Additive combinatorics generated by uniformly recurrent words

M. Bucci[1] N. Hindman[2]
S. Puzynina[1,3] L. Q. Zamboni[1,4]

[1]University of Turku, Finland

[2]Howard University, USA

[3]Sobolev Institute of Mathematics, Russia

[4]Université de Lyon, France

Introduction

additive Ramsey theory + combinatorics on words

- additive properties of sets of integers
- using infinite words for building sets with additive properties
- methods: ultrafilters, substitutive dynamics, numeration systems
- connections with Pisot conjecture

Introduction

Question

$\mathbb{N} = A_1 \cup \cdots \cup A_n$, $A_i \cap A_j = \emptyset$ for $i \neq j$.
Does there exist i and x, y such that x, y, $x + y \in A$?

Introduction

Question

$\mathbb{N} = A_1 \cup \cdots \cup A_n$, $A_i \cap A_j = \emptyset$ for $i \neq j$.
Does there exist i and x, y such that x, y, $x + y \in A$?

Answer: yes! Issai Schur (1916)

Question

$\mathbb{N} = A_1 \cup \cdots \cup A_n$, $A_i \cap A_j = \emptyset$ for $i \neq j$.
Does there exist i and x, y such that x, y, $x + y \in A$?

Answer: yes! Issai Schur (1916)

Moreover: for every k there exist i and $x_0 < x_1 < \cdots < x_{k-1}$ such that $\{\Sigma_{t \in F} x_t | F \subseteq \{0, 1, \ldots, k-1\}\} \subseteq A_i$
(Folkman-Rado-Sanders)

Introduction

Question

$\mathbb{N} = A_1 \cup \cdots \cup A_n$, $A_i \cap A_j = \emptyset$ for $i \neq j$.
Does there exist i and x, y such that x, y, $x + y \in A$?

Answer: yes! Issai Schur (1916)

Moreover: for every k there exist i and $x_0 < x_1 < \cdots < x_{k-1}$ such that $\{\Sigma_{t \in F} x_t | F \subseteq \{0, 1, \ldots, k-1\}\} \subseteq A_i$ (Folkman-Rado-Sanders)

Even more: there exist i and $x_0 < x_1 < \cdots < x_k < \ldots$ such that $\{\Sigma_{t \in F} x_t | F \subset \mathbb{N}, |F| \leq \infty\} \subseteq A_i$ (N. Hindman 1974)

IP-sets

Definition

$A \subseteq \mathbb{N}$ is an IP-set if there exist an infinite sequence $x_0 < x_1 < \cdots < x_k < \ldots$ such that A contains all its finite sums with distinct elements.

IP-sets

Definition

$A \subseteq \mathbb{N}$ is an IP-set if there exist an infinite sequence $x_0 < x_1 < \cdots < x_k < \ldots$ such that A contains all its finite sums with distinct elements.

Example

The set of even numbers is IP-set. The set of odd numbers is not.

IP-sets

Definition

$A \subseteq \mathbb{N}$ is an IP-set if there exist an infinite sequence $x_0 < x_1 < \cdots < x_k < \ldots$ such that A contains all its finite sums with distinct elements.

Example

The set of even numbers is IP-set. The set of odd numbers is not.

Question (typically difficult): Given $A \subseteq \mathbb{N}$, is A an IP-set?

IP-sets

Definition

$A \subseteq \mathbb{N}$ is an IP-set if there exist an infinite sequence $x_0 < x_1 < \cdots < x_k < \ldots$ such that A contains all its finite sums with distinct elements.

Example

The set of even numbers is IP-set. The set of odd numbers is not.

Question (typically difficult): Given $A \subseteq \mathbb{N}$, is A an IP-set?

reformulation in terms of the algebraic/topological properties of the Stone-Čech compactification of $\mathbb{N}$ (the set of all ultrafilters on $\mathbb{N}$):

Theorem (Theorem 5.12 in [1])

A subset $A \subseteq \mathbb{N}$ is an IP-set if and only if $A \in p$ for some idempotent ultrafilter p on $\mathbb{N}$.

[1] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, 2nd edition*, Walter de Gruyter & Co., Berlin, 2012.

Partition regularity

Definition

A collection of sets $\mathcal{S}$ is partition regular if for each $A \in \mathcal{S}$, whenever A is partitioned into finitely many sets, at least one set of the partition is in $\mathcal{S}$.

