Geometric representations for reducible Pisot substitutions

Milton Minervino

University of Leoben, Austria
Doctoral program in Discrete Mathematics

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Reducible Pisot substitutions

Hokkaido substitution associated with the minimal Pisot number:

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$$

$$M_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad f(x)g(x) = (x^3 - x - 1)(x^2 - x + 1)$$

Dominant root β of $f(x)$ is the smallest Pisot number. The substitution σ is a *reducible unit Pisot* substitution.

M_σ-invariant decomposition: $\mathbb{R}^5 = \mathbb{K}_\beta \oplus \mathbb{H}$.

$M_\sigma|_{\mathbb{K}_\beta}$ is hyperbolic and induces an expanding/contracting decomposition

$$\mathbb{K}_\beta = \mathbb{K}_e \times \mathbb{K}_c = \mathbb{R} \times \mathbb{C}.$$

Geometric representation

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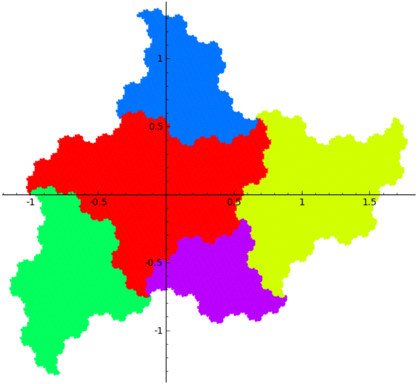
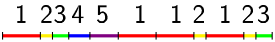


- Projection of vertices of a broken line.



- Embedded beta numeration integers:

$$\sum_{k \geq 0} \delta_c(d_k \beta^k),\ (d_k) \leq_{\text{lex}} (1)_\beta$$

Problems

Framework: **reducible** Pisot substitutions.

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2. No geometric representation for stepped surfaces.
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3. No periodic (multiple) tiling.

We show that indeed we can have all of them!!!

Dual substitutions

(Arnoux, Ito 01) formalism for *irreducible* Pisot substitutions.

Action of the substitution on 1-dimensional faces → broken line
For $(\mathbf{x}, a) \in \mathbb{Z}^d \times \mathcal{A}$

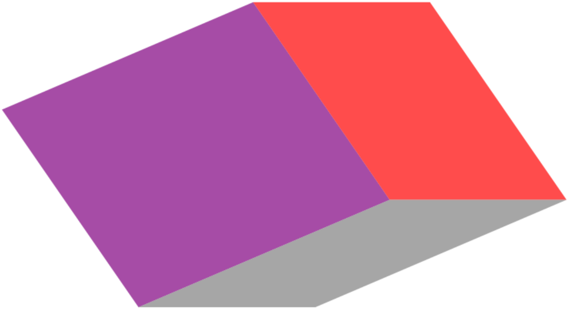
$$\mathbf{E}_1(\sigma)(\mathbf{x}, a) = \sum_{\sigma(a)=pbs} (M_\sigma \mathbf{x} + \mathbf{l}(p), b)$$

Dual action on $(d-1)$-dimensional faces:

$$\mathbf{E}_1^*(\sigma)(\mathbf{x}, a)^* = \sum_{\sigma(b)=pas} (M_\sigma^{-1}(\mathbf{x} - \mathbf{l}(p)), b)^*$$

Hausdorff limits

$$\mathcal{R}(a) = \lim_{k\to\infty} \pi_c(M_\sigma^k \, \mathbf{E}_1^*(\sigma)^k(\mathbf{0}, a)^*)$$



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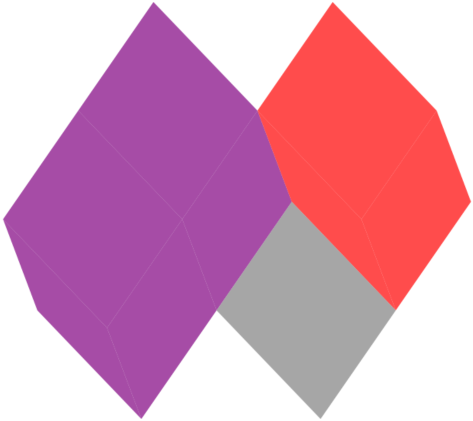

