Absolutely continuous invariant measure for complex continued fraction maps

Hiromi EI (Hirosaki Univ.)
Rie NATSUI (Japan Women's Univ.)

in collaboration with
Hitoshi NAKADA (Keio Univ.)

2016.3 Paris

Aim

Hurwitz complex continued fraction transformation

Define the domain U and the map T on U as follows:

$$U = \{z = x + iy : \quad -1/2 < x, y < 1/2\}$$

$$T(z) = \frac{1}{z} - \left[\frac{1}{z}\right] \text{ for } z \in U,$$

where $[x]_2$ for $x \in \mathbb{R}$ means the nearest integer of x, and $[x + iy] = [x]_2 + i[y]_2$ for $x, y \in \mathbb{R}$.

Define the map a on U and determine a_n $(n \in \mathbb{N})$ as follows:

$$a(z) = \left[\frac{1}{z}\right],$$

$$a_n(z)(= a_n) = a(T^{n-1}(z))$$

for $z \in U$, then we get the continued fraction expansion of $z \in U$:

$$z = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots .$$

Obviously, $|a_n| \geq \sqrt{2}$.

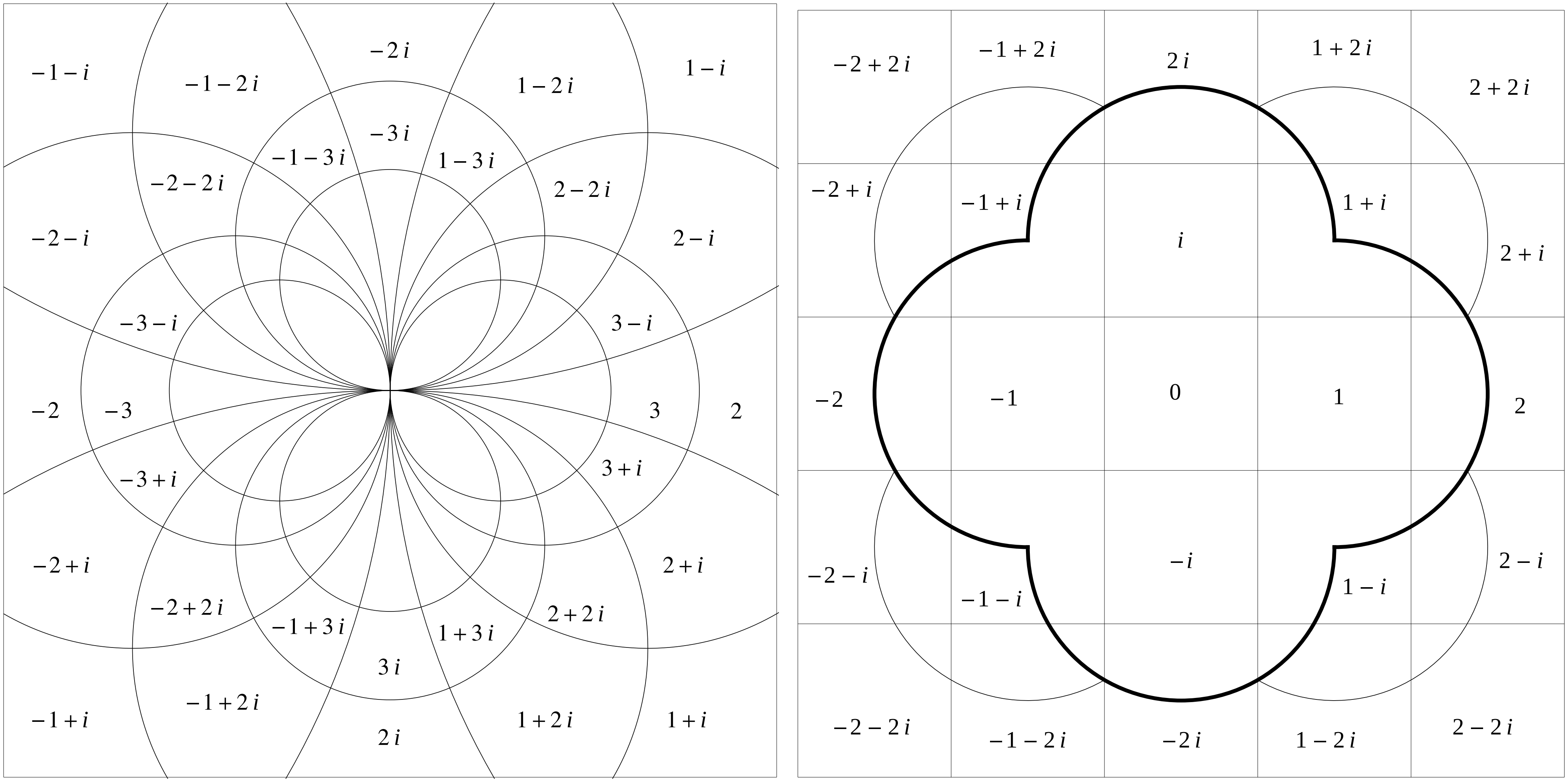


Fig. 1: The cylinder sets $\langle a \rangle = \{z \in U : [1/z] = a\}$ and $\{1/z \colon z \in \langle a \rangle\}$

Aim

Construct the natural extension of (U, T, μ) to determine the density function of the absolutely continuous invariant measure μ.

In the case of regular continued fraction transformation of $\mathbb{R}$

In the case of regular continued fraction transformation of $\mathbb{R}$

Define the map G on $I := [0, 1)$ by

$$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right].$$

Then it is known that an invariant measure is given by

$$\frac{1}{\log 2}\frac{1}{1+x}dx.$$

How do we get it?
$\rightarrow$ We construct a two dimensional map which is the natural extension of G.

Define

$$\hat{I} = [0, 1) \times (-\infty, -1],$$

$$\hat{G}(x, y) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y} - \left[\frac{1}{x}\right]\right) \text{ for } (x, y) \in \hat{I}.$$

Then $\hat{G}$ on $\hat{I}$ is 1-1 and onto except for a set of Lebesgue measure 0 and

$$\frac{1}{\log 2} \frac{dxdy}{(x-y)^2}$$

gives an invariant measure for $(\hat{I}, \hat{G})$. Then we get

$$\frac{1}{\log 2} \frac{1}{1+x} dx = \left(\int_{-\infty}^{-1} \frac{1}{\log 2} \frac{1}{(x-y)^2} dy\right) dx.$$

Construction of natural extension in real case

How do we determine $\hat{I} = [0, 1) \times \underline{(-\infty, -1]}$?

