On the discrepancy of random sequences generated by dynamical systems

Makoto Mori
Department of Mathematics,
College of Humanities and Sciences,
Nihon University

March 8, 2016

Random Sequences

Let us consider $I = [0,1]^d$. A random sequence $x_1, x_2, \ldots \in I$ is called uniformly distributed if $\lim_{N\to\infty} D_N(J) = 0$ for any interval $J = \prod_{i=1}^d [a_i, b_i)$ $(0 \le a_i < b_i \le 1)$. Here

$$D_N(J) = \left| \frac{1}{N} \#\{n \le N \colon x_n \in J\} - |J| \right|,$$

where $|J|$ is the Lebesgue measure of J.
The discrepancy D_N is defined by

$$D_N = \sup_J D_N(J),$$

where sup is taken over all the intervals J. When we restrict intervals only for $J = \prod_{i=1}^d [0, b_i)$, we call it $*$–discrepancy and denote it by D_N^*. There exists a constant C_* such that

$$D_N^* \le D_N \le C_* D_N^*.$$

Thus there exists no difference of meaning in both discrepancy.

low Discrepancy Sequence

A random sequence $x_1, x_2, \ldots \in I$ is called of low discrepancy if there exists a constant $C > 0$ such that

$$D_N \leq C \frac{(\log N)^d}{N}.$$

It is proved for $d = 1, 2$

$$D_N \geq O\left(\frac{(\log N)^d}{N}\right),$$

and is expected the above inequality holds even for $d \geq 3$. Namely, the low discrepancy sequence will be the best uniformly distributed sequence. Thus this is the best sequence to approximate an integration numerically

$$\frac{1}{N} \sum_{n=1}^{N} f(x_n) \sim \int_I f \, dx$$

by quasi Monte Carlo method.

Dynamical System

We consider a transformation F on I whcih is expanding:

$$\xi = \liminf_{n\to\infty} \frac{1}{n} \operatorname*{ess\,inf}_x \log |DF^n(x)| > 0,$$

and has the invariant measure absolutely continuous to the Lebesgue measure.
Let $\mathcal{A}$ be a finite set of symbols and a subinterval $\langle a \rangle \subset I$ corresponds to each $a \in \mathcal{A}$.

Example; β–transformation ($d = 1$, $\beta > 1$)

$$F(x) = \begin{cases} \beta x & x \in [0, \frac{1}{\beta}), \\ \beta x - 1 & x \in [\frac{1}{\beta}, 1]. \end{cases}$$

$\langle 0 \rangle = [0, \frac{1}{\beta})$, $\langle 1 \rangle = [\frac{1}{\beta}, 1]$ and $\mathcal{A} = \{0, 1\}$.

Words

For a finite sequence of symbols $a_1 \cdots a_n$ ($a_i \in \mathcal{A}$) is called a word and define

- $|w| = n$,
- $\langle w \rangle = \bigcap_{k=0}^{n-1} F^{-k} \langle a_{k+1} \rangle$,
- if $\langle w \rangle \neq \emptyset$, then w is called admissible, and $\mathcal{W}$ expresses the set of admissible words,
- we consider an empty word ϵ, and define $\langle \epsilon \rangle = I$.

van der Corput Sequence

For a word w and a point $x \in I$, we define wx such that $F^{|w|}(wx) = x$ if it exists,
We give an order to the alphabet $\mathcal{A}$, and we define for words $w = a_1 \cdots a_n$ and $w' = b_1 \cdots b_m$, we define $wx < w'x$, if both points exist,

1. $|w| < |w'|$,
2. $|w| = |w'|$ and there exists k such that $a_{k+1} \cdots a_n = b_{k+1} \cdots b_n$, and $a_k < b_k$.

We arrange all wx $(w \in \mathcal{W})$ in the above order, and we call this sequence the van der Corput sequence generated by the dynamical system.

Original van der Corput sequence

Let $F(x) = 2x \pmod 1$, and $\mathcal{A} = \{0, 1\}$. Then

$x, 0x, 1x, 00x, 10x, 01x, 11x, 000x, 100x, 010x, 110x, 001x, 101x, 010x, \ldots$

is our van der Corput sequence. The original sequence is $x = \frac{1}{2}$, and their binary expansions of points are

$0.1, 0.01, 0.11, 0.001, 0.101, 0.011, 0.111, 0.0001, 0.1001, 0.0101, 0.1101, \ldots$

which is the reversed sequence of

$$1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1100, \ldots,$$

and add 0.

Perron–Frobenius operator

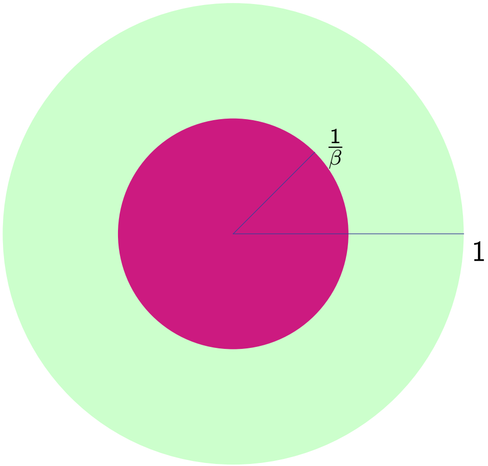
For a transformation $F\colon I \to I$,we define the Perron–Frobenius operator by

$$\begin{aligned} Pf(x) &= \sum_{y\colon F(y)=x} f(y)\,|DF(y)|^{-1} \\ &= \sum_{a\in\mathcal{A}} f(ax)\,|DF(ax)|^{-1}. \end{aligned}$$

P is an operator on L^1, but, in 1–dimensional cases, we restrict it to the set of functions with bounded variation BV.

essential spectrum

1–dimensional cases: $\beta = e^{\xi}$.



spectra of the Perron–Frobenius operator

- 1 is an eigenvalue and the dimension of eigenspace equals the number of ergodic components,
- If 1 is simple eigenvalue, then its eigenfuntion corresponds to the density function $\frac{d\mu}{dx}$ of the dynamical system,
- If there exists no eigenvalues modulus 1 except 1, then the dynamical system is mixing,
- the second greatest eigenvalue expresses the decay rate of correlation:

$$\int f(x)\, g(F^n x)\, dx \to \int f(x)\, dx \int g\, d\mu, \quad (f \in BV, g \in L^\infty)$$