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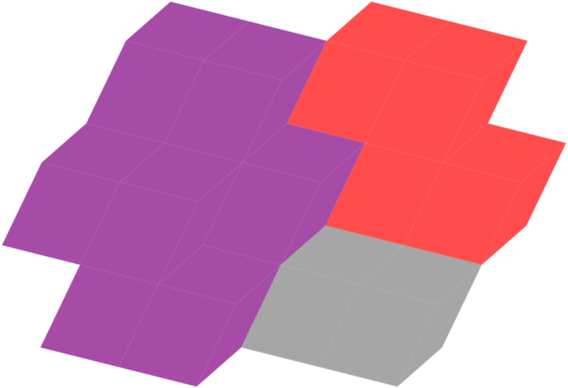

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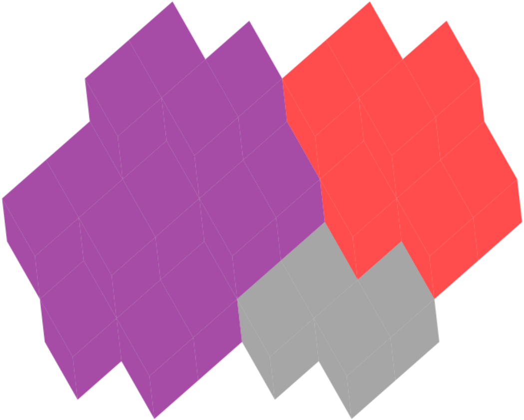

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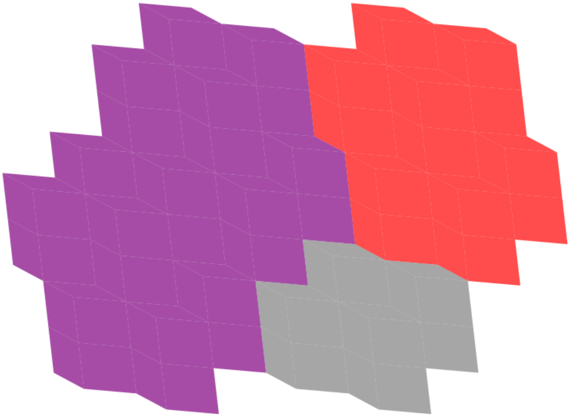

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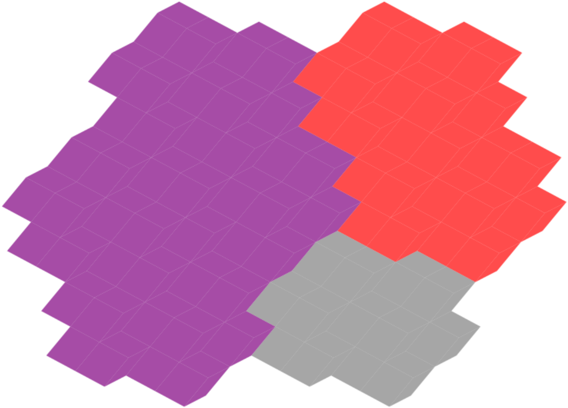

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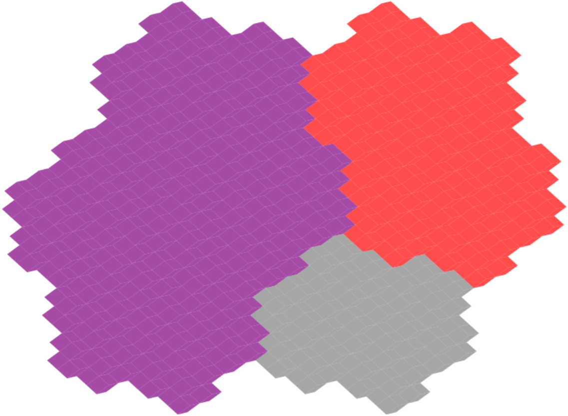

