Uniqueness of the ergodic measure with maximal dimension on self-affine sets

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4 novembre 2014

CHAP I

Self-affine sets and associated repeller maps

Self-affine repeller of the plane

- $A_1, \dots, A_r \in GL(\mathbb{R}^2)$ with $\|A_i\| < 1$ and $a_1, \dots, a_r \in \mathbb{R}^2$ and
$$R_i(x) = A_i(x) + a_i \ ;$$
- (Huchinson - 81) $\exists!$ compact $\emptyset \neq K \subset \mathbb{R}^2$, s.t.
$$K = R_1(K) \cup \cdots \cup R_r(K)$$
- (Folklore) For any proba $p = (p_1, \dots, p_r)$, $\exists!$ proba μ
$$\mu = p_1(R_1 \cdot \mu) + \cdots + p_r(R_r \cdot \mu)$$
- $K = \pi(\Sigma)$ where $\pi : \Sigma := \{1, \dots, r\}^{\mathbb{N}} \to \mathbb{R}^2$ is s.t.
$$\pi(\xi_1 \xi_2 \cdots) = R_{\xi_1} \circ \cdots R_{\xi_n}(0)$$

When the S_i are similarities

Theorem

If $\|R_i(x) - R_i(x)\| = c_i\|x - y\|$ and the OSC holds

- (Moran) The S_i are similarities with contraction $0 < c_1 < 1$:

$$\boxed{\dim_H K = \dim_M K = s \quad \text{where} \quad c_1^s + \cdots + c_r^s = 1}$$

- Moreover K is an s-set i.e. $0 < \mathcal{H}^s(K) < +\infty$ and

$$\boxed{\mathcal{H}^s(\cdot|K) = c_1^s\Big(S_1 \cdot \mathcal{H}^s(\cdot|K)\Big) + \cdots + c_r^s\Big(S_r \cdot \mathcal{H}^s(\cdot|K)\Big)}$$

When the R_i are not similarities

When the S_i are not similarities there are natural questions :

- When is $s_* = \dim_M K$ equal to $s_0 = \dim_H K$ a s-set ?
- When is it true that $0 < \mathcal{H}^{s_0}(K) < +\infty$?
- Suggestion : ? $\quad 0 < \mathcal{H}^{s_0}(K) < +\infty \iff s_0 = s_*$?
- Suggestion (Peres 94) : ? $\quad s_0 < s_* \Longrightarrow \mathcal{H}^{s_0}(K) = +\infty$?
- If K is a s_0-set, is $\mathcal{H}(\cdot|K)$ a s_0-Ahlfors regular measure ?
- If K is a s_0-set, how close is $\mathcal{H}(\cdot|K)$ from a self-affine measure ?
- more precisely, is there a probability measure μ s.t

$$\mu = \sum_i p_i R_1 \cdot \mu \quad \text{and} \quad \frac{1}{C}\mu(B) \leq \mathcal{H}(B|K) \leq C\mu(B)$$

When the R_i are not similarities

If $\mathcal{H}^{s_0}(K) = +\infty$ then $\mathcal{H}^{s_0}(\cdot)$ is not sigma finite. A question is :

is there a probability measure on K
which is *faithful* to the fractal nature of K ?

Possible answer :

- consider a suitable (measurable) transformation $S : K \to K$
- prove existence an uniqueness of a S-ergodic measure μ s.t.

$$\dim_H \mu = \dim_H K$$

- nature/properties (Gibbs...) of the ergodic measure of maximal dimension ?

This is the program of the variational principle for dimension.

When the R_i are not similarities

To define $S : K \to K$ we need separation conditions :

- $\{R_i\}_i$ satisfies the Strong-Separation-Condition (SSC) if

  $\exists U \neq \emptyset$ open s.t. $\quad U \subset R_l(U) \sqcup \cdots \sqcup R_r(U) \quad (\Longrightarrow K$ tot. disc.)

- $\{R_i\}_i$ satisfies the Open-Set-condition (OSC) if

  $$\exists U \neq \emptyset \text{ open s.t. } \quad U \supset R_l(U) \sqcup \cdots \sqcup R_r(U))$$

- If (SSC) or (OSC) then $\exists\ V_1, \ldots, V_r$ s.t.

  $$\mathbb{R}^2 = V_1 \sqcup \cdots \sqcup V_r \quad \text{and} \quad V_i \supset R_i(U)$$

- the repeller map $S : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $S_{|V_i} = R_i^{-1}$ and

  $$x \in K \iff x, S(x), S^2(x), \ldots \quad \text{is a bounded orbit.}$$

- $S(K) = K$ and $S : K \to K$ is the K-repeller map.

When the R_i are not similarities

- If (SSC) then $\begin{cases} S : K \to K & \text{continuous} \\ \pi : \Sigma \to K & \text{homeomorphism} \\ \pi \circ \sigma = S \circ \pi & \text{(top. conj.)} \end{cases}$

- If (OSC) then $\begin{cases} S : K \to K & \text{measurable} \\ \pi : \Sigma \to K & \text{continuous (surjective)} \\ \pi \circ \sigma(\xi) = S \circ \pi(\xi) & \text{(for } \xi \in \Sigma' \approx \Sigma\text{).} \end{cases}$

Falconer's model (88-92)

Falconer's work (88-92) about self-affine sets :

- $A \in GL_2(\mathbb{R})$ with singular values $\alpha(A) \geq \beta(A) > 0$
- $\phi^s(A) = \begin{cases} \alpha(A)^s & \text{if } 0 \leq s \leq 1 \\ \alpha(A)\beta(A)^{s-1} & \text{if } 1 \leq s \leq 2 \end{cases}$

Falconer 88

Let $B \subset \mathbb{R}^2$ and $R : x \mapsto Ax + a$ any non singular affine map : then

$$\frac{1}{C} \cdot \frac{1}{\phi^s(A^{-1})} \mathcal{H}^s(B) \leq \mathcal{H}^s(R(B)) \leq C \cdot \phi^s(A) \mathcal{H}^s(B) \qquad (1)$$

When the R_i are not similarities

The Falconer pressure of the affine-IFS $\{R_i : x \mapsto A_i x + a_i\}_{i=1}^r$ is

$$[0\,;+\infty[\ni s \mapsto P_F(s) = \lim_{n\to+\infty} \frac{1}{n} \log\left(\sum_{|w|=n} \phi^s(A_w)\right)$$

Falconer/Solomyak 88/98

Let K be the attractor of the affine IFS $\{R_i : x \mapsto A_i x + a_i\}_{i=1}^r$ satisfying (OSC) ; then $\exists!\ s \geq 0$ s.t. $P_F(s) = 0$ and