Markov cases

Let $\beta = \frac{1+\sqrt{5}}{2}$ and

$$F(x) = \beta x \pmod 1,$$

$\langle 0 \rangle = [0, \frac{1}{\beta})$,$\langle 1 \rangle = [\frac{1}{\beta}, 1]$, and $\mathcal{A} = \{0, 1\}$.

$$s^a(z, x) = (I - zP)^{-1} 1_{\langle a \rangle}(x) = \sum_{n=0}^{\infty} z^n P^n 1_{\langle a \rangle}(x) \quad (a \in \mathcal{A})$$

Then

$$\begin{aligned} s^0(z, x) &= 1_{\langle 0 \rangle}(x) + z \sum_{n=0}^{\infty} z^n P^n (\sum_{a \in \mathcal{A}} 1_{\langle 0 \rangle}(ax) \beta^{-1}) \\ &= 1_{\langle 0 \rangle}(x) + z\beta^{-1}(s^0(z, x) + s^1(z, x)) \\ s^1(z, x) &= 1_{\langle 1 \rangle}(x) + z \sum_{n=0}^{\infty} z^n P^n (\sum_{a \in \mathcal{A}} 1_{\langle 1 \rangle}(ax) \beta^{-1}) \\ &= 1_{\langle 1 \rangle}(x) + z\beta^{-1} s^0(z, x) \end{aligned}$$

Renewal equation

Let

$$\Phi(z) = \begin{pmatrix} z\beta^{-1} & z\beta^{-1} \\ z\beta^{-1} & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} s^0(z,x) \\ s^1(z,x) \end{pmatrix} &= \begin{pmatrix} 1_{\langle 0 \rangle}(x) \\ 1_{\langle 1 \rangle}(x) \end{pmatrix} + \Phi(z) \begin{pmatrix} s^0(z,x) \\ s^1(z,x) \end{pmatrix} \\ &= (I - \Phi(z))^{-1} \begin{pmatrix} 1_{\langle 0 \rangle}(x) \\ 1_{\langle 1 \rangle}(x) \end{pmatrix}. \end{aligned}$$

Moreover, we get

$$\zeta(z) = \frac{1}{\det(I - \Phi(z))}.$$

Theorem

We can generalize the above results to general piecewise linear cases, and get

Theorem. The reciprocals of the solutions of $\det(I - \Phi(z)) = 0$ is the eigenvalues in $\{z\colon |z| > e^{-\xi}\}$.

Even for higher dimensional cases, we can prove similar results.

Spectra and Discrepancy

When $|DF| = \beta$ (constant), then

$$P^n f(x) = \beta^{-n} \sum_{|w|=n} f(wx).$$

Thus for an indicator function 1_J,

$$P^n 1_J(x) = \beta^{-n} \times \#\{wx \in J\}.$$

Thus by the spectra of the Perron–Frobenius operator determine the discrepancy of the van der Corput sequences.

1–dimensional cases

Assume that $\det(I - \Phi(z)) = 0$ has no solution in the annulus $\{\frac{1}{\beta} < |z| \leq 1\}$ except 1, then the discrepancy of the van der Corput sequence equals

$$\frac{(\log N)^{k+1}}{N}$$

where k is the number of endpoints which is not Markov.

higher dimension

For a function $f \in L^1$ and $0 < r < 1$, we define a norm

$$||f||_r = \inf \sum_{n=1}^{\infty} \sum_{|w|=n} |C_w| r^n,$$

where inf is taken over all decomposition $f = \sum C_w 1_{\langle w \rangle}$.
We define a space $\mathcal{B}$ the set of functions f for which $||f||_r < \infty$ for any $0 < r < 1$.
In 1–dimensional case, this space is a slight extension of the set of functions with bounded variation.

Prime Field

Now we consider the prime field $\mathbb{F}_2$ of characteristic 2, and the irreducible polynomial

$$\beta^2 + \beta + 1 = 0 \quad \text{over } \mathbb{F}_2.$$

$\hat{\mathcal{A}}$ is an additive group generated by 1 and β. We identify 0 to $(0,0)$, 1 to $(1,0)$ and β to $(0,1)$. then $\gamma = 1 + \beta = \beta^2$ is identified with $(1,1)$.

Another type of Words

We also consider a set of words with this alphabet, and denote it by $\mathcal{W}$.

$\langle\beta\rangle = \langle 0\rangle \times \langle 1\rangle$	$\langle\gamma\rangle = \langle 1\rangle \times \langle 1\rangle$
$\langle 0\rangle = \langle 0\rangle \times \langle 0\rangle$	$\langle 1\rangle = \langle 1\rangle \times \langle 0\rangle$

Matrix U

To determine F, we introduce an infinite dimensional matrix U of the form $U = (u, 0u, 00u, 0^3 u, \ldots)$, where u is an infinite dimensional vector and the transpose of the vector $0^k u$ is given by

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix}, \quad 0^k u = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_1 \\ u_2 \\ \vdots \end{pmatrix}.$$

$$U = \begin{pmatrix} u_1 & 0 & 0 & \cdots \\ u_2 & u_1 & 0 & \cdots \\ u_3 & u_2 & u_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $U^{-1}FU$ be a shift operator, that is,

$$U^{-1}FU = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We will determine the components of u inductively. To make the notations simple, even when we restrict U and $\hat{F}$ and so on to finite dimensions, we use the same notations. Note that

$$0^{\ell} u \xrightarrow{U^{-1}} e_{\ell} \xrightarrow{U^{-1}F^{k}U} e_{\ell-k} \xrightarrow{U} 0^{\ell-k} u,$$

where for $\ell < k$, $0^{k-\ell} u$ is the zero vector.

Kernel

Thus the kernel of F^k is generated by $u, 0u, \ldots, 0^{k-1}u$. When we consider F^k, we restrict the vector space to $2k$ dimension, thus we need to construct vector u such that all the $2k$ dimensional vectors which belong to the restriction of the subspace generated by $u, 0u, \ldots, 0^{k-1}u$ contain both 1 and β. Note also for any vector x $F^k(0^k x) = x$.

