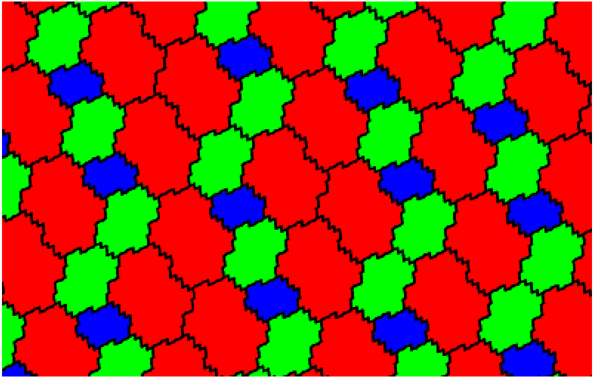
Tilings for Rauzy fractals

Arnaud Hilion

joint work with Nicolas Bédaride and Timo Jolivet

Admont – June 2015

First tiling



Tribonacci substitution

$$\sigma : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 13 \\ 3 \mapsto 1 \end{cases} \qquad \mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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- One real cubic eigenvalue $\beta \approx 1.839 \ (> 1)$
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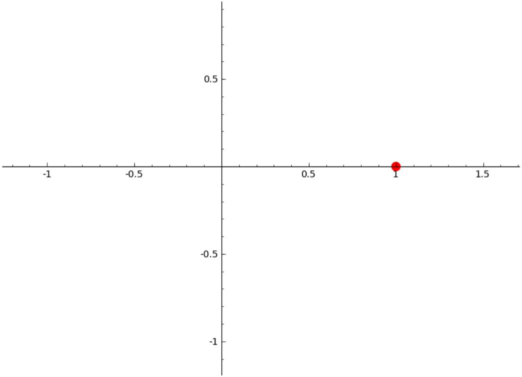
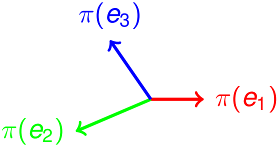
Action of $\mathbf{M}_\sigma$ on $\mathbb{R}^3$:

- **Expanding line $\mathbb{E}$**
- **Contracting plane $\mathbb{P}$**

Rauzy fractals

$\sigma^\infty(1) = 121312112131212131211\cdots$ $\pi : \mathbb{R}^3 \to \mathbb{P}$,
$\pi(e_1)$ projection along $\mathbb{E}$





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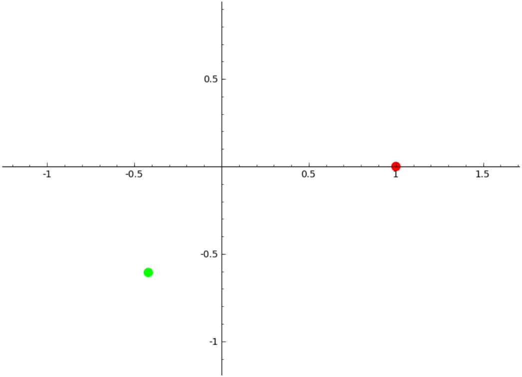

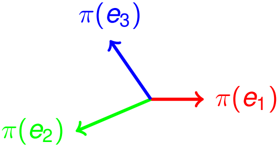

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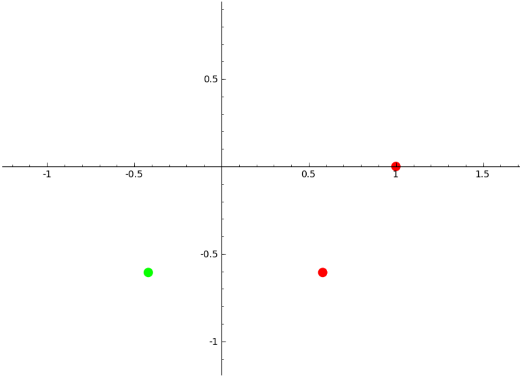

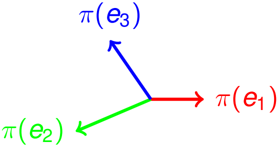

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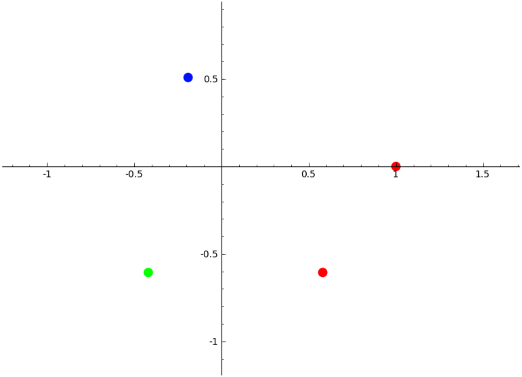

