Generalized carry process and riffle shuffle

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Adding 3 numbers,

10111	10210	11102	11122	01011	11210	2112
43443	07082	04401	15299	64642	73497	38426
00171	55077	11440	95932	91116	17255	19649
49339	70267	68885	98147	70311	43856	37376
92954	32426	84728	09380	26070	34608	95451

then 1 seems to appear frequently. $(\sharp 0 : \sharp 1 : \sharp 2 = 7 : 20 : 7)$。

Transition Probability 1

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Example 1 ($b = 2, n = 2$)

0	0	0	0	0	0	1	0
	0		1		0		1
	0		0		1		1
	0		1		1		0

$$\Longrightarrow \quad (P_{0,0}, P_{0,1}) = \frac{1}{2^2}(3, 1)$$

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$$\begin{array}{cc} \color{red}{0} & 0 \\ \hline & 0 \\ & 0 \\ \hline & 0 \end{array} \quad \begin{array}{cc} \color{red}{0} & 0 \\ \hline & 1 \\ & 0 \\ \hline & 1 \end{array} \quad \begin{array}{cc} \color{red}{0} & 0 \\ \hline & 0 \\ & 1 \\ \hline & 1 \end{array} \quad \begin{array}{cc} \color{red}{1} & 0 \\ \hline & 1 \\ & 1 \\ \hline & 0 \end{array} \quad \Longrightarrow \quad (P_{0,0}, P_{0,1}) = \frac{1}{2^2}(3, 1)$$

For $b = 2, n = 2$

$$P = \frac{1}{2^2}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad \Longrightarrow \quad \text{Stationary dist. } \pi = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Transition Probability 2

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$$\Longrightarrow \quad (P_{0,0}, P_{0,1}, P_{0,2}) = \frac{1}{2^3} \cdot (4, 4, 0)$$

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\qquad
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$$P = \frac{1}{2^3} \begin{pmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \\ 0 & 4 & 4 \end{pmatrix} \quad \Longrightarrow \quad \pi = \frac{1}{3!} \cdot (1, 4, 1)$$

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Add n base- b numbers $(b, n \in \mathbf{N}, b, n \geq 2)$

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$\mathbf{C_k}$: the carry coming out in the k-th digit.

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$\{C_k\}_{k=0}^{\infty}$ $(C_k \in \{0, \cdots, n-1\})$ is called the **carries process**.

Amazing Matrix : Holte(1997)

$$P_{ij} := \mathbf{P}\left(C_{k+1} = j \,|\, C_k = i\right), \quad i, j = \underline{0}, 1, \cdots, n-1$$

$$P_{ij} = b^{-n} \sum_{r=0}^{\lfloor z/b \rfloor} (-1)^r \binom{n+1}{r} \binom{n + z_{ij} - br}{n}$$

$$z_{ij} := (j+1)b - i - 1$$

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E-values and left E-vectors of Amazing Matrix

E-values/ E-vectors depends only on b / n.

$$P = L^{-1} D L, \quad D = \text{ diag } \left(1, \frac{1}{b}, \frac{1}{b^2}, \cdots, \frac{1}{b^{n-1}}\right)$$

$$L_{ij} = v_{ij}(n) = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} (j - r + 1)^{n-i}$$

Property of Left Eigenvectors

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$E(3,0) = \sharp\{(123)\} = 1$,
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[2] L is equal to the Foulkes character table of S_n (Diaconis-Fulman, 2012).

Foulkes character

Example

$$\sharp\{\sigma \in S_4 \mid \sigma(1) < \sigma(2) > \sigma(3) < \sigma(4)\}$$
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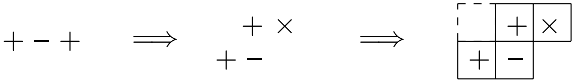

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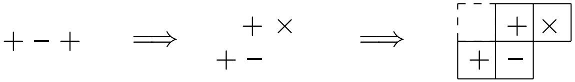

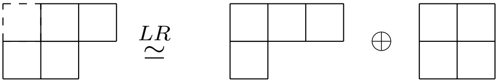

$\dim = 3$ $\quad\quad$ $\dim = 2$

Property of Right Eigenvectors

Right Eigenvector of P

$$P = RDR^{-1}$$

$$R_{ij} = \sum_{r=n-j}^{n} (-1)^{n-r} \left[\begin{array}{c} n \\ r \end{array} \right] \left(\begin{array}{c} r \\ n-j \end{array} \right) (n-1-i)^{r-(n-j)}$$