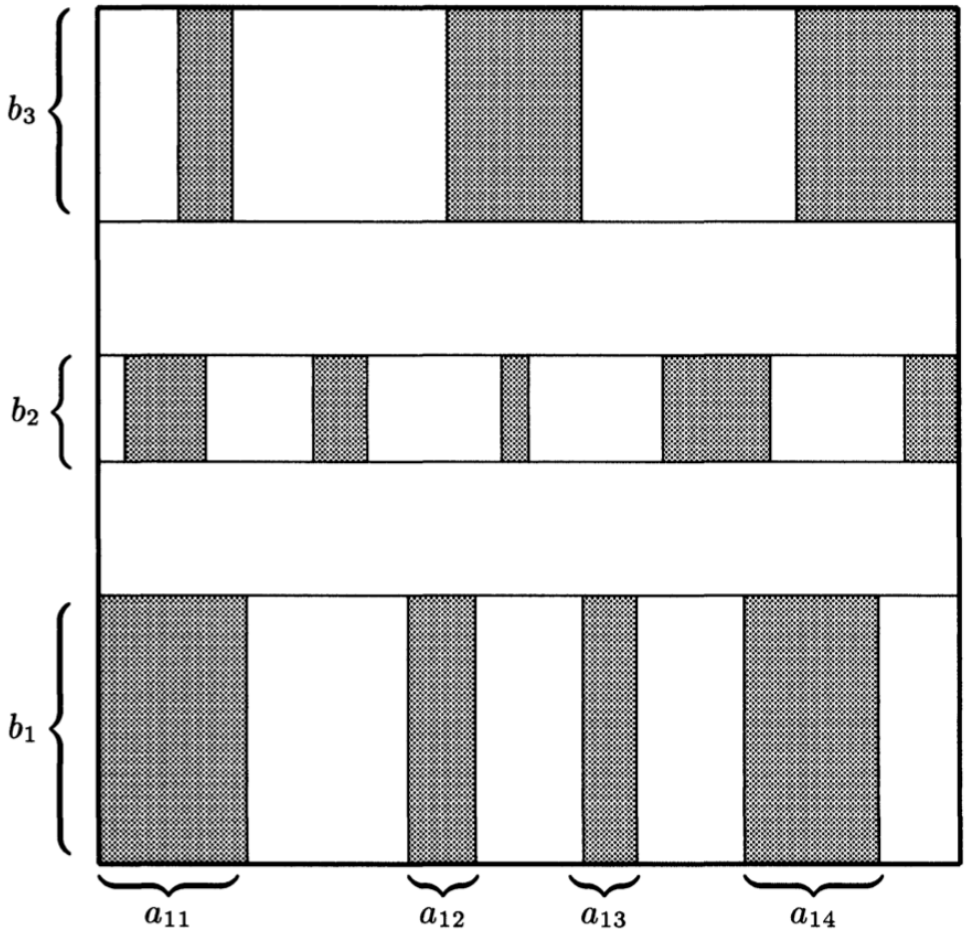
$$\dim_H K \leq \dim_M K \leq s$$

Moreover, if $\max_i\{\|A_i\|\} < 1/2$ then, for a.e. $(a_1, \ldots, a_r)$,

$$\dim_H K = \dim_M K = s$$

Solomyak's bound $1/2$ is sharp (see the Rademacher graphs G_β with $1 < \beta < 2$ related to "an Erdoös problem").

Gatzouras-Lalley model (1992)



Gatzouras-Lalley model (1992)

Theorem (Gatzouras-Lalley model – 1992)

- $K = \bigcup_{i,j} R_{i,j}(K)$ with

$$R_{i,j}X = \begin{pmatrix} a_{i,j} & 0 \\ 0 & b_i \end{pmatrix} X + c_{i,j}$$

- $s_0 = \dim_H K$ and $s_* = \dim_M K$;
- if $r > 0$ is s.t. $\sum_i b_i^r = 1$, then

$$0 < \mathcal{H}^{s_*}(K) < +\infty \iff s_0 = s_* \iff \sum_{i,j} a_{i,j}^{s_*-r} = 1$$

(in this model the dimensional hiatus is *generic*)

Gatzouras-Lalley model (1992)

Gatzouras-Lalley (1992)

Proposition 4.3. *Let r be the unique real number such that $\sum\sum b_i^r a_{ij}^{\delta-r} = 1$, and let $\mathbf{p} \in \Sigma$ be the probability vector defined by*

$$p_{ij} = b_i^r a_{ij}^{\delta-r} \qquad \forall (i,j) \in \mathcal{J}. \tag{4.5}$$

If

$$0 < H_\delta(\Lambda) < \infty, \tag{4.6}$$

then

$$\sum_{j=1}^{n_i} a_{ij}^{\delta-r} = 1 \qquad \forall i = 1, 2, \ldots, m \tag{4.7}$$

$$\sum_{i=1}^{m} b_i^r = 1, \tag{4.8}$$

and the restriction $H_\delta|\Lambda$ of δ-dimensional Hausdorff measure to δ is equivalent (mutually absolutely continuous) to $\tilde{\mu}_{\mathbf{p}}$.

CHAP II

The Kenyon-Peres family – 1996

The Kenyon-Peres family – 1996

- Parameter $0 \leq a \leq 1$ and OSC-IFS $\mathcal{R} := \{R_0, R_1, R_2\}$ with

$$R_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + U_i,$$

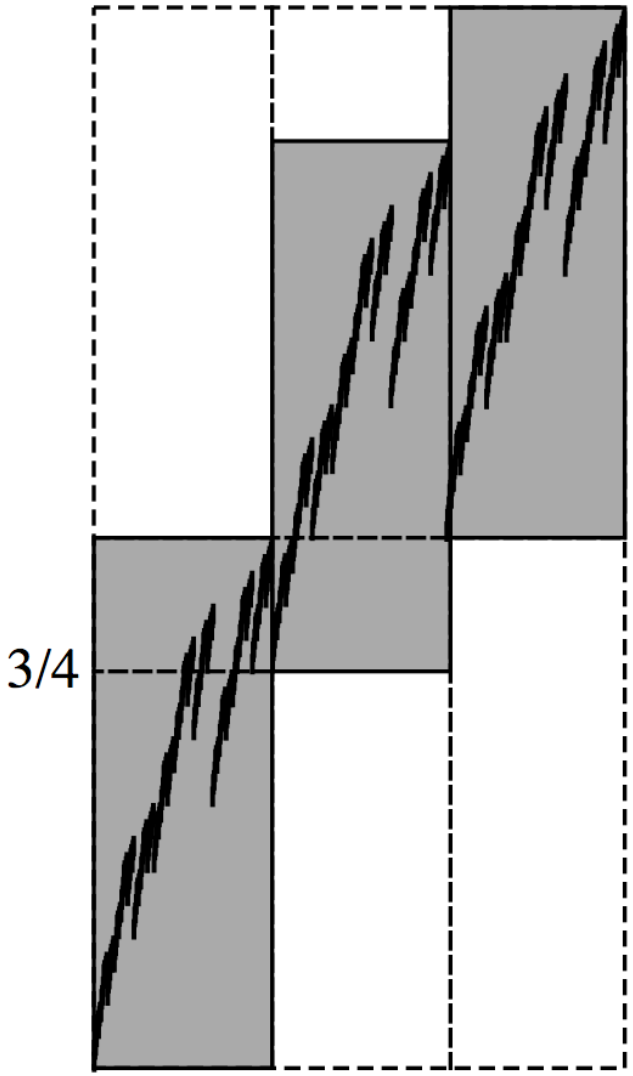
and

$$U_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; U_1 = \begin{pmatrix} 1/3 \\ a \end{pmatrix}, \; U_2 = \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

- The attractor $G = G(a) = \pi(X)$ where $X := \{0, 1, 2\}^{\mathbb{N}}$ and $\pi = (\pi_1, \pi_2) : X \to \mathbb{R}^2$ is the representation map s.t.

$$\pi(x = x_0 x_1 \cdots) = \lim_{n \to +\infty} R_{x_0} \circ \cdots \circ R_{x_n}(0, 0).$$

The Kenyon-Peres family – 1996



The Kenyon-Peres family – 1996

- $\mathbf{p}$ is the σ_X-ergodic measure with maximal entropy over X (i.e. $\mathrm{h}(\mathbf{p}, \sigma_X) = \mathrm{h}(X, \sigma_X) = \log 3$), so that

$$\pi \cdot \mathbf{p} = \frac{1}{3}\Big(R_0 \cdot (\pi \cdot \mathbf{p}) + R_1 \cdot (\pi \cdot \mathbf{p}) + R_2 \cdot (\pi \cdot \mathbf{p}) \Big)$$

Theorem A

For a.e. parameter $0 < a < 1$, $\mathbf{p}$ is the unique measure in $\mathcal{E}(X, \sigma_X)$ s.t. $\pi_2 \cdot \mathbf{p}$ is of maximal dimension over $G = G(a)$.

Proof of Theorem A

Theorem Feng-Hu – 2009

Given any $\mu \in \mathcal{E}(X, \sigma_X)$ the measure $\pi \cdot \mu$ is exact dimensional and

$$\dim_H \pi \cdot \mu = \mathrm{h}(\mu, \sigma_X)/\log 3 + (1-\alpha)\dim_H \pi_2 \cdot \mu \qquad (2)$$

Theorem (Kenyon-Peres 96)

$\pi_2 \cdot \mathbf{p}$ is absolutely continuous for a.e. $0 < a < 1$.