Definition of u

When $k = 1$, we put

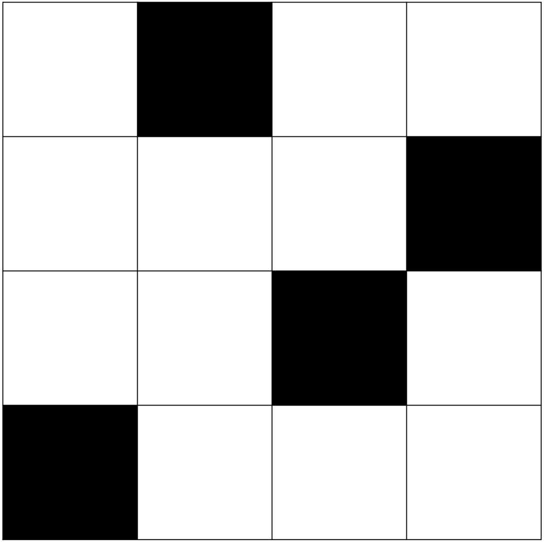
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \end{pmatrix}.$$

Thus the kernel of F from $\hat{\mathcal{A}}^2$ to $\hat{\mathcal{A}}$ are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \gamma \\ 1 \end{pmatrix},$$

and they contain both 1 and β.

Kernel, length=2



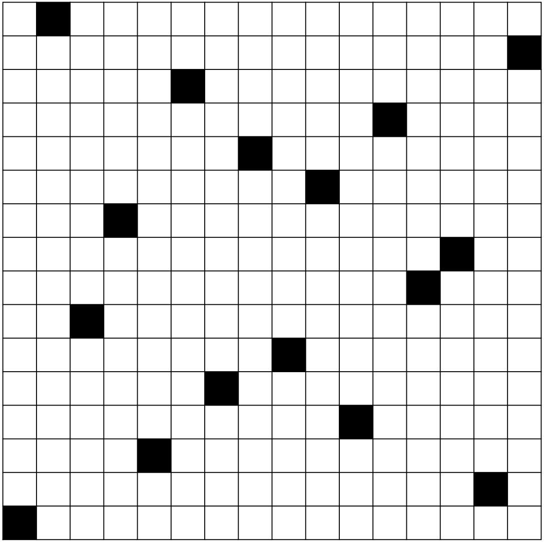
We can continue the procedure and get

$$u = \begin{pmatrix} 1 \\ \beta \\ 0 \\ \beta \\ \vdots \end{pmatrix},$$

and the kernel F^2 are 16 vectors generated by

$$\begin{pmatrix} 1 \\ \beta \\ 0 \\ \beta \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ \beta \\ 0 \end{pmatrix}.$$

Kernel, length=4



We can continue the procedure and get

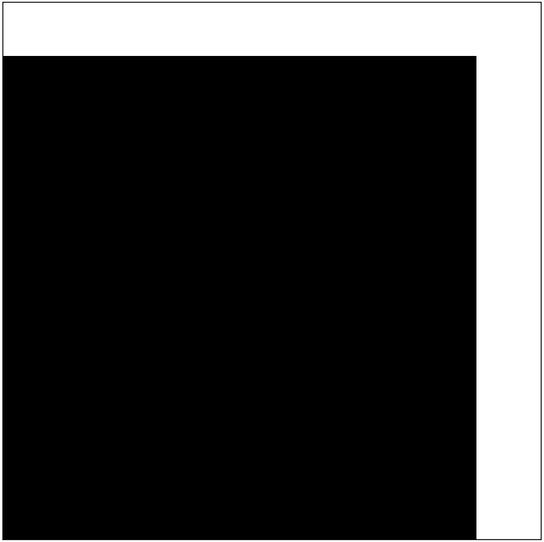
$$u = \begin{pmatrix} 1 \\ \beta \\ 0 \\ \beta \\ 0 \\ 0 \\ 0 \\ \beta \\ \vdots \end{pmatrix}.$$

Matrix U

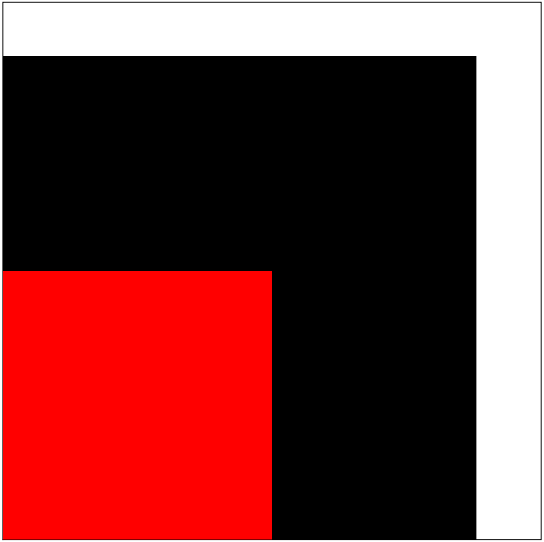
$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \beta & 1 & 0 & 0 & \cdots \\ 0 & \beta & 1 & 0 & \cdots \\ \beta & 0 & \beta & 1 & \cdots \\ 0 & \beta & 0 & \beta & \cdots \\ 0 & 0 & \beta & 0 & \cdots \\ 0 & 0 & 0 & \beta & \cdots \\ \beta & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From this theorem, we can prove all the indicator functions of rectangular belong to $\mathcal{B}$, that is, the space $\mathcal{B}$ is rich enough. Moreover, we can calculate the spectra of the Perron–Frobenius operator restricted to this space, and get the essential spectrum radius equals $\frac{1}{4}$, and there exists no eigenvalues in $|z| > \frac{1}{4}$ except 1 which is simple. Therefore, the dynamical system is mixing and the decay rate of correlation equals $\frac{1}{4}$.

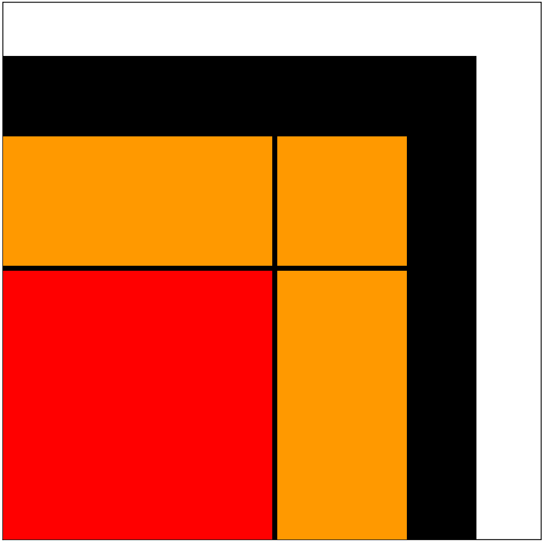
rectangular



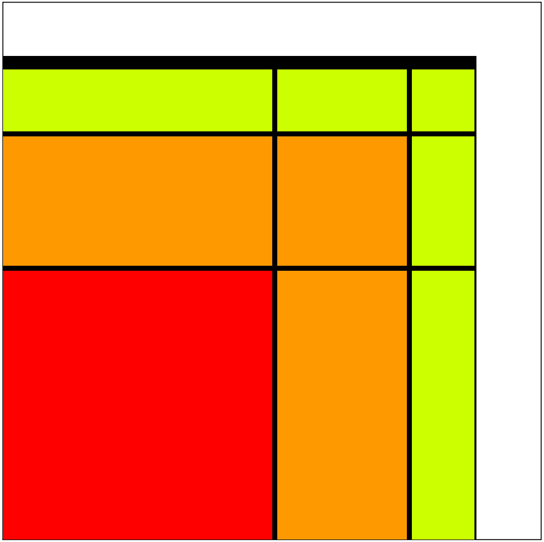
1st approximation



2nd approximation



3rd approximation



Number of rectangles

There exists two types of rectangles:

- first type(strip) generates one first type.
- second type(rectangle) generates two first types and one second type.