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$R(0, n-1-j) = S(n, j)$

$S(n, j) := \sharp\{\sigma \in S_n$ with j-cycles $\}$ Stirling num. of 1st kind

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Relation to Riffle Shuffles (Diaconis-Fulman, 2009)

$$\{C_k\}_{k=1}^{\infty} \stackrel{d}{=} \{d(\sigma_k)\}_{k=1}^{\infty}, \quad d(\sigma) : \text{ the descent of } \sigma \in S_n.$$

In other words,

$$\mathbf{P}\left(C_1 = j_1, C_2 = j_2, \cdots, C_k = j_k \,|\, C_0 = 0\right)$$
$$= \mathbf{P}\left(d(\sigma_1) = j_1, d(\sigma_2) = j_2, \cdots, d(\sigma_k) = j_k \,|\, \sigma_0 = id\right).$$

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Since the stationary dist. of $\{\sigma_k\}$ is uniform on S_n,

$$P_{0j} = \mathbf{P}_{unif}(d(\sigma) = j) \propto E(n, j),$$

explaining why Eulerian num. appears.

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and / or

(2) taking the negative base

$$b \quad \Longrightarrow \quad -b$$

Generalized $(+b)$-expansion

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Then $\forall x \in \mathbf{N}$ can be represented uniquely as

$$x = a_n(+b)^n + a_{n-1}(+b)^{n-1} + \cdots + a_0, \quad a_k \in \mathcal{D}_d$$

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$\{C_k\}$ $(C_k \in \mathcal{C}(b, n))$ is called a **(generalized) Carries Process**.

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(2) $\sharp \mathcal{C}(b,n) = \begin{cases} n & (n-1)l_+ \in \mathbf{Z} \quad (\supset \text{ Holte's case }) \\ n+1 & (n-1)l_+ \notin \mathbf{Z} \end{cases}$

Why ?

$$F := \left\{ \frac{a_1}{b} + \cdots + \frac{a_N}{b^N} \,\middle|\, N \in \mathbf{N},\ a_j \in \mathcal{D}_d \right\} \hookrightarrow (l_+, l_+ + 1)$$

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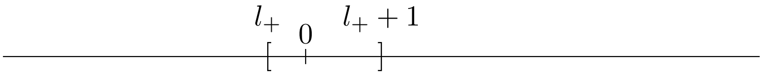

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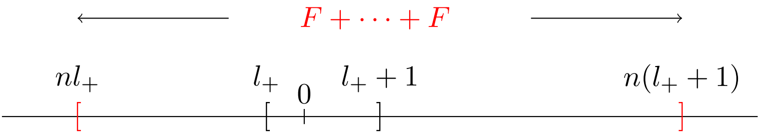

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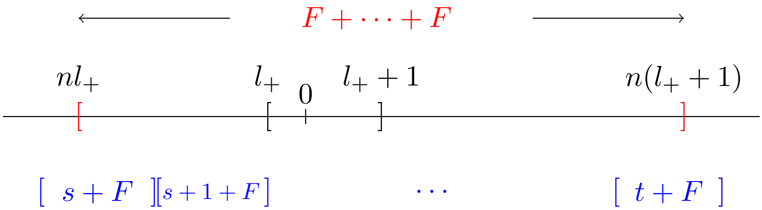

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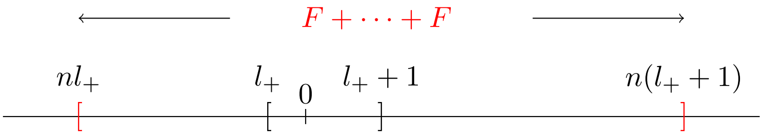

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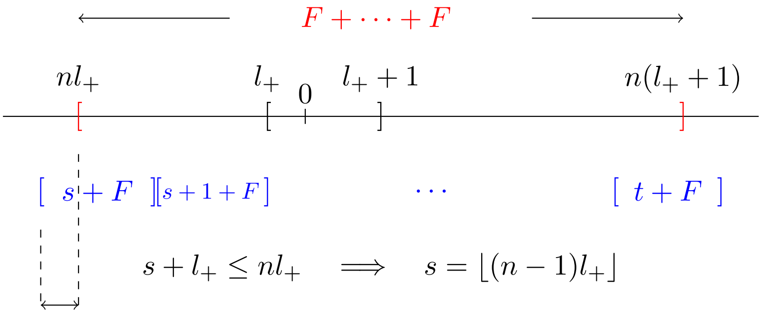

