The Lagrange spectrum of some square-tiled surfaces

Joint with P. Hubert, S. Lelièvre, C. Ulcigrai

Luca Marchese

LAGA, Université Paris 13

Université Paris 7, 7 Mars 2016

1. Classical Lagrange spectrum.
2. Classical Lagrange spectrum and $\mathrm{SL}(2, \mathbf{R})/\mathrm{SL}(2, \mathbf{Z})$.
3. A new example: Lagrange spectrum of a special $\mathrm{SL}(2, \mathbf{R})/\Gamma$.
4. Translation Surfaces.
5. Lagrange spectrum of an affine manifold $\mathcal{M}$.
6. Lower part of the spectrum in the special case.
7. Proof.

1) Classical Lagrange Spectrum

i) Continued fraction expansion: $\alpha = [a_1, a_2, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$

ii) Function $L : \mathbf{R} \setminus \mathbf{Q} \to \mathbf{R}_+$ defined by

$$L(\alpha) := \limsup_{q,p\to\infty} \frac{1}{q \cdot |q\alpha - p|}$$

$$= \limsup_{n\to+\infty} [a_n, \dots, a_1] + a_{n+1} + [a_{n+2}, a_{n+3}, \dots]$$

iii) α is *badly approximable* $\Leftrightarrow L(\alpha) < +\infty \Leftrightarrow \exists L > 0$ such that $\forall \epsilon > 0$ and all but finitely many p, q:

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{L+\epsilon} \cdot \frac{1}{q^2}.$$

iv) $L(\alpha)$ is the smallest L such that the condition is satisfied.

v) The set of such α is a *thick* subset of $\mathbf{R}$ with measure zero.

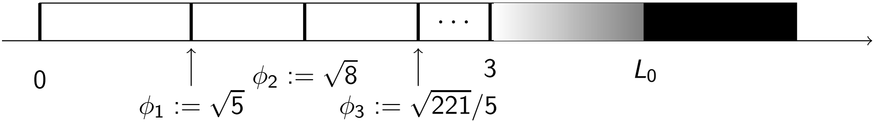
vi) The classical *Lagrange Spectrum* is the set $\mathcal{L} \subset \mathbf{R}_+$ defined by

$$\mathcal{L} := \{L(\alpha); \alpha \text{ badly approx.}\}$$

Cusick:
$\mathcal{L}$ is a closed subset of $\mathbf{R}$

Cusick:
$L(\beta)$, β quad. irrat. dense subset of $\mathcal{L}$

Moreira: for $3 \leq t \leq L_0$
$t \mapsto HD(\mathcal{L} \cap (0, t))$
continuous funct.



Hurwitz constant:
$\phi_1 = \sqrt{5} = \min \mathcal{L}$

Markoff numbers:
$\phi_n \in [\sqrt{5}, 3)$
$\phi_n \to 3$
The discrete part

Hall ray:
$\mathcal{L} \supset [L_0, +\infty)$

$$L_0 := 4 + \frac{253589820 + 283748 \cdot \sqrt{462}}{491993569} \sim 4,52783\ldots$$

Example of use of formula with continued fraction

-**Theorem [Hall].** We have $[6,+\infty) \subset \mathcal{L}$.

Proof. Consider the Cantor set

$$\mathbf{K} := \{\alpha = [a_1, a_2, \dots]; 1 \le a_n \le 4 \text{ for any } n\}.$$

-Hall: $\mathbf{K} + \mathbf{K} = \left[2 \cdot [\overline{4,1}], 2 \cdot [\overline{1,4}]\right] \supset [0.414\dots, 1.656\dots]$

-Consider any $L \ge 6$ and write it as

$$L = N + [a_1, a_2, \dots] + [b_1, b_2, \dots]$$

with $[a_1, a_2, \dots], [b_1, b_2, \dots] \in \mathbf{K}$ and $N \in \mathbf{N}$ with $5 \le N \le L$.