Let

$$M = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Then the number of rectangles in n–th approximation is at most of order

$$M^n \sim n.$$

This shows the indicator function of any rectangle belongs to $\mathcal{B}$.

3–dimensional transformation

For three dimensional cases, we consider

$$\beta^3 + \beta + 1 = 0.$$

In 2–dimensional case, we can determine a matrix U by one vector u. However, in 3–dimensional cases, $U = (u_{ij})$ has a fractal structure.
We consider

$$\mathcal{W}_3 = \{w = (w_1, w_2, w_3) \colon w_1, w_2, w_3 \in \mathcal{W}_1, |w_i| = 0 \pmod 3\},$$

and define $|w| = \frac{|w_1|+|w_2|+|w_3|}{3}$.
We want to construct F such that for all $w \in \mathcal{W}_3$

$$F^{|w|} \colon \langle w \rangle \to I$$

is 1 to 1 and onto.

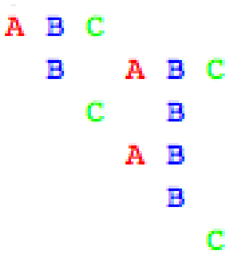
Let

$$A = \begin{pmatrix} 1 \\ \beta \\ \beta^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ \beta^2 \\ \beta^4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ \beta^4 \\ \beta^8 \end{pmatrix}.$$

Then a matrix (ABC) has inverse $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$,where

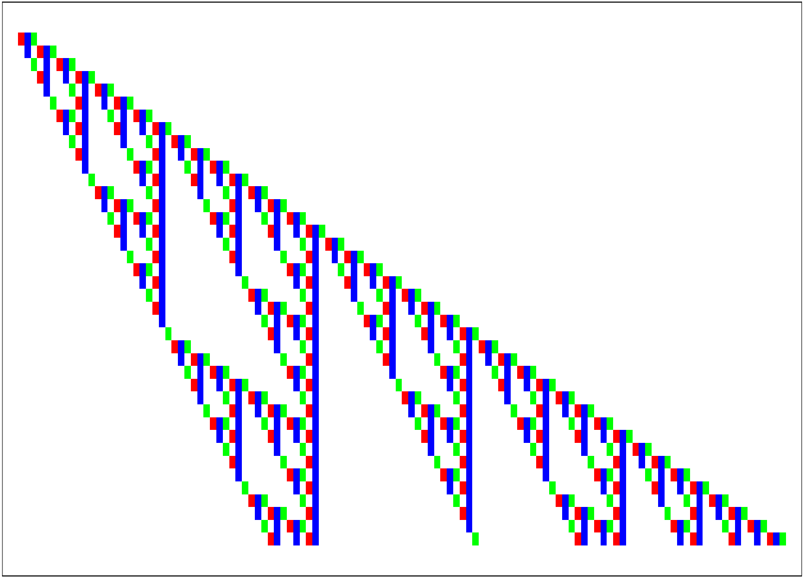
$$X = (1, \beta^2, \beta), \quad Y = (1, \beta + \beta^2, \beta^2), \quad Z = (1, \beta, \beta + \beta^2).$$

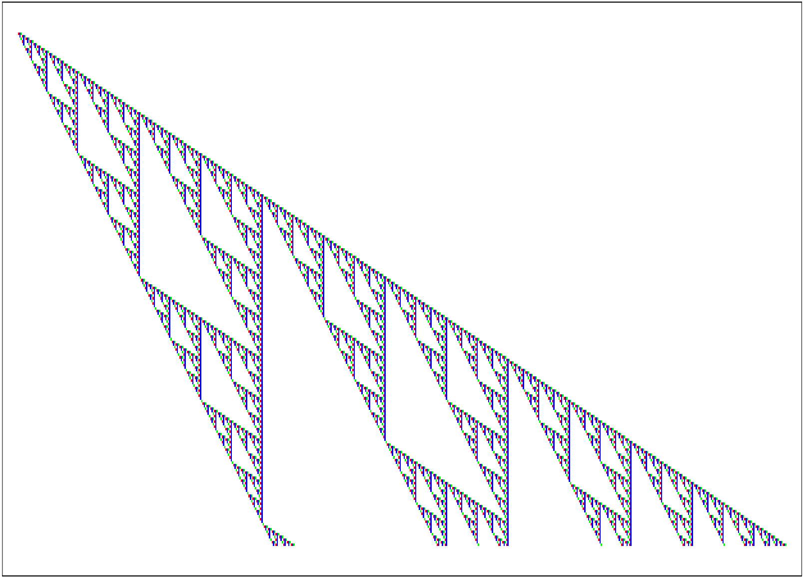
We define U as



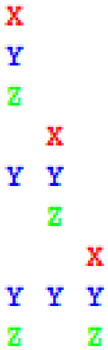
Rule to determine U

$$\tilde{u}_{ij} = \begin{cases} \tilde{u}_{i-1,j-1} + \tilde{u}_{i.j-1} \pmod 2 & j = 0, 2 \pmod 3, \\ \tilde{u}_{i-1,j-1} & j = 1 \pmod 3. \end{cases}$$