Proposition

Consider the propositions : (i) : $\dim_H \pi_2 \cdot \mathbf{p} = 1$; (ii) : $\mathbf{p}$ is the unique shift-ergodic measure for which $\dim_H \pi \cdot \mathbf{p} = \dim_H G$; (iii) : $\dim_H G = \dim_M G = s_*$; then

$$(i) \Longrightarrow (ii) \Longrightarrow (iii)$$

The McMullen-Bedford model

The the (a, b)-McMullen-Bedford carpet for $(a, b) = (2, 3)$:

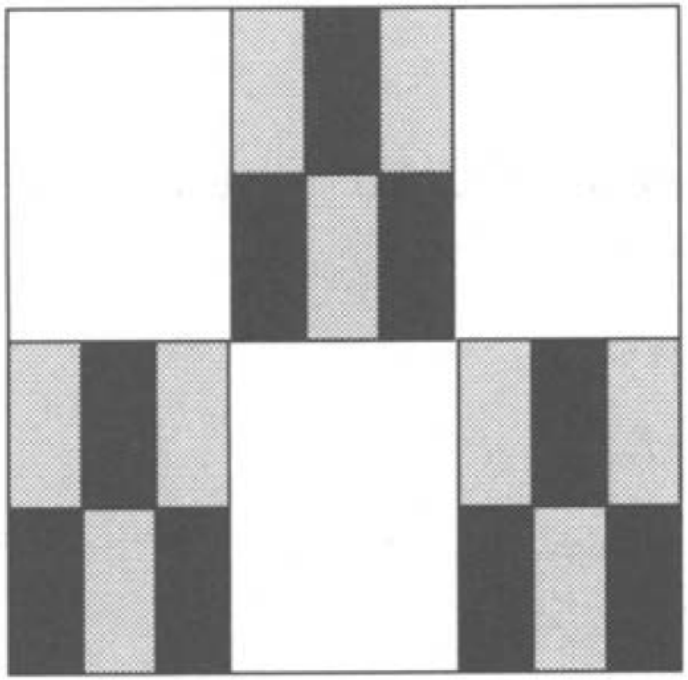
- $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \equiv [0\,;1[\times[0\,;1[\,;$
- $T : \mathbb{T}^2 \to \mathbb{T}^2$ s.t. $T(x, y) = \Big(3x \pmod 1, 2y \pmod 1\Big)$
- $\Omega := \Big\{(0,0), (1,0), (2,0), (0,1), (1,1), (2,1)\Big\}^{\mathbb{N}}$
- $\Pi : \Omega \to \mathbb{T}^2$ s.t.

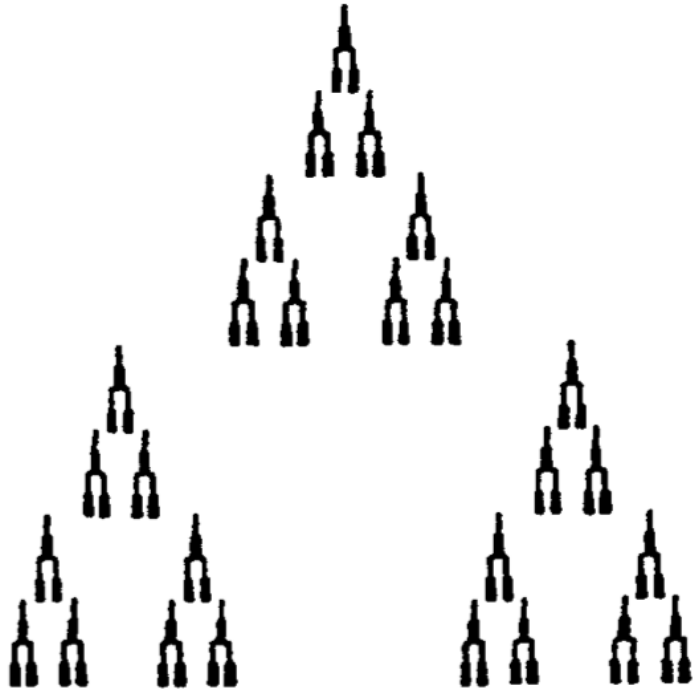
$$\Pi\Big((x_1, y_1)(x_2, y_2)\cdots\Big) = \Big(\sum_{k=1}^{\infty} \frac{x_k}{3^k}, \sum_{k=1}^{\infty} \frac{y_k}{2^k}\Big) \pmod 1$$

- Any $\emptyset \neq \mathcal{D} \subset \Big\{(0,0), \ldots, (2,1)\Big\}$ is associated with

$$K(\mathcal{D}) = \Pi(\mathcal{D}^{\mathbb{N}}) \quad \text{(McMullen-Bedford carpet).}$$

The McMullen-Bedford model





The McMullen-Bedford model

Theorem (McMullen 1984)

Let $s_* = \dim_M K(\mathcal{D})$; the following are equivalent

- $\dim_H K(\mathcal{D}) = s_*$;
- $\forall 1 \leq j', j'' \leq 2, \quad \#\{i \ ; \ (i,j') \in \mathcal{D}\} = \#\{i \ ; \ (i,j'') \in \mathcal{D}\}$

moreover $\dim_H K(\mathcal{D}) = s_* \Longrightarrow 0 < \mathcal{H}^{s_*}(K(\mathcal{D})) < +\infty$

THE HAUSDORFF DIMENSION OF GENERAL SIERPIŃSKI CARPETS

Nagoya Math. J.
Vol. 96 (1984), 1-9

CURT McMULLEN

Furthermore one easily obtains the stronger statement that the Hausdorff measure of $\overline{R}$ satisfies $0 < \mu_s \overline{R} < \infty$. It would be interesting to know if this last assertion holds in general.

The McMullen-Bedford model

Theorem (Peres -1994)

Let $K = K(\mathcal{D})$ be a McMullen-Bedford carpet : then

$$s_0 = \dim_H K < \dim_M K \Longrightarrow \mathcal{H}^{s_0}(K) = +\infty$$

Theorem (McMullen -1984)

$K = K(\mathcal{D})$ is a OSC-s.a.s and $S : K \to K$ is the repeller map ;

- $\exists!$ S-ergodic measure μ s.t. $\dim_H \mu = \dim_H K$;

Generalization (Kenyon-Peres 1996)

Let Z be a subshift of Ω (i.e. Z is compact and $\sigma(Z) = Z$) : then $K(Z) := \Pi(Z)$ is a called an affine T-invariant set (in general an affine T-invariant set is not a self-affine set !)

T-affine-invariant sets

Kenyon-Peres (1996)

Measures of Full Dimension on Affine-Invariant Sets

R. Kenyon* and Y. Peres†

April 19, 1996

Theorem 1.1 *Let T be an expanding linear endomorphism of the d-torus which is a direct sum of conformal endomorphisms. Let K be a compact T-invariant set. Then K supports a T-invariant probability measure μ of full Hausdorff dimension; more precisely, any subset of K with positive μ-measure has the same dimension as K.*

What about uniqueness of the measure with maximal dimension ?

T-affine-invariant sets

(Olivier 2010)

Under the condition (**H**) (Gibbs property of the y-axis projection of the Parry measure over K) :

- η is a T-ergodic measure of maximal dimension over K iff $\eta = \Pi \cdot \tilde{\eta}$, where $\tilde{\eta}$ is a shift-ergodic equilibrium state of some (continuous potential) $\Phi : Z \to \mathbb{R}$:
- the uniqueness (if it holds !) follows from the uniqueness of the equilibrium state of Φ ;
- η is a T-conformal measure...

T-affine-invariant sets

Theorem (Feng - 2011) (Gatzouras-Peres question - 1996)

- (Σ, σ) and $(\mathbf{Y}, \sigma_{\mathbf{Y}})$ are subshifts over finite alphabets ;
- $\mathbf{y} : \Sigma \to \mathbf{Y}$ is a factor map (continuous surjective) ;
- Σ satisfies weak specification ;
- $a > 0$ and $b \geq 0$;
- $f : \Sigma \to \mathbf{R}$ is continuous with sufficient regularity ;

then, $\exists!$ shift-invariant measure μ over Σ that maximizes

$$\boxed{\mu(f) + a \cdot \mathrm{h}(\sigma, \mu) + b \cdot \mathrm{h}(\sigma_{\mathbf{Y}}, \mathrm{p} \cdot \mu)}$$

in particular, for $f \equiv 0$, $\exists!$ $\mu \in \mathcal{M}(\sigma, \Sigma)$ that maximizes

$$\boxed{a \cdot \mathrm{h}(\sigma, \mu) + b \cdot \mathrm{h}(\sigma_{\mathbf{Y}}, \mathrm{p} \cdot \mu)}$$

T-affine-invariant sets

Application with :

- $\Sigma = Z \subset \Omega := \Big\{(0,0),(1,0),\dots,(2,1)\Big\}^{\mathbb{N}}$ (w-spec) ;
- $Y = \{0,1\}^{\mathbb{N}}$ and $\mathbf{y} : Z \to Y$ s.t.