Partition regularity

Definition

A collection of sets $\mathcal{S}$ is partition regular if for each $A \in \mathcal{S}$, whenever A is partitioned into finitely many sets, at least one set of the partition is in $\mathcal{S}$.

Example

Partition regular collections of sets:

- sets having positive upper density
- sets having arbitrarily long arithmetic progressions

Definitions

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ natural numbers
$A \subseteq \mathbb{N}$ a subset of naturals

$FS(A) = \{\sum_{x \in F} x \mid F \subset A, |F| < \infty\}$ finite sums of elements of A

Definitions

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ natural numbers
$A \subseteq \mathbb{N}$ a subset of naturals

$FS(A) = \{\sum_{x \in F} x \,\big|\, F \subset A, |F| < \infty\}$ finite sums of elements of A

$FS_{\leq k}(A) = \{\sum_{x \in F} x \,\big|\, F \subset A, |F| \leq k\}$ finite sums of at most k elements of A

Definitions

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ natural numbers
$A \subseteq \mathbb{N}$ a subset of naturals

$FS(A) = \{\sum_{x \in F} x \,\big|\, F \subset A, |F| < \infty\}$ finite sums of elements of A

$FS_{\leq k}(A) = \{\sum_{x \in F} x \,\big|\, F \subset A, |F| \leq k\}$ finite sums of at most k elements of A

Definition

- A is finite FS-big if for every positive integer k there exists $x_0 < x_1 < \cdots < x_{k-1}$ such that $FS\langle x_n \rangle_{n=0}^{k-1} \subseteq A$.

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ natural numbers
$A \subseteq \mathbb{N}$ a subset of naturals

$FS(A) = \{\sum_{x \in F} x \,\big|\, F \subset A, |F| < \infty\}$ finite sums of elements of A

$FS_{\leq k}(A) = \{\sum_{x \in F} x \,\big|\, F \subset A, |F| \leq k\}$ finite sums of at most k elements of A

Definition

- A is finite FS-big if for every positive integer k there exists $x_0 < x_1 < \cdots < x_{k-1}$ such that $FS\langle x_n \rangle_{n=0}^{k-1} \subseteq A$.
- A is infinite FS-big if for every positive integer k there exists $x_0 < x_1 < \cdots < x_n < \ldots$ such that $FS_{\leq k}\langle x_n \rangle_{n=0}^{\infty} \subseteq A$.

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ natural numbers
$A \subseteq \mathbb{N}$ a subset of naturals

$FS(A) = \{\sum_{x \in F} x \,\big|\, F \subset A, |F| < \infty\}$ finite sums of elements of A

$FS_{\leq k}(A) = \{\sum_{x \in F} x \,\big|\, F \subset A, |F| \leq k\}$ finite sums of at most k elements of A

Definition

- A is finite FS-big if for every positive integer k there exists $x_0 < x_1 < \cdots < x_{k-1}$ such that $FS\langle x_n \rangle_{n=0}^{k-1} \subseteq A$.
- A is infinite FS-big if for every positive integer k there exists $x_0 < x_1 < \cdots < x_n < \ldots$ such that $FS_{\leq k}\langle x_n \rangle_{n=0}^{\infty} \subseteq A$.
- A is an IP-set if there exists $x_0 < x_1 < \cdots < x_n < \ldots$ A such that $FS\langle x_n \rangle_{n=0}^{\infty} \subseteq A$.

Definitions

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ natural numbers
$A \subseteq \mathbb{N}$ a subset of naturals

$FS(A) = \{\sum_{x \in F} x \,\middle|\, F \subset A, |F| < \infty\}$ finite sums of elements of A

$FS_{\leq k}(A) = \{\sum_{x \in F} x \,\middle|\, F \subset A, |F| \leq k\}$ finite sums of at most k elements of A

Definition

- A is finite FS-big if for every positive integer k there exists $x_0 < x_1 < \cdots < x_{k-1}$ such that $FS\langle x_n \rangle_{n=0}^{k-1} \subseteq A$.
- A is infinite FS-big if for every positive integer k there exists $x_0 < x_1 < \cdots < x_n < \dots$ such that $FS_{\leq k}\langle x_n \rangle_{n=0}^{\infty} \subseteq A$.
- A is an IP-set if there exists $x_0 < x_1 < \cdots < x_n < \dots$ A such that $FS\langle x_n \rangle_{n=0}^{\infty} \subseteq A$.