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$1 - \frac{1}{p}$

Define p s.t. $1 - \frac{1}{p} = nl_+ - (s + l_+)$

(b, n, p)-process

Let $s := \min \mathcal{C}(b, n)$, $p := (1 - \{(n-1)l_+\})^{-1} \in \mathbf{Q}$, and let $\tilde{P} := \{\tilde{P}_{ij}\}_{i,j}$, $\tilde{P}_{ij} := \mathbf{P}\left(C_{k+1} - s = j \,|\, C_k - s = i\right)$.

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Remark : Holte's process $\Longrightarrow p = 1$.

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[3] For $p \notin \mathbf{N}$, we do not know...

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No hits on OEIS...

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Theorem 2

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Descent on $G_{p,n}$

Define a ordering on Σ

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" $\sigma \in G_{p,n}$ has a descent at i "
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(i) $(\sigma(i), \sigma^c(i)) > (\sigma(i+1), \sigma^c(i+1))$ (for $i = 1, 2, \cdots, n-1$)
(ii) $\sigma^c(n) \neq 0$ (for $i = n$).

Descent on $G_{p,n}$

Define a ordering on Σ

$$
\begin{aligned}
&(1,0) < (2,0) < \cdots < (n,0) \\
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$$

" $\sigma \in G_{p,n}$ has a descent at i "
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(i) $(\sigma(i), \sigma^c(i)) > (\sigma(i+1), \sigma^c(i+1))$ (for $i = 1, 2, \cdots, n-1$)
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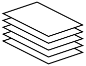
$d(\sigma)$: the number of descents of σ.

Generalized Riffle Shuffle



n cards
with p colors

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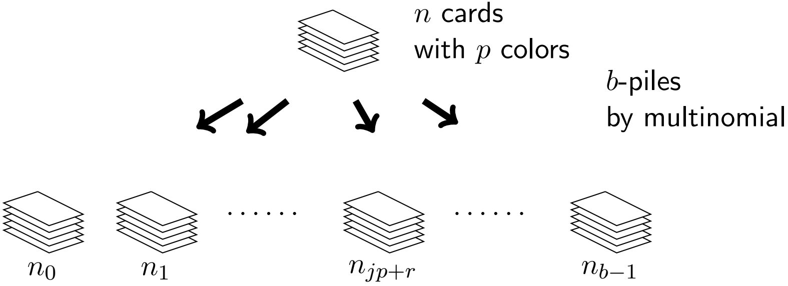

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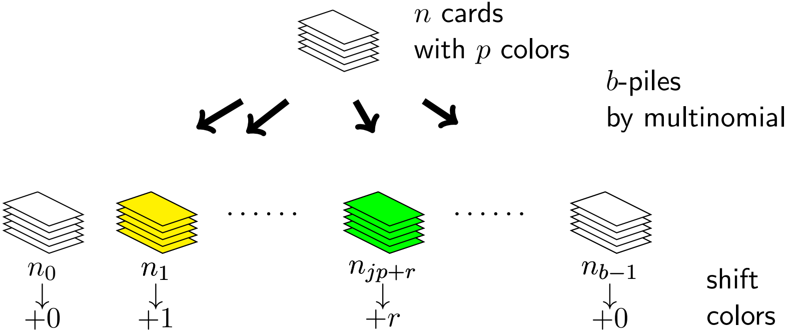