$$\mathbf{y}\Big((x_1,y_1)(x_2,y_2)\cdots\Big) = y_1 y_2 \cdots$$

Ledrappier-Young Formula (see Kenyon-Peres proof)

For any shift-ergodic measure in $\mathcal{M}(\sigma,\Omega)$ the T-ergodic measure $\Pi\cdot\mu$ is exact dimensional with Hausdorff dimension and

$$\dim_H \Pi\cdot\mu = \frac{1}{\log 3}\cdot \mathrm{h}(\sigma,\mu) + \left(\frac{1}{\log 2} - \frac{1}{\log 3}\right)\cdot \mathrm{h}(\sigma_Y, \mathbf{y}\cdot\mu)$$

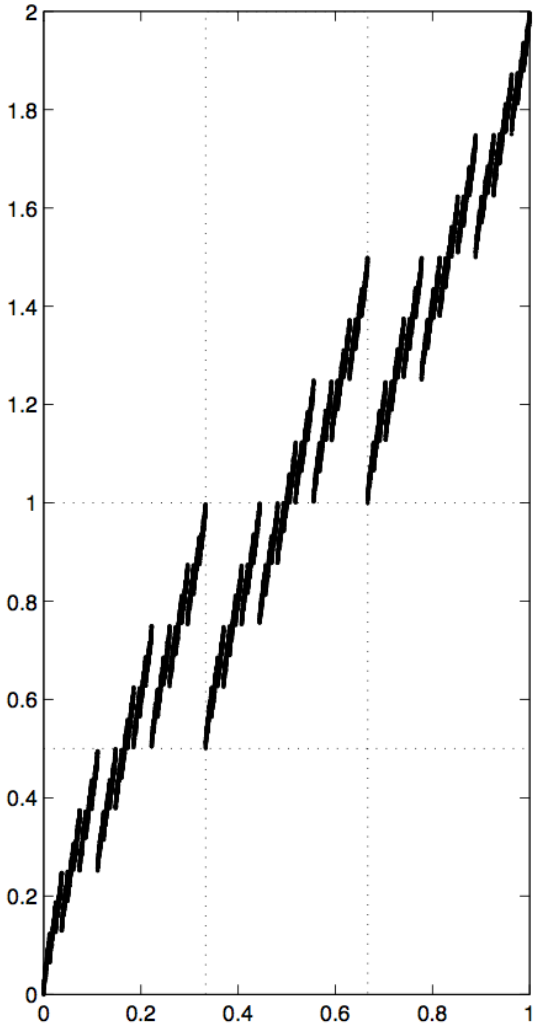
T-affine-invariant sets

Corollary of Feng's Theorem

If $K = \Pi(Z)$ is a T-affine invariant subset of $\mathbb{T}^2$ with symbolic model Z satisfying the weak-specification ; then, there exists a unique T-ergodic measure with maximal dimension over K.

Under which conditions is it possible to transfer the above Variational Principle for dimension concerned with T-affine-invariant sets, to prove existence and uniqueness of the S-ergodic measure with maximal dimension for S being the repeller associated with a self-affine set ?

An example : $G(1/2)$ for the Kenyon-Peres family



An example : $G(1/2)$ for the Kenyon-Peres family

Some results/questions about $G(1/2)$

- (Kenyon-Peres 96) $s_0 = \dim_H G \approx 1.3662969\cdots < \dim_M G$
- for probability vector $p = (p_0, p_1, p_2)$ let μ_p s.t.

$$\mu_p = p_0(R_0 \cdot \mu_P) + p_1(R_1 \cdot \mu_P) + p_2(R_2 \cdot \mu_P)$$

then Kenyon & Peres gives the estimation

$$\max_p \left\{ \dim_H \mu_p \right\} \approx 1.364987\cdots < \dim_H G.$$

Proposition (KP96)

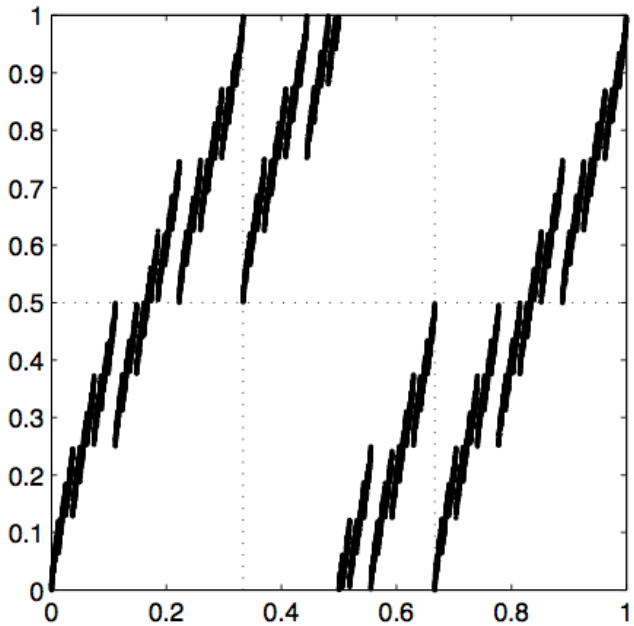
If there exists a S-ergodic measure μ_* with maximal dimension over G then μ_* cannot be a self-affine measure.

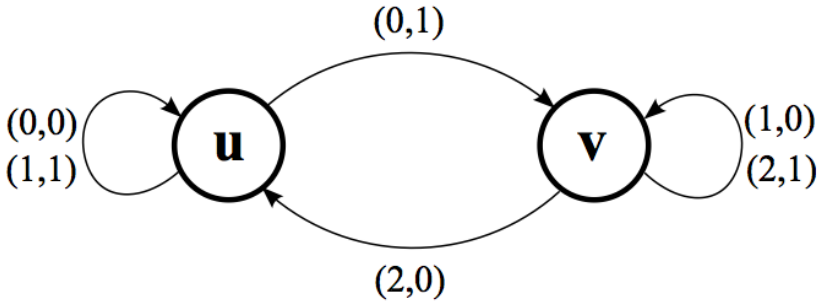
- $\exists$? S-ergodic μ_* having maximal dimension over G ?
- Uniqueness ? Nature of μ_* ?
- $\mathcal{H}^{s_0}(G(1/2)) = +\infty$?

An example : $G(1/2)$ for the Kenyon-Peres family

Let $G' := G(1/2) \cap [0\,;1[\times[0\,;1[$ and consider the intermediate T-affine invariant set K :

$$\mathbf{K} = \overline{\bigcup_{k=0}^{\infty} T^k(G')} = G' \cup T(G')$$





An example : $G(1/2)$ for the Kenyon-Peres family

Olivier 2010

The unique T-ergodic measure with maximal dimension over K is $\Pi \cdot \mu$ for μ being the unique equilibrium state of $\Phi : Z \to \mathbb{R}$, s.t.

$$\Phi\Big((x_1, y_1)(x_2, y_2)\cdots\Big) = \left(\frac{\log 2}{\log 3} - 1\right)\phi(y_1 y_2 \cdots)$$

where for $y_1 y_2 \cdots = y_1^{a_1}(1-y_1)^{a_2}\cdots \quad (a_1, a_1, \cdots > 0)$:

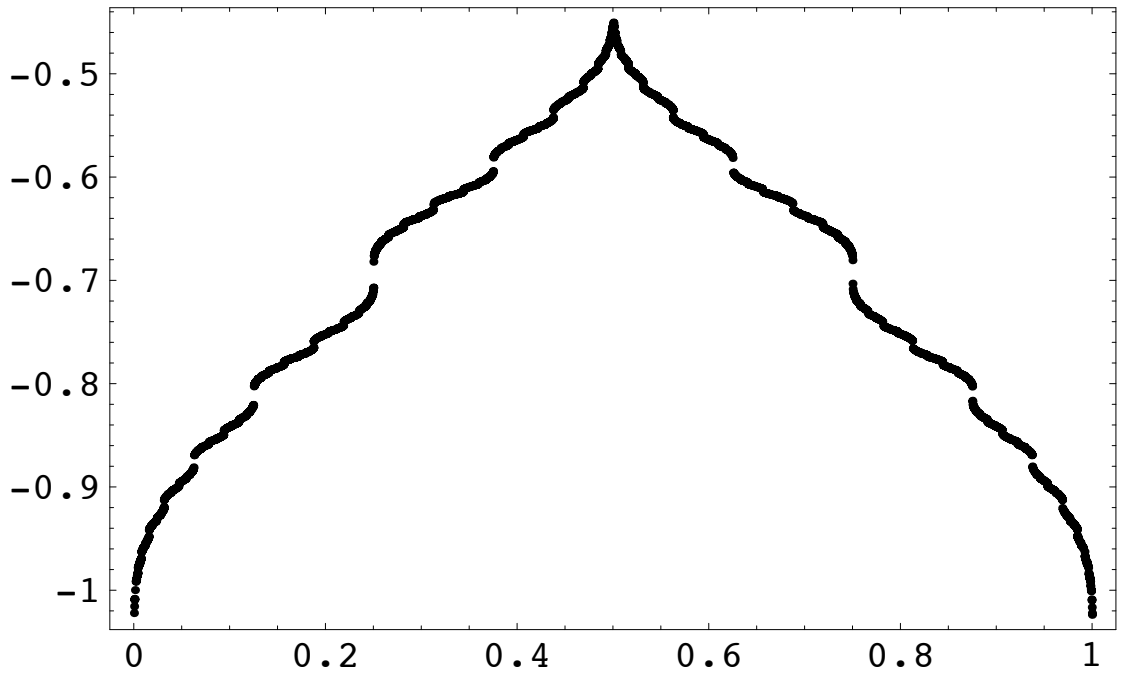
$$\phi(y_1 y_2 \cdots) = \log\left(\frac{1}{3} + \frac{1}{3}\cdot\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}\right)$$

An example : $G(1/2)$ for the Kenyon-Peres family

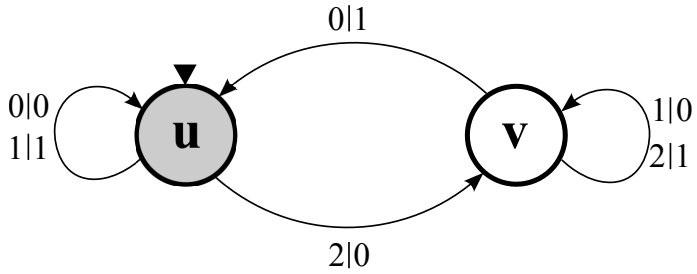
It is possible to plot the graph of

$$[0\,;1[\ni y=\sum_{k=1}^{\infty}\frac{y_k}{2^k}\mapsto\phi(y_0y_1\cdots)$$

(to be related with the so-called Conway and Minkowski functions).



An example : $G(1/2)$ for the Kenyon-Peres family

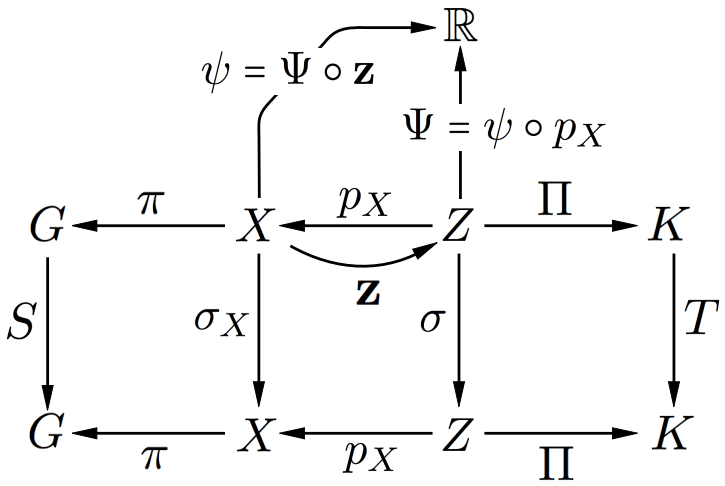


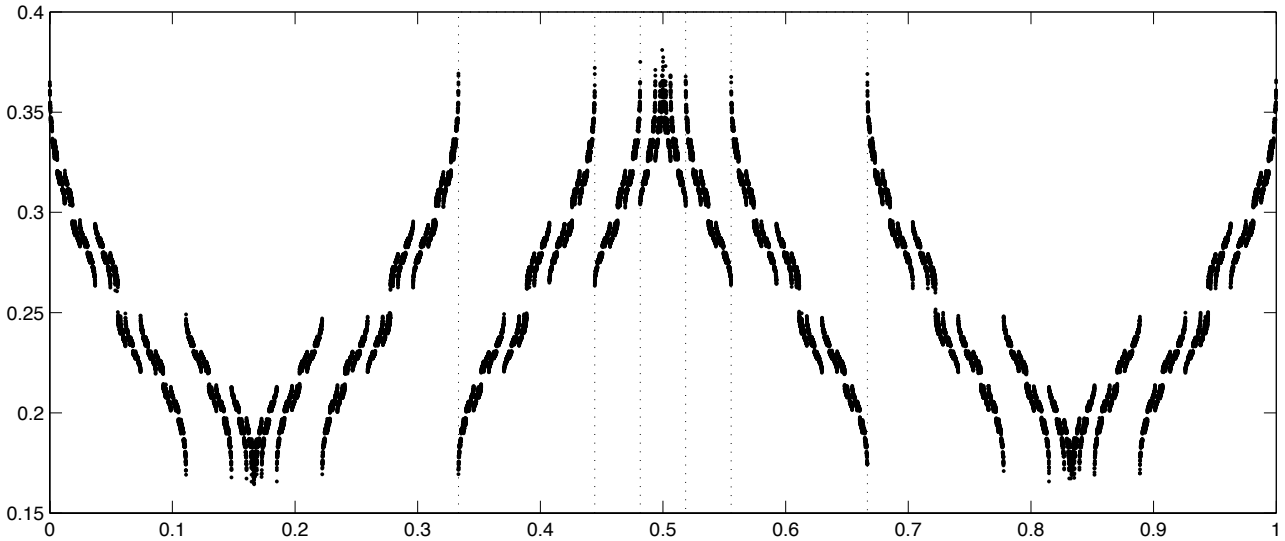
$$\mathbf{z} : \mathbf{X} \to \mathbf{Y} \quad \text{s.t.} \quad \mathbf{z}(x_0 x_1 \cdots) = \lim_{n \to +\infty} \mathbf{z}(x_0 \cdots x_{n-1})00 \cdots .$$

Theorem B

The unique S-ergodic measure with maximal dimension over $G(1/2)$ is the unique $\exp(\Psi)$-conformal measure with continuous potential $\Psi : \{0,1,2\}^{\mathbb{N}} \to \mathbb{R}$ s.t. $\Psi = \Phi \circ \mathbf{z}$

An example : $G(1/2)$ for the Kenyon-Peres family





CHAP III

The Erdős-Rademacher family

The Erdős-Rademacher family

- Let $\sigma_X : X \to X$ the shift maps over $\{0,1\}^{\mathbb{N}}$;
- Let $1 < \beta < 2$ and $Z : X \to \mathbb{T}^2 := [0\,;1[\times[0\,;1[$ s.t.

$$Z(\xi) = \left(\left\{ \sum_{k=0}^{\infty} \frac{\xi_k}{2^{k+1}} \right\}, \left\{ (\beta-1) \sum_{k=0}^{\infty} \frac{\xi_k}{\beta^{k+1}} \right\} \right)$$

- $\mathbf{p}$ is the σ_X-ergodic measure with maximal entropy over X (i.e. $\mathrm{h}(\mathbf{p}, \sigma_X) = \mathrm{h}(X, \sigma_X) = \log 2$), so that

$$\pi \cdot \mathbf{p} = \frac{1}{2}\Big(R_0 \cdot (\pi \cdot \mathbf{p}) + R_1 \cdot (\pi \cdot \mathbf{p}) \Big)$$

Theorem A′

For a.e. parameter $1 < \beta < 2$, $\mathbf{p}$ is the unique σ_X-ergodic measure s.t. $Z \cdot \mathbf{p}$ is of maximal dimension over $G(\beta) = Z(X)$.