Take $(x, -\infty) \in [0, 1) \times [-\infty, -1]$ and let

$$x = \frac{1}{|a_1} + \frac{1}{|a_2} + \frac{1}{|a_3} + \cdots .$$

Then,

$$\hat{G}(x, -\infty) = \left(\frac{1}{|a_2} + \frac{1}{|a_3} + \frac{1}{|a_4} + \cdots , -a_1 \right)$$

$$\hat{G}^2(x, -\infty) = \left(\frac{1}{|a_3} + \frac{1}{|a_4} + \cdots , -\left(a_2 + \frac{1}{|a_1} \right) \right)$$

By induction, we have

$$\hat{G}^n(x,-\infty) = \left(\frac{1|}{|a_{n+1}} + \frac{1|}{|a_{n+2}} + \cdots, -(a_n + \frac{1|}{|a_{n-1}} + \cdots + \frac{1|}{|a_1})\right).$$

By the set of the reversed sequences of $\{a_n(x)\}$,

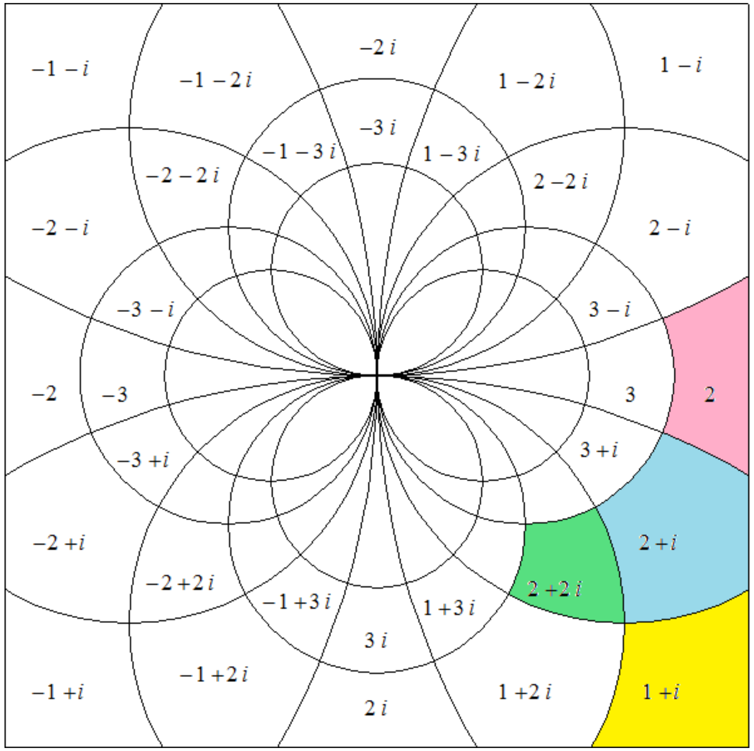
we obtain the domain

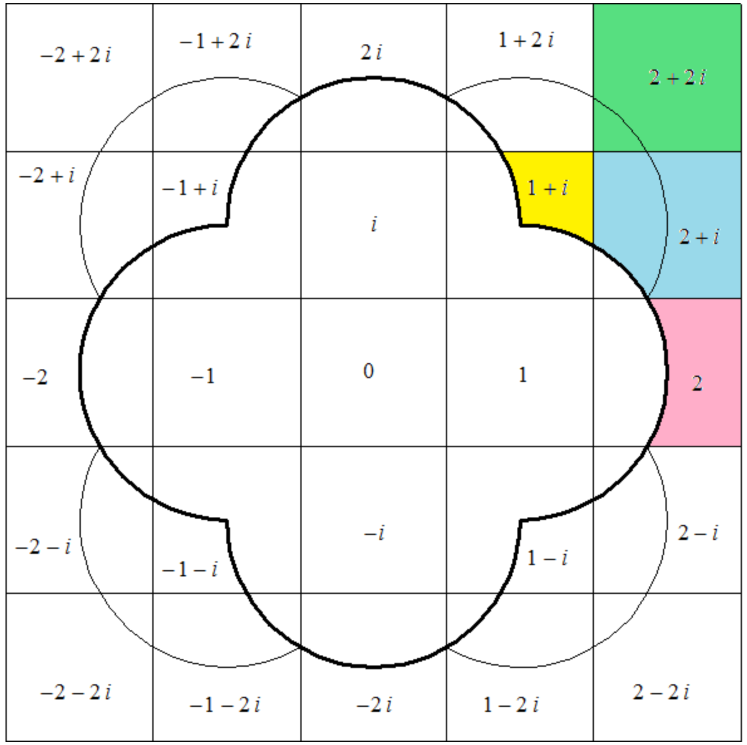
$$\left\{ -\left(a_n(x) + \frac{1|}{|a_{n-1}(x)} + \cdots + \frac{1|}{|a_1(x)} \right) : \begin{array}{l} x \in (0,1) \\ n \in \mathbb{N} \end{array} \right\}.$$

