, P_j) = k\}$ are contained in the straight line $\gamma_{ij}\left(-\frac{k\sqrt{3}}{2^{n+1}}\right)$.

Proof. (By a previous result)

$$
\begin{aligned}
& \{x \in V(G_n) \mid d_n(x, P_i) - d_n(x, P_j) = k\} \\
= \; & \{x \in V(G_n) \mid d_{eucl}(x, \sigma_i) - d_{eucl}(x, \sigma_j) = -\frac{k\sqrt{3}}{2^{n+1}}\} \\
\subseteq \; & \{x \in \mathbb{R}^2 \mid d_{eucl}(x, \sigma_i) - d_{eucl}(x, \sigma_j) = -\frac{k\sqrt{3}}{2^{n+1}}\},
\end{aligned}
$$

which is contained in the straight line $\gamma_{ij}\left(-\frac{k\sqrt{3}}{2^{n+1}}\right)$ (by previous results).

Sierpiński graphs in the Euclidean plane. Graph distances

- $h_1^n(i) = \{x \in (\overline{A_1 A_3 P_2})_n \mid d_n(x, A_1) - d_n(x, A_3) = i\}$ and
- $h_2^n(i) = \{x \in (\overline{A_2 A_1 P_3})_n \mid d_n(x, A_1) - d_n(x, A_2) = 2^{n-1} - i\}$.

Proposition

1. *For $-2^{n-1} \leq i_1 \leq 2^{n-1}$, the points in $h_1^n(i_1)$ are contained in the straight line $\rho_1^n(i_1) : -x_1 + \sqrt{3}x_2 = \frac{i_1}{2^n}$.*
2. *For $0 \leq i_2 \leq 2^n$, the points in $h_2^n(i_2)$ are contained in the straight line $\rho_2^n(i_2) : x_1 + \sqrt{3}x_2 = \frac{3}{2} - \frac{i_2}{2^n}$.*

Proof. By the last proposition, the straight line $\rho_1^n(i_1)$ is the straight line $\gamma_{13}\left(\frac{i_1\sqrt{3}}{2^{n+1}}\right)$ and $\rho_2^n(i_2)$ is $\gamma_{12}\left(\frac{(-i_2+2^{n-1})\sqrt{3}}{2^{n+1}}\right)$. Then apply the proposition before.

Sierpiński graphs in the Euclidean plane. Graph distances

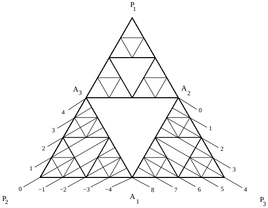


Figure: On the left side are segments of the straight lines $\rho_1^3(i_1)$, for $-4 \leq i_1 \leq 4$, and on the right side are segments of the straight lines $\rho_2^3(i_2)$, for $0 \leq i_2 \leq 8$.

$\rho_1^n(i_1) : -x_1 + \sqrt{3}x_2 = \frac{i_1}{2^n}$
$\rho_2^n(i_2) : x_1 + \sqrt{3}x_2 = \frac{3}{2} - \frac{i_2}{2^n}$

Sierpiński graphs in the Euclidean plane. Graph distances.

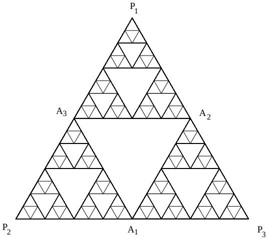
$\rho_1^n(i_1) : -x_1 + \sqrt{3}x_2 = \frac{i_1}{2^n}$
$\rho_2^n(i_2) : x_1 + \sqrt{3}x_2 = \frac{3}{2} - \frac{i_2}{2^n}$

Theorem

Let $x \in (\overline{A_1 A_3 P_2})_n$ and $y \in (\overline{A_2 A_1 P_3})_n$. Then $x \in \rho_1^n(i_1)$ and $y \in \rho_2^n(i_2)$, for one and only one i_1 with $-2^{n-1} \leq i_1 \leq 2^{n-1}$ and one and only one i_2 with $0 \leq i_2 \leq 2^n$. Furthermore,

1. *if $i_1 \leq i_2$ then $d_n(x, y) = d_n(x, A_1) + d_n(A_1, y)$ and*
2. *if $i_1 > i_2$ then $d_n(x, y) = d_n(x, A_3) + d_n(A_3, A_2) + d_n(A_2, y)$.*

Sierpiński graphs in the Euclidean plane. Graph distances.
Main result: geometric criterion



Theorem

Let $x \in (\overline{A_1 A_3 P_2})_n$ *and* $y \in (\overline{A_2 A_1 P_3})_n$.

1. *If* $\frac{3}{2} + x_1 - \sqrt{3}x_2 \geq y_1 + \sqrt{3}y_2$ *then* $d_n(x,y) = d_n(x, A_1) + d_n(A_1, y)$ *and*
2. *otherwise* $d_n(x,y) = d_n(x, A_3) + d_n(A_3, A_2) + d_n(A_2, y)$.