-Set

$$\alpha := [b_1, N, a_1, b_2, b_1, N, a_1, a_2, b_3, b_2, b_1, N, a_1, a_2, a_3 \dots]$$

-We have

$$L(\alpha) = \lim_{n\to\infty} [b_1, \dots, b_n] + N + [a_1, \dots, a_n] = L.$$

2) Geometric-dynamical interpretation

i) Standard flat torus $\mathbf{T}^2 := \mathbf{R}^2/\mathbf{Z}^2$. In general a flat torus has the form $X := \mathbf{R}^2/G \cdot \mathbf{Z}^2$ with $G \in \mathrm{SL}(2, \mathbf{R})$.

ii) $\mathrm{Hol}(\mathbf{T}^2) = \{(p, q); \gcd(p, q) = 1\}$: planar development of closed geodesics. In general $\mathrm{Hol}(X) = G \cdot \mathrm{Hol}(\mathbf{T}^2)$.

iii) $\mathrm{Sys}(X) := \min\{|v|; v \in \mathrm{Hol}(X), v \neq 0\}$.

iv) Moduli space $\mathcal{H}(0) := \mathrm{SL}(2, \mathbf{R})/\mathrm{SL}(2, \mathbf{Z})$ is a punctured sphere.

v) Proper function $X \mapsto \mathrm{Sys}(X)^{-1}$.

vi) Group action: $X \mapsto A \cdot X := \mathbf{R}^2/A \cdot G \cdot \mathbf{Z}^2$ for $A \in \mathrm{SL}(2, \mathbf{R})$.

-**Lemma.** No vertical $v \in \mathrm{Hol}(X) \Leftrightarrow g_t \cdot X \not\to \infty$ for $t \to +\infty \Rightarrow$ exist unique $s, \alpha \in \mathbf{R}$, $t > 0$ with $X = h_s g_t u_\alpha \cdot \mathbf{T}^2$, where

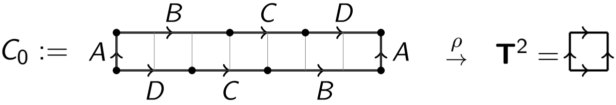
$$h_s := \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \quad g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad u_\alpha := \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.$$

-**Proposition.** For $X = h_s g_t u_\alpha \cdot \mathbf{T}^2$ let $v \in \mathrm{Hol}(X)$ vary. We have

$$L(\alpha) = \limsup_{|\mathrm{Im}(v)| \to \infty} \frac{1}{|\mathrm{Re}(v)| \cdot |\mathrm{Im}(v)|} = \limsup_{t \to +\infty} \frac{2}{\mathrm{Sys}^2(g_t \cdot X)}.$$

3) A new example of Lagrange spectrum

i) Degree 7 covering $\rho : C_0 \to \mathbf{T}^2$ by genus 2 flat surface C_0, one conical angle $6\pi = (2+1)\pi$ at $p = \bullet$



ii) $\mathrm{Hol}(C_0)$ set of planar developments of flat loops γ at $p = \bullet$.

iii) $\mathbf{Q}$ is the set of directions p/q of $(p,q) \in \mathrm{Hol}(C_0)$.

iv) Multiplicity $m(C_0, \cdot) : \mathbf{Q} \to \{1,2\}$, where $m(C_0, p/q)$ minimal degree of maps $t \mapsto \rho \circ \gamma(t)$ over flat loops γ in direction p/q.

Example: $\gamma_1 = (2,0)$, $\gamma_2 := (3,0)$, thus $m(C_0, \infty = 1/0) = 2$.

v) Deformations $C_0 \mapsto X := G \cdot C_0$ for $G \in \mathrm{SL}(2,\mathbf{R})$.

vi) Covariance: $\mathrm{Hol}(C_0) \mapsto \mathrm{Hol}(X) = G \cdot \mathrm{Hol}(C_0)$.

vii) Stabilizer: $\mathrm{Stab}(C_0) := \{G \in \mathrm{SL}(2,\mathbf{R}); G \cdot C_0 = C_0\}$.

viii) Moduli space: $\mathcal{B}_7 = \mathrm{SL}(2,\mathbf{R})/\mathrm{Stab}(C_0)$ is a finite volume Riemann surface with cusps.

ix) Proper function $X \mapsto \mathrm{Sys}^{-1}(X)$ on moduli space, where

$$\mathrm{Sys}(X) := \min\{|v|; v \in \mathrm{Hol}(X), v \neq 0\}.$$

x) Parametrize by $X = h_s g_t u_\alpha \cdot C_0$ those $X \in \mathcal{B}_7$ such that $g_t \cdot X \not\to \infty$ for $t \to +\infty$.