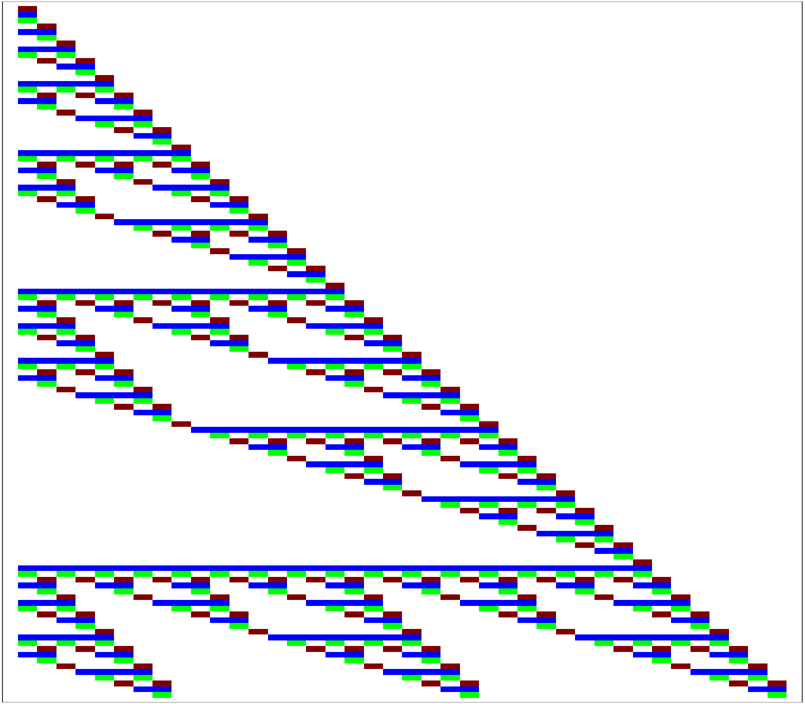


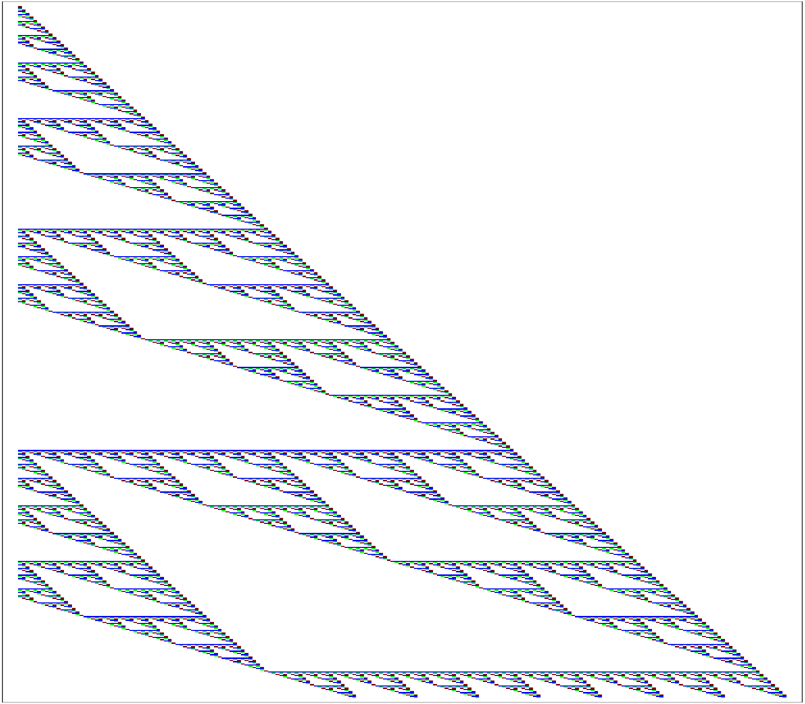
Then its inverse can be expressed as



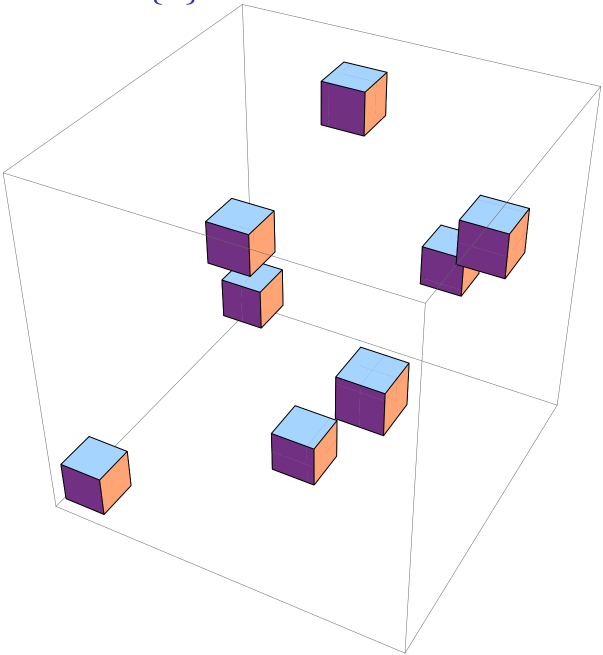
Rule to determine inverse matrix

$$\tilde{v}_{ij} = \begin{cases} \tilde{v}_{i-1,j} + \tilde{v}_{i.j+1} \pmod 2 & j = 0, 2 \pmod 3, \\ \tilde{v}_{i-1,j-1} & j = 1 \pmod 3. \end{cases}$$

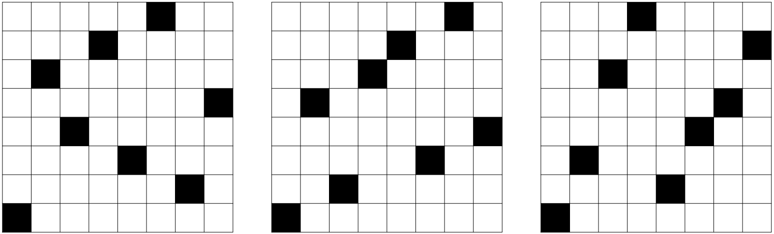




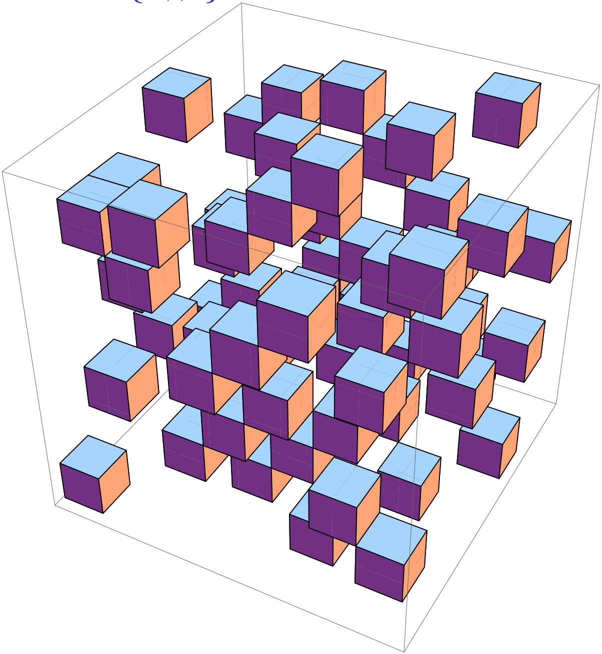
Kernel, $F^2 : \{1\}^3 \to \mathcal{A}$



Kernel, $F^3 : \{1\}^3 \to \mathcal{A}$



Kernel, $F\colon \{1,\beta\}^2 \to \mathcal{A}^2$



As in 2 dimensional case, there exists 3 types of cubes.