$$= (-\infty, -1].$$

In the case of Hurwitz complex continued fraction transformation

In the case of Hurwitz complex continued fraction transformation





$$\mapsto$$

$$\frac{1}{z}$$

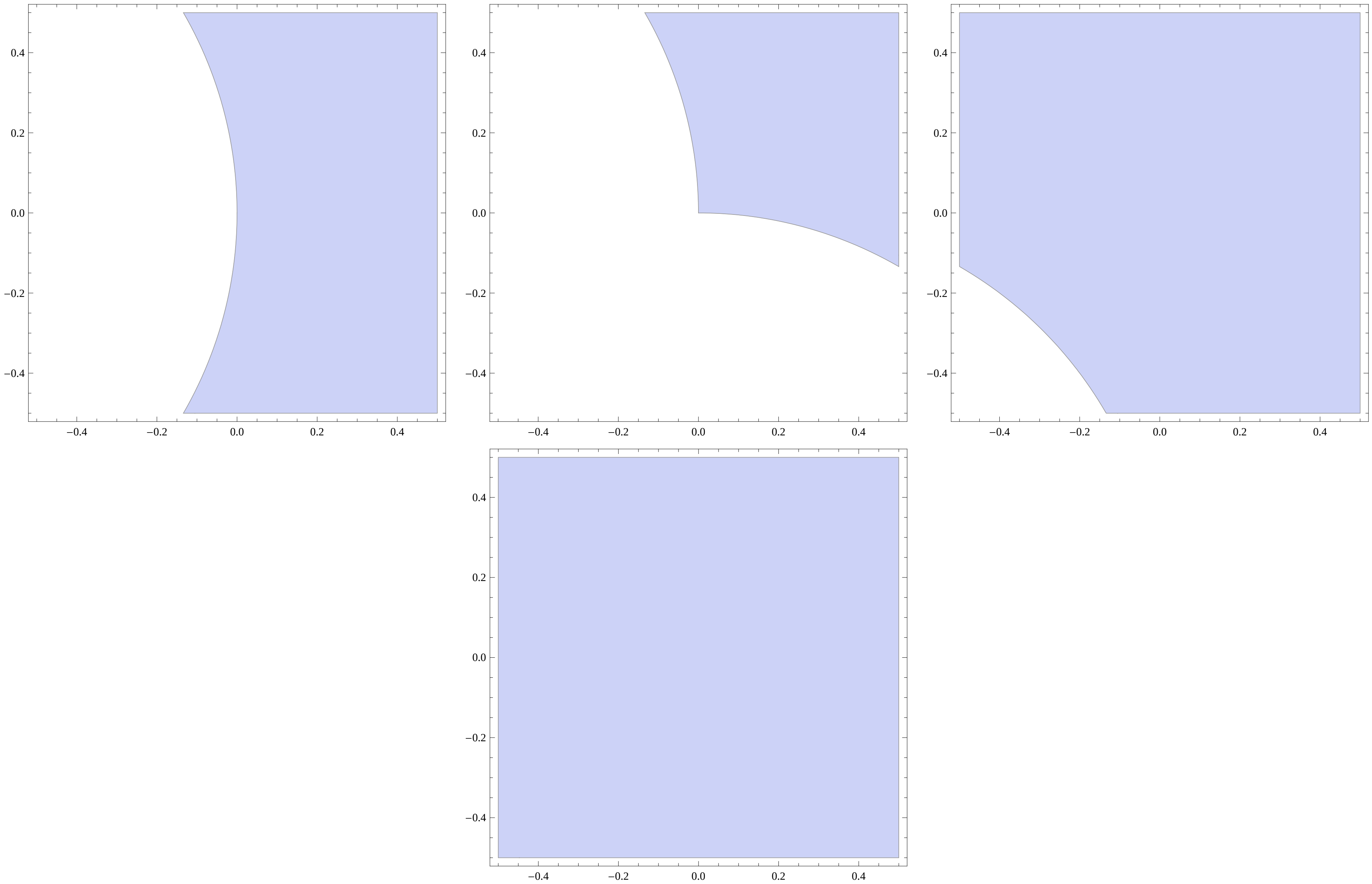


Fig. 2: 13 domains by $T\langle a\rangle$

In the case of regular continued fraction of real number

$\rightarrow a_n$ (resp. a_{n+1}) is not restricted from a_{n+1} (resp. a_n).

In the case of Hurwitz continued fraction of complex number

$\rightarrow a_n$ (resp. a_{n+1}) is restricted from a_{n+1} (resp. a_n).

$\rightarrow$ We decompose U and get the following partition $\{V_{k,\ell}\}$ which is a Markov partition of T:

$$V_{1,\ell} = (i)^{\ell-1} \cdot \{z \in U : |z+i| > 1, |z-i| > 1, \, Re\, z > 0\}$$

$$V_{2,\ell} = (i)^{\ell-1} \cdot \{z \in U : |z-1| < 1, |z-i| < 1, |z-(1+i)| > 1\}$$

$$V_{3,\ell} = (i)^{\ell-1} \cdot \{z \in U : |z-(1+i)| < 1\}$$

$1 \leq \ell \leq 4$.

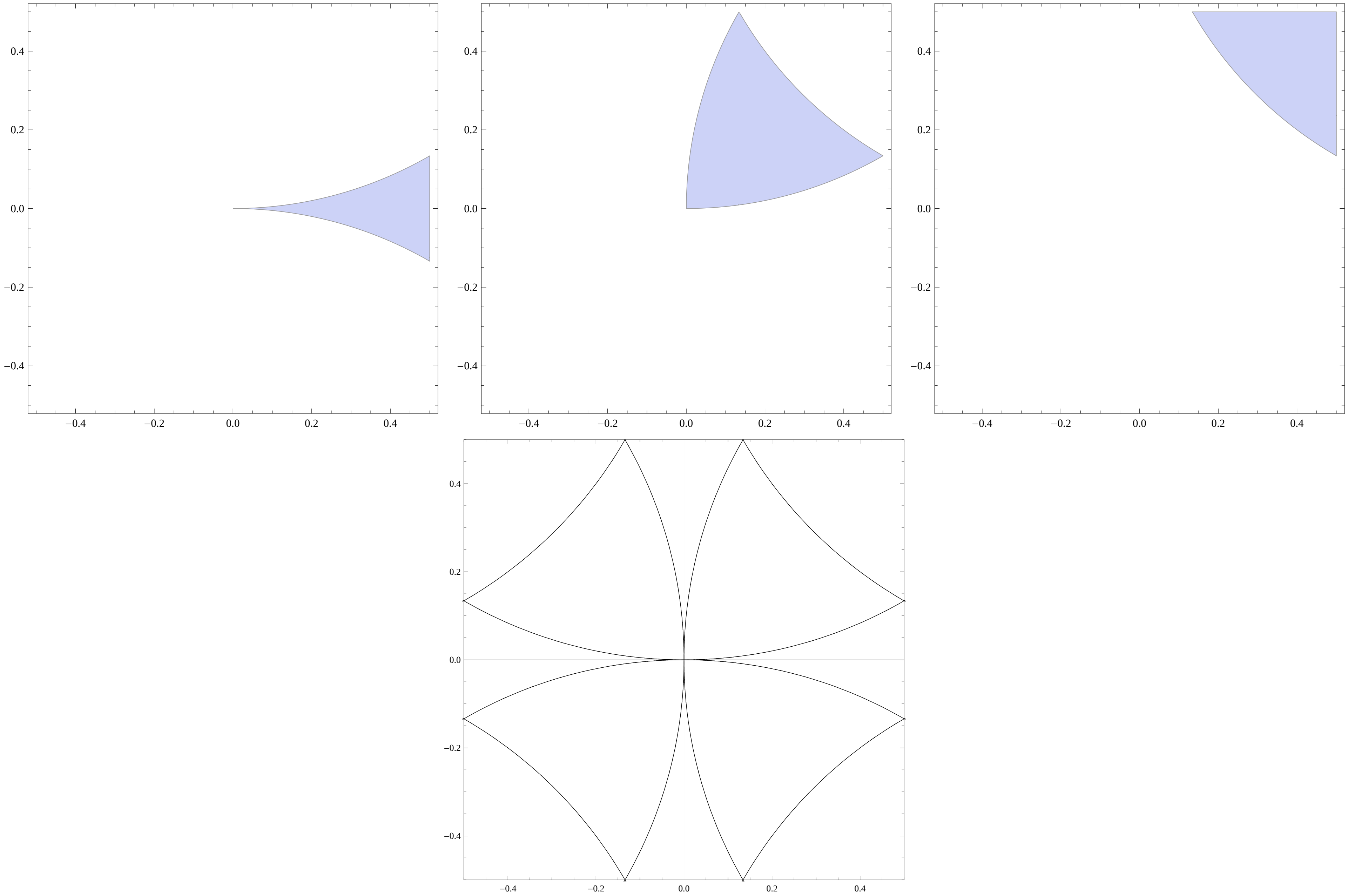


Fig. 3: $V_{1,1}$, $V_{2,1}$, $V_{3,1}$ and the partition of U

Construction of the natural extension in Hurwitz case

Construction of the natural extension in Hurwitz case

Computer experience by Shunji ITO (Kokyuroku 496 (1983).)