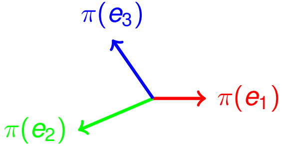

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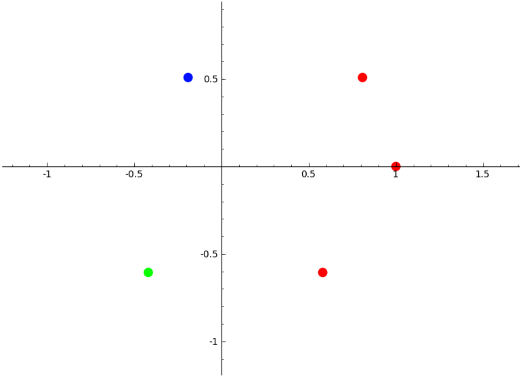

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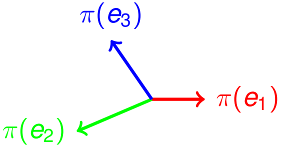

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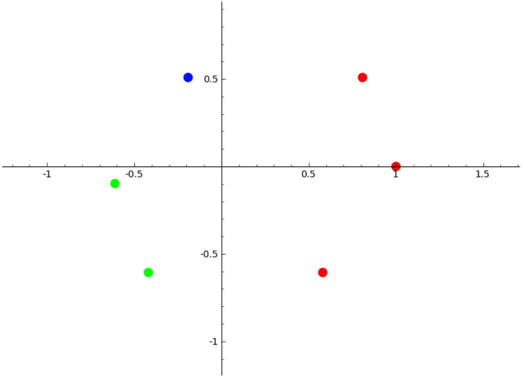

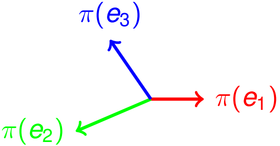

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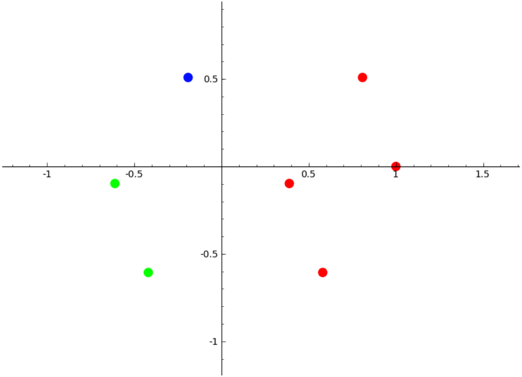

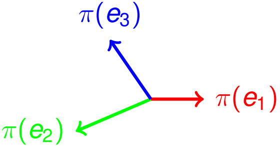

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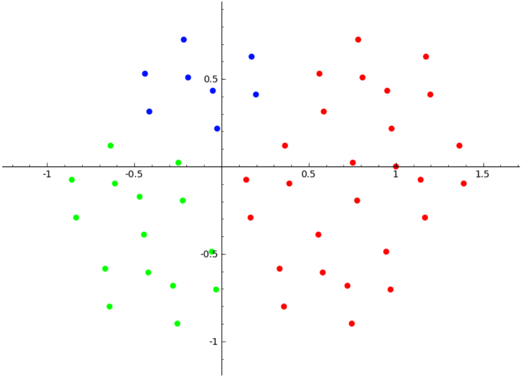

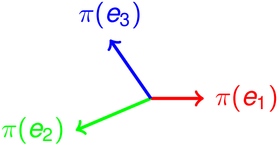

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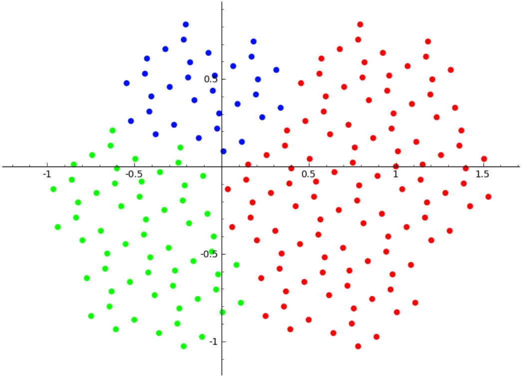