IP-sets $\subset$ infinite FS-big $\subset$ finite FS-big

Partition regularity of additive properties

Theorem (Hindman, 1974)

The collection of all IP-sets is partition regular.

Partition regularity of additive properties

Theorem (Hindman, 1974)

The collection of all IP-sets is partition regular.

Theorem (Bucci, Hindman, P., Zamboni, 2013)

The collection of all finite FS-big sets is partition regular.

Partition regularity of additive properties

Theorem (Hindman, 1974)

The collection of all IP-sets is partition regular.

Theorem (Bucci, Hindman, P., Zamboni, 2013)

The collection of all finite FS-big sets is partition regular.

Theorem (Bucci, Hindman, P., Zamboni, 2013)

The collection of all infinite FS-big sets is not partition regular.

Sets defined by words

alphabet: a finite non-empty set Σ
infinite word: $w = w_0 w_1 \dots w_i \dots$, where $w_i \in \Sigma$
the set of infinite words: $\Sigma^{\mathbb{N}}$
factor u of w: $u = w_i \dots w_{i+j}$ for some $i, j \in \mathbb{N}$

Subset of $\mathbb{N}$ defined by occurrence of factor u in w:

$$w\big|_u = \{n \in \mathbb{N} \mid u_n u_{n+1} \dots u_{n+|u|-1} = u\}$$

alphabet: a finite non-empty set Σ
infinite word: $w = w_0 w_1 \dots w_i \dots$, where $w_i \in \Sigma$
the set of infinite words: $\Sigma^{\mathbb{N}}$
factor u of w: $u = w_i \dots w_{i+j}$ for some $i, j \in \mathbb{N}$

Subset of $\mathbb{N}$ defined by occurrence of factor u in w:

$$w\big|_u = \{n \in \mathbb{N} \mid u_n u_{n+1} \dots u_{n+|u|-1} = u\}$$

w an aperiodic uniformly recurrent word.
Is $w\big|_u$ an IP-set?

Thue-Morse word

the Thue-Morse word

$$\mathbb{T} = 0110100110010110 1001 \ldots$$

a fixed point of the morphism $0 \mapsto 01$ and $1 \mapsto 10$.

Thue-Morse word

the Thue-Morse word

$$\mathbb{T} = 0110100110010110 1001\ldots$$

a fixed point of the morphism $0 \mapsto 01$ and $1 \mapsto 10$.

Example

$\mathbb{T}|_{01} = \{0, 3, 6, 10, 12, 15, \ldots\}$

Recurrent words

$\omega \in \Sigma^{\mathbb{N}}$ is recurrent if for each prefix u of ω the set $\omega\big|_u$ is infinite.
$\omega \in \Sigma^{\mathbb{N}}$ is uniformly recurrent if for each prefix u of ω the set $\omega\big|_u$ is syndetic, i.e., of bounded gap.

Proposition

Let $\omega = \omega_0\omega_1\omega_2 \ldots \in \Sigma^{\mathbb{N}}$ be recurrent and set $a = \omega_0$. Then $\omega\big|_a$ is an IP-set.

Recurrent words

$\omega \in \Sigma^{\mathbb{N}}$ is recurrent if for each prefix u of ω the set $\omega\big|_u$ is infinite.
$\omega \in \Sigma^{\mathbb{N}}$ is uniformly recurrent if for each prefix u of ω the set $\omega\big|_u$ is syndetic, i.e., of bounded gap.

Proposition

Let $\omega = \omega_0\omega_1\omega_2\ldots \in \Sigma^{\mathbb{N}}$ be recurrent and set $a = \omega_0$. Then $\omega\big|_a$ is an IP-set.

Proof:

$\omega = \overset{a}{\bullet}$

$x_0 = 0$

Recurrent words

$\omega \in \Sigma^{\mathbb{N}}$ is recurrent if for each prefix u of ω the set $\omega\big|_u$ is infinite.
$\omega \in \Sigma^{\mathbb{N}}$ is uniformly recurrent if for each prefix u of ω the set $\omega\big|_u$ is syndetic, i.e., of bounded gap.

Proposition

Let $\omega = \omega_0\omega_1\omega_2\ldots \in \Sigma^{\mathbb{N}}$ be recurrent and set $a = \omega_0$. Then $\omega\big|_a$ is an IP-set.