Sierpiński graphs in the Euclidean plane. Graph distances

Final remark. The above results obtained for $x, y \in G_n$, where $n \geq 0$, $x \in (\overline{A_1 A_3 P_2})_n$ and $y \in (\overline{A_2 A_1 P_3})_n$, can be applied to any $x, y \in V(G_n)$:

By the construction of the graphs G_n, $n \geq 0$, it follows that for any integer $n \geq 0$, and for any vertices $x, y \in V(G_n)$, $x \neq y$ there exists an integer m, with $1 \leq m \leq n$ such that x, y lie in distinct elementary triangles of level m that have a common vertex $z \in V(G_m)$, and lie inside the same elementary triangle of level $m-1$. We write $x \in \Delta_m(x)$, $y \in \Delta_m(y)$, $\Delta_m(x) \cap \Delta_m(y) = \{z\}$, and $\Delta_m(x), \Delta_m(y) \subseteq \Delta_{m-1}(x, y) \in \mathcal{T}_{m-1}$. Then, the subgraph of G_n induced by the vertex set $V(G_n) \cap \Delta_{m-1}(x, y)$ is isomorphic to G_{n-m+1}. By applying a similarity f with factor 2^{n-m+1}, $f(x) \in (\overline{A_1 A_3 P_2})_{n-m+1}$, $f(y) \in (\overline{A_2 A_1 P_3})_{n-m+1}$, and $d_n(x, y) = d_{n-m+1}(f(x), f(y))$.

The Sierpiński gasket in the Euclidean plane. The geodesic distance. Geometric aspects

$h_1(i) = \{x \in (\overline{A_1 A_3 P_2})_\infty \mid d_{geod}(x, A_1) - d_{geod}(x, A_3) = i\}$ and
$h_2(i) = \{x \in (\overline{A_2 A_1 P_3})_\infty \mid d_{geod}(x, A_1) - d_{geod}(x, A_2) = \frac{1}{2} - i\}$

Proposition

1. *For* $-\frac{1}{2} \leq i_1 \leq \frac{1}{2}$, *the points in* $h_1(i_1)$ *are contained in the straight line* $\rho_1(i_1) : -x_1 + \sqrt{3}x_2 = i_1$.
2. *For* $0 \leq i_2 \leq 1$, *the points in* $h_2(i_2)$ *are contained in the straight line* $\rho_2(i_2) : x_1 + \sqrt{3}x_2 = \frac{3}{2} - i_2$.

The Sierpiński gasket in the Euclidean plane. The geodesic distance. **The main result: geometric criterion**

Theorem

Let $x \in (\overline{A_1 A_3 P_2})_\infty$ *and* $y \in (\overline{A_2 A_1 P_3})_\infty$. *Then* $x \in \rho_1(i_1)$ *and* $y \in \rho_2(i_2)$, *for one and only one* i_1 *with* $-\frac{1}{2} \le i_1 \le \frac{1}{2}$ *and one and only one* i_2 *with* $0 \le i_2 \le 1$. *Furthermore,*

1. *if* $i_1 \le i_2$ *then* $d_{geod}(x, y) = d_{geod}(x, A_1) + d_{geod}(A_1, y)$ *and*
2. *if* $i_1 > i_2$ *then* $d_{geod}(x, y) = d_{geod}(x, A_3) + d_{geod}(A_3, A_2) + d_{geod}(A_2, y)$.

Theorem

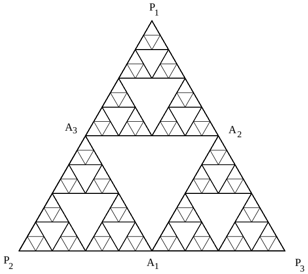
Let $x \in (\overline{A_1 A_3 P_2})_\infty$ *and* $y \in (\overline{A_2 A_1 P_3})_\infty$.

1. *If* $-x_1 + \sqrt{3}x_2 \le \frac{3}{2} - y_1 - \sqrt{3}y_2$ *then* $d_{geod}(x, y) = d_{geod}(x, A_1) + d_{geod}(A_1, y)$ *and*
2. *otherwise* $d_{geod}(x, y) = d_{geod}(x, A_3) + d_{geod}(A_3, A_2) + d_{geod}(A_2, y)$.

Average distances in Sierpiński graphs

Proposition

Let $P \in V(G_0) \subseteq \mathbb{R}^2$ and $x \in V(G_n)$.
The average value of the distance $d_n(P, x)$ is $\frac{2}{3} \cdot 2^n$.



Thank you!

Merçi!

Danke!