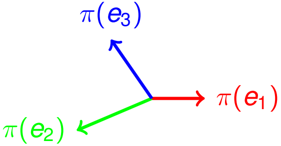

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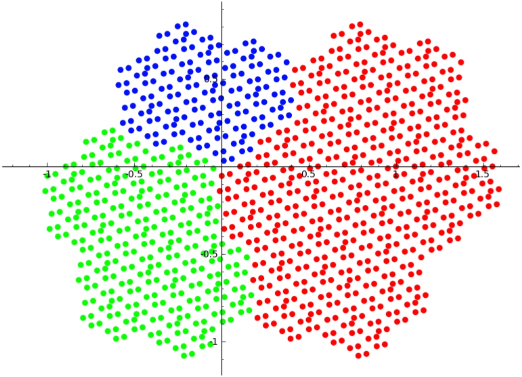

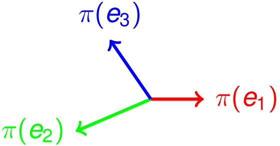

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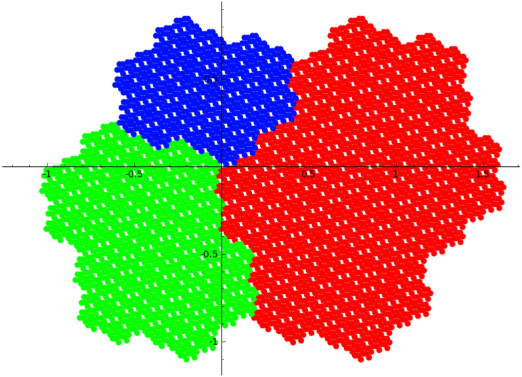

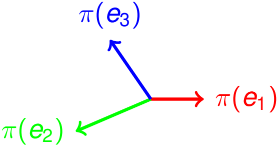

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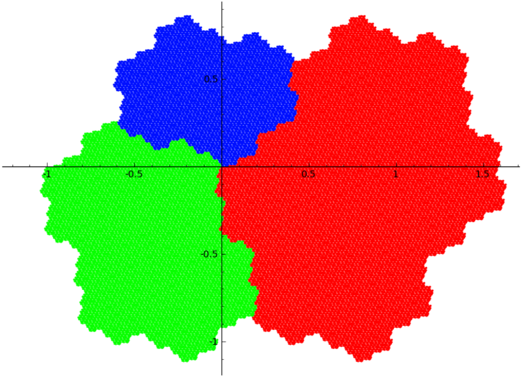

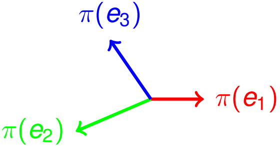

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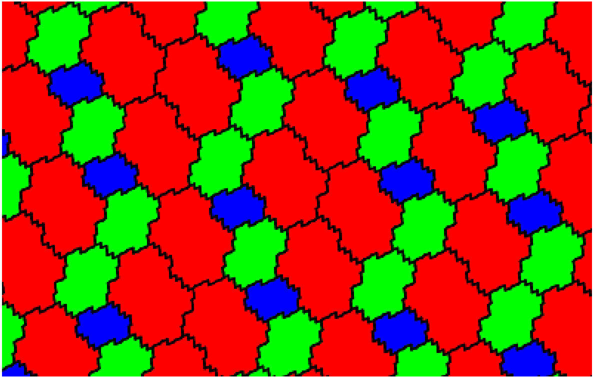

Second tiling...

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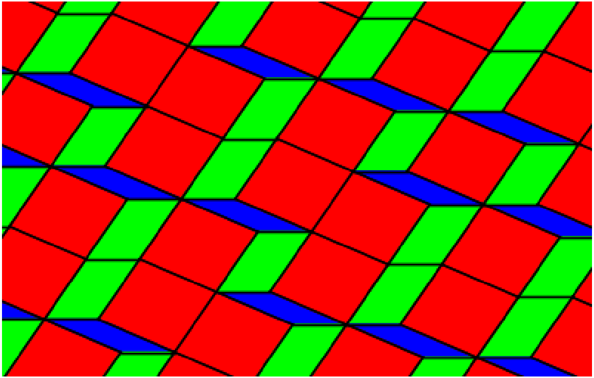
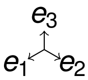