Proof:

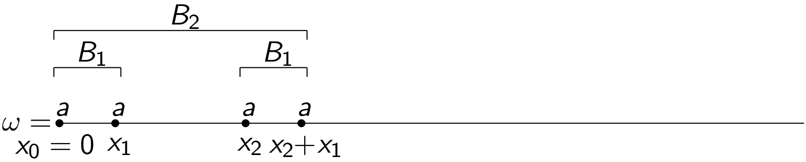
Recurrent words

$\omega \in \Sigma^{\mathbb{N}}$ is recurrent if for each prefix u of ω the set $\omega\big|_u$ is infinite.
$\omega \in \Sigma^{\mathbb{N}}$ is uniformly recurrent if for each prefix u of ω the set $\omega\big|_u$ is syndetic, i.e., of bounded gap.

Proposition

Let $\omega = \omega_0\omega_1\omega_2 \ldots \in \Sigma^{\mathbb{N}}$ be recurrent and set $a = \omega_0$. Then $\omega\big|_a$ is an IP-set.

Proof:



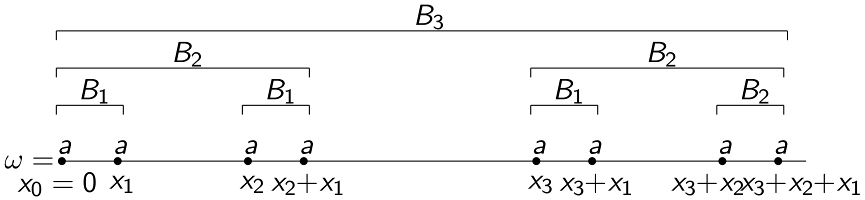
Recurrent words

$\omega \in \Sigma^{\mathbb{N}}$ is recurrent if for each prefix u of ω the set $\omega\big|_u$ is infinite.
$\omega \in \Sigma^{\mathbb{N}}$ is uniformly recurrent if for each prefix u of ω the set $\omega\big|_u$ is syndetic, i.e., of bounded gap.

Proposition

Let $\omega = \omega_0\omega_1\omega_2\ldots \in \Sigma^{\mathbb{N}}$ be recurrent and set $a = \omega_0$. Then $\omega\big|_a$ is an IP-set.

Proof:



Definition

$k \in \mathbb{N}$, $A \subseteq \mathbb{N}$
A is k-summable, if there exists $x_0 < x_1 < \ldots < x_{k-1}$ such that $FS\langle x_n \rangle_{n=0}^{k-1} \subseteq A$.

Definition

$k \in \mathbb{N}$, $A \subseteq \mathbb{N}$
A is k-summable, if there exists $x_0 < x_1 < \ldots < x_{k-1}$ such that $FS\langle x_n \rangle_{n=0}^{k-1} \subseteq A$.

finite FS-big $=$ k-summable for every k

Definition

$k \in \mathbb{N}$, $A \subseteq \mathbb{N}$
A is k-summable, if there exists $x_0 < x_1 < \ldots < x_{k-1}$ such that $FS\langle x_n \rangle_{n=0}^{k-1} \subseteq A$.

finite FS-big = k-summable for every k

Theorem

Let u be a factor of the Thue-Morse word $\mathbb{T}$. Then

1. *If u is a prefix of $\mathbb{T}$ then $\mathbb{T}\big|_u$ is an IP-set.*
2. *If u is a prefix of $\tilde{\mathbb{T}}$ then $\mathbb{T}\big|_u$ is infinite FS-big but is not an IP-set.*
3. *If u is neither a prefix of $\mathbb{T}$ nor a prefix of $\tilde{\mathbb{T}}$ then $\mathbb{T}\big|_u$ is not 3-summable. Moreover $\mathbb{T}\big|_u$ is 2-summable if and only if u is a prefix of $\tau^n(010)$ or of $\tau^n(101)$ for some $n \geq 0$.*

Example

$\mathbb{T}|_0$ is an IP-set

Example: $\mathbb{T}|_0$ is an IP-set

Example

$\mathbb{T}|_0$ is an IP-set

$[n]_2$ the binary expansion of n in base 2

$$\mathbb{T}|_0 = \{n | \text{ the number of 1's in } [n]_2 \text{ is even}\}$$

Example: $\mathbb{T}|_0$ is an IP-set

Example

$\mathbb{T}|_0$ is an IP-set

$[n]_2$ the binary expansion of n in base 2

$$\mathbb{T}|_0 = \{n | \text{ the number of 1's in } [n]_2 \text{ is even}\}$$

x_i is defined by:

$$[x_i]_2 = 11(00)^i$$

Example: $\mathbb{T}|_0$ is an IP-set

Example

$\mathbb{T}|_0$ is an IP-set

$[n]_2$ the binary expansion of n in base 2

$$\mathbb{T}|_0 = \{n | \text{ the number of 1's in } [n]_2 \text{ is even}\}$$

x_i is defined by:

$$[x_i]_2 = 11(00)^i$$

$$[x_{i_0} + \cdots + x_{i_{k-1}}]_2 = 11(00)^{i_{k-1}-i_{k-2}}11\ldots11(00)^{i_1-i_0}$$

