# Ergodic properties of $\beta$-adic Halton sequences

(joint work with M. Hofer and R. Tichy)

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### Definition

A sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1[$ is uniformly distributed if and only if for every interval $[a,b[\subset [0,1[$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{[a,b[}(x_i) = \lambda([a,b[)$$

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Given a sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1[$, the quantity

$$D_N(x_n) = \sup_{0\le a<b\le 1} \left| \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[a,b[}(x_n) - \lambda([a,b[) \right| \tag{1}$$

is called discrepancy of $(x_n)_{n\in\mathbb{N}}$.

# Classical example: van der Corput sequence in base $b$

Fix an integer $b \geq 2$. Then, every $n \in \mathbb{N}$ has a unique expansion

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with $\epsilon_j \in \{0, 1, \ldots, b-1\}$.

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Define the map $\phi_b : \mathbb{N} \longrightarrow [0, 1[$ by

$$\phi_b \left( \sum_{j \geq 0} \epsilon_j b^j \right) = \sum_{j \geq 0} \epsilon_j b^{-j-1}.$$

Then the sequence

$$(\phi_b(n))_{n \in \mathbb{N}}$$

is the van der Corput sequence in base $b$.

Fix $b = 2$.

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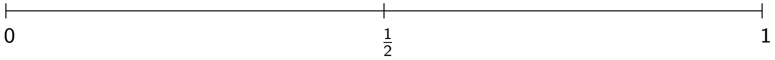

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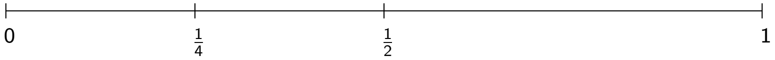

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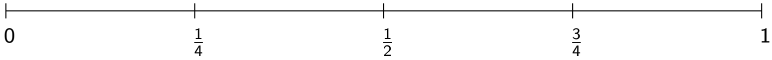

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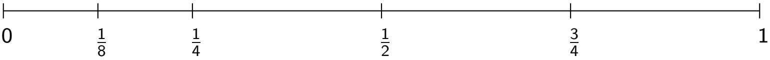

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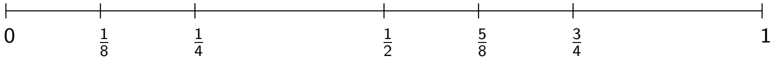

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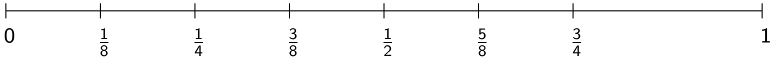

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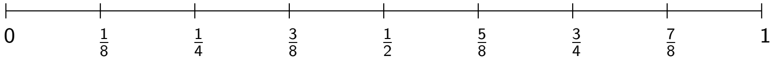

# Extensions of the van der Corput sequence

- Multidimensional version
- Irrational base $\beta$

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  - Fix $b_1, b_2 > 1$ positive integers. If $\gcd(b_1, b_2) = 1$, then the sequence

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    is called Halton sequence and it is low-discrepancy.
- Irrational base $\beta$
  - Fix an irrational $\beta > 1$. If $\beta$ is Pisot and all its conjugates belong to $\{z \in \mathbb{C} : |z| < 1\}$, then the sequence
    $$(\phi_\beta(n))_{n\in\mathbb{N}}$$
    is low-discrepancy.

Let $\beta > 1$ and $\beta \in \mathbb{R}$. Let $f_\beta : [0,1[ \to [0,1[$ be the $\beta$-adic transformation. Let $A = \mathbb{Z} \cap [0, \beta[$. Then we have the following fibred system $([0,1[, f_\beta)$:

$$\begin{array}{ccc} [0,1[ & \xrightarrow{f_\beta} & [0,1[ \\ \phi \downarrow & & \downarrow \phi \\ \Omega & \xrightarrow[\sigma]{} & \Omega \end{array}$$

$\sigma$ is the one-sided shift and $\phi$ is the representation map defined by

$$\phi(x)(n) = \epsilon_i \ , \qquad \text{if } \frac{\epsilon_i}{\beta} \le f_\beta^n(x) < \frac{(\epsilon_i + 1)}{\beta}.$$

Let $d_\beta(1)$ the $\beta$-expansion of 1 and

$$X_\beta = \{\omega \in \Omega : \forall n \in \mathbb{N}\ \sigma^n\omega \prec d_\beta(1)\}.$$

Define $\rho_\beta : X_\beta \to [0,1[$ by

$$\rho_\beta(a) = \sum_{i=0}^{\infty} \epsilon_i(a)\beta^{-n-1}\ .$$

Let $\beta = \frac{\sqrt{5}+1}{2}$. Then $A = \{0, 1\}$ and $d_\beta = 110^\omega$.