The Sierpiński gasket in the Euclidean plane. The geodesic distance

Notations

- for three points A, B, and C in the Euclidean plane: $(\overline{ABC})_\infty$ the set of all points in $\mathcal{G}$, that are contained in the convex hull of A, B and C
- For real i, let $h_1(i) = \{x \in (\overline{A_1 A_3 P_2})_\infty \mid d_{geod}(x, A_1) - d_{geod}(x, A_3) = i\}$ and $h_2(i) = \{x \in (\overline{A_2 A_1 P_3})_\infty \mid d_{geod}(x, A_1) - d_{geod}(x, A_2) = \frac{1}{2} - i\}$

Proposition

Let $x \in (\overline{A_1 A_3 P_2})_\infty$ and $y \in (\overline{A_2 A_1 P_3})_\infty$.

1. *There is one and only one i_1 such that $-\frac{1}{2} \leq i_1 \leq \frac{1}{2}$ and $x \in h_1(i_1)$.*
2. *There is one and only one i_2 such that $0 \leq i_2 \leq 1$ and $y \in h_2(i_2)$.*

The Sierpiński gasket in the Euclidean plane. The geodesic distance

$$h_1(i) = \{x \in (\overline{A_1 A_3 P_2})_\infty \mid d_{geod}(x, A_1) - d_{geod}(x, A_3) = i\}$$
$$h_2(i) = \{x \in (\overline{A_2 A_1 P_3})_\infty \mid d_{geod}(x, A_1) - d_{geod}(x, A_2) = \frac{1}{2} - i\}$$

Proposition

Let $x \in (\overline{A_1 A_3 P_2})_\infty$, $y \in (\overline{A_2 A_1 P_3})_\infty$, $x \in h_1(i_1)$, *and* $y \in h_2(i_2)$, *for some real numbers* i_1 *and* i_2. *Then we have the equality*
$d_{geod}(x, A_1) + d_{geod}(A_1, y) =$
$d_{geod}(x, A_3) + d_{geod}(A_3, A_2) + d_{geod}(A_2, y) + i_1 - i_2$.

Proof. The proof is analogue to that for the Sierpiński graph.

The Sierpiński gasket in the Euclidean plane. The geodesic distance

Corollary

Let $x \in (\overline{A_1 A_3 P_2})_\infty$, $y \in (\overline{A_2 A_1 P_3})_\infty$, $x \in h_1(i_1)$, *and* $y \in h_2(i_2)$, *for some real numbers* i_1 *and* i_2. *Then we have*

1. $d_{geod}(x, A_1) + d_{geod}(A_1, y) = d_{geod}(x, A_3) + d_{geod}(A_3, A_2) + d_{geod}(A_2, y)$ *if and only if* $i_1 = i_2$,
2. $d_{geod}(x, A_1) + d_{geod}(A_1, y) > d_{geod}(x, A_3) + d_{geod}(A_3, A_2) + d_{geod}(A_2, y)$ *if and only if* $i_1 > i_2$, *and*
3. $d_{geod}(x, A_1) + d_{geod}(A_1, y) < d_{geod}(x, A_3) + d_{geod}(A_3, A_2) + d_{geod}(A_2, y)$ *if and only if* $i_1 < i_2$.

The Sierpiński gasket in the Euclidean plane. The geodesic distance. Geometric aspects

Proposition

Let $P \in V(G_0)$ and σ be the straight line that contains the side of T_0 that lies opposite the vertex P. Then, for all $x \in \mathcal{G}$,

$$d_{eucl}(x, \sigma) = \frac{\sqrt{3}}{2}(1 - d_{geod}(x, P)).$$

Corollary

Let t be a real number such that $0 \leq t \leq 1$, $P \in V(G_0)$ and σ be the straight line that contains the side of T_0 that lies opposite the vertex P.

1. *The points $x \in \mathcal{G}$ with $d_{geod}(x, P) = t$ lie on a straight line $\omega_t = \omega_t(P)$, where ω_t is parallel to σ.*
2. *The Euclidean distance between σ and ω_t is $\frac{\sqrt{3}}{2}(1 - t)$.*

The Sierpiński gasket in the Euclidean plane. The geodesic distance. Geometric aspects

Corollary

Let $P \in V(G_0)$ and $x \in \mathcal{G}$. Then $d_{geod}(x, P) = 1$ if and only if x lies on the side of the triangle $\overline{P_1 P_2 P_3}$ opposite to P.

Proposition

Let $n \geq 0$. For any $x \in \mathcal{G}$, we have

$$\sum_{k=1,2,3} d_{geod}(x, P_k) = 2.$$

Proposition

Let $1 \leq i < j \leq 3$. For any real k, with $-1 \leq k \leq 1$, the points in $\{x \in \mathcal{G} \mid d_{geod}(x, P_i) - d_{geod}(x, P_j) = k\}$ are contained in the straight line $\gamma_{ij}\left(-\frac{k\sqrt{3}}{2}\right)$, where $\gamma_{ij}(a)$ is defined as before.