Example: $\mathbb{T}|_0$ is an IP-set

Example

$\mathbb{T}|_0$ is an IP-set

$[n]_2$ the binary expansion of n in base 2

$$\mathbb{T}|_0 = \{n | \text{ the number of 1's in } [n]_2 \text{ is even}\}$$

x_i is defined by:

$$[x_i]_2 = 11(00)^i$$

$$[x_{i_0} + \cdots + x_{i_{k-1}}]_2 = 11(00)^{i_{k-1} - i_{k-2}} 11 \ldots 11(00)^{i_1 - i_0}$$

so $[x_{i_0} + \cdots + x_{i_{k-1}}]_2$ has even number of 1's and hence is in $\mathbb{T}|_0$

Infinite FS-big is not partition regular

Definition

$k \in \mathbb{N}$, $A \subseteq \mathbb{N}$

A is k^∞-summable, if there exists $x_0 < x_1 < \ldots$ such that $FS_{\leq k}\langle x_n \rangle_{n=0}^{\infty} \subseteq A$.

Infinite FS-big is not partition regular

Definition

$k \in \mathbb{N}$, $A \subseteq \mathbb{N}$
A is k^∞-summable, if there exists $x_0 < x_1 < \ldots$ such that $FS_{\leq k}\langle x_n \rangle_{n=0}^{\infty} \subseteq A$.

infinite FS-big = k^∞-summable for every k

Infinite FS-big is not partition regular

Definition

$k \in \mathbb{N}$, $A \subseteq \mathbb{N}$
A is k^∞-summable, if there exists $x_0 < x_1 < \ldots$ such that $FS_{\leq k}\langle x_n \rangle_{n=0}^{\infty} \subseteq A$.

infinite FS-big $=$ k^∞-summable for every k

Lemma

There exists a partition $\mathbb{T}\big|_1 = A_0 \cup A_1$ *such that* A_0, A_1 *are not* 2^∞*-summable.*

Since $\mathbb{T}\big|_1$ is infinite FS-big, we get:

Corollary

The collection of all infinite FS-big sets is not partition regular.

A_0 and A_1 are finite FS-big, but not infinite FS-big

An open problem

An open problem of Imre Leader

The property k-summable or k^∞-summable is not partition regular.

Consider the set

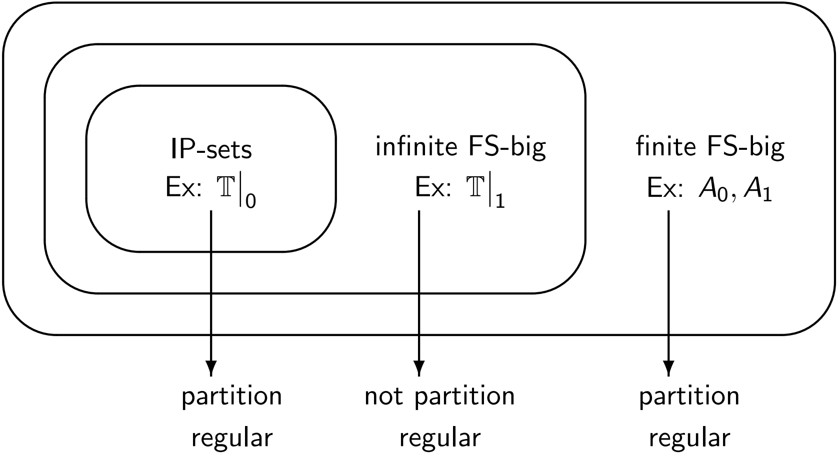
$$\mathcal{R}^\infty(k) = \{A \subseteq \mathbb{N} \,|\, \text{whenever } r \in \mathbb{N} \text{ and } A = \bigcup_{i=0}^{r} A_i,$$
$$\exists\, 0 \le i \le r \text{ such that } A_i \text{ is } k^\infty\text{-summable}\}$$

Then $\mathcal{R}^\infty(k) \neq \emptyset$ (e.g., contains all IP-sets).