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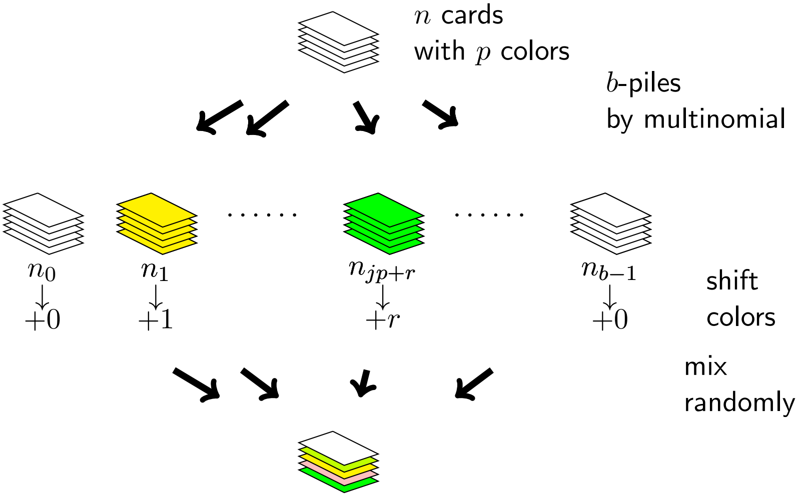
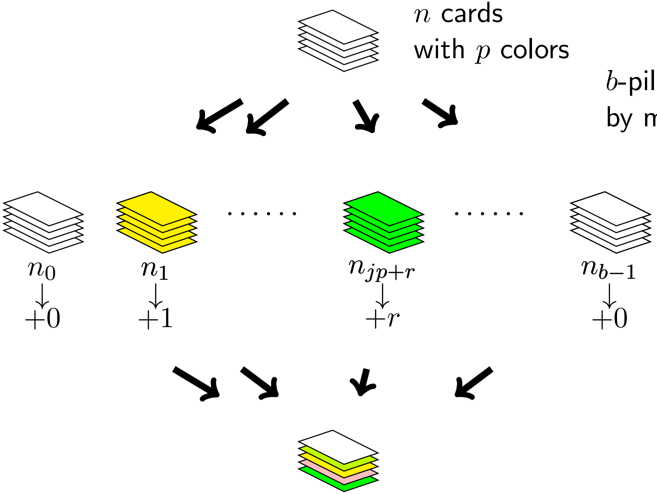


Generalized Riffle Shuffle

n cards
with p colors

b-piles
by multinomial



shift
colors

mix
randomly

This process defines a Markov chain $\{\sigma_r\}_{r=0}^{\infty}$ on $G_{p,n}$.
(called the (b, n, p)-shuffle)

Carries Process and Riffle Shuffle

$\{\kappa_r := C_r - s\}_{r=1}^{\infty}$: (b, n, p) - process

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Theorem 3

$$\{\kappa_r\} \stackrel{d}{=} \{d(\sigma_r)\}$$

In other words,

$$\mathbf{P}\left(\kappa_1 = j_1, \kappa_2 = j_2, \cdots, \kappa_k = j_k \,|\, \kappa_0 = 0\right)$$
$$= \mathbf{P}\left(d(\sigma_1) = j_1, d(\sigma_2) = j_2, \cdots, d(\sigma_k) = j_k \,|\, \sigma_0 = id\right)$$

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Theorem 3 explains why the descent statistics of $G_{p,n}$ appears in the stationary distribution of (b, n, p) - process.

What about $(-b)$-case ?

Any $x \in \mathbf{Z}$ can be expanded uniquely as

$$x = a_n(-b)^n + a_{n-1}(-b)^{n-1} + \cdots + a_0, \quad a_k \in \mathcal{D}_d$$

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Ev : $1, \frac{1}{b}, \frac{1}{b^2}, \cdots \quad \Longrightarrow \quad 1, \left(-\frac{1}{b}\right), \left(-\frac{1}{b}\right)^2, \cdots,$

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$L_\pm$, $R_\pm$ have the same dependence on p.

Dash - Descent on $G_{p,n}$

(1) " Dash - order " $<'$ on Σ:

$$\begin{aligned}
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$\overset{def}{\Longleftrightarrow}$

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$\sigma^c(n) = p - 1$ (for $i = n$).

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(3) $d'(\sigma)$: the number of dash-descents of $\sigma \in G_{p,n}$.

$d(\sigma) = d'(\sigma)$ for $p = 1$
$E'_p(n,k) = E_p(n, n-k)$.

