Topology of of tiles associated with numeration systems

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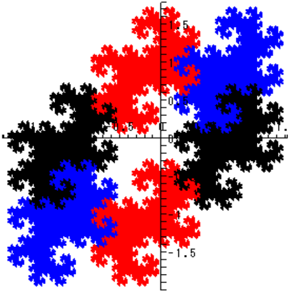
FAN-Workshop

Paris, 02.05.2013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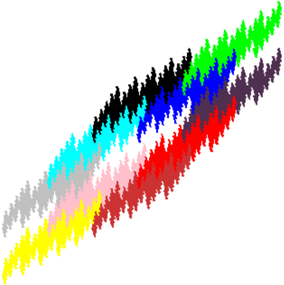
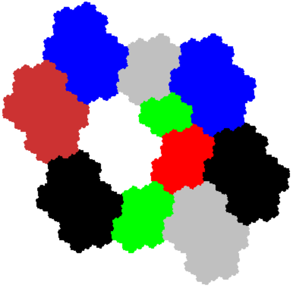




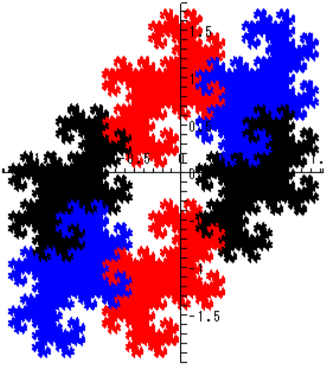
Motivations

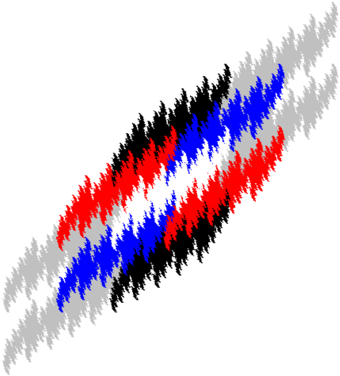
- Canonical number systems, β-numeration :
  - finiteness property $\leftrightarrow$ 0 interior point
  - rationals with periodic expansions $\leftrightarrow$ cut with real axes (Akiyama-Scheicher, Akiyama-Barat-Berthé-Siegel, Adamczewski-Frougny-Siegel)
  - simultaneous diophantine approximation (Hubert-Messaoudi,...)
- Substitutive dynamical systems : Markov partitions for hyperbolic toral automorphisms (Praggastis, ...)
- **Questions** : tiling property, connectivity, homeomorphy to the closed disk, fundamental group,...

Summary

- Boundary parametrization of self-affine tiles
- Application to canonical number systems
- Application to a family of cubic Rauzy fractals

Introduction





$\alpha = -1 + i,\ \mathcal{D} = \{0, 1\}$. $\alpha = -2 + i,\ \mathcal{D} = \{0, 1, 2, 3, 4\}$.

$$T = \left\{ a_1\alpha^{-1} + a_2\alpha^{-2} + \ldots ; a_1, a_2, \ldots \in \mathcal{D} \right\}.$$

Self-affine tiles

T is a planar *integral self-affine tile* if

$$\mathbf{A}T = \bigcup_{a\in\mathcal{D}} (T + a)$$

with

- $\mathbf{A} \in \mathbb{Z}^{2\times 2}$ *expanding* matrix : $|\mathrm{sp}(\mathbf{A})| \subset (1, +\infty)$.
- $\mathcal{D}$ is a complete residue system of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$.

Under these conditions, the compact $T = \overline{T^o}$ is called *tile* and

$$T = \left\{ \sum_{j\geq 1} \mathbf{A}^{-j} a_j ; a_j \in \mathcal{D} \right\}.$$

T connected $\Rightarrow \partial T$ locally connected continuum (Tang 2005).
Aim : parametrization $C : \mathbb{S}^1 \to \partial T$ with "good properties".

Tiling property

- T integral self-affine tile : $\mathbf{A}T = \bigcup_{a\in\mathcal{D}}(T+a)$.
- Then there is a $\mathcal{J} \subset \mathbb{Z}^2$ such that

$$\bigcup_{s\in\mathcal{J}}(T+s) = \mathbb{R}^2,$$

$$\lambda_2((T+s)\cap(T+s')) = 0 \text{ for } s \neq s' \in \mathcal{J}.$$

  This is the *tiling property* (Lagarias-Wang, Groechenig-Haas).
- We will assume that $\mathcal{J} = \mathbb{Z}^2$.

Boundary of $\mathbb{Z}^2$-self-affine tiles

Neighbors of the tile T :

$$\mathcal{S} = \{s \in \mathbb{Z}^2 \setminus \{0\}; T \cap (T + s) \neq \emptyset\}$$

is finite.