-**Proposition/Definition.** We have a function $L : \mathcal{B}_7 \to \mathbf{R}_+$, where $L(X) = L(\alpha)$ is defined for $X = h_s g_t u_\alpha \cdot C_0$ by

$$7 \cdot \limsup_{t \to +\infty} \frac{2}{\mathrm{Sys}^2(g_t \cdot X)} =$$

$$7 \cdot \limsup_{v \in \mathrm{Hol}(C_0), |\mathrm{Im}(v) \to \infty|} = \frac{1}{|\mathrm{Re}(v)| \cdot |\mathrm{Im}(v)|}$$

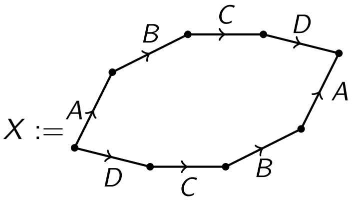
$$7 \cdot \limsup_{p,q \to +\infty} \frac{1}{q^2 \cdot |\alpha - p/q| \cdot m^2(C_0, p/q)}.$$

-**New Lagrange spectrum:** $\mathcal{L}(\mathcal{B}_7) := \{L(X); X \in \mathcal{B}_7\}$.

-**Thm** [Hubert-Lelièvre-M.-Ulcigrai]. $\mathcal{L}(\mathcal{B}_7)$ has Hall's ray and isolated minimum, but **no discrete part**. It is a closed set. A dense subset of $\mathcal{L}(\mathcal{B}_7)$ is given by the values $L(X)$ such that $t \mapsto g_t \cdot X$ is a closed geodesic in $\mathcal{B}_7$.

4) Translation Surfaces

i) Vectors $\zeta_1, \ldots, \zeta_d$ in $\mathbf{R}^2$ defining a polygon $P \subset \mathbf{R}^2$ with $2d$ sides, corresponding to two rearrangements of the ζ_js.
ii) *Translation surface*: $X = P/\partial P$, where any two sides in ∂P are identified iff they correspond to the same ζ_j.



iii) Genus g flat surface, conical points $p_1, \ldots, p_r$, with angles $2(k_1+1)\pi, \ldots, 2(k_r+1)\pi$.
iv) $2g - 2 = k_1 + \cdots + k_r$.
-Example: $C_0, X \in \mathcal{H}(2)$.
-In particular $\mathcal{B}_7 \subset \mathcal{H}(2)$.

v) *Saddle connection:* straight segment γ in X connecting p_i to p_j with no other $p_k \in \mathrm{Int}(\gamma)$. Its planar development: $\mathrm{Hol}(\gamma) \in \mathbf{R}^2$.
vi) Discrete set $\mathrm{Hol}(X) := \{\mathrm{Hol}(\gamma); \gamma \text{ s.c. }\} \subset \mathbf{R}^2$.
vii) *Moduli space:* the orbifold $\mathcal{H}(k_1, \ldots, k_r)$ of all X whose conical points have prescribed angle. Local coordinates $\zeta_1, \ldots, \zeta_d$.
viii) Normalization: $\mathrm{Area}(X) = 1$.

ix) Proper function $X \mapsto \mathrm{Sys}(X)^{-1}$ defined on $\mathcal{H}(k_1, \dots, k_r)$, where

$$\mathrm{Sys}(X) := \min\{|v|, v \in \mathrm{Hol}(X)\}.$$

x) Well defined *group action:* for $X = P/\partial P$ and $G \in \mathrm{SL}(2, \mathbf{R})$ define

$$G \cdot X := GP/\partial GP.$$

-**Eskin-Mirzakhani, Eskin-Mirzakhani-Mohammadi:** any $\mathrm{SL}(2, \mathbf{R})$-orbit closure is an affine submanifold $\mathcal{M} \subset \mathcal{H}(k_1, \dots, k_r)$ carrying a nice $\mathrm{SL}(2, \mathbf{R})$-ergodic and g_t-ergodic proba μ.

