Dirichlet uniformly well-approximated numbers

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Substitutions and continued fractions
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Dirichlet's Theorem

Theorem (Dirichlet (1842))

Let θ, Q *be real numbers with* $Q \geq 1$.
There exists an integer q *with* $1 \leq q \leq Q$, *such that*

$$\|q\theta\| < \frac{1}{Q}.$$

where $\|t\| = \min_{n \in \mathbb{Z}} |t - n|$.

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where $\|t\| = \min_{n \in \mathbb{Z}} |t - n|$.

there exist infinitely many q satisfying $\quad \|q\theta\| < \frac{1}{q}.$

there exist infinitely many p/q's such that $\quad \left|\theta - \frac{p}{q}\right| < \frac{1}{q^2}.$

Uniform approximation and asymptotic approximation

- Uniform approximation :

  For all $\theta \in \mathbb{R}$

  $$\text{for all large } Q, \ \|n\theta\| < \frac{1}{Q} \ \text{ has a solution } 1 \le n \le Q.$$

- Asymptotic approximation :

  For all $\theta \in \mathbb{R}$

  $$\|n\theta\| < \frac{1}{n} \quad \text{for infinitely many } n.$$

Inhomogeneous approximation

Theorem (Minkowski (1907))

Let θ be an irrational. Let y be a real number which is not equal to any $m\theta + \ell$ with $m, \ell \in \mathbb{N}$. Then there exist infinitely many integers n such that

$$\|n\theta - y\| < \frac{1}{4|n|}.$$

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The inhomogeneous analogy of the uniform Dirichlet theorem does NOT exist.

Measure theoretic results of asymptotic approximation

- Khinchine (1924):

$$\mathrm{Leb}(\{\theta : \|n\theta\| < \psi(n) \text{ infinitely often}\}) = 1 \iff \sum_{n=1}^{\infty} \psi(n) = \infty.$$

- Szüsz (1958); Schmidt (1964):

$$\mathrm{Leb}(\{\theta : \|n\theta - y\| < \psi(n) \text{ infinitely often}\}) = 1 \iff \sum_{n=1}^{\infty} \psi(n) = \infty.$$

We fix irrational θ and find the set of y.

- Fuchs-K (2016):

$$\mathrm{Leb}(\{y : \|n\theta - y\| < \psi(n) \text{ for infinitely often}\}) = 1$$
$$\iff \sum_{k=0}^{\infty} \sum_{n=q_k}^{q_{k+1}-1} \min\left(\psi(n), \|q_k\theta\|\right) = \infty,$$

where q_k is the denominator of the principal convergents of θ.

Kurzweil (1955), Cassels (1957), Berend-William (1992), Tseng (2008), Chaika-Constantine (2013), Simons (2015) ...,

Also, K-Nakada (2011), K-Nakada-Natsui (2013)

For the example, Kurzweil (1955)

$$\{\theta :\ \exists c > 0 \text{ with } \|n\theta\| \geq c/n \text{ for all } n \geq 1\}$$
$$= \left\{\theta : \|n\theta - y\| < \psi(n) \text{ i.o. } \forall \psi(n), \sum_{n\geq 1} \psi(n) = \infty\right\}.$$

Tseng (2008)

$$\{\theta :\ \exists c > 0 \text{ with } \|n\theta\| \geq c/n^{\tau} \text{ for all } n \geq 1\}$$
$$= \left\{\theta : \|n\theta - y\| < \psi(n) \text{ i.o. } \forall \psi(n), \sum_{n\geq 1} \psi(n)^{\tau} = \infty\right\}.$$

and many others.

Hausdorff dimension for the asymptotic approximation

- Jarník (1929); Besicovitch (1934): For $\tau \geq 1$,

$$\dim_H(\{\theta : \|n\theta\| < n^{-\tau} \text{ infinitely often}\}) = \frac{2}{1+\tau}.$$

- Levesley (1998): For any $y \in \mathbb{R}$, and $\tau \geq 1$:

$$\dim_H(\{\theta : \|n\theta - y\| < n^{-\tau} \text{ infinitely often}\}) = \frac{2}{1+\tau}.$$