- By the tiling property,

$$\partial T = \bigcup_{s \in \mathcal{S}} \underbrace{T \cap (T + s)}_{B_s}.$$

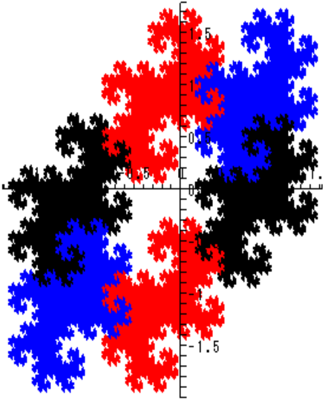
- Since $T = \bigcup_{a \in \mathcal{D}} \mathbf{A}^{-1}(T + a)$, we have

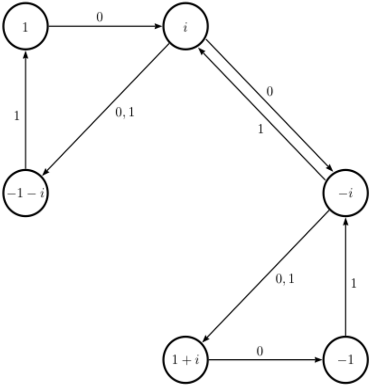
$$B_s = \bigcup_{a,a'} \mathbf{A}^{-1}(\underbrace{T \cap (T + a' + \mathbf{A}s - a)}_{=B_{s'}} + a).$$

- Graph :

$$s \xrightarrow{a|a'} s' \text{ iff } \mathbf{A}s + d' - d \; = \; s'.$$

Example : boundary automaton of Knuth tile





$$T \cap (T+1) = \left\{ \sum_{k \geq 1} \alpha^{-k} a_k ; 1 \xrightarrow{a_1 = 0} i \xrightarrow{a_2} s_1 \xrightarrow{s_3} \cdots \in G \right\}.$$

Boundary automaton

- G can be computed algorithmically from $(\mathbf{A}, \mathcal{D})$.
- We have a graph iterated function system ($GIFS$) :

$$B_s = \bigcup_{s \xrightarrow{a} s'} \mathbf{A}^{-1}(B_{s'} + a),$$

$$\partial T = \bigcup_{s \in \mathcal{S}} B_s$$

- Boundary points : $x = \sum_{j=1}^{\infty} \mathbf{A}^{-j} a_j \in T \cap (T + s)$ iff there is an infinite walk $w = (a_1, a_2, \ldots)$ in G

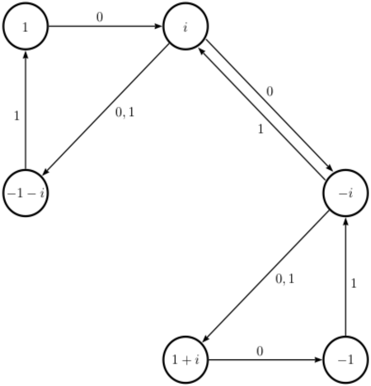
$$s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_3} \ldots$$

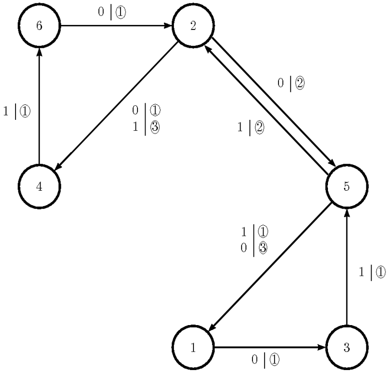
- Surjective mapping :

$$\begin{array}{rccl} \psi : & G & \to & \partial T \\ & w & \mapsto & \mathbf{A}^{-1} a_1 + \mathbf{A}^{-2} a_2 + \ldots \end{array}$$

- Aim : find ϕ such that $[0, 1] \xrightarrow{\phi} G \xrightarrow{\psi} \partial T$ is continuous surjection.

Ordered automaton G^o (boundary substitution)





$$\begin{array}{llll} \sigma: & 1 \rightarrow 3 & 2 \rightarrow 454 & 3 \rightarrow 5 \\ & 4 \rightarrow 6 & 5 \rightarrow 121 & 6 \rightarrow 2 \end{array}$$

Dumont-Thomas number system

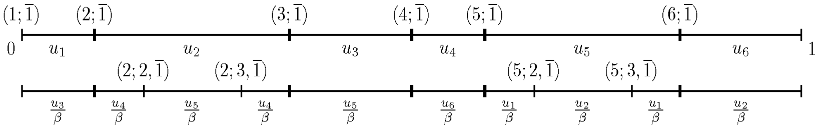
- Let $\mathbf{L}$ the incidence matrix and u be defined by

$$(u_1, u_2, u_3, u_4, u_5, u_6)^T \mathbf{L} = \beta(u_1, u_2, u_3, u_4, u_5, u_6)$$

and $u_1 + \ldots + u_6 = 1$. We have $u_j > 0$ and $u_j \in \mathbb{Q}(\beta)$.

- Define

$$\begin{array}{cccc} \Phi : & G^o & \to & [0,1] \\ & (i; o_1, o_2, o_3, \ldots) & \mapsto & t \end{array}$$



$\ldots$

Properties of $\Phi : G^o \to [0,1]$

- The above procedure defines :

$$\Phi : \begin{array}{ccc} G^o & \to & [0,1] \\ v = (i; o_1, o_2, \ldots) & \mapsto & x(v) \end{array}$$

- Since $\mathbf{u} > 0$, Φ is almost $1:1$.
- The identifications are trivial :

$$\Phi((1; o_{max}, o_{max}, \ldots)) = \Phi((2; 1, 1, \ldots))$$
$$\ldots$$
$$\Phi((6; o_{max}, o_{max}, \ldots)) = \Phi((1; 1, 1, \ldots))$$
$$\Phi((v_0, o, o_{max}, o_{max}, \ldots)) = \Phi((v_0, o+1, 1, 1, \ldots)).$$

- We define $\phi(t) = \max^{lex}\{\Phi^{-1}(t)\} \in G^o$.