xi) *Stabilizer:* fon any $X \in \mathcal{H}(k_1, \dots, k_r)$ set

$$\mathrm{Stab}(X) := \{G \in \mathrm{SL}(2, \mathbf{R}); G \cdot X = X\}.$$

-$\mathrm{Stab}(G \cdot \mathbf{T}^2) \sim \mathrm{Stab}(\mathbf{T}^2) = \mathrm{SL}(2, \mathbf{Z})$ for any $G \in \mathrm{SL}(2, \mathbf{R})$.

-$\mathrm{Stab}(C_0)$ is a finite index subgroup of $\mathrm{SL}(2, \mathbf{Z})$.

-$\mathcal{M} := \overline{\mathrm{SL}(2, \mathbf{R}) \cdot X}$ is a finite volume Riemann surface with cusps iff $\mathrm{Stab}(X)$ is a lattice in $\mathrm{SL}(2, \mathbf{R})$.

-$\mathrm{Stab}(X) = \{\mathrm{Id}\}$ for generic $X \in \mathcal{H}(k_1, \dots, k_r)$.

5) Lagrange spectrum of an affine manifold $\mathcal{M}$

-**Proposition/Definition.** We have a function $L : \mathcal{M} \to \mathbf{R}_+$, where for any $X \in \mathcal{M}$, letting $v \in \mathrm{Hol}(X)$ vary, $L(X)$ is given by

$$L(X) := \limsup_{|\mathrm{Im}(v)| \to \infty} \frac{1}{|\mathrm{Re}(v)| \cdot |\mathrm{Im}(v)|} = \limsup_{t \to +\infty} \frac{2}{\mathrm{Sys}^2(g_t \cdot X)}.$$

-The *Lagrange Spectrum* of the affine manifold $\mathcal{M}$ is the set

$$\mathcal{L}(\mathcal{M}) := \{L(X); X \in \mathcal{M}\} \subset \mathbf{R}_+.$$

-**Tool:** express $L(X)$ in terms of adapted *renormalization algorithm* and generalize the classical formula

$$L(\alpha) := \limsup_{n \to +\infty} [a_n, \dots, a_1] + a_{n+1} + [a_{n+2}, a_{n+3}, \dots].$$

-**Hubert-M.-Ulcigrai:** $\mathcal{L}(\mathcal{M})$ is closed for any $\mathcal{M}$. Dense subset of values provided by $L(X)$ for those $X \in \mathcal{M}$ such that $g_t \cdot X$ is a closed geodesic. Tool: *Rauzy-Veech* induction.

-**Artigiani-M.-Ulcigrai:** If $\exists X \in \mathcal{M}$ such that $\mathrm{Stab}(X)$ is a lattice in $\mathrm{SL}(2, \mathbf{R})$ then $\mathcal{L}(\mathcal{M})$ has Hall's ray. Tool: $\mathrm{Stab}(X)$ acting by homographies. Previously proved for X square-tiled in H.-M.-U.

6) Lower part of the Lagrange spectrum $\mathcal{L}(\mathcal{B}_7)$

i) Consider the finite words $a := 1, 4, 2, 4$ and $b := 1, 3$. Let $\tilde{\sigma} : \Xi \to \Xi$ be the shift on $\Xi := \{a, b\}^{\mathbf{Z}}$.

ii) Consider the subset $\Xi_0 \subset \Xi$ of those $\bar{\xi} = (\xi_k)_{k \in \mathbf{Z}}$ such that

$$\xi_0 = a \quad \text{and} \quad \xi_n = a \text{ for infinitely many} n \in \mathbf{Z}_+$$

iii) Let $\sigma : \Xi_0 \to \Xi_0$ be the first return of $\tilde{\sigma}$ to Ξ_0.