- Bugeaud (2003); Schmeling-Troubetzkoy (2003): For all $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\tau \geq 1$,

$$\dim_H(\{y : \|n\theta - y\| < n^{-\tau} \text{ infinitely often}\}) = \frac{1}{\tau}.$$

Uniformly well approximated numbers

- Khintchine (1926)

$$\{\theta : \text{ for all large } Q,\ 1 \le \exists n \le Q,\ \|n\theta\| < Q^{-\tau}\}$$
$$= \begin{cases} \mathbb{Q} & \text{if } \ \tau > 1 \\ \mathbb{R} & \text{if } \ \tau \le 1. \end{cases}$$

- K-Seo (2003)

$$\mathrm{Leb}(\{y : \text{ for all large } Q,\ 1 \le \exists n \le Q,\ \|n\theta - y\| < Q^{-\tau}\})$$
$$= \begin{cases} 0 & \text{if } \ \tau > 1/w \\ 1 & \text{if } \ \tau < 1/w. \end{cases}.$$

Diophantine exponent

For an irrational number θ let

$$w(\theta) = \sup\left\{ s > 0 : \liminf_{j\to\infty} j^s \|j\theta\| = 0 \right\}.$$

- For every irrational θ

$$w(\theta) \geq 1.$$

- The set of irrational numbers of $w = 1$ has measure 1 and include the set of irrational numbers with bounded partial quotients, which is of measure 0.
- There exist numbers of $w = \infty$, called the Liouville numbers.

Result on uniformly approximated numbers

- Cheung (2011), Cheung-Chevallier (2015)

$$\Big\{(\theta_1, \theta_2) : \text{ for all large } Q, 1 \le \exists n \le Q, \\ \max\{\|n\theta_1\|, \|n\theta_2\|\} < \delta Q^{-1/2}\Big\}$$

is of Hausdorff dimension $4/3$
($\frac{d^2}{d+1}$ for higher dimensional cases).

- Kleinbock-Wadleigh (in more general form)

$$\{(y, \theta) : \text{ for all large } Q,\ 1 \le \exists n \le Q, \|n\theta - y\| < \psi(Q)\}$$

has full Lebesgue measure if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \psi(n)} < \infty.$$

Main problem

Let an irrational θ be fixed.

What is the size (Hausdorff dimension) of

$$\mathcal{U}_\tau[\theta] := \Big\{ y : \text{ for all large } Q,\ \|n\theta - y\| < Q^{-\tau} \\ \text{has a solution } 1 \le n \le Q \Big\}$$

Question of Bugeaud and Laurent (2005)

Theorem (K-Liao)

Let θ be an irrational with $w = w(\theta) > 1$. Then

$$\dim_H(\mathcal{U}_\tau[\theta]) = \begin{cases} \varliminf_{k\to\infty} \dfrac{\log\left(n_k^{1/\tau+1}\prod_{j=1}^{k-1} n_j^{1/\tau}\|n_j\theta\|\right)}{\log(n_k\|n_k\theta\|^{-1})}, & \dfrac{1}{w} < \tau < 1, \\ \varliminf_{k\to\infty} \dfrac{-\log\left(\prod_{j=1}^{k-1} n_j\|n_j\theta\|^{1/\tau}\right)}{\log(n_k\|n_k\theta\|^{-1})}, & 1 < \tau < w, \end{cases}$$

where n_k is the (maximal) subsequence of (q_k) such that

$$\begin{aligned} n_k\|n_k\theta\|^\tau &< 1, & \text{if } 1/w < \tau < 1, \\ n_k^\tau\|n_k\theta\| &< 2, & \text{if } 1 < \tau < w. \end{aligned}$$

Theorem (K-Liao)

Let θ be an irrational with $w = w(\theta) \geq 1$. Then

$$\begin{aligned} \mathcal{U}_\tau[\theta] &= \mathbb{T} && \text{if } \tau < 1/w, \\ \mathcal{U}_\tau[\theta] &= \{i\theta \in \mathbb{T} : i \geq 1\} && \text{if } \tau > w. \end{aligned}$$

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Moreover, for $\tau = 1$

$$\frac{1}{w+1} \leq \dim_H(\mathcal{U}_1[\theta]) \leq \frac{2}{w+1}$$

Theorem (K-Liao)

For each irrational θ, $\dim_H(\mathcal{U}_\tau[\theta])$ is a continuous function of τ on $(0,1) \cup (1,\infty)$.