Boundary parametrization

- We have a natural transition mapping :

$$\begin{array}{ccccccc} [0,1] & \overset{\phi}{\underset{\Phi}{\rightleftarrows}} & G^o & \xrightarrow{\text{digits}} & G & \xrightarrow{\psi} & \partial T \\ t & \mapsto & v & \mapsto & w = (a_1, a_2, \dots) & \mapsto & \sum_{k \geq 1} \mathbf{A}^{-k} a_k \end{array}$$

- **Compatibility Lemma** :

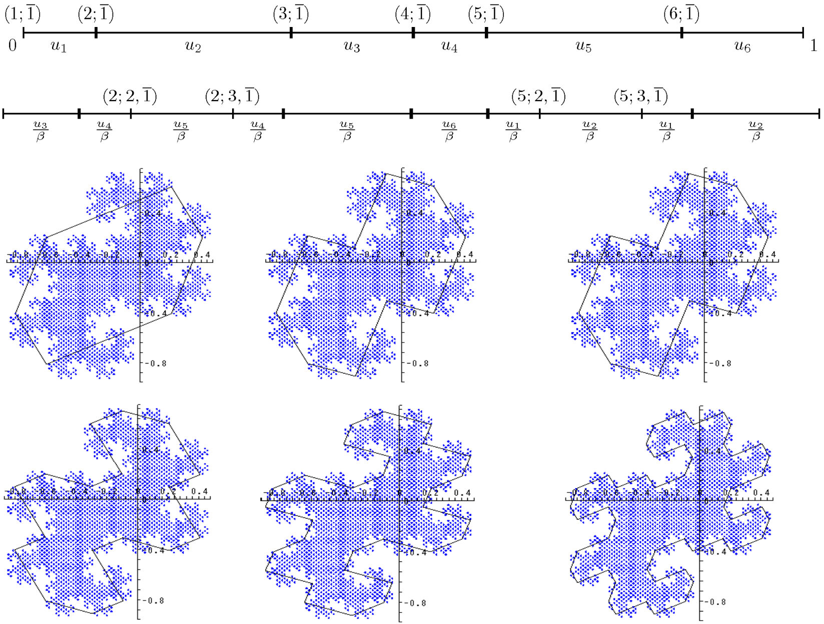
$$\Phi(v) = \Phi(v') \Rightarrow \psi(\text{digits}(v)) = \psi(\text{digits}(v')).$$

This is by choice of ordering. By GIFS property, finitely many equalities have to be checked.

- **Proposition**. *$C : \mathbf{S}^1 \xrightarrow{\phi} G^o \xrightarrow{digits} G \xrightarrow{\psi} \partial T$ is Hölder continuous.*

The exponent is $\alpha = 1/\dim_{\mathrm{H}} \partial T$ if $\mathbf{A}$ is a similarity.

Boundary approximations



A class of self-affine tiles

Theorem (Akiyama-L.)

Let T be the integral self-affine tile satisfying $\mathbf{A}T = T + \mathcal{D}$, where $\mathcal{D}$ is a collinear digit set : $\mathcal{D} = \{0, \mathbf{v}, \dots, (|\det(\mathbf{A})| - 1)\mathbf{v}\}$. Then there exists $C : [0,1] \to \partial T$ continuous onto mapping and an hexagon $Q \subset \mathbb{R}^2$ with the following properties. Let $T_0 := Q$ and

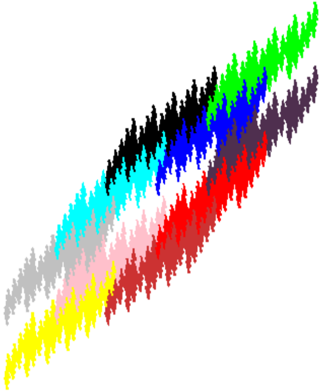
$$\mathbf{A}T_n = \bigcup_{a \in \mathcal{D}} (T_{n-1} + a).$$

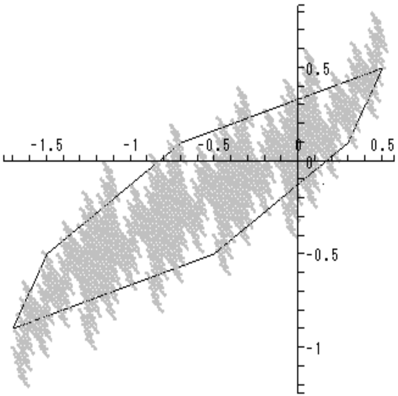
be the sequence of approximations of T associated to Q. Then :

1. $\lim_{n\to\infty} \partial T_n = \partial T$ *(Hausdorff metric).*
2. *For all $n \in \mathbb{N}$, ∂T_n is a polygonal simple closed curve.*
3. *Denote by V_n the set of vertices of ∂T_n. For all $n \in \mathbb{N}$, $V_n \subset V_{n+1} \subset C(\mathbb{Q}(\beta) \cap [0,1])$.*

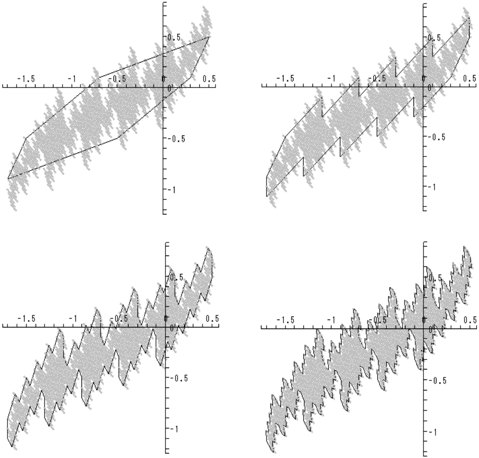
Note : β is the dominant eigenvalue of the incidence matrix of a strongly connected **contact graph** (Gröchenig-Haas).