Fumihiko NAKANO, Taizo SADAHIRO

Shuffles for $(-b, n, p)$ - process

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Theorem 4

$$\{\kappa_r^-\}_r \overset{d}{=} \{d_r^-\}_r$$

Limit Theorem

For any $p \geq 1$, and for $n \geq 2$, $k = 0, 1, \cdots, n$, let

$$\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_p := \sum_{r=0}^{k} (-1)^r \left(\begin{array}{c} n+1 \\ r \end{array} \right) \{p(k-r)+1\}^n,$$

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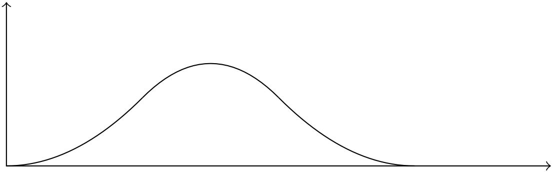
Theorem 5

$$\mathbf{P}\left(S_n \in \frac{1}{p} + [k-1, k]\right) = \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_p (p^n n!)^{-1}$$

for $k = 0, 1, \cdots, n$.

Example

(1) $n = 3$, $p = 1$: (Eulerian number)



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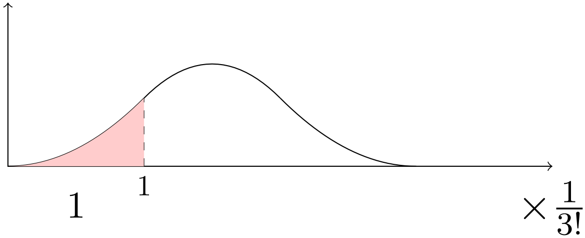

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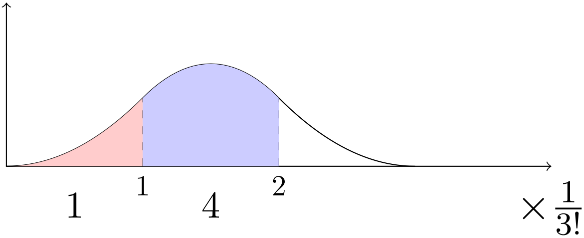

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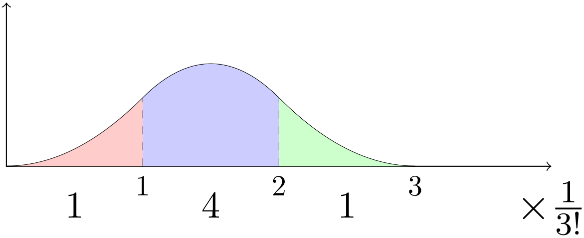

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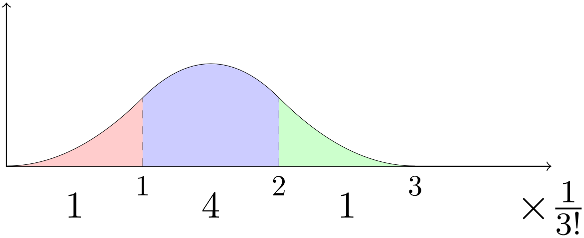

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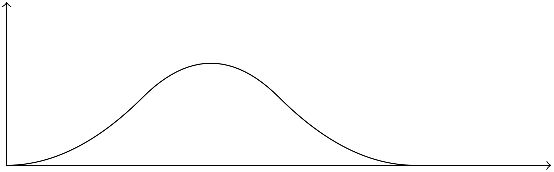

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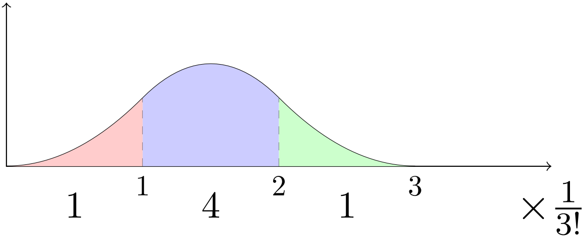

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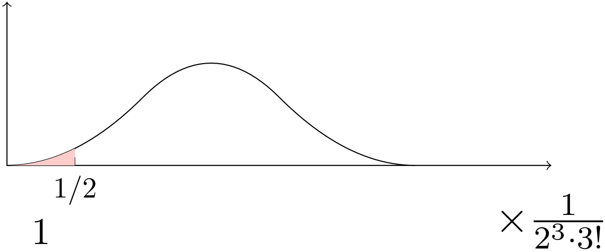

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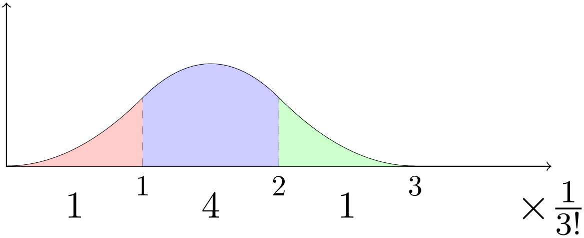