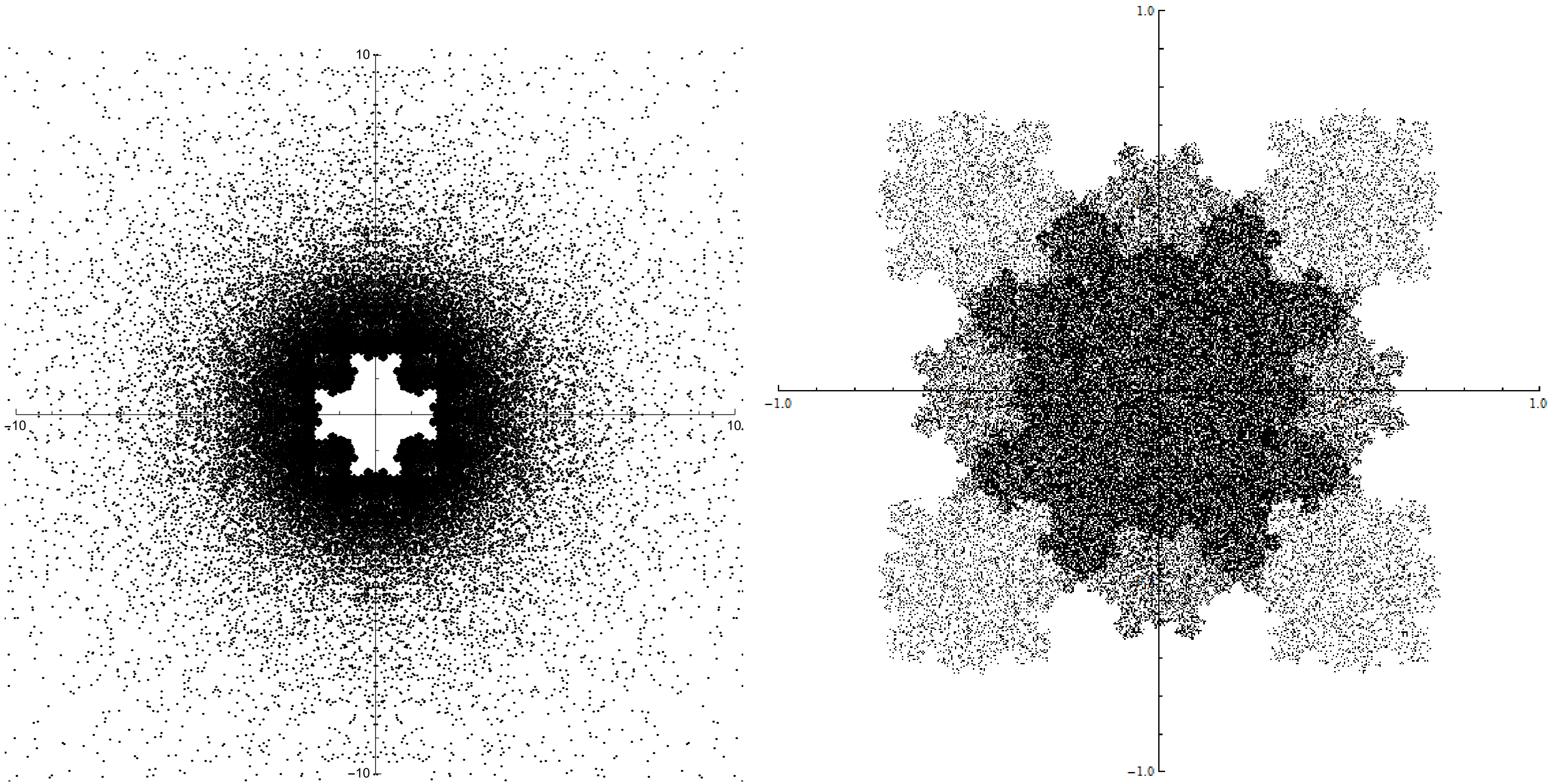


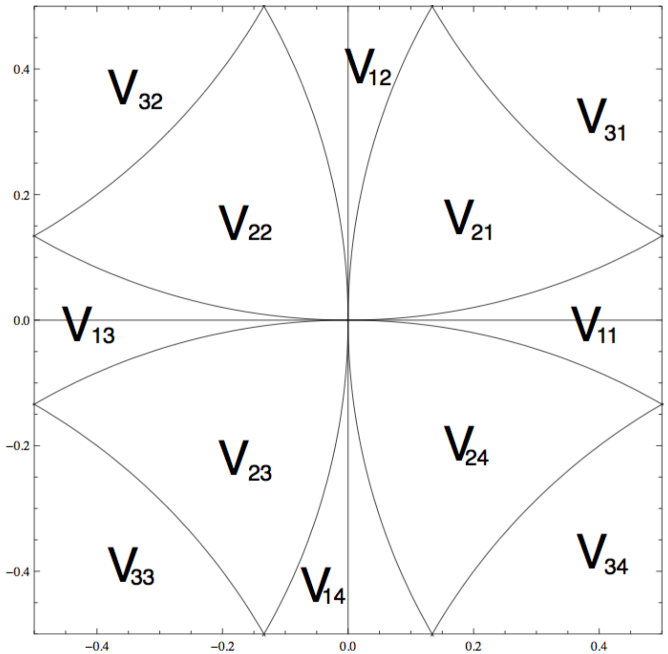
Fig. 4: $$\overline{\left\{ -\left(a_n(z) + \frac{1}{|a_{n-1}(z)} + \cdots + \frac{1}{|a_1(z)} \right) : \begin{array}{l} z \in U, \\ n \in \mathbb{N} \end{array} \right\}}$$

We define

$$V_{k,\ell}^{*} = \overline{\bigcup_{n=1}^{\infty} \left\{ -\left(a_n(z) + \frac{1}{|a_{n-1}(z)} + \cdots + \frac{1}{|a_1(z)} \right) \; : \; \begin{array}{l} z \in U, \\ T^n(z) \in V_{k,\ell} \end{array} \right\}}$$

$$X_{k,\ell} = \{ \frac{1}{w} : \; w \in V_{k,\ell}^{*} \}$$

for $1 \leq k \leq 3$, $1 \leq \ell \leq 4$.