Example $-2+i$



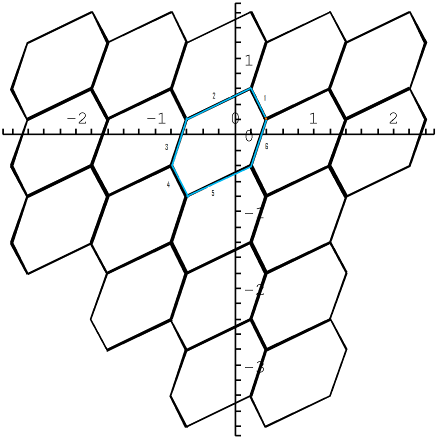


$-2+i$



Relation to Dekking's recurrent set method

Lemma. $Q + \mathbb{Z}^2$ is a tiling of $\mathbb{R}^2$.



Relation to Dekking's recurrent set method : compatible homomorphism

To each letter $1, 2, 3, \dot{1}, \dot{2}, \dot{3}$, we associate a side of the hexagon Q :

$$\begin{array}{lll} \mathbf{v}_1 := C_2 - C_1, & \mathbf{v}_2 := C_3 - C_2, & \mathbf{v}_3 := C_4 - C_3, \\ \mathbf{v}_{\dot{1}} := C_5 - C_4 = -\mathbf{v}_1, & \mathbf{v}_{\dot{2}} := C_6 - C_1 = -\mathbf{v}_2, & \mathbf{v}_{\dot{3}} := C_1 - C_6 = -\mathbf{v}_3. \end{array}$$

Directed curves in the plane. Let the homomorphism

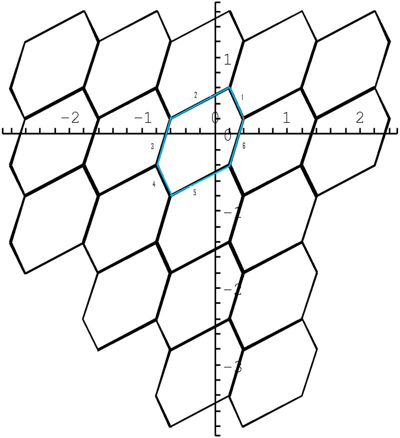
$$\begin{array}{rcl} g \ : < 1, 2, 3 > & \rightarrow & \mathbb{R}^2 \\ o_1 o_2 \ldots o_n & \rightarrow & \mathbf{v}_{o_1} + \ldots + \mathbf{v}_{o_n}. \end{array}$$

Then g connects the action of σ on the words and the action of $\mathbf{A}$ on the plane.

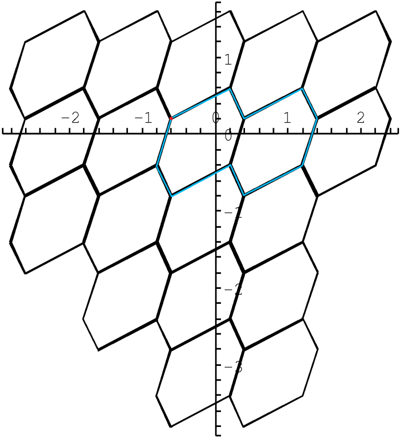
Lemma

For all words $w \in < 1, 2, 3 >, \quad g(\sigma(w)) = \mathbf{A} g(w)$.

Iterations of σ

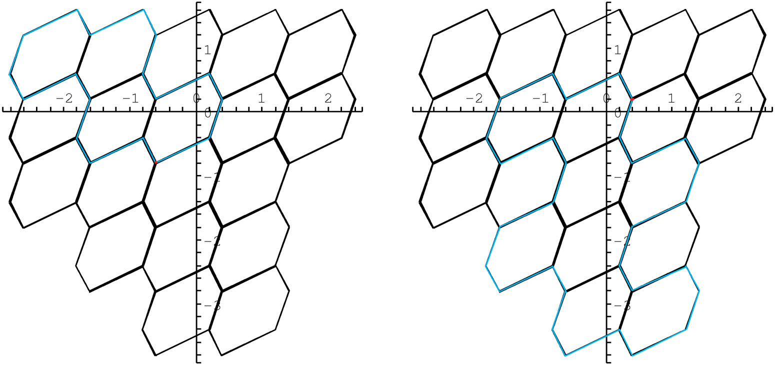


$p(123456)$.



$p(\sigma(123456)) = p(3454561212)$.

Iterations of σ



$p(\sigma^2(123456))$ $\qquad$ $p(\sigma^3(123456))$.