iv) Define a function $L^\sigma : \Xi_0 \to \mathbf{R}_+$ by

$$L^\sigma(\bar{\xi}) := \limsup_{n \to +\infty} [\sigma^n(\bar{\xi})]_{(-)} + [\sigma^n(\bar{\xi})]_{(+)} \quad \text{where}$$

$$[\bar{\xi}]_{(-)} := [1, 4, \xi_{-1}, \xi_{-2}, \dots] \ , \ [\bar{\xi}]_{(+)} := [1, 4, \xi_1, \xi_2, \dots]$$

-**Main Theorem**[Lelièvre-Hubert-M.-Ulcigrai] ϕ_0 is the minimum of $\mathcal{L}(\mathcal{B}_7)$, moreover (ϕ_0, ϕ_1) is a gap in $\mathcal{L}(\mathcal{B}_7)$. The lower part $[\phi_1, \phi_\infty] \cap \mathcal{L}(\mathcal{B}_7)$ is the set $\mathbf{K}$ of values of $L^\sigma : \Xi_0 \to \mathbf{R}_+$, where

$$\phi_0 := 7 + 14 \cdot [\overline{3, 1}] = 10,696277 \pm 10^{-6}$$

$$\phi_1 := 14 \cdot [1, 4, \overline{1, 3}] = 11,582576 \pm 10^{-6}$$

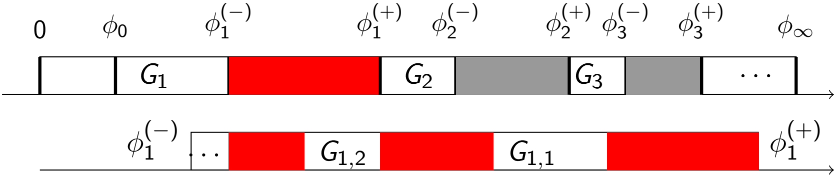
$$\phi_\infty := 14 \cdot [1, 4, \overline{1, 4, 2, 4}] = 11,593101 \pm 10^{-6}.$$

(v) A *gap* in $\mathcal{L}(\mathcal{B}_7)$ is an open interval $G = (a, b)$ with $a, b \in \mathcal{L}(\mathcal{B}_7)$ and $(a, b) \cap \mathcal{L}(\mathcal{B}_7) = \emptyset$.

(vi) For $k \geq 0$ set $I_k := [\phi_k^{(-)}, \phi_k^{(+)}]$ and $G_k := (\phi_k^{(+)}, \phi_{k+1}^{(-)})$, where, forgetting the factor 7, we set

$$\phi_0 = \phi_0^{(-)} = \phi_0^{(+)} := 1 + 2 \cdot [\overline{3, 1}] = 1 + 2 \cdot [3, \overline{b}]$$
$$\phi_1^{(-)} := 2 \cdot [1, 4, \overline{b}], \quad \phi_1^{(+)} := 2 \cdot [1, 4, \overline{b, a}]$$
$$\phi_2^{(-)} := [1, 4, a, \overline{b}] + [1, 4, \overline{b}], \quad \phi_2^{(+)} := [1, 4, a, \overline{b, a^2}] + [1, 4, \overline{b, a^2}]$$
$$\phi_3^{(-)} := 2 \cdot [1, 4, a, \overline{b}], \quad \phi_3^{(+)} := 2 \cdot [1, 4, a, \overline{b, a^3}]$$

1st generation of holes $G_k \nearrow \phi_\infty$ as $k \to +\infty$	Inside any I_k, 2nd gen. $G_{k,n} \searrow \phi_k^{(-)}$ as $n \to +\infty$	No discrete part!