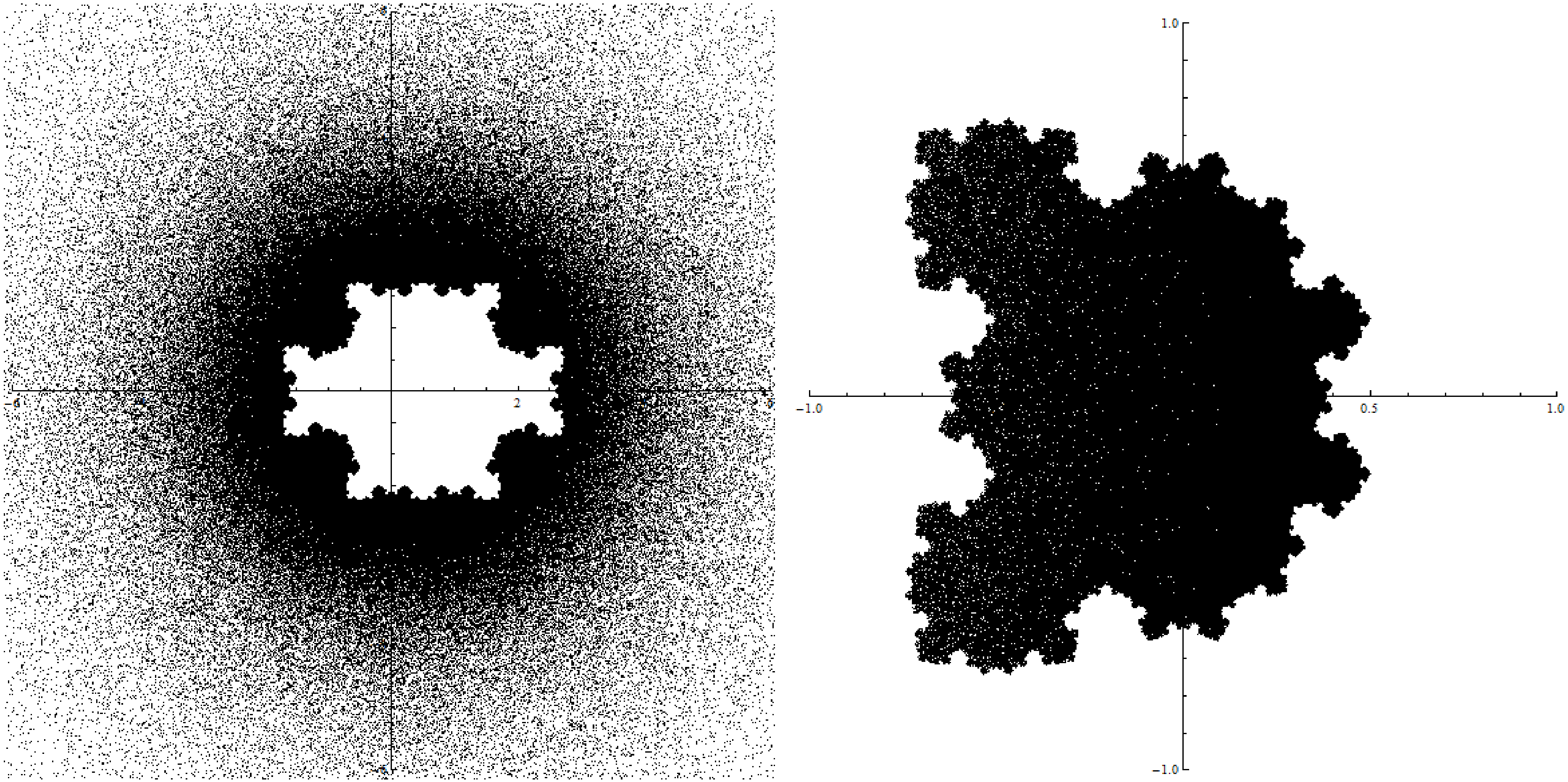


Fig. 5: $V^*_{1,1}$ and $X_{1,1}$ (Gremlin)

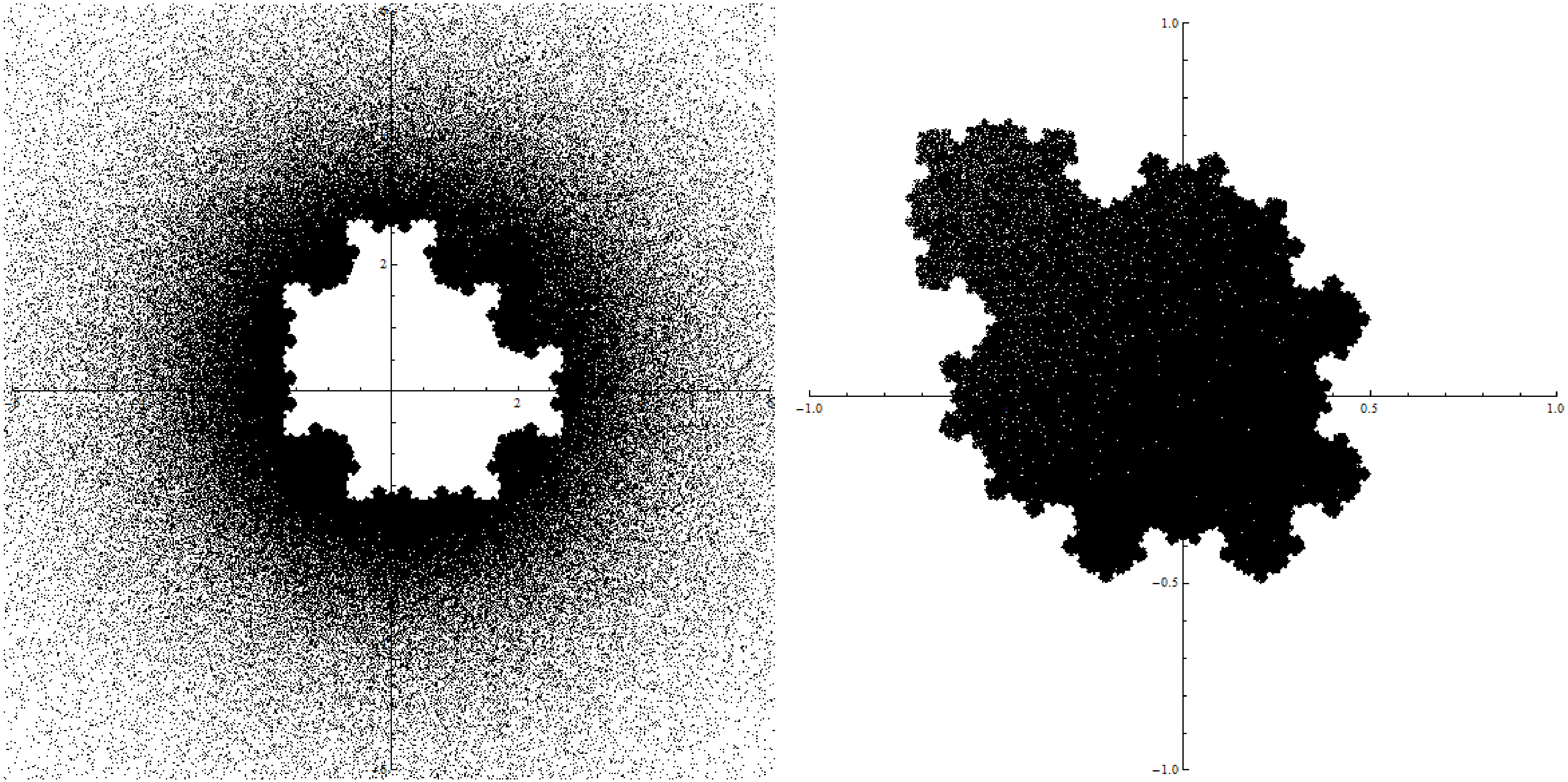
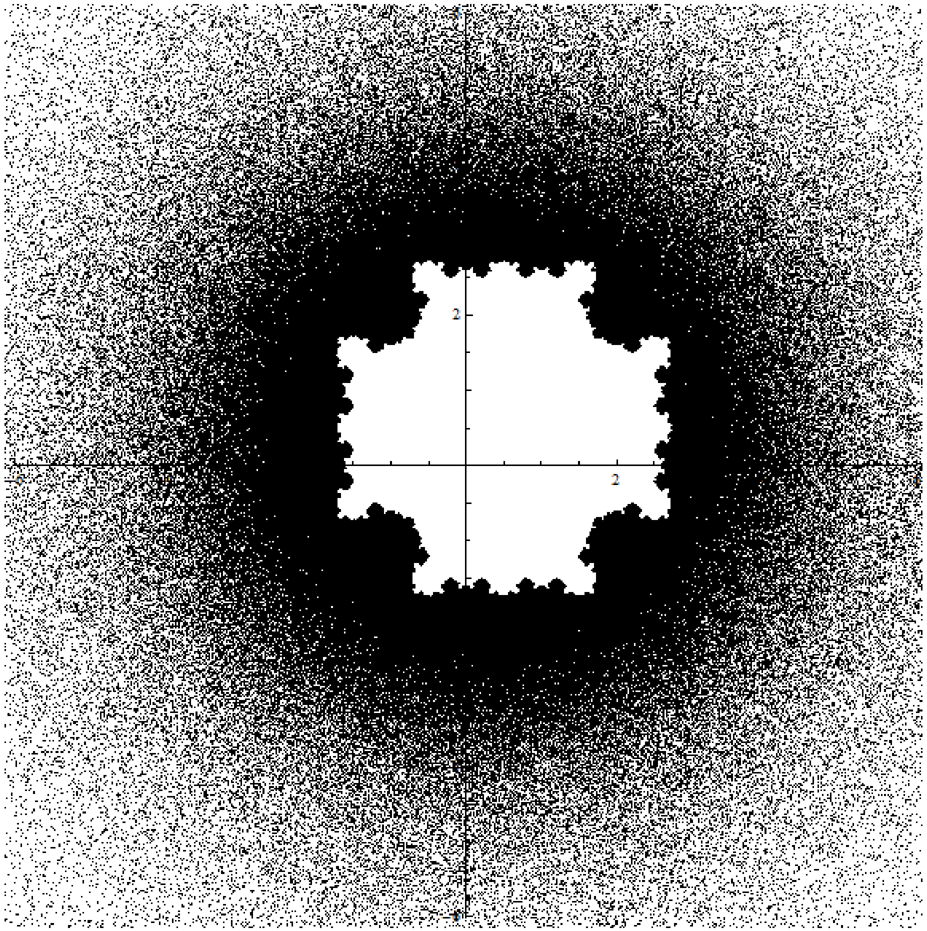