Dual substitution E



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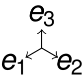

$$[\mathbf{x}, 1]^* = \{\mathbf{x} + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$ 

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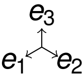

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$$\mathbf{E} : \begin{cases} [\mathbf{x}, 1]^* \mapsto \mathbf{M}_\sigma{}^{-1}\mathbf{x} + \left([\mathbf{0}, 1]^* \cup [\mathbf{0}, 2]^* \cup [\mathbf{0}, 3]^*\right) \\ [\mathbf{x}, 2]^* \mapsto \mathbf{M}_\sigma{}^{-1}\mathbf{x} + [\mathbf{e}_3, 1]^* \\ [\mathbf{x}, 3]^* \mapsto \mathbf{M}_\sigma{}^{-1}\mathbf{x} + [\mathbf{e}_3, 2]^* \end{cases}$$

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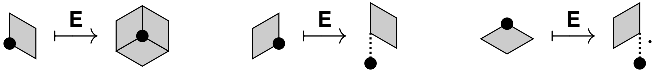

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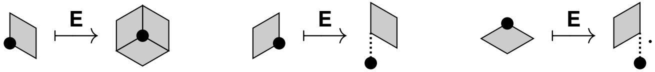

in the preimages : black dots stand for **x**

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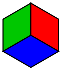

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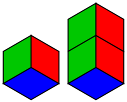

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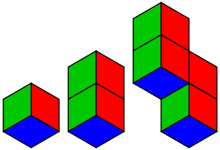

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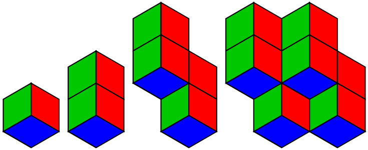

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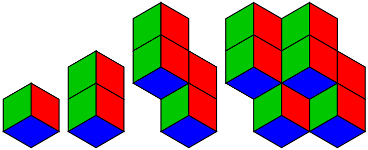

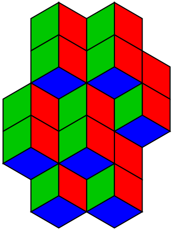

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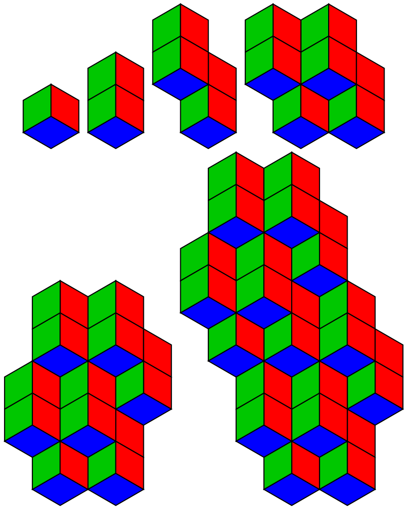
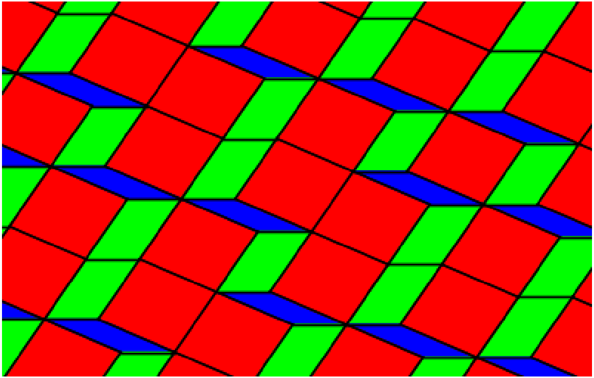
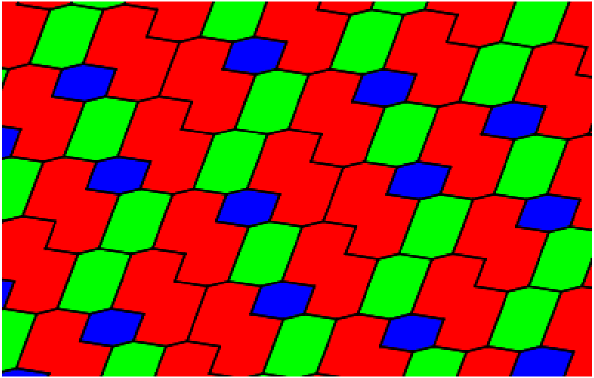