- first type (face) generates one first type.
- second type(stick) generates 2 first types and one second type.
- third type(cube) generates 3 first types, 3 second types and one third type.

Let

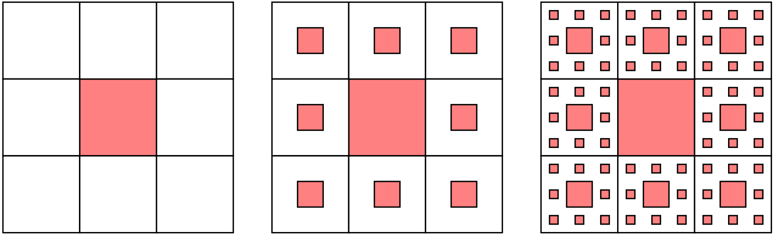
$$M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix}.$$

Thus, the number of cubes to approximate any cube, its order is

$$M^n \sim n^2.$$

This shows any indicator of any cube belongs to $\mathcal{B}$.

Cantor carpet



$\mathcal{A}$ is generated by 1 and $\beta \mod 3$

$-1+\beta$	β	$1+\beta$
-1	0	1
$-1-\beta$	$-\beta$	$1-\beta$

First we construct U from $F(x) = 3x \pmod 1$ just in the same way as before with the irreducible polynomial

$$\beta^2 + 1 = 0 \text{ on } \mathbb{F}_3.$$

The essential spectrum radius of this dynamical system equals 9^{-1}, and we can construct low discrepancy sequences.

We want to remake this transformation to the dynamical system on the cantor carpet.

We denote by ν the Hausdorff measure on the cantor carpet. Then the Perron–Frobenius operator defined by

$$\int Pf(x)\, g(x)\, d\nu = \int f(x)\, g(F(x))\, d\nu$$

satisfies

$$Pf(x) = \sum_{a \in \mathcal{A}} f(ax)\, 8^{-1}.$$

Construction of $\hat{F}^n\colon \mathcal{C} \to \mathcal{C}$

Let $\mathcal{A}_0 = \mathcal{A}\backslash\{0\}$. For a m–dimensional vector $\mathbf{u} \in \mathcal{A}_0^m$ ($m > n$), we consider $(F_m)^n(\mathbf{u}) \in \mathcal{A}^{m-n}$. Then

- if $(F_m)^n(\mathbf{u}) \in \mathcal{A}_0^{m-n}$, then $\mathbf{u}' = \mathbf{u}$.
- Otherwise, let i be the smallest i such that $((F_m)^n(\mathbf{u}))_i = 0$, then

  $$u_j^1 = \begin{cases} u_j & j \neq n+i, \\ 0 & j = n+i, \end{cases}$$

  and if $(F_m)^n(\mathbf{u}^1) \in \mathcal{A}_0^{m-n}$, we define $\mathbf{u}' = \mathbf{u}^1$.
- We have made $\mathbf{u}^k$ but still $(F_m)^n(\mathbf{u}^k) \notin \mathcal{A}_0^{m-n}$, we again do the same procedure as above, and define $\mathbf{u}^{k+1}$, and if $(F_m)^n(\mathbf{u}^{k+1}) \in \mathcal{A}_0^{m-n}$, we define $\mathbf{u}' = \mathbf{u}^{k+1}$. Otherwise, we continue the same procedure.

Now for $a_1 a_2 \cdots$ ($a_i \in \mathcal{A}_0$), we define

$$\hat{F}^n(a_1 a_2 \cdots) = \lim_{m\to\infty} (F_m)^n(a_1 \cdots a_m) = \lim_{m\to\infty} (F_m)^n((a_1 \cdots a_m)').$$

F and $\hat{F}$ on $\langle\beta\rangle$

$$\binom{1}{\beta}\binom{\beta}{a} = \beta + a \cdot \beta.$$

F

-1	$-1+\beta$	$-1-\beta$
0	β	$-\beta$
1	$1+\beta$	$1-\beta$

$\hat{F}$

-1	$-1+\beta$	$-1-\beta$
β	undefined	$-\beta$
1	$1+\beta$	$1-\beta$

From the construction, for a square $\langle w \rangle$ corresponding to a word with length n $\hat{F}^n$ maps $\langle w \rangle$ to I one to one and onto.
For $J = \langle w_1 \rangle \times \langle w_2 \rangle$ such that $|w_1| = k$ and $|w_2| = -k + 2n$, we consider squares inside J with length k. There exists $2n - 2k$ such words, and $F^n(J) = I$.
We divide it into two types

- $w \in A_J$, if w has no zero, and $w \in A_J^i$ if the number of 0 in $F^{k+n}(\langle w \rangle)$ equals i.
- $w \in B_J$, if there exists 0 in w, and $w \in B_J^{l,m}$ if the number of 0 in w equals l and the number of 0 in $F^{k+n}(\langle w \rangle)$ equals m.

The worst case: J(a word with length n) consists words only of type A_J. Then

$$\begin{aligned} s^J(z,x) &= \sum_{m=0}^{\infty} z^m P_{\hat{F}^m} 1_J(x) \\ &= \sum_{m=0}^{\infty} z^m \sum_{|w|=m} 1_J(wx) 8^{-m} \\ &= \sum_{m=0}^{\infty} z^m \sum_{k=0}^{n} \sum_{w\in A_J^k} 1_J(wx) 8^{-m} \\ &= \sum_{k=0}^{n} \sum_{m=0}^{n+k-1} z^m \sum_{w\in A_J^k} 1_J(wx) 8^{-m} \\ &\quad + \sum_{k=0}^{n} \sum_{m=n+k-1}^{\infty} z^m \sum_{w\in A_J^k} 1_J(wx) 8^{-m}. \end{aligned}$$

Second Term

$$\begin{aligned}\text{The Second Term} &= z^n 8^{-n} \sum_{k=0}^{n} z^k 8^{-k} \binom{n}{k} s^I(z,x). \\ &= z^n 8^{-n} (1 + z8^{-1})^n s^I(z,x).\end{aligned}$$