Theorem (K-Liao)

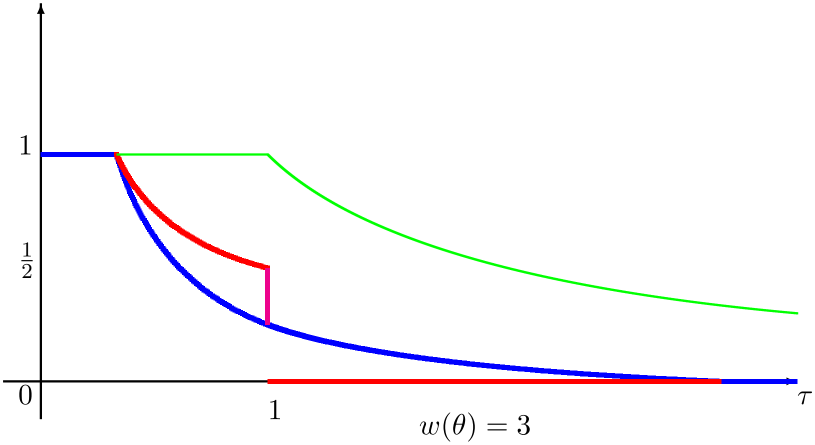
Let θ be an irrational with $w(\theta) = 1$. Then

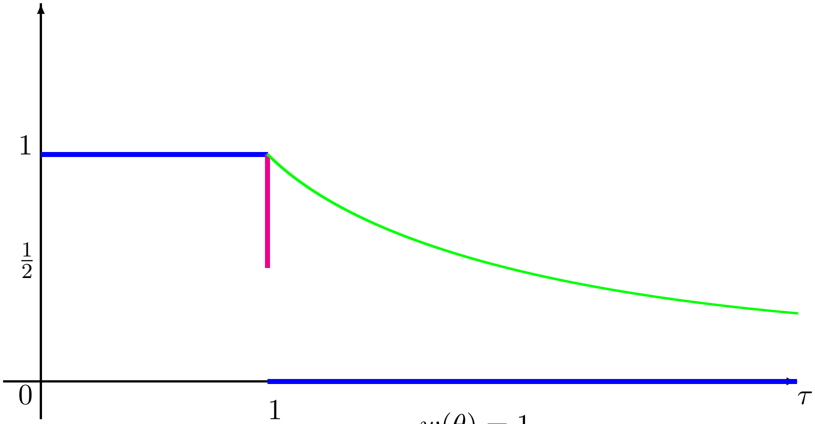
$$\frac{1}{2} \leq \dim_H (\mathcal{U}_1[\theta]) \leq 1.$$

For any irrational θ with $w(\theta) = w > 1$ we have

$$\begin{aligned} \frac{w/\tau - 1}{w^2 - 1} &\leq \dim_H (\mathcal{U}_\tau[\theta]) \leq \frac{1/\tau + 1}{w + 1}, & \frac{1}{w} \leq \tau \leq 1, \\ 0 &\leq \dim_H (\mathcal{U}_\tau[\theta]) \leq \frac{w/\tau - 1}{w^2 - 1}, & 1 < \tau \leq w. \end{aligned}$$

If $w(\theta) = \infty$, then $\dim_H (\mathcal{U}_\tau[\theta]) = 0$.


1
$\frac{1}{2}$
0
1
τ
$w(\theta) = 3$



$w(\theta) = 1$

Remarks

Remark 1: The results depend on $w(\theta)$.

Remark 2: For the case $\tau > 1$,

$$\dim_H \left(\mathcal{U}_\tau[\theta]\right) \leq \frac{1}{2\tau\left(\tau + \sqrt{\tau^2 - 1}\right)} < \frac{1}{2\tau}.$$

- $\mathcal{U}_\tau[\theta] \subset \mathcal{L}_\tau[\theta]$ except for a countable set of points.
- $\dim_H(\mathcal{L}_\tau[\theta]) = \tau$.

Remark 3: No mass transference principle for uniform approximations.