Question: does there exist a member of $\mathcal{R}^\infty(2)$ which is not an IP-set?

V. Bergelson and B. Rothschild, *A selection of open problems*, Topology Appl. **156** (2009), 2674–2681.

Summary


IP-sets
Ex: $\mathbb{T}\big|_0$
infinite FS-big
Ex: $\mathbb{T}\big|_1$
finite FS-big
Ex: A_0, A_1
partition
regular
not partition
regular
partition
regular

Definition

A set $\mathcal{U}$ of subsets of $\mathbb{N}$ is called an ultrafilter if

- $\emptyset \notin \mathcal{U}$.
- If $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$.
- $A \cap B \in \mathcal{U}$ whenever both A and B belong to $\mathcal{U}$.
- For every $A \subseteq \mathbb{N}$ either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$ where A^c denotes the complement of A.

Example

$\forall n \in \mathbb{N}$, the set $\mathcal{U}_n = \{A \subseteq \mathbb{N} \mid n \in A\}$ is a principal ultrafilter.

This defines an injection $i : \mathbb{N} \hookrightarrow \beta\mathbb{N}$ by: $n \mapsto \mathcal{U}_n$.

By way of Zorn's lemma, one can show the existence of non-principal (or *free*) ultrafilters.

Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$: the set of all ultrafilters on $\mathbb{N}$ with the *Stone topology.*

- $A \subseteq \mathbb{N}$, we set $A^\circ = \{p \in \beta\mathbb{N} | A \in p\}$.
- $\mathcal{B} = \{A^\circ | A \subseteq \mathbb{N}\}$: a basis for the open sets of $\beta\mathbb{N}$, defines a topology on $\beta\mathbb{N}$
- $\beta\mathbb{N}$ is both compact and Hausdorff.

addition of ultrafilters p, q

$$p + q = \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} | A - n \in p\} \in q\}.$$

$p + q$ is an ultrafilter

for each fixed $p \in \beta\mathbb{N}$, the mapping $q \mapsto p + q$ defines a continuous map from $\beta\mathbb{N}$ into itself

for principal ultrafilters: $\mathcal{U}_m + \mathcal{U}_n = \mathcal{U}_{m+n}$

in general addition of ultrafilters is associative and non-commutative

$\beta\mathbb{N}$ is a compact *left-topological semigroup*
(i.e., $\forall x \in \beta\mathbb{N}$ the mapping $y \mapsto x + y$ is continuous)

Idempotent ultrafilters

$(\mathcal{S}, +)$ a semigroup
An element $p \in \mathcal{S}$ is called an idempotent if $p + p = p$

Theorem (Ellis, 1958)

Let $(\mathcal{S}, +)$ be a compact left-topological semigroup. Then $\mathcal{S}$ contains an idempotent.

$\implies \beta\mathbb{N}$ contains a non-principal ultrafilter p

IP*-sets

Definition

A subset $A \subseteq \mathbb{N}$ is called an *IP*-set* if $A \cap B \neq \emptyset$ for every IP-set $B \subseteq \mathbb{N}$.

Every IP*-set is IP-set (follows from Hindman's theorem).

link between IP-sets and idempotents in $\beta\mathbb{N}$:

Theorem (Theorem 5.12 in [1])

A subset $A \subseteq \mathbb{N}$ is an IP-set if and only if $A \in p$ for some idempotent $p \in \beta\mathbb{N}$.

Corollary

A is an IP*-set if and only if $A \in p$ for every idempotent $p \in \beta\mathbb{N}$

[1] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, 2nd edition, Walter de Gruyter & Co., Berlin, 2012.

Σ a non-empty finite set

$p \in \beta\mathbb{N}$ an ultrafilter

Definition

Define a mapping $p^* : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ as follows:
For each $\omega \in \Sigma^{\mathbb{N}}$, $u \in \Sigma^*$ is a prefix of $p^*(\omega) \iff \omega\big|_u \in p$.

Σ a non-empty finite set
$p \in \beta\mathbb{N}$ an ultrafilter

Definition

Define a mapping $p^* : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ as follows:
For each $\omega \in \Sigma^{\mathbb{N}}$, $u \in \Sigma^*$ is a prefix of $p^*(\omega) \iff \omega\big|_u \in p$.