Proposition. *Let* $T_0 := Q$ *and* $T_{n+1} = \mathbf{A}^{-1}(T_n + \mathcal{D})$ *the associated sequence of approximations. Then*

$$\mathbf{A}^{-n} p(\sigma^n(123456)) = \partial T_n + k_n.$$

Relation to the Hausdorff measure $\mathcal{H}^\alpha$

Consider the strongly connected GIFS in $\mathbb{R}^d$

$$\partial T = \bigcup_{s \in \mathcal{S}} B_s,$$

$$B_s = \bigcup_{s \xrightarrow{a} s' \in G} \mathbf{A}^{-1}(B_{s'} + a). \tag{1}$$

We say that the GIFS satisfies the *open set condition* if there are open sets V_s ($s \in \mathcal{S}$) such that for all $s \in \mathcal{S}$

$$V_s \supset \bigcup_{s \xrightarrow{a} s' \in G} \mathbf{A}^{-1}(V_{s'} + a) \ \text{(disjoint)}.$$

Relation to the Hausdorff measure $\mathcal{H}^\alpha$ (He-Lau, Luo-Yang)

If $\mathbf{A}$ is a similarity, OSC implies that

(i) $\dim_{\mathrm{H}}\partial T = d\frac{\log(\beta)}{\log(\det(\mathbf{A}))}$.

(ii) $0 < \mathcal{H}^\alpha(\partial T) < \infty$.

(iii) The union (1) is disjoint for the Hausdorff measure $\mathcal{H}^\alpha$.

(iii)' Let $f_a(x) := \mathbf{A}^{-1}(x+a)$ and $f_{a_1\dots a_n} := f_{a_1} \circ \dots \circ f_{a_n}$. Then

$$\mathcal{H}^\alpha(f_{a_1 a_2 \dots a_n}(B_{s_n}) \cap f_{a'_1 a'_2 \dots a'_n}(B_{s'_n})) = 0,$$

for $s \xrightarrow{a_1} s_1 \dots \xrightarrow{a_n} s_n$ and $s \xrightarrow{a'_1} s_1 \dots \xrightarrow{a'_n} s'_n$ distinct.

If $\mathbf{A}$ is just affinity, the above results remain true for a Hausdorff measure $\mathcal{H}^\alpha_w$ with respect to a pseudo norm w :

$$w(\mathbf{A}x) = |\det(\mathbf{A})|^{1/2} w(x) \ \ (x \in \mathbb{R}^2).$$

Lemma

OSC holds for integral self-affine tiles.

Result for the class with collinear digit set

Theorem (Akiyama-L.)

Let T as in the previous theorem and C the corresponding parametrization. Furthermore, let w be a pseudo-norm for $\mathbf{A}$ and $\alpha := 2\frac{\log(\beta)}{\log(|\det(\mathbf{A})|)}$. Then, for each boundary part B_s,

$$\infty > \mathcal{H}_w^\alpha(B_s) > 0.$$

Moreover, there is a subdivision of the interval $[0,1]$, $t_0 := 0 < t_1 < \ldots < t_6 := 1$ such that

$$\frac{1}{c}\mathcal{H}_w^\alpha(\ C([t_i,t))\ )\ =\ t - t_i\ \ (t_i \le t \le t_{i+1}),$$

where $c := \sum_s \mathcal{H}_w^\alpha(B_s)$. If $\det(\mathbf{A}) \geq 2$, then $\mathcal{H}^\alpha(B_s \cap B_{s'}) = 0$ for $s \neq s'$ and

$$t = \frac{\mathcal{H}^\alpha(C([0,t)))}{\mathcal{H}^\alpha(\partial T)}.$$

Class of substitutions over the alphabet $\mathcal{A} = \{1, 2, 3\}$

Let $a \geq b \geq 1$ and

$$\sigma_{a,b}(1) = \underbrace{1 \ldots 1}_{a} 2 \qquad \sigma_{a,b}(2) = \underbrace{1 \ldots 1}_{b} 3 \qquad \sigma_{a,b}(3) = 1,$$

$$\sigma(uv) = \sigma(u)\sigma(u) \text{ for all } u, v \in \mathcal{A}^*$$

The *incidence matrix* $\mathbf{M} = \begin{pmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

-is primitive

-has characteristic polynomial $x^3 - ax^2 - bx - 1$. The roots are

$$\beta > 1, \quad \alpha, \alpha' \text{ with } |\alpha|, |\alpha'| < 1$$

(β is a Pisot unit).

Decomposition of the space.

$$\mathbb{R}^3 = \underbrace{\mathbb{H}_e}_{\mathbb{R}\cdot\mathbf{v}_\beta} \oplus \mathbb{H}_c,$$

and $\mathbf{M}_{|\mathbb{H}_c} =: \mathbf{h}$ is a contraction.

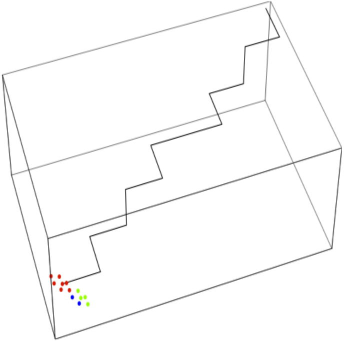
Tribonacci substitution

$$\sigma(1) = 12 \qquad \sigma(2) = 13 \qquad \sigma(3) = 1$$

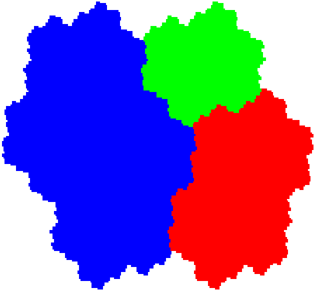
Let $\quad f : \{1, 2, 3\}^* \to \mathbb{Z}^3,\ 1 \mapsto \mathbf{e}_1,\ 2 \mapsto \mathbf{e}_2,\ 3 \mapsto \mathbf{e}_3,$

and $\quad u = \sigma^\infty(1) = \lim_{n\to\infty} \sigma^n(1) = 1213121121312... =: u_0 u_1 u_2 \ldots$

$$\mathbb{R}^3 \quad \xrightarrow{\pi} \quad \mathbb{H}_c$$



Approximation of $\mathbb{H}_e = \mathbb{R} \cdot \mathbf{v}_\beta$



$\mathcal{R} = \bigcup_{i=1}^3 \mathcal{R}_i$ (Rauzy, 1982)