Fig. 6: $V^*_{2,1}$ and $X_{2,1}$ (Suppon)



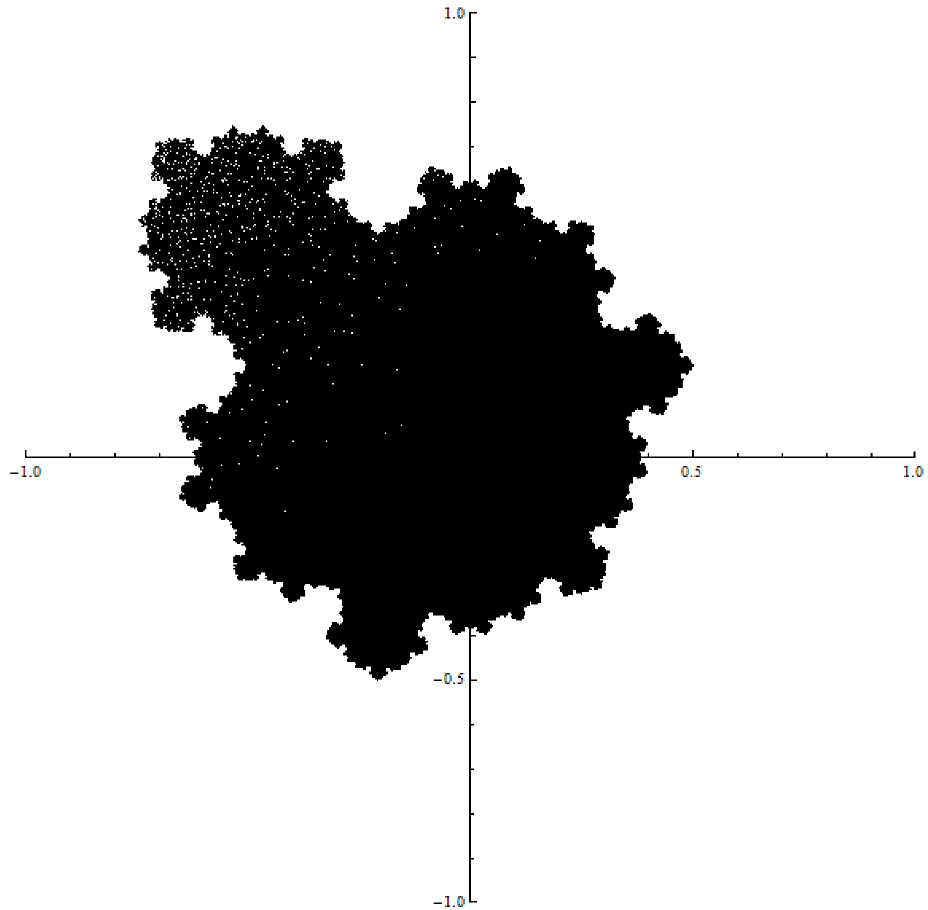


Fig. 7: $V^*_{3,1}$ and $X_{3,1}$ (Turtle)

We put

$$\hat{U} = \bigcup_{k=1}^{3} \bigcup_{\ell=1}^{4} V_{k,\ell} \times V^*_{k,\ell}$$

and define

$$\hat{T}(z,w) = \left(\frac{1}{z} - a,\ \frac{1}{w} - a\right) = \left(\frac{-aiz+i}{iz},\ \frac{-aiw+i}{iw}\right)$$

for $(z,w) \in \hat{U}$ where $a = [1/z]$.

We define a measure $\hat{\mu}$ on $\mathbb{C} \times \mathbb{C}$ as follows

$$d\hat{\mu} = \frac{dx_1 dx_2 dw_1 dw_2}{|z-w|^4}$$

for $(z,w) \in \mathbb{C} \times \mathbb{C}$ with $z = x_1 + ix_2$ and $w = w_1 + iw_2$.

Theorem 1 (Ei-Nakada-Natsui)

1. $\hat{U}$ has positive 4-dimensional Lebesgue measure.
2. $\hat{T}$ is 1-1 and onto except for a set of 4-dimensional Lebesgue measure 0.
3. $\hat{\mu}$ is $\hat{T}$-invariant measure.
   i. e. $(\hat{U}, \hat{T}, \hat{\mu})$ is a natural extension of (U, T, μ) where μ is an absolutely continuous invariant measure which is unique.

Corollary

$$d\mu(z) = \left(\int_{V_{k,\ell}^{*}} \frac{1}{|z - w|^4} dw_1 dw_2 \right) dx_1 dx_2$$

for $z \in V_{k,\ell}$.