Examples

(i) Let θ be an irrational with $w(\theta) = w > 1$ and $q_{k+1} > q_k^w$ for all k. Then for each $1/w < \tau < w$ we have

$$\dim_H(\mathcal{U}_\tau[\theta]) = \frac{w/\tau - 1}{w^2 - 1}.$$

(ii) Assume that θ is an irrational of $w(\theta) = w > 1$ with subsequence $\{k_i\}$ of $q_{k_i+1} > q_{k_i}^w$ satisfying that

$$a_{n+1} = 1 \text{ for } n \neq k_i \quad \text{and} \quad q_{k_{i+1}} > (q_{k_i})^{2^i}.$$

Then we have

$$\dim_H(\mathcal{U}_\tau[\theta]) = \begin{cases} \dfrac{1/\tau + 1}{w+1}, & \text{for } \dfrac{1}{w} < \tau \leq 1, \\ 0, & \text{for } \tau > 1. \end{cases}$$

Examples-continued

(iii) Let $\theta = \frac{\sqrt{5}-1}{2}$, of which partial quotients $a_k = 1$ for all k. Note that $w(\theta) = 1$. Then $\mathcal{U}_\tau[\theta] = \mathbb{T}$ for $\tau = 1$. Thus, we have

$$\dim_H(\mathcal{U}_\tau[\theta]) = \begin{cases} 1, & \tau \leq 1, \\ 0 & \tau > 1. \end{cases}$$

(iv) Let θ be the irrational with partial quotient $a_k = k$ for all k. Then $w(\theta) = 1$ and

$$\dim_H(\mathcal{U}_\tau[\theta]) = \begin{cases} 1, & \tau < 1, \\ \frac{1}{2}, & \tau = 1, \\ 0 & \tau > 1. \end{cases}$$

Construction of Cantor sets

$$\mathcal{U}_\tau[\theta] = \{y : \text{for all large } Q,\ 1 \le \exists n \le Q,\ \|n\theta - y\| < Q^{-\tau}\}.$$

Let

$$G_n = \bigcup_{i=1}^{n} B\left(i\theta, \frac{1}{n^\tau}\right), \qquad F_k = \bigcap_{n=q_k}^{q_{k+1}-1} G_n.$$

Then

$$\mathcal{U}_\tau[\theta] = \bigcup_{\ell=1}^{\infty} \bigcap_{n=\ell}^{\infty} G_n = \bigcup_{\ell=1}^{\infty} \bigcap_{k=\ell}^{\infty} F_k.$$

We calculate the Hausdorff dimensions of $\bigcap_{k=\ell}^{\infty} F_k$.

Take

$$E_m := \bigcap_{k=1}^{m} F_k.$$

Then E_m is the union of the intervals at level m.

Dimension calculation (from Falconer's book)

Let

$$E_0 \supset E_1 \supset E_2 \supset \dots \text{ and } F = \bigcap_{n=0}^{\infty} E_n.$$

Suppose each interval of E_{k-1} contains at least m_k intervals of E_k $(k = 1, 2, \dots)$ which are separated by gaps of at least ε_k, where $0 < \varepsilon_{k+1} < \varepsilon_k$ for each k. Then

$$\dim_H(F) \geq \varliminf_{k \to \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}.$$

Suppose F can be covered by ℓ_k sets of diameter at most δ_k with $\delta_k \to 0$ as $k \to \infty$. Then

$$\dim_H(F) \leq \varliminf_{k \to \infty} \frac{\log \ell_k}{-\log \delta_k}.$$

Distribution of orbit points

Given an irrational number θ and a positive integer N, if one arranges the points $\{\theta\}, \{2\theta\}, \ldots, \{N\theta\}$ in ascending order, the distance between consecutive points can have at most three different lengths, and if there are three, one will be the sum of the other two. (Sós (1958) ...)

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If $N = q_k$, for some $k \geq 0$, then there are

$$q_k - q_{k-1} \ \text{ intervals of length } \ \|q_{k-1}\theta\|$$

and

$$q_{k-1} \ \text{ intervals of length } \ \|q_{k-1}\theta\| + \|q_k\theta\|.$$

Structure of n-th level intervals

Lemma

If

$$2\left(\frac{1}{q_{k+1}}\right)^{\tau} \geq \|q_{k-1}\theta\| + \|q_k\theta\|,$$

then

$$F_k = \mathbb{T}.$$

Consequently,

- If $1/w < \tau < 1$ and $q_k\|q_k\theta\|^{\tau} \geq 1$, then $F_k = \mathbb{T}$.
- If $\tau < 1/w$, then $F_k = \mathbb{T}$.
- If $\tau = 1$ and $a_{k+1} = 1$, then $F_k = \mathbb{T}$.