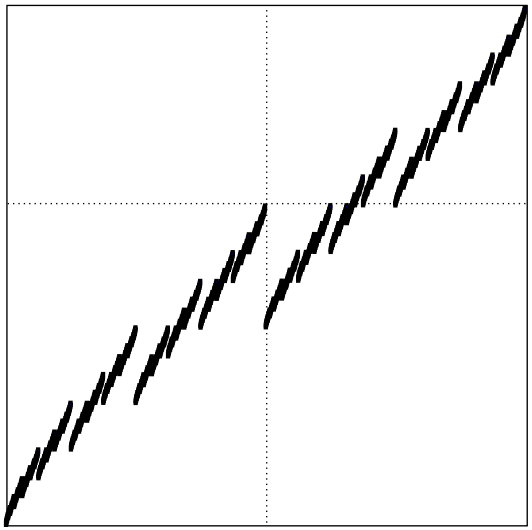
The Erdős-Rademacher Golden Graph

- Let $\beta = (1+\sqrt{5})/2$ and $X : X \to \mathbb{T}^2 := [0\,;1[\times[0\,;1[$ s.t.

$$X(\xi) = \left(\left\{ \sum_{k=0}^{\infty} \frac{\xi_k}{2^{k+1}} \right\}, \left\{ (\beta-1) \sum_{k=0}^{\infty} \frac{\xi_k}{\beta^{k+1}} \right\} \right)$$

- $T_\beta : \mathbb{T}^2 \to \mathbb{T}^2$ s.t. $T_\beta(x,y) = (\{2x\}, \{\beta y\})$

$$\mathbf{G} := Z(\{0,1\}^{\mathbb{N}})$$



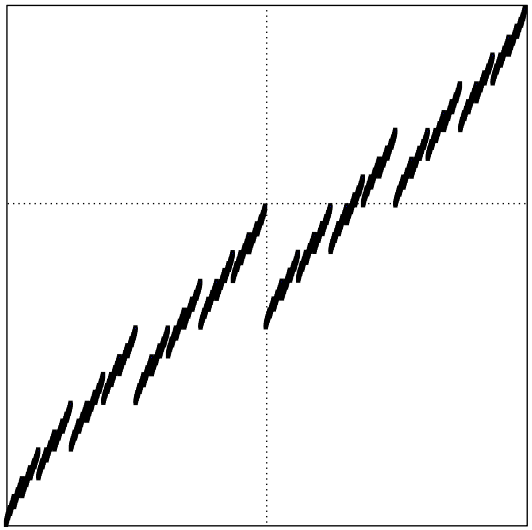
The Erdős-Rademacher Golden Graph

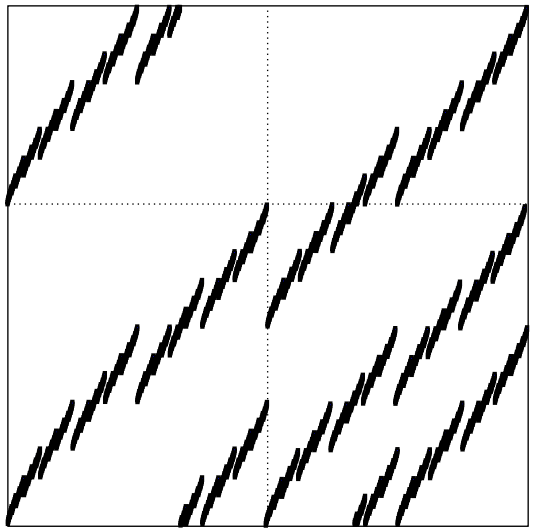
- Let $\beta = (1+\sqrt{5})/2$ and $Z : X \to \mathbb{T}^2 := [0\,;1[\times[0\,;1[$ s.t.

$$Z(\xi) = \left(\left\{ \sum_{k=0}^{\infty} \frac{\xi_k}{2^{k+1}} \right\}, \left\{ (\beta-1) \sum_{k=0}^{\infty} \frac{\xi_k}{\beta^{k+1}} \right\} \right)$$

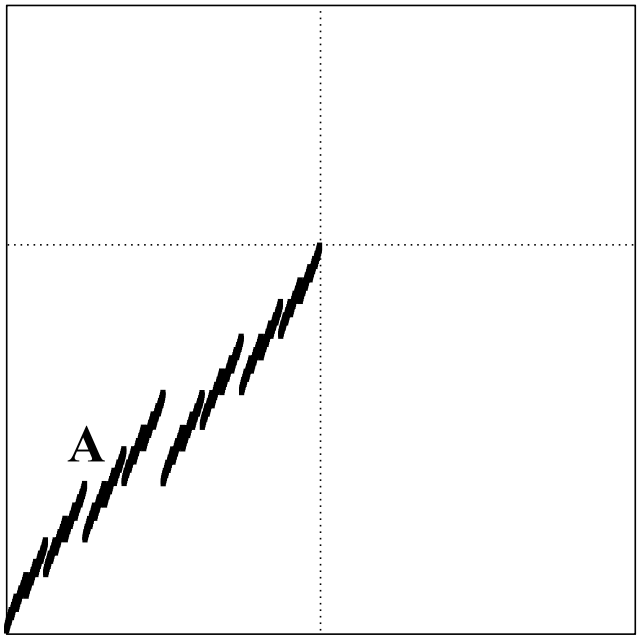
- $T_\beta : \mathbb{T}^2 \to \mathbb{T}^2$ s.t. $T_\beta(x,y) = (\{2x\}, \{\beta y\})$

$$\mathbf{G} := Z(\{0,1\}^{\mathbb{N}}) \qquad \mathbf{K} := \overline{\bigcup_{k=0}^{\infty} T_\beta^k(\mathbf{G})}$$





The Erdős-Rademacher Golden Graph



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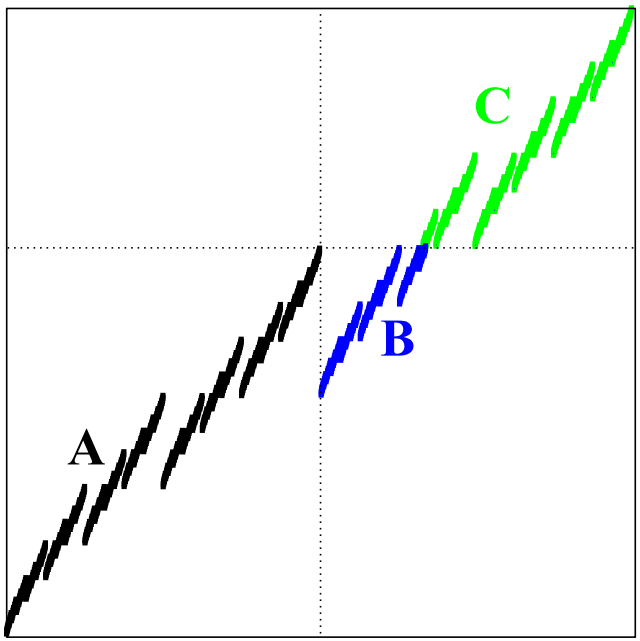
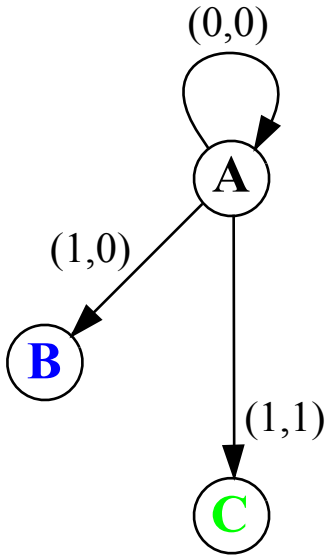
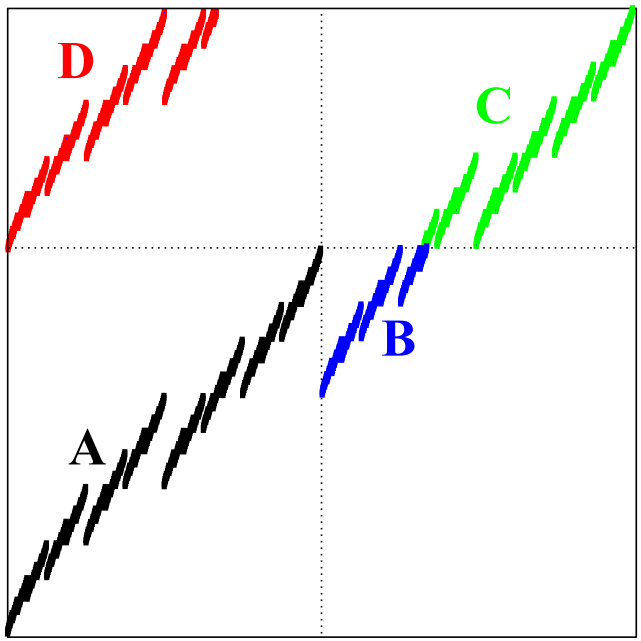
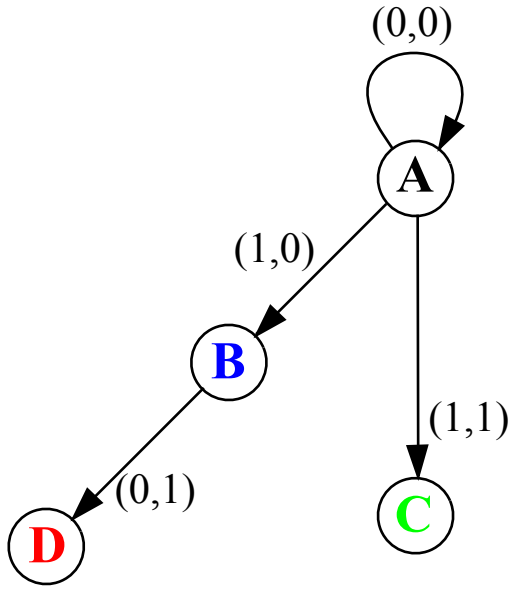
The Erdős-Rademacher Golden Graph