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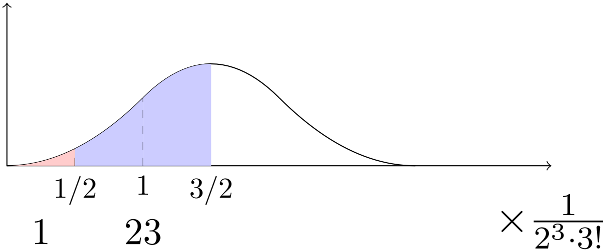

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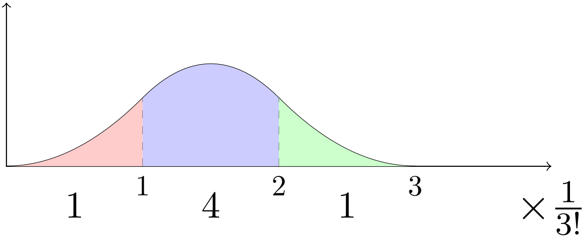

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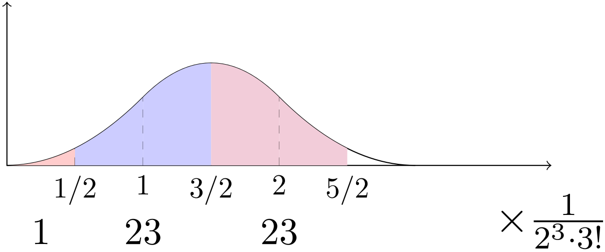

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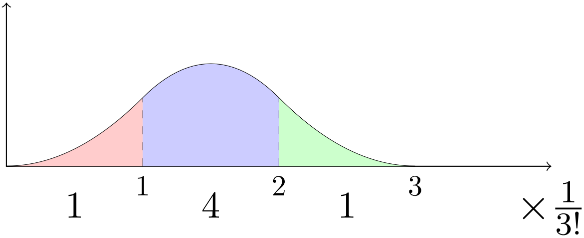

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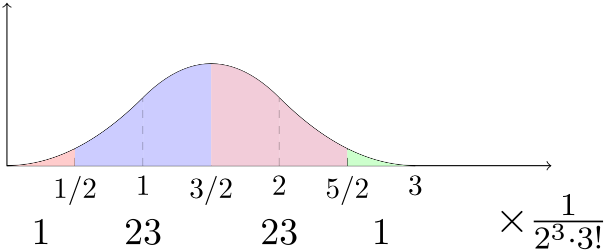

Idea of Proof

Let $X'_1, \cdots, X'_m$ be independent, uniformly distributed r.v.'s on $[l, l+1]$, and let $S'_m := X'_1 + \cdots + X'_m$.

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		$\vdots$		$\vdots$	$\vdots$	$\vdots$
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\pi(j) & & \mathbf{P}(S'_m \in [l, l+1] + j)
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Generalized carry process and riffle shuffle

Fumihiko NAKANO, Taizo SADAHIRO

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(4) $\{\kappa_r\}_r \overset{d}{=} \{d(\sigma_r)\}_r$ or $\overset{d}{=} \{d_r^-\}_r$, which explains (1).

Summary

[1] We study the generalization of the carries process $\{\kappa_r\}_r$, called $(\pm b, n, p)$ - process, and derived the left/right eigenvectors of its transition probability matrix.

For $\underline{p \in \mathbf{N}}$,
(1) Stationary distribution gives the descent statistics of $G_{p,n}$

(2) Left eigenvector matrix is equal to the Foulkes character table of $G_{p,n}$

(3) Stirling - Frobenius cycle number and the number of (b, n, p)-shuffles appear in the right eigenvector matrix

[2] We consider a generalization of riffle shuffle $\{\sigma_r\}$ on $G_{p,n}$, called (b, n, p) - shuffle, for $\underline{p \in \mathbf{N}}$.

(4) $\{\kappa_r\}_r \overset{d}{=} \{d(\sigma_r)\}_r$ or $\overset{d}{=} \{d_r^-\}_r$, which explains (1).

[3] for $\underline{p \notin \mathbf{N}}$, no combinatorial meaning is known so far...

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