Structure of n-th level intervals

Lemma

For any $\tau \leq 1$, we have

$$\bigcup_{i=1}^{q_k} \left(i\theta - q_{k+1}^{-\tau},\ i\theta + C_\tau \left(\frac{\|q_k\theta\|}{q_k} \right)^{\frac{\tau}{\tau+1}} - 2\|q_k\theta\| \right) \subset F_k \subset \bigcup_{i=1}^{q_k} \left(i\theta - q_{k+1}^{-\tau},\ i\theta + C_\tau \left(\frac{\|q_k\theta\|}{q_k} \right)^{\frac{\tau}{\tau+1}} \right),$$

where $C_\tau = \tau^{\frac{1}{\tau+1}} + \tau^{-\frac{\tau}{\tau+1}}$. Note that $1 < C_\tau \leq 2$.

Structure of n-th level intervals

Lemma

If $q_{k+1}^{-\tau} + q_k^{-\tau} \leq \|q_k\theta\|$, *then we have*

$$F_k = \bigcup_{i=1}^{q_k} B\left(i\theta, q_{k+1}^{-\tau}\right).$$

Lemma

If $\tau > 1$, *then for large* q_k,

$$\bigcup_{i=1}^{\max(c_k,1)\cdot q_k} B\left(i\theta, q_{k+1}^{-\tau}\right) \subset F_k \subset \bigcup_{i=1}^{(c_k+2)q_k} B\left(i\theta, q_{k+1}^{-\tau}\right).$$

where $c_k = \left\lfloor (\|q_k\theta\| q_k^{\tau})^{-1/(\tau+1)} \right\rfloor$.

Lower bound for $1/w < \tau < 1$

Take $(k_i)_{i\geq 1}$ those that $q_{k_i}\|q_{k_i}\|^\tau < 1$. Then

$$\bigcap_{k=1}^{\infty} F_k = \bigcap_{i=1}^{\infty} F_{k_i}.$$

Define

$$E_i = \bigcap_{j=1}^{i} F_{k_j}, \text{ and } F = \bigcap_{i=1}^{\infty} E_i.$$

Let m_{i+1} be the number of intervals of E_{i+1} contained in E_i. Then

$$m_{i+1} \geq \frac{q_{k_{i+1}}}{5}\left(\frac{\|q_{k_i}\theta\|}{q_{k_i}}\right)^{\frac{\tau}{\tau+1}}.$$

Let ε_i be the smallest gap between the intervals in E_i. Then

$$\varepsilon_i \geq \frac{1}{2}\|q_{k_i-1}\theta\|.$$

Upper bound for $1/w < \tau < 1$

The set E_i can be covered by ℓ_i sets of diameter at most δ_i, with

$$\ell_i \leq \left(2C_\tau\Big(\frac{1}{q_{k_1}^\tau \|q_{k_1}\theta\|}\Big)^{\frac{1}{\tau+1}} + 2\right) \cdots \left(2C_\tau\Big(\frac{1}{q_{k_{i-1}}^\tau \|q_{k_{i-1}}\theta\|}\Big)^{\frac{1}{\tau+1}} + 2\right),$$

$$\delta_i = 2C_\tau \left(\frac{\|q_{k_i}\theta\|}{q_{k_i}}\right)^{\frac{\tau}{\tau+1}} .$$

Some words for the other cases

More efforts are needed for $\tau = 1$.

Lemma

Put

$$\tilde{r}_{k+1} := \begin{cases} 1, & \text{if } a_{k+1} = 2, \\ 2, & \text{if } a_{k+1} = 4, a_{k+2} \geq 2, \\ \lfloor \sqrt{4a_{k+1} + 5} \rfloor - 3, & \text{otherwise.} \end{cases}$$

Then we have

$$\bigcup_{1 \leq i \leq q_k} (i\theta - q_k\theta,\ i\theta + \tilde{r}_{k+1} q_k \theta - q_{k+1}\theta) \subset F_k.$$

Thank you for attention!!