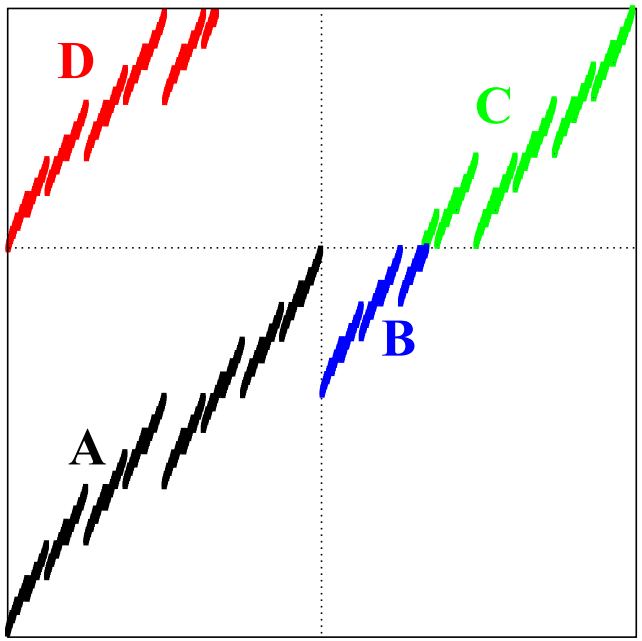


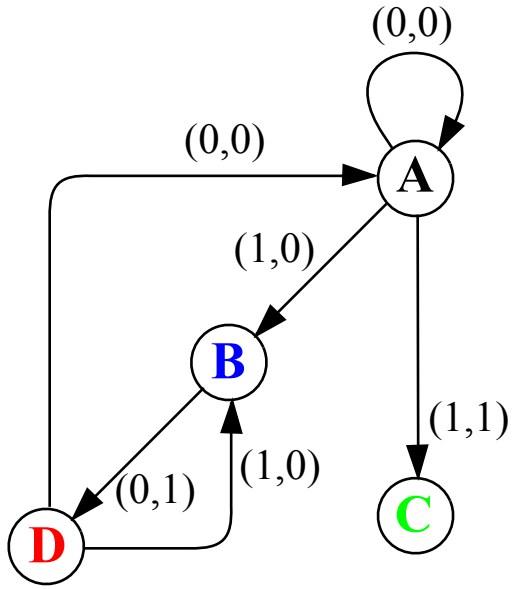
The Erdős-Rademacher Golden Graph



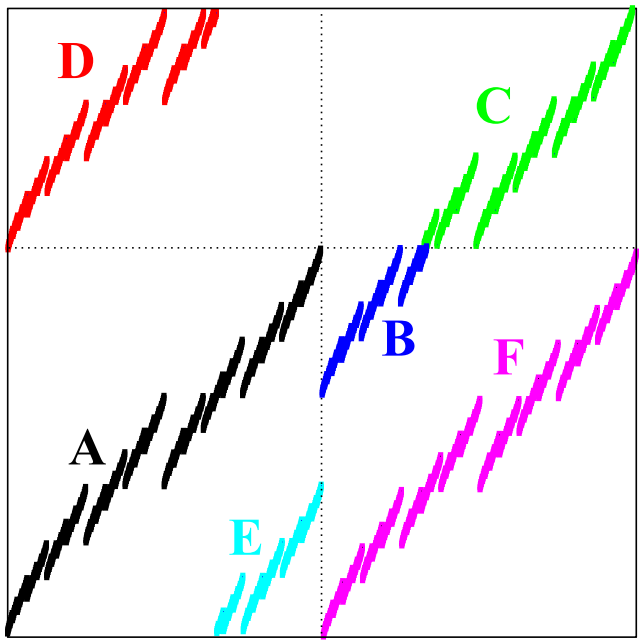


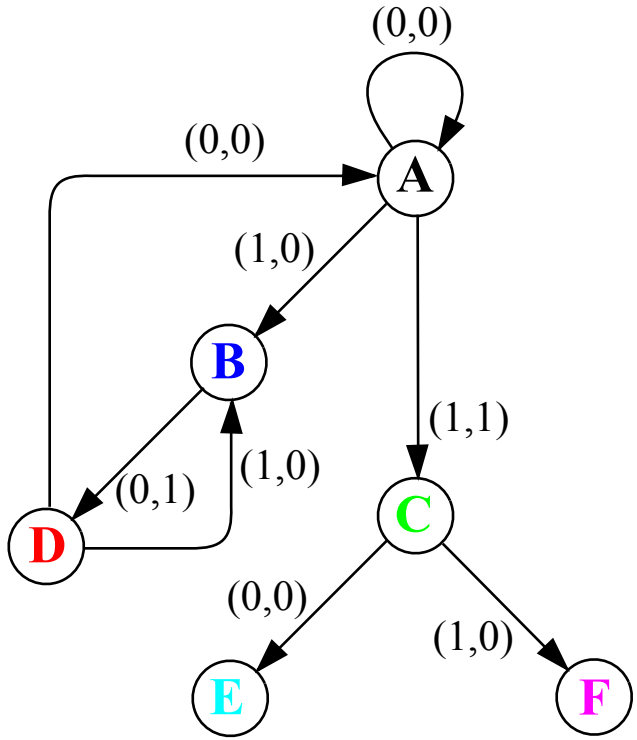
The Erdős-Rademacher Golden Graph



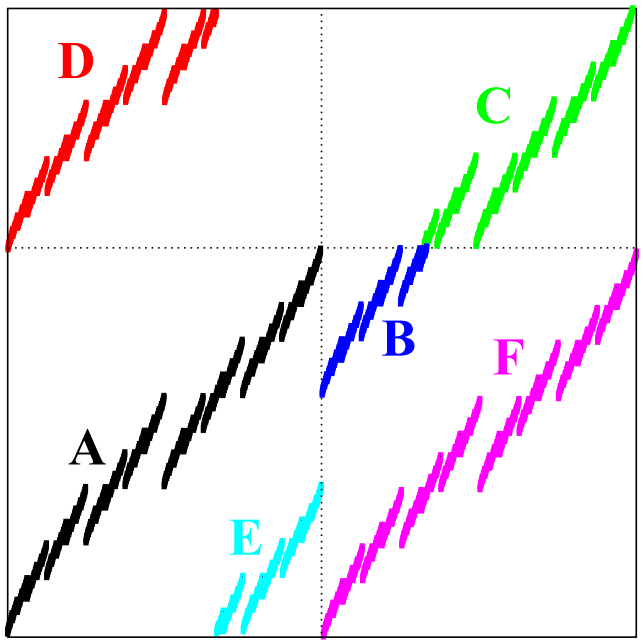


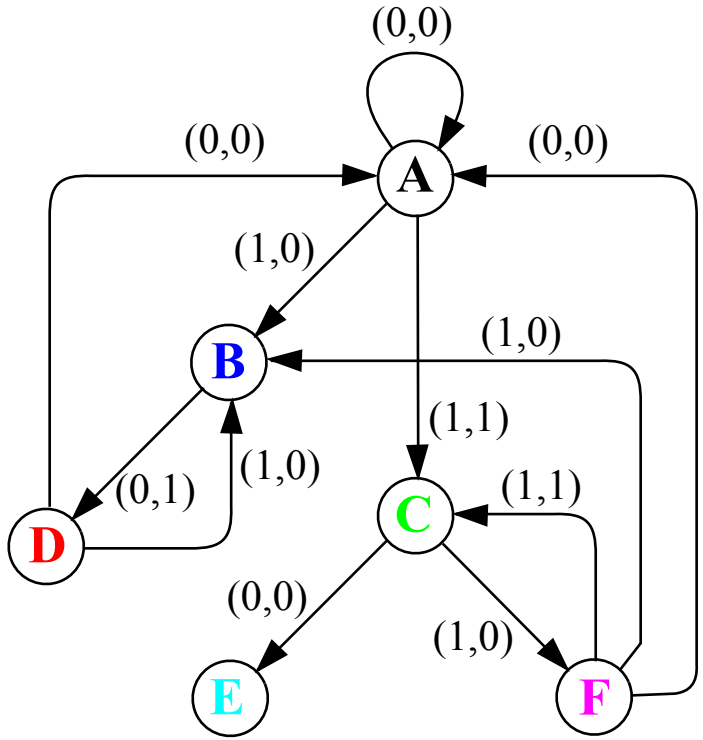
The Erdős-Rademacher Golden Graph



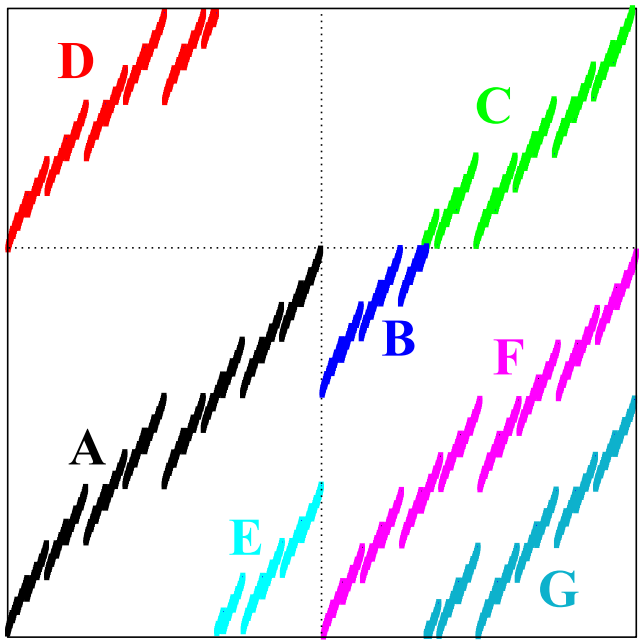