Second tiling



Third tiling



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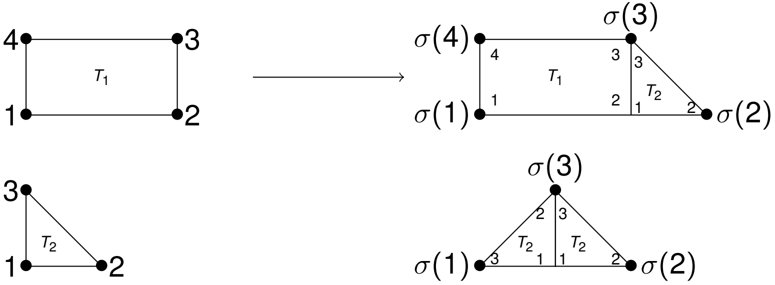

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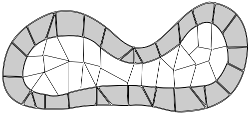

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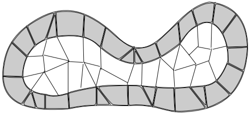

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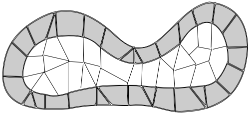

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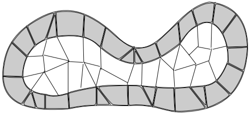

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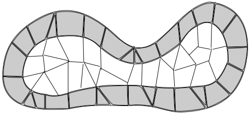

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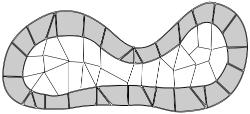

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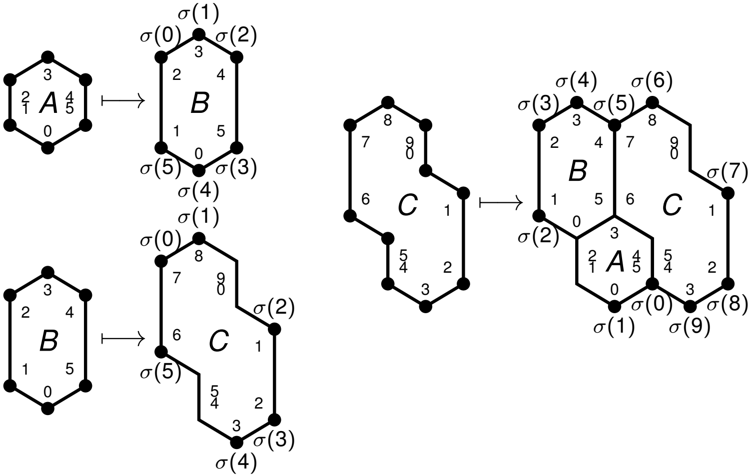
Tribonacci topological substitution



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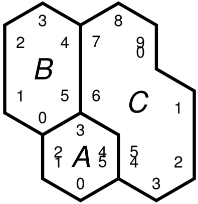
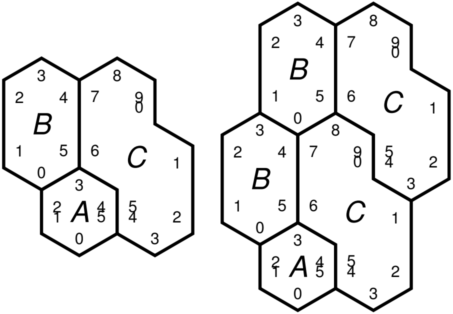
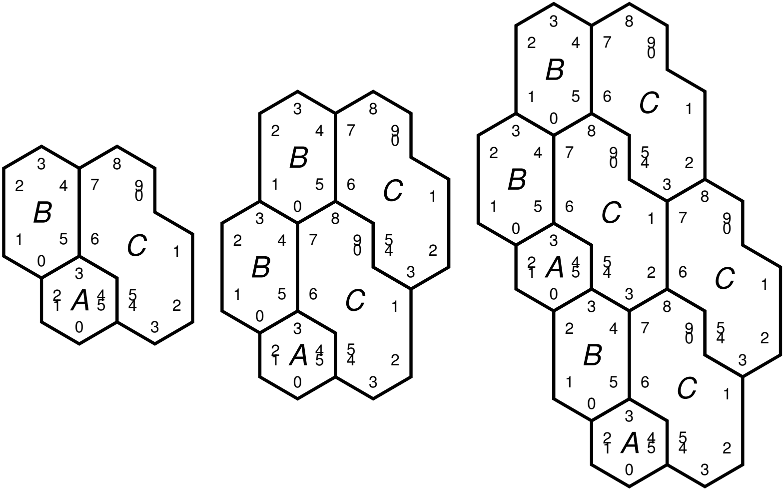