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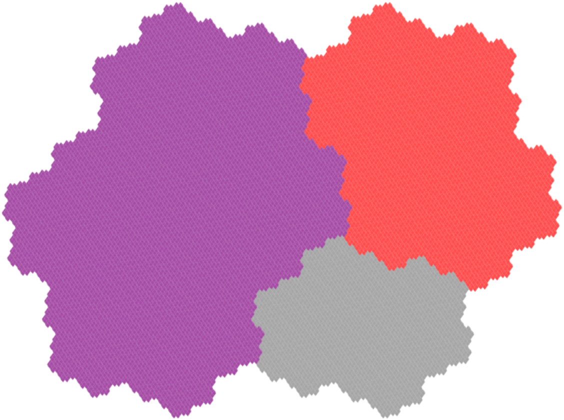

Stepped surfaces

Set of coloured points "near" to $\mathbb{K}_c$:

$$\Gamma = \{(\mathbf{x}, a) \in \mathbb{Z}^d \times \mathcal{A} : \mathbf{x} \in (\mathbb{K}_c)^{\geq}, \mathbf{x} - \mathbf{e}_a \in (\mathbb{K}_c)^{<}\}$$

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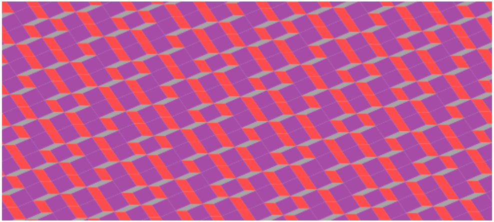
- $\mathbf{E}_1^*(\sigma)(\Gamma) = \Gamma \;\rightarrow$ self-replicating property (Kenyon).
- Aperiodic translation set (Delone set) for a self-replicating multiple tiling made of Rauzy fractals.
- Geometric representation as an arithmetic discrete model of the hyperplane $\mathbb{K}_c$, whose projection is a polygonal tiling.



Higher dimensional dual maps

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We want to work with $(d-1)$-dimensional faces! The dual map $\mathbf{E}^*_{n-d+1}(\sigma)$ will suit:

$$\mathbf{E}^*_{n-d+1}(\sigma)(\mathbf{x}, \underline{a})^* = \sum_{\underline{b} \xrightarrow{\underline{p}} \underline{a}} \left(M_\sigma^{-1}(\mathbf{x} - \mathbf{l}(\underline{p})), \underline{b}\right)^*$$

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Remarks:

- $\mathbf{E}^*_{n-d+1}(\sigma)$ acts on $\binom{n}{n-d+1}$ oriented faces.
- If σ is irreducible $n = d$ and $\mathbf{E}^*_{n-d+1}(\sigma) = \mathbf{E}^*_1(\sigma)$.
- $\mathbf{E}_k(\sigma)$ and $\mathbf{E}^*_k(\sigma)$ commute in general with boundary and coboundary operators (Sano, Arnoux, Ito 2001).
- Similar approach for the study of a free group automorphism associated with a complex Pisot root (Arnoux, Furukado, Harriss, Ito 2011).

Stepped surfaces

Let $\mathcal{U} = \{(\mathbf{0}, 2 \wedge 3), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 3 \wedge 4)\}$. We have $\mathcal{U} \subset \mathbf{E}_3^*(\sigma)^5(\mathcal{U})$.
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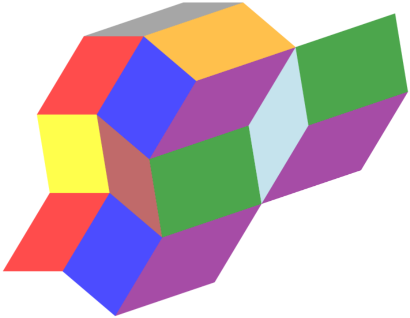

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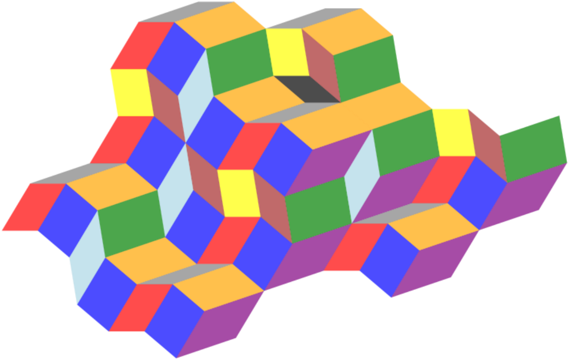