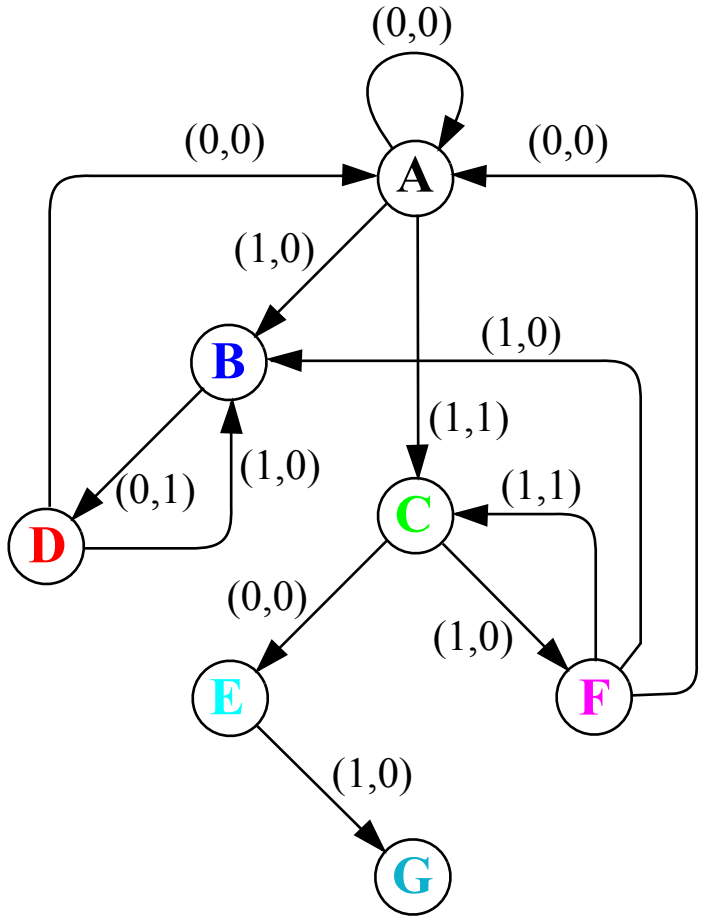
The Erdős-Rademacher Golden Graph



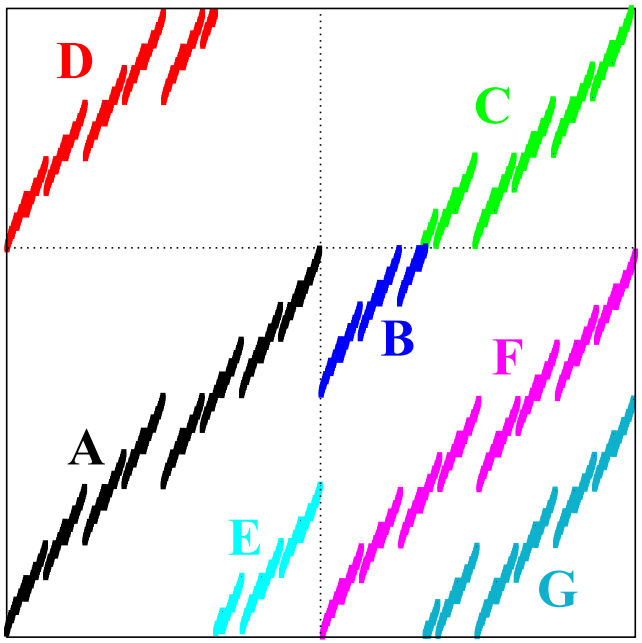


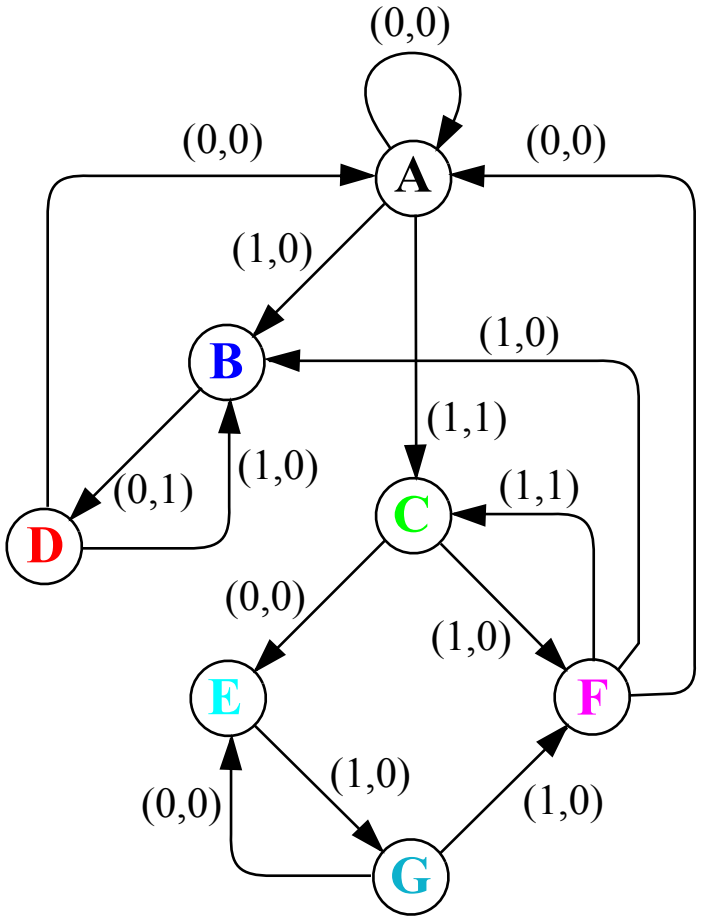
The Erdős-Rademacher Golden Graph





The Erdős-Rademacher Golden Graph





The Erdős-Rademacher Golden Graph

Theorem B′

There exists a unique $\sigma\chi$-ergodic measure s.t. $Z \cdot \mathbf{p}$ has maximal dimension over $G((1+\sqrt{5})/2)$.