Especially,

$$s^I(z,x) = 1 + z8^{-1} \times 8 s^I(z,x) = 1 + z s^I(z,x).$$

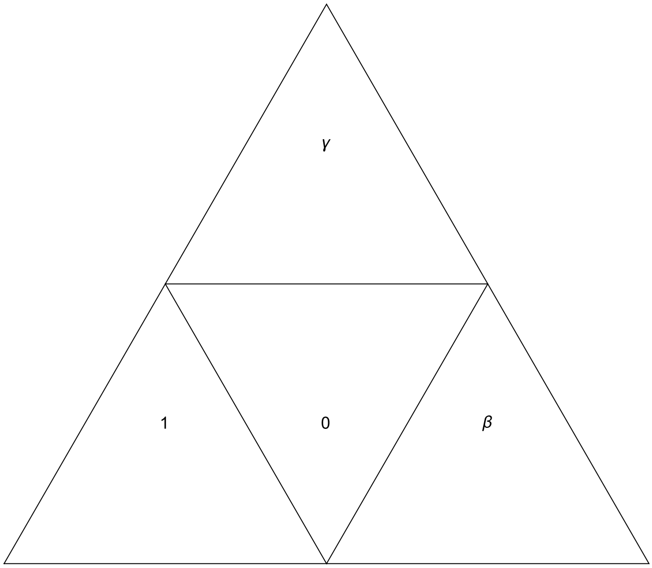
Thus for a rectangular J with length n, the main term equals

$$s^J(z,x) = z^n 8^{-n} (1 + z8^{-1})^n \frac{1}{1-z}.$$

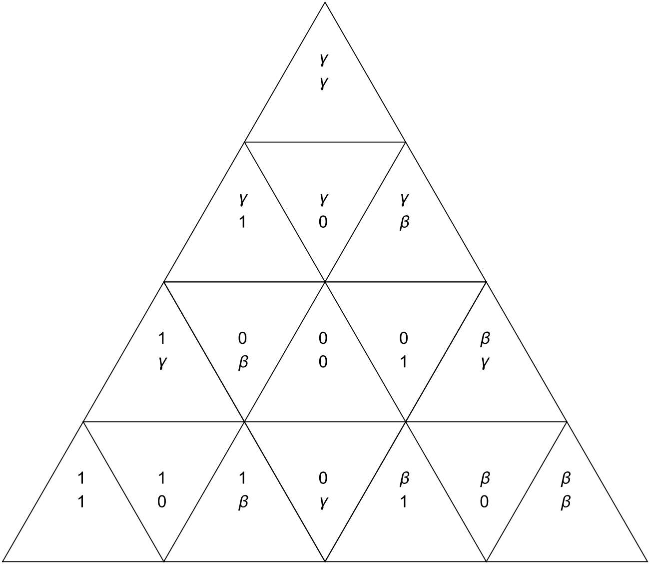
$$\sum_{n=0}^{\infty} z^n 8^{-n}(1+z8^{-1})^n \times n$$

has the minimal singularity at $\frac{-1+\sqrt{5}}{2} \times 8 < 8$. Thus the discrepancy of the random number generated by this dynamical system is not of low discrepancy.