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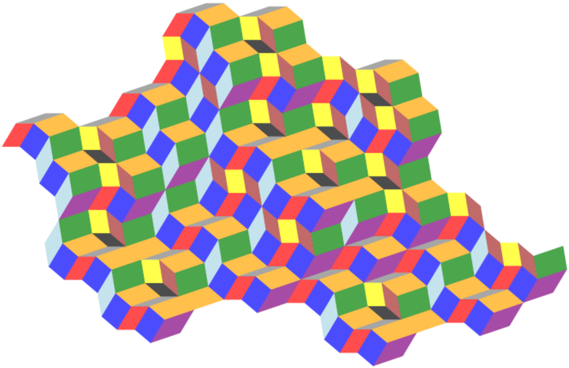

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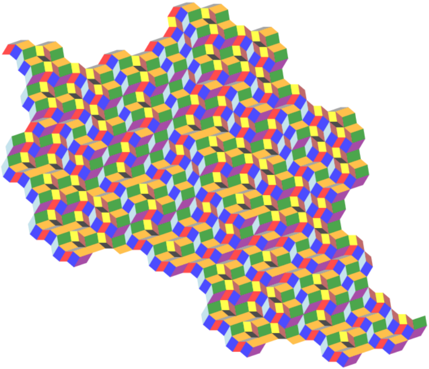

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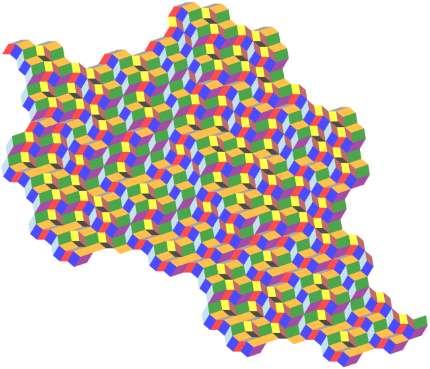

- **Regularity**: $\mathbf{E}_3^*(\sigma)(\mathbf{0}, \underline{a})^*$ in good position, $\forall \underline{a}$.
- **Geometric finiteness property**: $\pi_c(\Gamma_{\mathcal{U}})$ covers $\mathbb{K}_\beta^c$.
- $\pi_c(\Gamma_{\mathcal{U}})$ is a polygonal tiling.

Rauzy fractals and tilings

Rauzy fractals: $\mathcal{R}(\underline{a}) + \pi_c(\mathbf{x}) = \lim_{k\to\infty} \pi_c(M_\sigma^k \, \mathbf{E}_{n-d+1}^*(\sigma)^k(\mathbf{x}, \underline{a})^*)$.

Properties:

- if neutral polynomial has only roots of modulus one

$$\mathcal{R}(\underline{a}) + \pi_c(\mathbf{x}) = \bigcup_{(\mathbf{y},\underline{b})\in \mathbf{E}_{n-d+1}^*(\sigma)(\mathbf{x},\underline{a})} \beta \cdot \big(\mathcal{R}(\underline{b}) + \pi_c(\mathbf{y})\big),$$

  where the union is measure disjoint.
- compact with nonzero measure.
- closure of the interior.
- boundary has zero measure.

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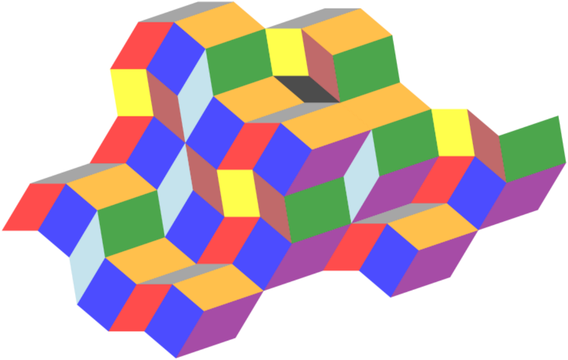
The collection $\{\mathcal{R}(\underline{a}) + \pi_c(\mathbf{x}) : (\mathbf{x}, \underline{a})^* \in \Gamma_{\mathcal{U}}\}$ is a *self-replicating tiling*.