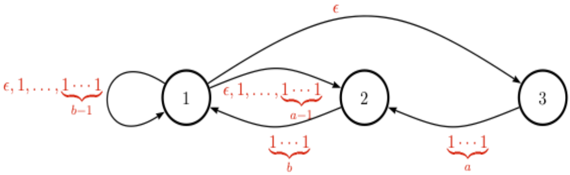
Substitution tiles as graph directed self-affine sets

Theorem (Sirvent and Wang, 1999)

Let σ be a primitive unit Pisot substitution over the alphabet $\mathcal{A}$. The central tile $\mathcal{R}$ and the subtiles $\mathcal{R}_i$ are compact sets satisfying $\mathcal{R} = \overline{\mathcal{R}^o}$ and $\mathcal{R}_i = \overline{\mathcal{R}_i{}^o}$. Moreover, the subtiles satisfy

$$\begin{aligned} \mathcal{R}_i &= \bigcup_{\underbrace{\sigma(j) = pis}_{i \xrightarrow{p} j \in \mathcal{P}}} \mathbf{h}\mathcal{R}_j + \pi f(p) \\ &= \left\{ \sum_{k \geq 0} \mathbf{h}^k \pi f(p_k) \; ; \; i \xrightarrow{p_0} j_1 \xrightarrow{p_1} \ldots \text{ infinite walk in } \mathcal{P} \right\} \end{aligned}$$

($\mathbf{h}$-ary representation).



Tiling property

Let σ be a primitive unit Pisot substitution over the alphabet $\mathcal{A}$. Define

$$\Gamma_{srs} := \{[\pi(\mathbf{x}), i] \in \pi(\mathbb{Z}^n) \times \mathcal{A} \; ; \; 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\} \, .$$

Theorem (Ito-Rao 2006)

Let σ be a primitive unit Pisot substitution over the alphabet $\mathcal{A}$. Then

$$\mathbb{H}_c = \bigcup_{[\gamma, i] \in \Gamma_{srs}} \mathcal{R}_i + \gamma$$

is a tiling if and only if σ satisfies the super coincidence condition.

Remark : $\sigma_{a,b}$ satisfies the super coincidence condition (Solomyak, Barge-Kwapisz).

Boundary graph

Define the graph $\mathcal{G}$ with :

- nodes $[i, \gamma, j] \in \mathcal{A} \times \pi(\mathbb{Z}^n) \times \mathcal{A}$ s.t.
$$||\gamma|| \leq 2\frac{\max\{||\pi f(p)||; p \text{ label of } \mathcal{P}\}}{1 - \max\{|\alpha_1|, |\alpha_2|\}}.$$
- edges $[i, \gamma, j] \xrightarrow{p|p'} [i_1, \gamma_1, j_1]$ iff $i \xrightarrow{p} i_1 \in \mathcal{P}$, $j \xrightarrow{p'} j_1 \in \mathcal{P}$ and
$$\mathbf{h}\gamma_1 = \gamma + \pi(f(p') - f(p)).$$

Remove the states of $\mathcal{G}$ that are not the starting state of an infinite walk $\longrightarrow$ finite boundary graph $\mathcal{G}(\mathcal{S})$.

Lemma

$$\sum_{k \geq 0} \mathbf{h}^k \pi f(p_k) = \gamma + \sum_{k \geq 0} \mathbf{h}^k \pi f(p'_k) \in \mathcal{R}_i \cap (\gamma + \mathcal{R}_j)$$

if and only if there is an infinite walk

$$[i, \gamma, j] \xrightarrow{p_0|p'_0} [i_1, \gamma_1, j_1] \xrightarrow{p_1|p'_1} \ldots \in \mathcal{G}(\mathcal{S}).$$

Boundary graph

Let $\mathcal{G}(\mathcal{S})^+$be the graph with *initial states* $[i,\gamma,j]$ s.t. $[\gamma,j] \in \Gamma_{srs}$.