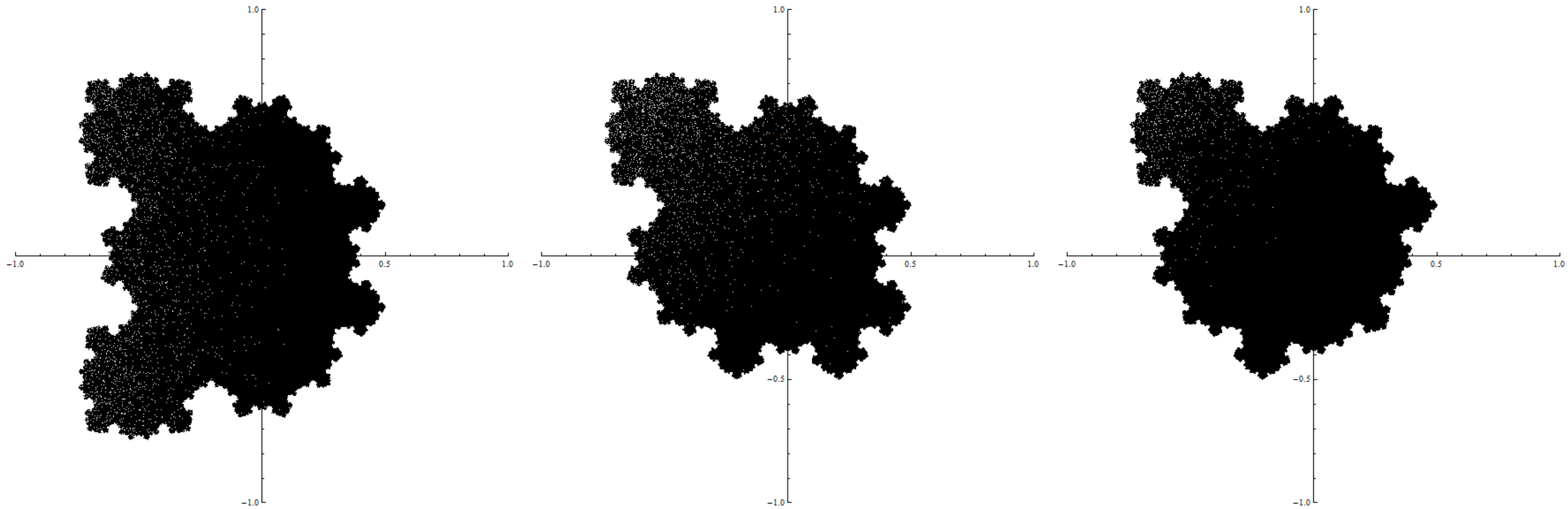


Fig. 8: Tiles $X_{1,1}$, $X_{2,1}$, $X_{3,1}$

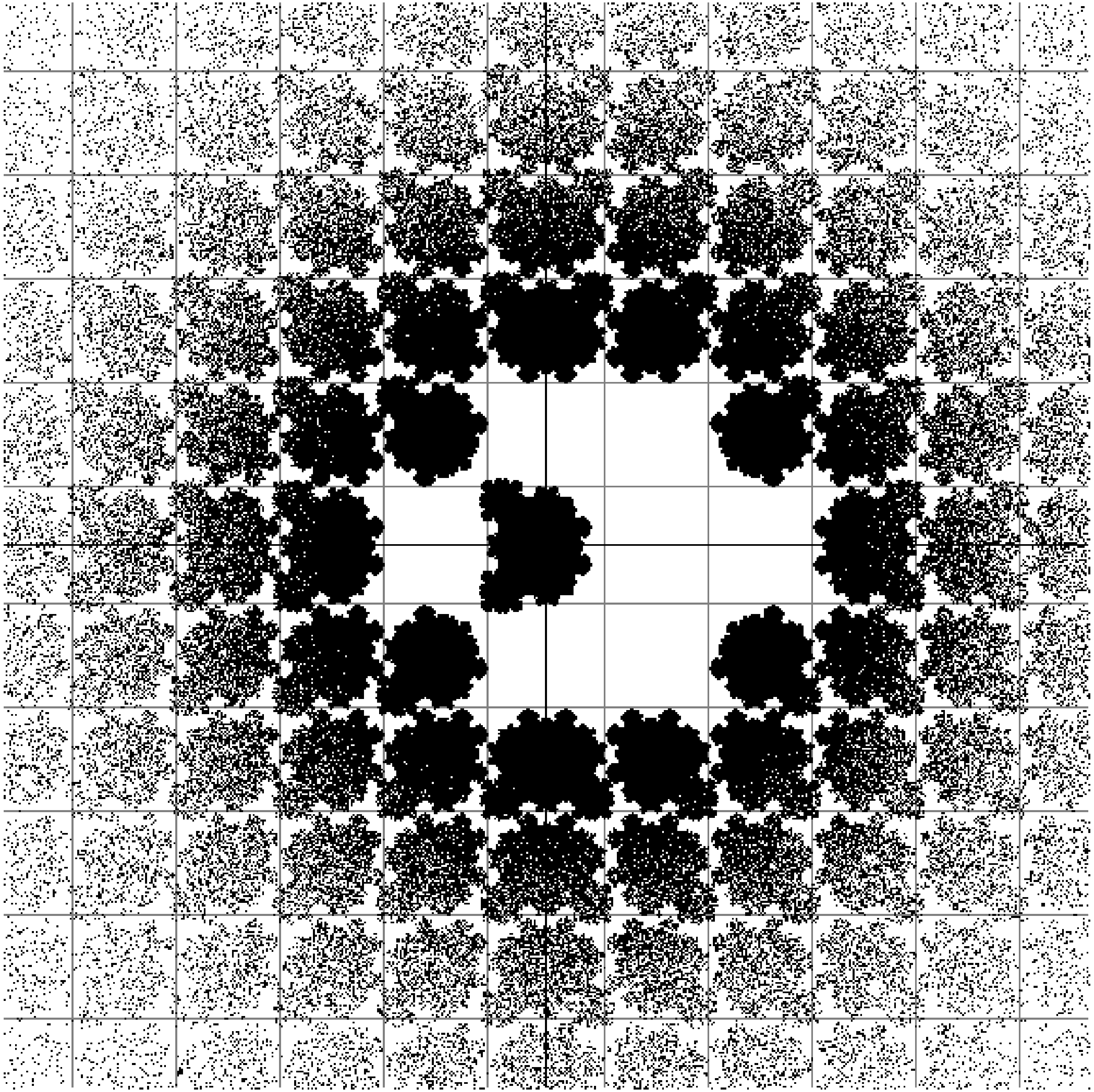


Fig. 9: Tiling of $V_{1,1}^{*}$ (The original picture was found by S. Ito.)