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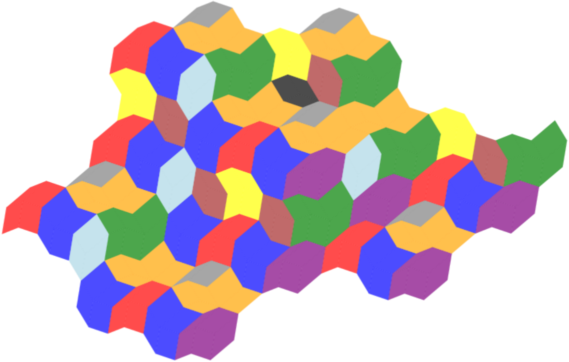

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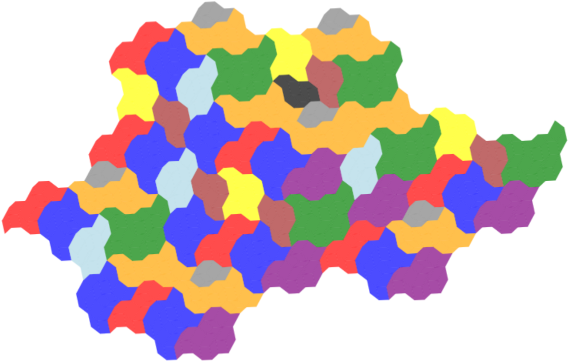

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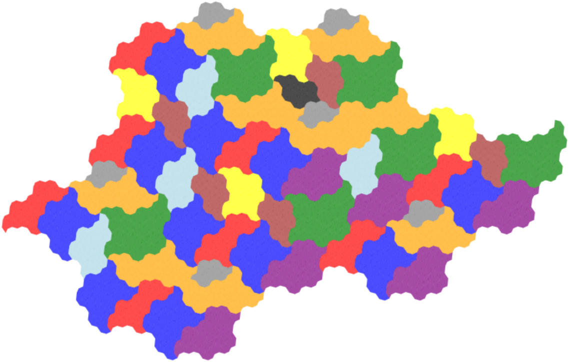

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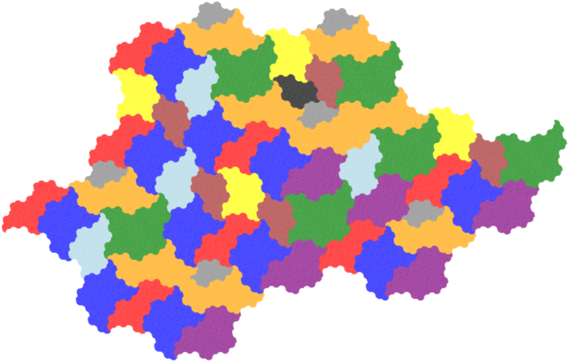

Periodic tilings

Recall: the original Hokkaido tile can not tile periodically (Ei, Ito 2005)

$$\mathcal{U} = \{(\mathbf{0}, 2 \wedge 3), (\mathbf{0}, 2 \wedge 4), (\mathbf{0}, 3 \wedge 4)\}.$$

- The patch $\pi_c(\mathcal{U})$ tiles periodically by the lattice $\Lambda_{\mathcal{U}} = \pi_c((\mathbf{e}_4 - \mathbf{e}_3)\mathbb{Z} + (\mathbf{e}_4 - \mathbf{e}_2)\mathbb{Z})$.