Triangle



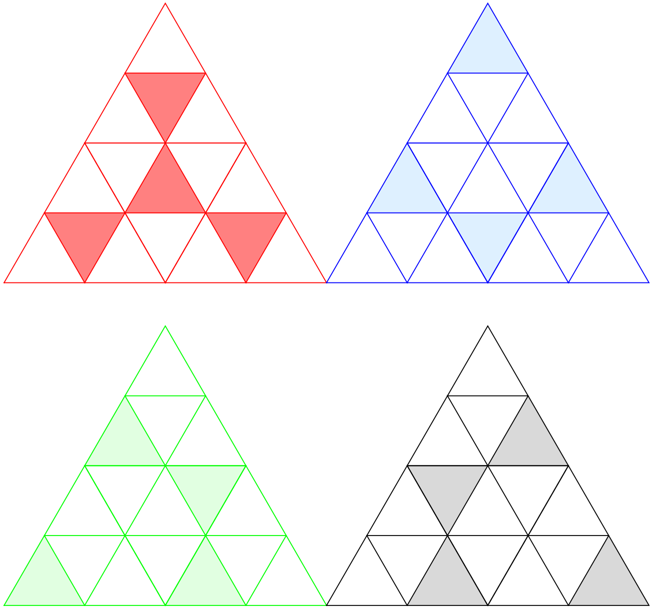
Words with Length 2

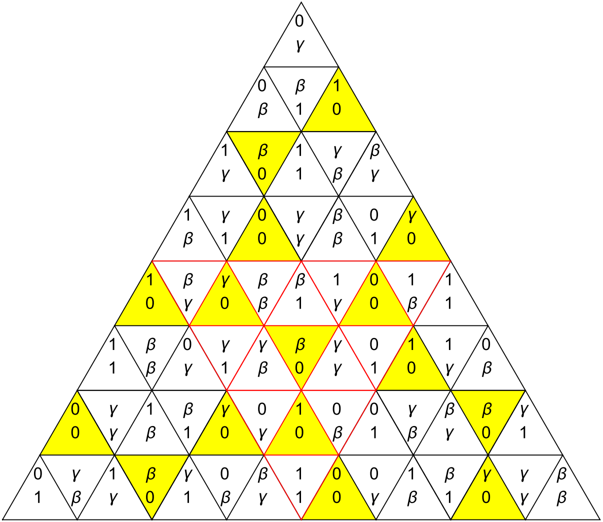


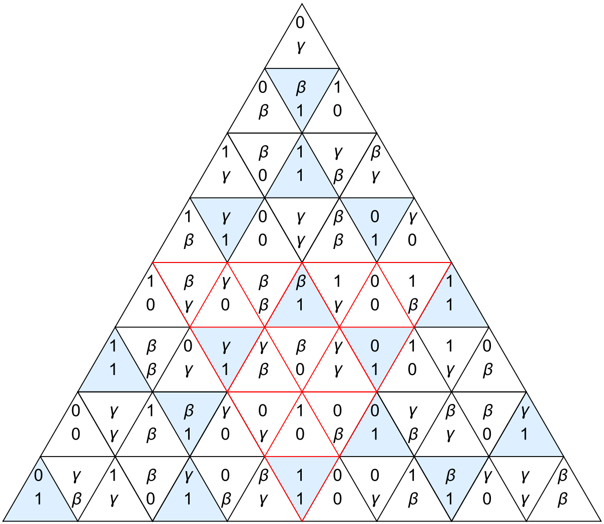
$$
\begin{aligned}
A &= \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\\beta\end{pmatrix}, \begin{pmatrix}0\\\gamma\end{pmatrix}, \begin{pmatrix}\beta\\1\end{pmatrix}, \begin{pmatrix}\beta\\0\end{pmatrix}, \begin{pmatrix}\beta\\\beta\end{pmatrix}, \begin{pmatrix}\gamma\\0\end{pmatrix} \right\}\\
A' &= \left\{ \begin{pmatrix}1\\\gamma\end{pmatrix}, \begin{pmatrix}0\\\beta\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\beta\\\gamma\end{pmatrix}, \begin{pmatrix}\gamma\\\beta\end{pmatrix}, \begin{pmatrix}\gamma\\0\end{pmatrix}, \begin{pmatrix}\gamma\\1\end{pmatrix} \right\}\\
B' &= \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\\gamma\end{pmatrix}, \begin{pmatrix}0\\\beta\end{pmatrix}, \begin{pmatrix}\gamma\\1\end{pmatrix}, \begin{pmatrix}\gamma\\0\end{pmatrix}, \begin{pmatrix}\gamma\\\gamma\end{pmatrix}, \begin{pmatrix}\beta\\\beta\end{pmatrix} \right\}\\
B &= \left\{ \begin{pmatrix}1\\\beta\end{pmatrix}, \begin{pmatrix}0\\\gamma\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\gamma\\\beta\end{pmatrix}, \begin{pmatrix}\beta\\\gamma\end{pmatrix}, \begin{pmatrix}\beta\\0\end{pmatrix}, \begin{pmatrix}\beta\\1\end{pmatrix} \right\}\\
C &= \left\{ \begin{pmatrix}\beta\\\beta\end{pmatrix}, \begin{pmatrix}\beta\\0\end{pmatrix}, \begin{pmatrix}\beta\\\gamma\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\gamma\\\beta\end{pmatrix}, \begin{pmatrix}\gamma\\0\end{pmatrix}, \begin{pmatrix}\gamma\\\gamma\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}\\
C' &= \left\{ \begin{pmatrix}\beta\\1\end{pmatrix}, \begin{pmatrix}0\\\gamma\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\\beta\end{pmatrix}, \begin{pmatrix}\gamma\\1\end{pmatrix}, \begin{pmatrix}1\\\gamma\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\\beta\end{pmatrix} \right\}
\end{aligned}
$$

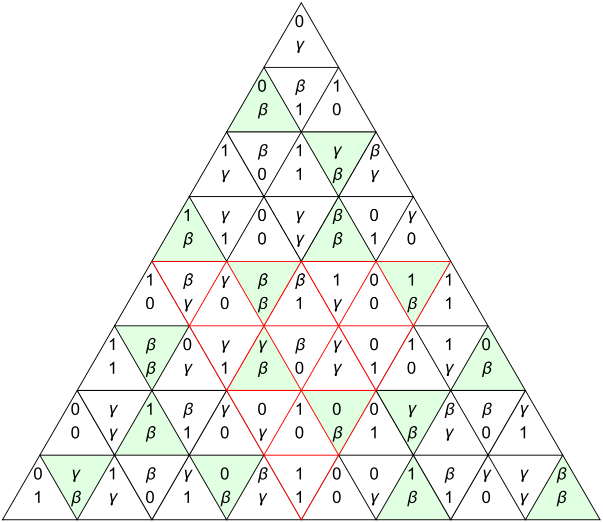
$$\begin{aligned}
ABC &= \left\{ \begin{pmatrix} \beta \\ 0 \end{pmatrix} \right\} \\
AB'C' &= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \\
A'BC' &= \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\
A'B'C &= \left\{ \begin{pmatrix} \gamma \\ 0 \end{pmatrix} \right\} \\
A'B'C' &= \left\{ \begin{pmatrix} \gamma \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \right\} \\
A'BC &= \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \gamma \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \right\} \\
AB'C &= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma, \\ \gamma \end{pmatrix} \right\} \\
ABC' &= \left\{ \begin{pmatrix} \beta \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \beta \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \right\}
\end{aligned}$$

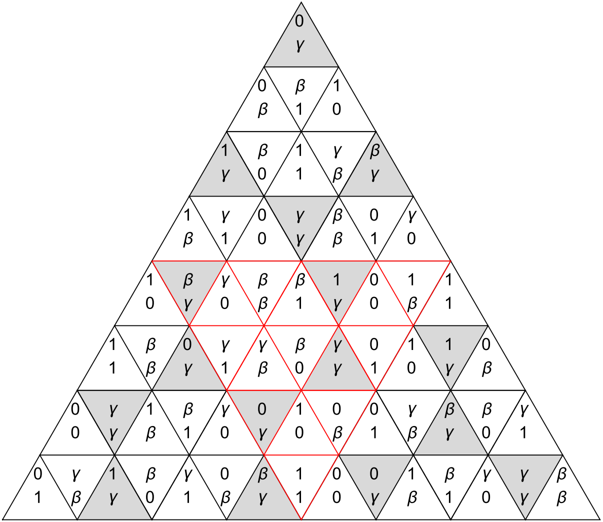
$$
\begin{aligned}
AB &= \left\{ \begin{pmatrix} \beta \\ 0 \end{pmatrix}, \begin{pmatrix} \beta \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \beta \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \right\} \\
AB' &= \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ \gamma \end{pmatrix} \right\} \\
A'B &= \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \gamma \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \right\} \\
A'B' &= \left\{ \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \begin{pmatrix} \gamma \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \right\}
\end{aligned}
$$










0 γ
0 β, β 1, 1 0
1 γ, β 0, 1 1, γ β, β γ
1 β, γ 1, 0 0, γ γ, β β, 0 1, γ 0
1 0, β γ, γ 0, β β, β 1, 1 γ, 0 0, 1 β, 1 1
1 1, β β, 0 γ, γ 1, γ β, β 0, γ γ, 0 1, 1 0, 1 γ, 0 β
0 0, γ γ, 1 β, β 1, γ 0, 0 γ, 1 0, 0 β, 0 1, γ β, β γ, β 0, γ 1
0 1, γ β, 1 γ, β 0, γ 1, 0 β, β γ, 1 1, 0 0, 0 γ, 1 β, β 1, γ 0, γ γ, β β