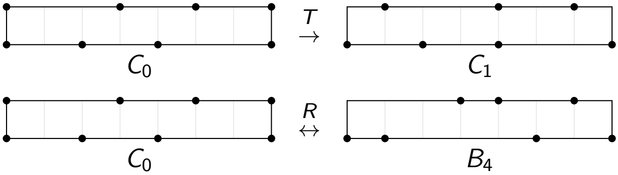


7) Proof

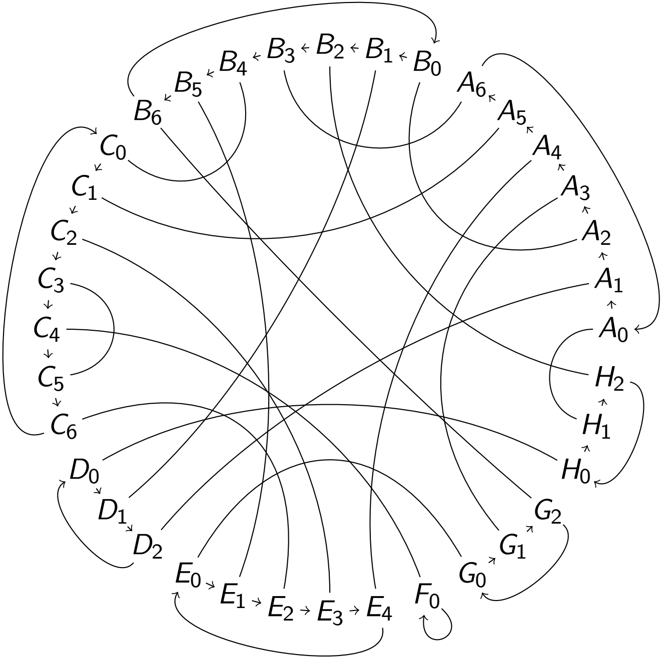
i) The group $\mathrm{SL}(2,\mathbf{Z})$ is generated by

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

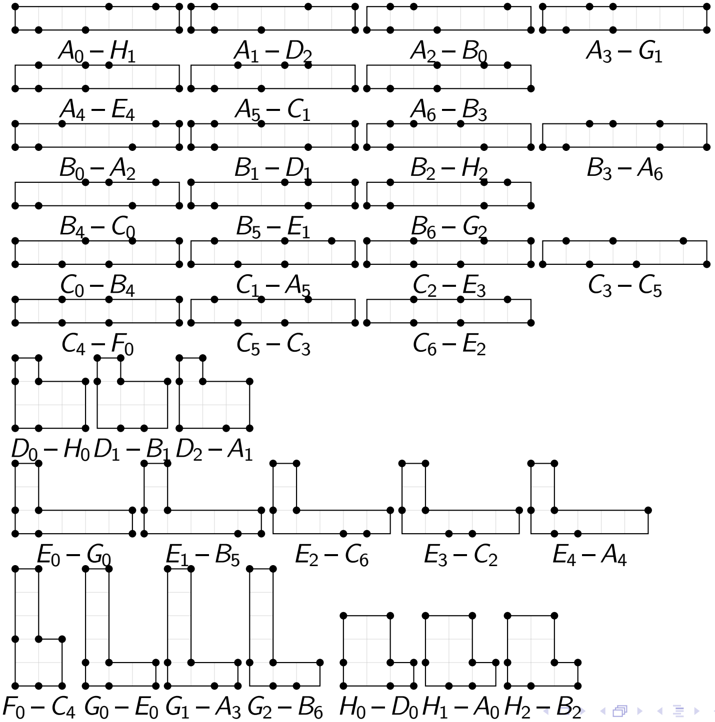
ii) Elements of $\mathrm{SL}(2,\mathbf{Z})$ permute the surfaces with 7 squares:



iii) We obtain a **graph structure on** $\mathrm{SL}(2,\mathbf{Z}) \cdot C_0$:



B_0 B_1 B_2 B_3 B_4 B_5 B_6
A_0 A_1 A_2 A_3 A_4 A_5 A_6
C_0 C_1 C_2 C_3 C_4 C_5 C_6
D_0 D_1 D_2
E_0 E_1 E_2 E_3 E_4
F_0
G_0 G_1 G_2
H_0 H_1 H_2