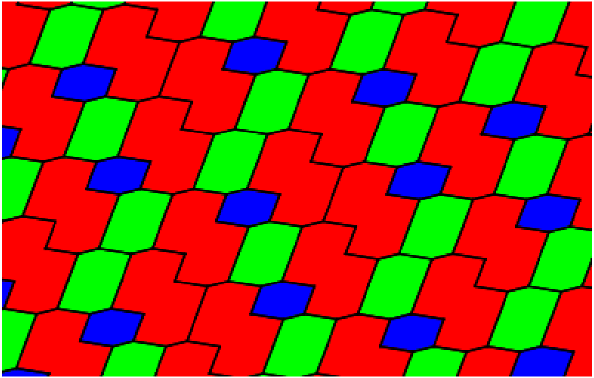
Tribonacci topological substitution



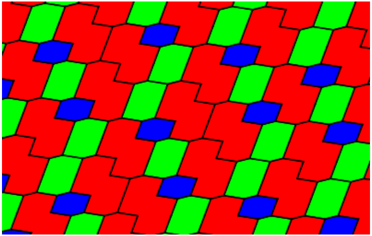
Tribonacci topological substitution

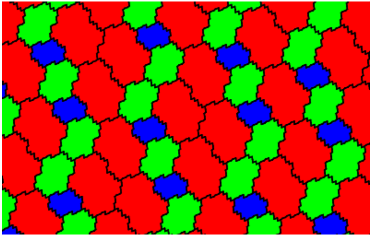


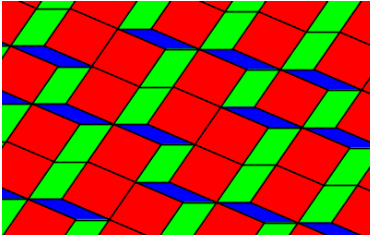
Third tiling



Are they "the same" ?







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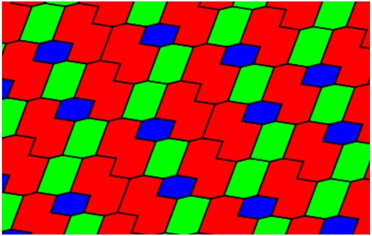 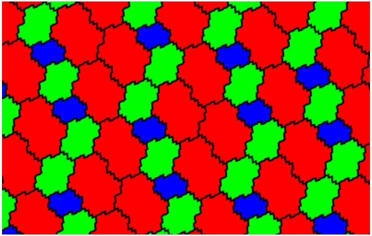

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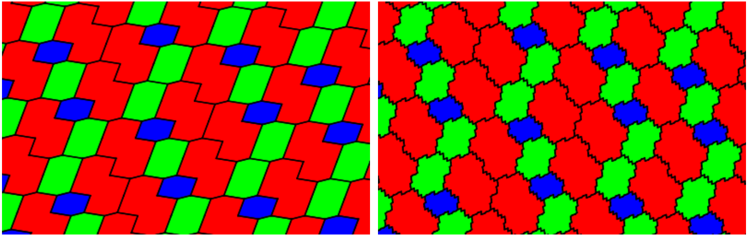
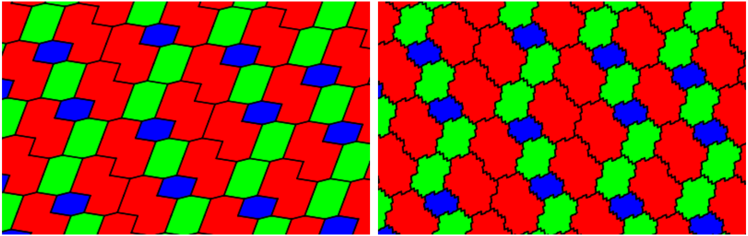


2 different geometric realizations of the same complex $\sigma^{\infty}(C)$

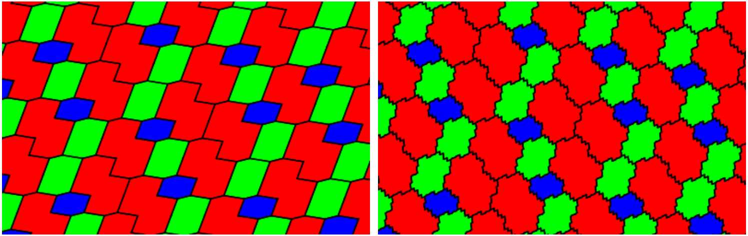
Are they "the same" ?