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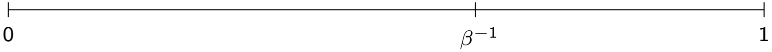

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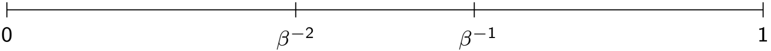

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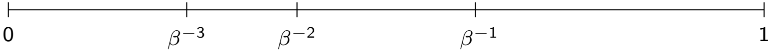

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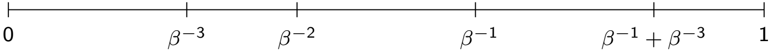

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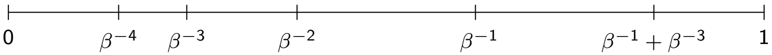

### Definition

Let $(G_n)_{n\geq 0}$ be an increasing sequence with $G_0 = 1$ and $G_n \in \mathbb{N}$. Then every $n \in \mathbb{N}$ can be written as

$$n = \sum_{k=0}^{\infty} \varepsilon_k(n) G_k,$$

where $\varepsilon_k(n) \in \{0, \ldots, \lfloor G_{k+1}/G_k \rfloor\}$. This expansion is unique and finite, provided that for every $K > 0$

$$\sum_{k=0}^{K-1} \varepsilon_k(n) G_k < G_K. \tag{2}$$

$\mathcal{K}_G$ is the subset of sequences that verify (2) and the elements in $\mathcal{K}_G$ are called G-admissible.

Let $G_0 = 1$ and $G_k = a_0 G_{k-1} + \cdots + a_{k-1} G_0 + 1$ for $k < d$ and

$$G_{n+d} = a_0 G_{n+d-1} + \cdots + a_{d-1} G_n, \quad n \geq d \ ,$$

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and $a_0 \geq \ldots \geq a_{d-1} \geq 1$. The solution $\beta$ of the characteristic equation of the numeration system $G$

$$x^d = a_0 x^{d-1} + \ldots + a_{d-1}.$$

is a Pisot number.

Theorem (Grabner-Liardet-Tichy 1995)

$(\mathcal{K}_G, \mathcal{A}, \mu, \tau)$ *is uniquely ergodic.*

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{\scriptstyle ?}\big\downarrow & & \big\downarrow{\scriptstyle ?} \\
\mathcal{K}_G & \xrightarrow[\tau]{} & \mathcal{K}_G
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- Fix an irrational basis $\beta > 1$ and consider

$$\phi_\beta \colon \mathcal{K}_G \to \mathbb{R}^+$$

$$\phi_\beta(n) = \phi_\beta\left(\sum_{j\geq 0} \epsilon_j(n) G_j\right) = \sum_{j\geq 0} \epsilon_j(n)\beta^{-j-1} \ .$$

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- Is the image of $\mathcal{K}_G$ under $\phi_\beta$ contained and dense in $[0,1[$?

## Lemma

*Let* $\mathbf{a} = (a_0, \ldots, a_{d-1})$ *be the decreasing coefficients defining* $G$ *and*

$$x^d = a_0 x^{d-1} + \ldots + a_{d-1}$$

*be the minimal polynomial of* $\beta$*. Then* $\phi_\beta(\mathbb{N}) \subset [0,1)$ *is dense in* $[0,1]$ *if and only if either*

$$\mathbf{a} = (a_0, \ldots, a_0)$$

*or*

$$\mathbf{a} = (a_0, a_0 - 1, \ldots, a_0 - 1, a_0)\ ,$$

*where* $a_0 > 0$*.*

### Lemma

*Let*

$$G_{n+d} = a(G_{n+d-1} + \cdots + G_n), \quad n \geq d \ ,$$

*with* $a > 0$ *and let* $\beta$ *denote the corresponding characteristic root. Then* $\mu(Z) = \lambda(\phi_\beta(Z))$ *for every cylinder set* $Z$ *in* $\mathcal{K}_G$*. Moreover* $\phi_\beta$ *is bijective in* $[0, 1[$*.*

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The system $([0, 1[, \lambda, T)$, with $T = \phi_\beta \circ \tau \circ \phi_\beta^{-1}$ is uniquely ergodic.

## Theorem

*Let $G^1, \dots, G^s$ be numeration systems given by*

$$G^1_{n+d} = b_1(G_{n+d-1} + \cdots + G_n), \quad n \geq d \ ,$$

$$G^2_{n+d} = b_2(G_{n+d-1} + \cdots + G_n), \quad n \geq d \ ,$$

$$\vdots$$

$$G^s_{n+d} = b_s(G_{n+d-1} + \cdots + G_n), \quad n \geq d \ ,$$

*with pairwise coprime, positive integers $b_i$. Furthermore let $\frac{\beta_i^k}{\beta_j^l} \notin \mathbb{Q}$, for all $l, k \in \mathbb{N}$, where $\beta_1, \dots, \beta_s$ denote the characteristic roots of the numerations systems. Then*

$$((\mathcal{K}_{G^1}, \tau_1) \times \ldots \times (\mathcal{K}_{G^s}, \tau_s)) \ ,$$

*is uniquely ergodic.*

The $\boldsymbol{\beta}$-adic Halton sequence is given as

$$(\phi_{\boldsymbol{\beta}}(n))_{n\in\mathbb{N}} = (\phi_{\beta_1}(n), \ldots, \phi_{\beta_s}(n))_{n\in\mathbb{N}} \ ,$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$.

### Theorem

*Let the numeration system $G$ be defined by the coefficients $(a_0, a_1, a_2) = (1, 0, 1)$, $\beta$ be its characteristic root and $\tau$ the odometer on $G$. Then $\mu(Z) = \lambda(\phi_\beta(Z))$ for all cylinder sets $Z$. Thus $T(x) = \phi_\beta \circ \tau \circ \phi_\beta^{-1}(x)$ is uniquely ergodic and $(T^n x)_{n \in \mathbb{N}}$ is u.d. for all $x$ in $[0, 1)$.*