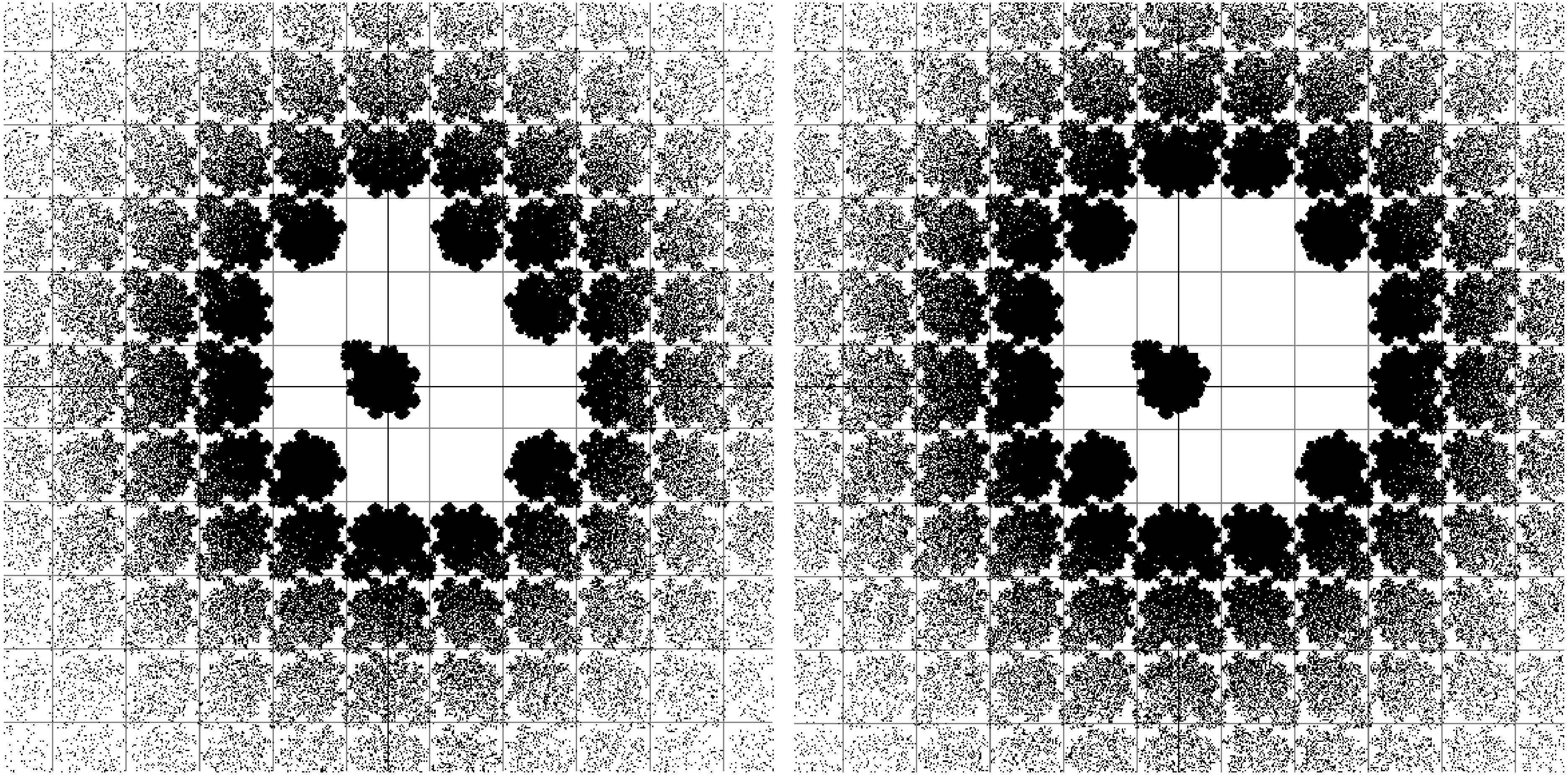


Fig. 10: Tiling of $V^*_{2,1}$ and $V^*_{3,1}$

Theorem 2 (Ei-Nakada-Natsui)

1. $V^*_{k,\ell}$ is tiled by $\{X_{k,\ell} : k = 1, 2, 3,\ \ell = 1, 2, 3, 4\}$.
   Concretely for any $1 \le k_0 \le 3$ and $1 \le \ell_0 \le 4$,

$$V^*_{k_0,\ell_0} = \bigcup_{k=1}^{3} \bigcup_{\ell=1}^{4} \bigcup_{a \in D_{k_0,\ell_0,k,\ell}} (X_{k,\ell} - a)$$

   where

$$D_{k_0,\ell_0,k,\ell} = \left\{ a \in \mathbb{Z}[i] : \begin{array}{l} \text{there exists } w \in \langle a \rangle \cap V_{k,\ell} \\ \text{such that } Tw \in V_{k_0,\ell_0} \end{array} \right\}.$$

2. The boundary of $X_{k,\ell}$ is a Jordan curve and has
   2-dimensional Lebesgue measure 0.
   $\rightarrow X_{k,\ell}$ is a topological disk.

The other cases

There are some other nearest type complex continued fractions for $-2, -7$ and -11. However, they do not have the best approximation property.

The best approximation property: p/q is a best approximation to x if

$$|q'| < |q| \Longrightarrow |q'x - p'| > |qx - p|.$$

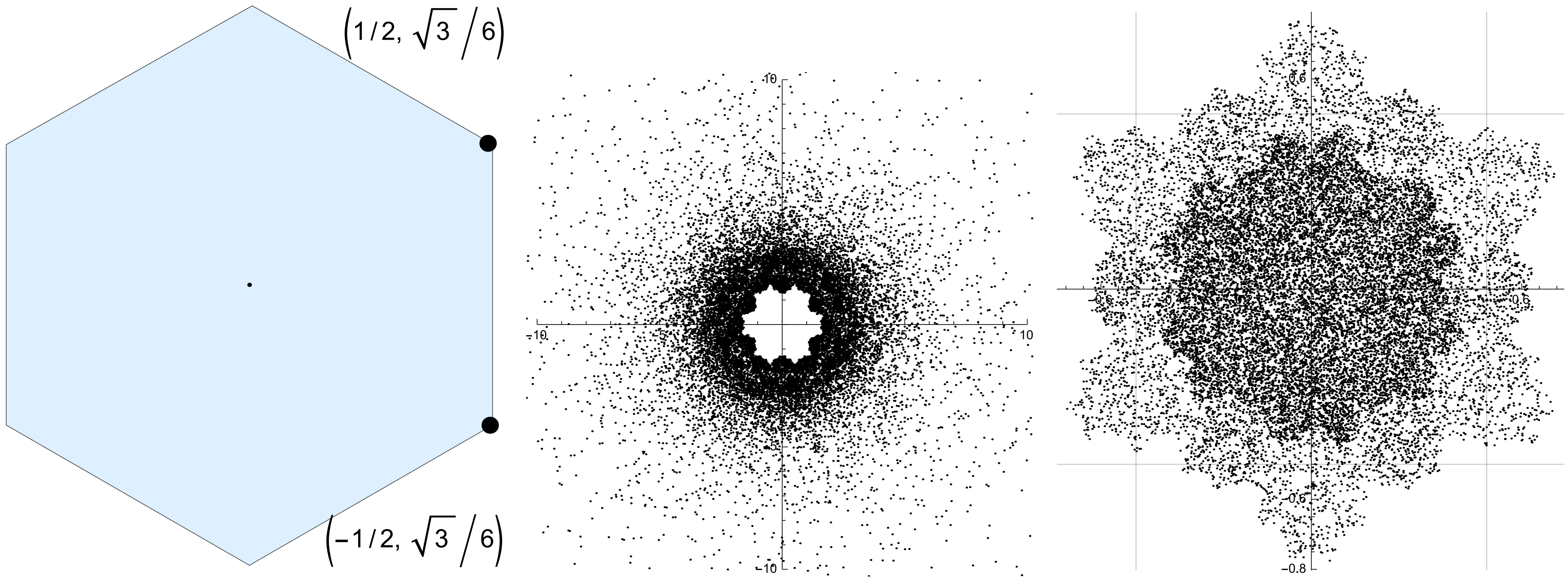


Fig. 11: In the case of $\mathbb{Q}(\sqrt{-3})$

Thank you very much.