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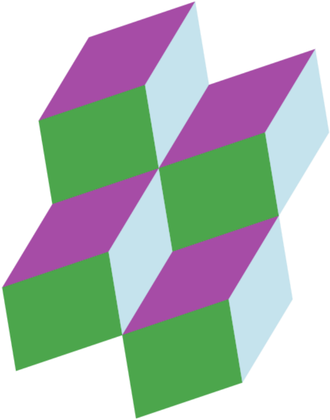

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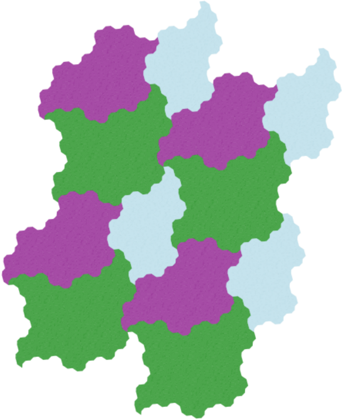

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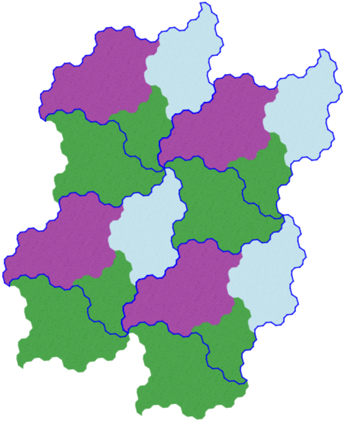

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- $\mathcal{R}_{\mathcal{U}} + \Lambda_{\mathcal{U}}$ is a *periodic tiling*.
- Do you see the original Hokkaido tile?

Broken lines and codings

We have a broken line in $\mathbb{R}^5$ which is the geometrical interpretation of the fixed point u of σ:

$$\overline{u} = \bigcup_{i \geq 1} \{(\mathbf{l}(u_0 u_1 \cdots u_{i-1}), u_i)\},$$

where $(\mathbf{x}, i)$ denotes the segment from $\mathbf{x}$ to $\mathbf{x} + \mathbf{e}_i$.

Being reducible means that some linear dependencies arise when we project the basis vectors $\{\mathbf{e}_a\}_{a \in \mathcal{A}}$ from $\mathbb{R}^5$ to $\mathbb{R}^3$:

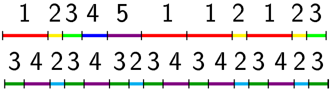
$$\pi(\mathbf{e}_1) = \pi(\mathbf{e}_3) + \pi(\mathbf{e}_4), \quad \pi(\mathbf{e}_5) = \pi(\mathbf{e}_2) + \pi(\mathbf{e}_3)$$

Combinatorially this is equivalent to applying the coding

$$\chi : \ 1 \mapsto 34, \quad 2 \mapsto 2, \quad 3 \mapsto 3, \quad 4 \mapsto 4, \quad 5 \mapsto 32.$$

Broken lines and codings

Effect of the coding χ:



In this process we converted the substitution into an irreducible one!

Project now the vertices of the new broken line. . .

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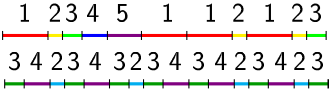

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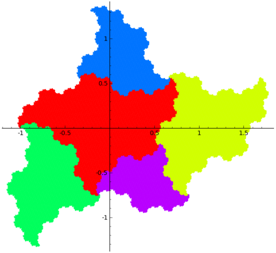
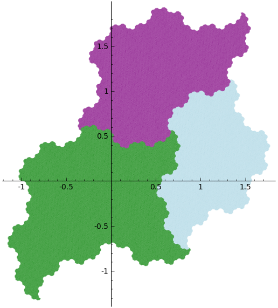
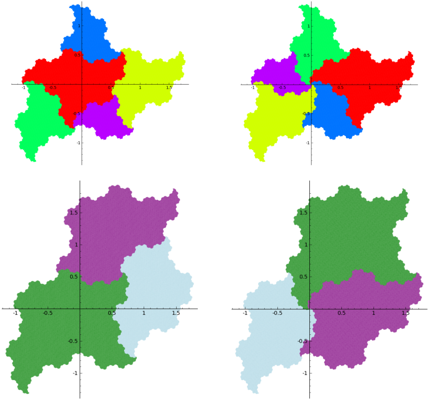




Domain exchange



- $(\mathcal{T}, E_{\mathcal{T}})$ is a *domain exchange* on the original Hokkaido tile.

$$E_{\mathcal{T}} : \mathcal{T}(a) \mapsto \mathcal{T}(a) + \pi_c(\mathbf{e}_a), \; a \in \mathcal{A}$$

- $(\mathcal{R}, E)$ is a *toral translation*, since it induces a periodic tiling of $\mathbb{C}$.

$$E : \mathcal{R}(a) \mapsto \mathcal{R}(a) + \pi_c(\mathbf{e}_a), \; a \in \{2, 3, 4\}$$

- $E_{\mathcal{T}}$ is the *first return* of E on $\mathcal{T}$.