Theorem (Siegel-Thuswaldner 2010)

Let σ be a primitive unit Pisot substitution satisfying the super coincidence condition. Let $B_{[i,\gamma,j]}$ be the solution of the graph directed system :

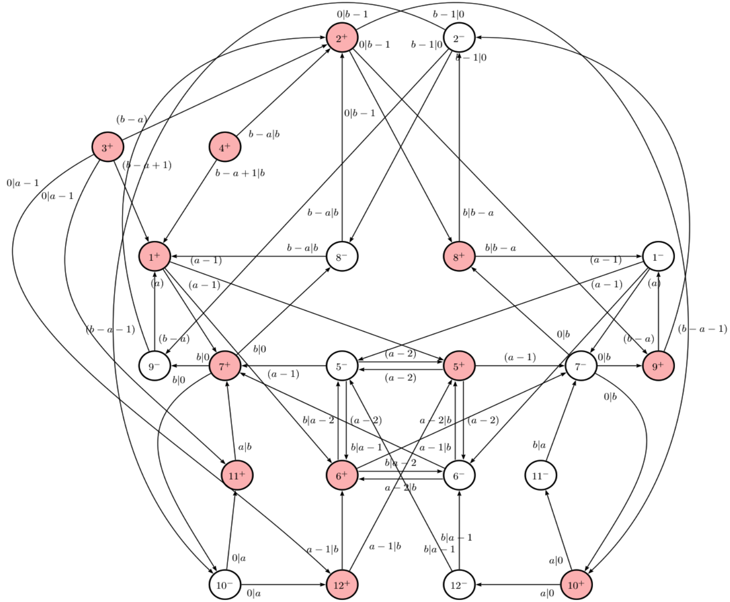
$$B_{[i,\gamma,j]} = \bigcup_{[i,\gamma,j] \xrightarrow{p} [i_1,\gamma_1,j_1] \in \mathcal{G}(\mathcal{S})} \mathbf{h} B_{[i_1,\gamma_1,j_1]} + \pi f(p).$$

Then

$$\partial \mathcal{R} = \bigcup_{[i,\gamma \neq \mathbf{0},j] \in \mathcal{G}(\mathcal{S})^+} B_{[i,\gamma,j]}$$

and $B_{[i,\gamma,j]} = \mathcal{R}_i \cap (\mathcal{R}_j + \gamma)$ *.*

A subgraph of the boundary graph for the substitution $\sigma_{a,b}$



Boundary parametrization

We obtain a mapping $C : [0,1] \to \mathcal{G} \to \partial\mathcal{R}$.

Continuity.

$$t = u_{1^+} \to \begin{array}{lcl} (1^+; o_{max}, o_{max}, \ldots) & \to & \sum_{k\geq 0} \mathbf{h}^k \pi f(p_k) \\ (2^+; 1, 1, \ldots) & \to & \sum_{k\geq 0} \mathbf{h}^k \pi f(p'_k) \end{array} \Bigg\} = ?$$

By self-affinity, only finitely many equalities to be checked

$\hookrightarrow$ use boundary graph.

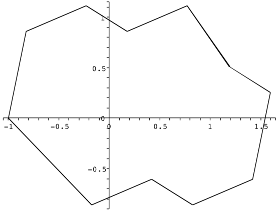
Theorem

Consider the substitution $\sigma_{a,b}$ ($a \geq b \geq 1$). Let λ be the largest root of $x^4 + (1-b)x^3 + (b-a)x^2 - (a+1)x - 1$.

Then there exists a surjective Hölder continuous mapping $C : [0,1] \to \partial\mathcal{R}$ with $C(0) = C(1)$ and a sequence of polygonal curves $(\Delta_n)_{n\geq 0}$ such that

- $\lim_{n\to\infty} \Delta_n = \partial\mathcal{R}$ *(Hausdorff metric).*
- *Denote by V_n the set of vertices of Δ_n. Then*

$$V_n \subset V_{n+1} \subset C(\mathbb{Q}(\lambda) \cap [0,1]).$$



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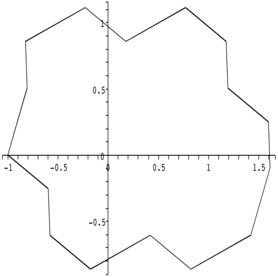

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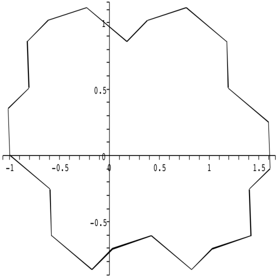

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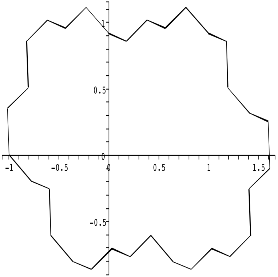

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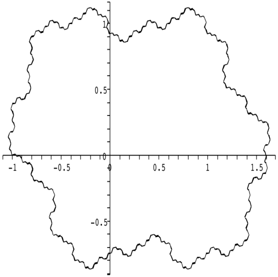
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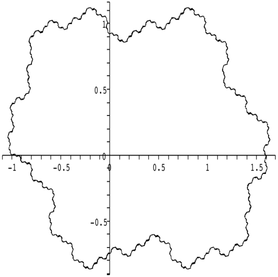
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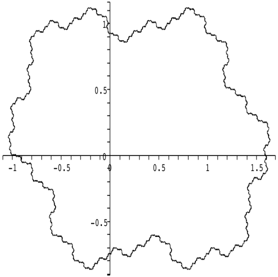
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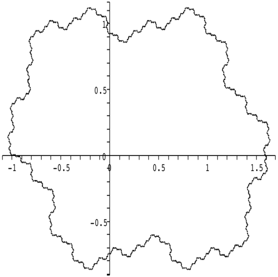
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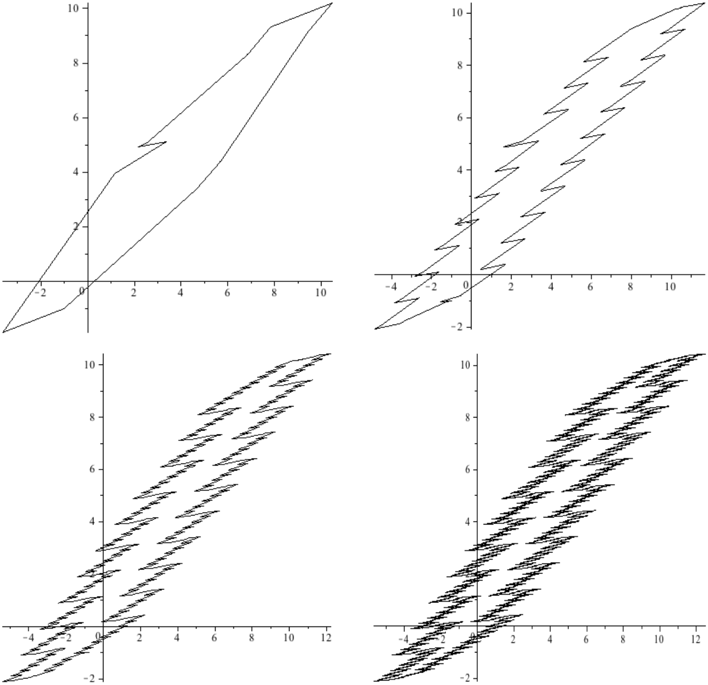