2 different geometric realizations of the same complex $\sigma^{\infty}(C)$

- by hexagons and decagons,

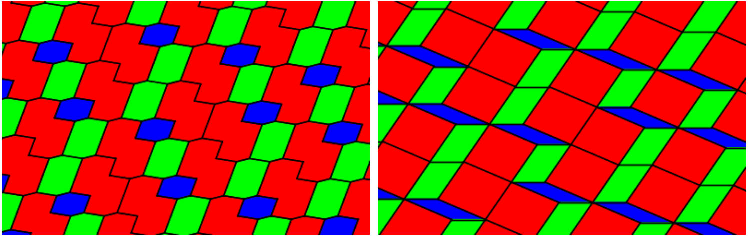
Are they "the same" ?



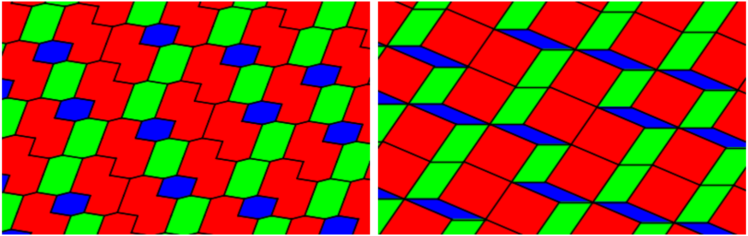
2 different geometric realizations of the same complex $\sigma^{\infty}(C)$

- by hexagons and decagons,
- by Rauzy fractals.

Are they "the same" ?

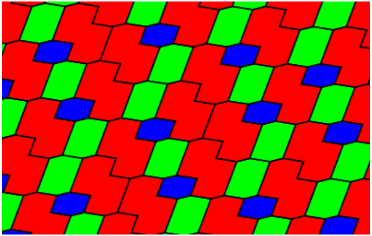
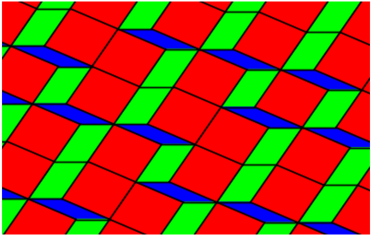


Are they "the same" ?



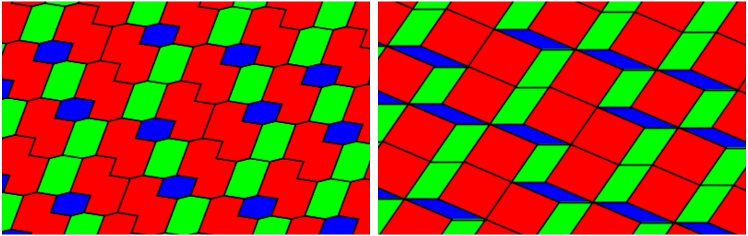
- not the same underlying complex

Are they "the same" ?

- not the same underlying complex
- no basic trick, such as taking some edges of length 0

Are they "the same" ?



- not the same underlying complex
- no basic trick, such as taking some edges of length 0
- but we guess the "obvious" bijection between tiles

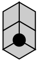
Morphism "tile to tile"

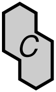
Morphism "tile to tile"

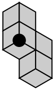
$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}^3([\mathbf{0}, 2]^*)$

$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}^3([\mathbf{0}, 3]^*)$

$\Phi\Big($ 

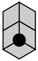
$\Big) =$  $= \mathbf{E}^3([\mathbf{0}, 1]^*)$

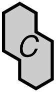
Morphism "tile to tile"

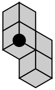
$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}^3([\mathbf{0}, 2]^*)$

$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}^3([\mathbf{0}, 3]^*)$

$\Phi\Big($ 

$\Big) =$  $= \mathbf{E}^3([\mathbf{0}, 1]^*)$

$\mathbf{E}^{-3} \circ \Phi$ is a morphism "tile to tile"