Codings of the domain exchange

Let $\Omega = \overline{\{S^k w : k \in \mathbb{N}\}}$, where $w = \chi(u)$ is the coded fixed point of σ.

We have the following commutative diagram:

$$\begin{array}{ccccccc}
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \mathcal{R} & \longrightarrow & \mathbb{C}/\Lambda \\
S \downarrow & & S \downarrow & & E \downarrow & & E \downarrow \\
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \mathcal{R} & \longrightarrow & \mathbb{C}/\Lambda
\end{array}$$

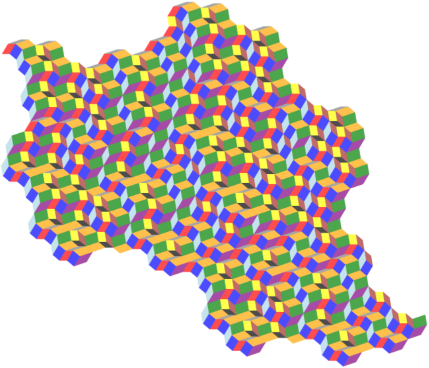
ϕ measure conjugation.

We can generalize what shown for the family of substitutions

$$\sigma_t : \ 1 \mapsto 1^{t+1}2, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 1^t 5, \ 5 \mapsto 1$$

Irreducibilifying

Guiding philosophy: try to turn the substitution into an irreducible one!



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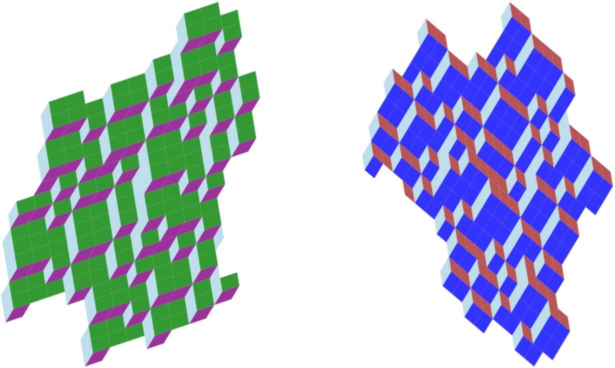
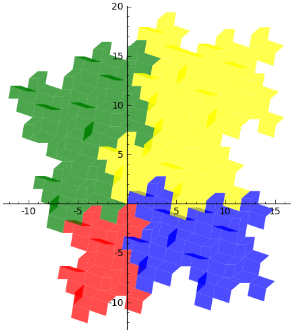


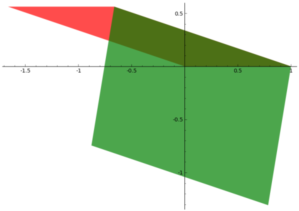
Figure : Changing suitably the projection we get different polygonal tilings by some faces of three different types.

Remarks and perspectives

Important hypotheses:

- Regularity $\rightarrow$ projection of patches onto $\mathbb{K}_c$ behaves well.





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- Regularity $\rightarrow$ projection of patches onto $\mathbb{K}_c$ behaves well.
- Geometric finiteness property $\rightarrow$ covering property for the stepped surface.
- Roots of the neutral polynomial of modulus one $\rightarrow$ measure disjointness in the set equation.
- Strong coincidence condition $\rightarrow$ domain exchange.

Remarks and perspectives

Perspectives:

- Can we generalize these constructions to every reducible Pisot substitution?
- Characterization of the points of the stepped surfaces as in the irreducible case?
- Is $\chi \circ \sigma$ a new (irreducible) substitution?
- Influence of the neutral space in the dynamics?
- When are first returns of rotations on compact groups again rotations?
- Cohomology? (Barge, Bruin, Jones, Sadun 2012)
- Pisot conjecture for reducible substitutions?