Topological result

Theorem

Let $\mathcal{R}$ be the tile associated to the substitution $\sigma_{a,b}$ $(a \geq b \geq 1)$. If $2b - a \geq 4$, then $\mathcal{R}$ is not homeomorphic to a closed disk.

Proof.
$$\begin{cases} t : 5^{+} \xrightarrow{a-1} 6^{-} \underset{\xleftarrow{a-2}}{\overset{a-2}{\longrightarrow}} 5^{+} \\ t' : 12^{+} \xrightarrow{a-1} 5^{+} \underset{\xleftarrow{a-2}}{\overset{a-2}{\longrightarrow}} 6^{-} \end{cases}$$

Thus $t < t'$ and $C(t) = C(t')$: C is not a simple closed curve.

Remark.

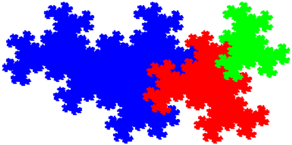
- Siegel and Thuswaldner (2010) give a criterion of homeomorphy to the closed disk algorithmically checkable.
- Messaoudi 2006 : if $a = 1$, then $\mathcal{R}$ is disk-like.

Topological result

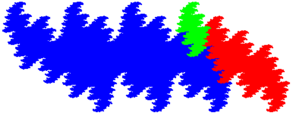
Theorem

Let $\mathcal{R}$ be the tile associated to the substitution $\sigma_{a,b}$ ($a \geq b \geq 1$). If $2b - a \leq 3$, then $\partial\mathcal{R}$ is a simple closed curve. Therefore, $\mathcal{R}$ is homeomorphic to a closed disk.

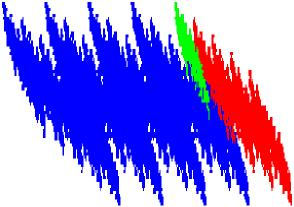
Proof. Check that all identifications are trivial in the parametrization (use boundary automaton from L. - Messaoudi - Surer - Thuswaldner 2013).



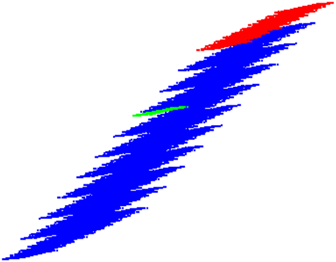
$\sigma_{1,2}$



$\sigma_{3,3}$



$\sigma_{4,4}$



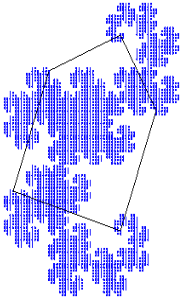
$\sigma_{7,10}$

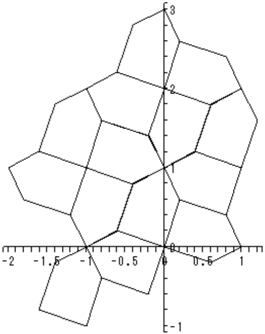
Further questions

- Other classes, for example among the substitutions generated by the Arnoux-Rauzy substitutions.
  See V. Berthé, T. Jolivet and A. Siegel for the connectedness of the fractals associated to $\sigma = \tau_1 \cdots \tau_r$, where $r \geq 3$ and $\{\tau_1, \ldots, \tau_r\} = \{\sigma_1, \sigma_2, \sigma_3\}$ (σ_i are the Arnoux-Rauzy substitutions).

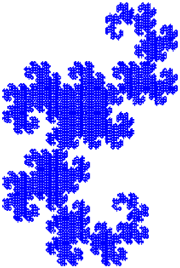
- Further topological information (description of cut points, interior components). Difficulty : complementation of Büchi automata.

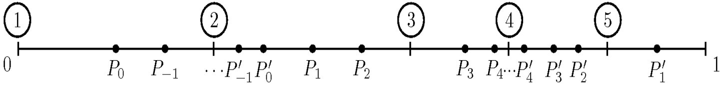
Heighway dragon





Non trivial identifications : cutpoints





Non trivial identifications (P, P') in the number system.

Thank you !