$A_0 - H_1$
$A_1 - D_2$
$A_2 - B_0$
$A_3 - G_1$
$A_4 - E_4$
$A_5 - C_1$
$A_6 - B_3$
$B_0 - A_2$
$B_1 - D_1$
$B_2 - H_2$
$B_3 - A_6$
$B_4 - C_0$
$B_5 - E_1$
$B_6 - G_2$
$C_0 - B_4$
$C_1 - A_5$
$C_2 - E_3$
$C_3 - C_5$
$C_4 - F_0$
$C_5 - C_3$
$C_6 - E_2$
$D_0 - H_0$
$D_1 - B_1$
$D_2 - A_1$
$E_0 - G_0$
$E_1 - B_5$
$E_2 - C_6$
$E_3 - C_2$
$E_4 - A_4$
$F_0 - C_4$
$G_0 - E_0$
$G_1 - A_3$
$G_2 - B_6$
$H_0 - D_0$
$H_1 - A_0$
$H_2 - B_2$

iv) Recall $m(C_6, p/q) :=$ minimal degree of maps $t \mapsto \rho \circ \gamma(t)$ over saddle connections γ in direction p/q (on C_6).

v) *Covariance of multiplicity:* let $\alpha \mapsto G(\alpha)$ be the homographic action of $G \in \mathrm{SL}(2, \mathbf{Z})$ on $\mathbf{R}$. We have

$$m(G \cdot C_6, G \cdot (p/q)) = m(C_6, p/q).$$

vi) Forget the area factor 7 and parametrize $\mathcal{L}(\mathcal{B}_7)$ by

$$\alpha \mapsto L(C_6, \alpha) = \limsup_{p,q \to +\infty} \frac{1}{q \cdot |q\alpha - p|} \cdot \frac{1}{m^2(C_6, p/q)}$$

$$\overset{(*)}{=} \limsup_{n \to \infty} \max_{1 \le i \le a_n} D(n, i, \alpha) \cdot \frac{1}{m^2(R \cdot g(a_1, \ldots, a_{n-1}, i) \cdot C_6, \infty)}$$

where for any $a_1, \ldots, a_n$ and any i with $1 \le i \le a_n$ we set

$$D(n, i, \alpha) := [a_n, \ldots, a_1] + a_{n+1} + [a_{n+2}, a_{n+3}, \ldots] \text{ if } i = a_n$$

$$D(n, i, \alpha) := [i, \ldots, a_1] + [a_n - i, a_{n+1}, \ldots] \text{ if } 1 \le i < a_n.$$

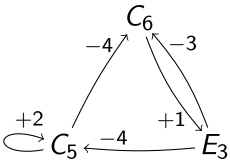
$$g(a_1, \ldots, a_{n-1}, i) := (T^{-i}R) \ldots (T^{-a_2}R)(T^{a_1}R) \text{ if } n \text{ even}$$

$$g(a_1, \ldots, a_{n-1}, i) := (T^{i}R) \ldots (T^{-a_2}R)(T^{a_1}R) \text{ if } n \text{ odd .}$$

vii) Elementary operations $T^a R \cdot X_j = Y_k$ and $T^{-a} R \cdot X_j = Z_l$ for $a \in \mathbf{N}^*$ and X_j, Y_k, Z_l in $\mathrm{SL}(2, \mathbf{Z}) \cdot C_6$ are represented by

$$X_j \quad \stackrel{+a}{\longrightarrow} \quad Y_k \quad \text{and} \quad X_j \quad \stackrel{-a}{\longrightarrow} \quad Z_l.$$

-**Proposition**[End of the proof] Condition $L(X_j, \alpha) \leq \phi_\infty$ implies that the path in the orbit graph is encoded by the subset of operations



viii) Any such path with $L(X_j, \alpha) > \phi_0$ must pass through C_5 infinitely often.

ix) For this class of paths, the expression in $(*)$ is maximal when $a_n = 2$ and $i = 1$ (a Farey step, not Gauss).

x) $m(RTR \cdot C_5) = 1$, thus the denominator in $(*)$ disappears.

Thank you!