Lemma

The set $\omega\big|_u$ is an IP-set if and only if u is a prefix of $p^*(\omega)$ for some idempotent $p \in \beta\mathbb{N}$.

Remark: our definition of p^* coincides with the definition of p-lim

mapping $p*$

Theorem (Theorem 19.26 in [1])

Given two infinite words $x, y \in \Sigma^{\mathbb{N}}$. If x and y are proximal with y uniformly recurrent, then there exists an idempotent $p \in \beta\mathbb{N}$ such that $p^*(x) = y$.

words and subshifts

- topology in $\Sigma^{\mathbb{N}}$ generated by the metric

$$d(x,y) = \frac{1}{2^n} \text{ where } n = \inf\{k : x_k \neq y_k\}$$

  whenever $x, y \in \Sigma^{\mathbb{N}}$
- $T : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ the shift transformation defined by $T : (x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$.
- $\mathcal{F}_\omega$ the set of factors of ω
  ω is uniformly recurrent if for every $u \in \mathcal{F}_\omega$ the set $\omega\big|_u$ is syndedic, i.e., of bounded gap.
- subshift on Σ: a pair (X, T) where X is a closed and T-invariant subset of $\Sigma^{\mathbb{N}}$.
- (X, T) is minimal whenever X and the empty set are the only T-invariant closed subsets of X.

- each $\omega \in \Sigma^{\mathbb{N}}$ is associated to the subshift (X, T) where X is the shift orbit closure of ω.
- If ω is uniformly recurrent, then the associated subshift (X, T) is minimal.

  Thus $\forall x, y \in X$ have the same set of factors, i.e., $\mathcal{F}_x = \mathcal{F}_y$.

  Denote by $\mathcal{F}_X$ the set of factors of any word $x \in X$.
- Two points x, y in X are proximal if and only if for each $N > 0$ there exists $n \in \mathbb{N}$ such that

$$x_n x_{n+1} \cdots x_{n+N} = y_n y_{n+1} \cdots y_{n+N}.$$

Sturmian words

- factor complexity of $w \in \Sigma^{\mathbb{N}}$: $p_w(n)$ = the number of distinct factors of w of length n
- Sturmian words are infinite words having the factor complexity $p(n) = n + 1$ for all $n \geq 0$.
- $p(1) = 2 \Rightarrow$ Sturmian words are binary
- equivalent definitions:
  - via an irrational rotation on the circle of circumference one
  - mechanical words and cutting sequences
  - Sturmian morphisms
  - balance property
  - palindromic closure
  - standard factors
- Example of Sturmian word: the Fibonacci word

$$0100101001001010010100100101001 0\ldots$$

fixed by the morphism $0 \mapsto 01$ and $1 \mapsto 0$.

special factor

A factor v of a word w is called right (resp., left) special if va and vb (resp., av and bv) are factors of w for $a \neq b \in \Sigma$.

Example

Fibonacci word 0100101001001010010100100101001 $\ldots$,
10 is right special (both 100 and 101 are factors), 01 is not (011 is not a factor).

characteristic word

$\omega \in \{0,1\}^{\mathbb{N}}$ a Sturmian word,
Ω the shift orbit closure of ω.
Ω contains a unique word all of whose prefixes are left special factors of ω. Such a word is called the characteristic word $\tilde{\omega}$.
Hence both $0\tilde{\omega}, 1\tilde{\omega} \in \Omega$.

In the definition via mechanical words characteristic words correspond to $\rho = \alpha$.

singular word

A Sturmian word ω is called singular if $T^n(\omega) = \tilde{\omega}$ for some $n \geq 1$. Otherwise it is said to be nonsingular.

IP-sets and Sturmian words

Theorem (Bucci, P., Zamboni, 2013)

Let $\omega \in \Omega$ be a nonsingular Sturmian word, and u a factor of ω. Then $\omega\big|_u$ is an IP-set if and only if u is a prefix of ω.

Theorem (Bucci, P., Zamboni, 2013)

Let $\omega \in \Omega$ be a singular Sturmian word such that $T^{n_0}(\omega) = \tilde{\omega}$ with $n_0 \geq 1$. Then $\omega\big|_u$ is an IP-set if and only if either u is a prefix of ω or a prefix of ω' where ω' is the unique other element of Ω with $T^{n_0}(\omega') = \tilde{\omega}$.

Corollary

For every prefix v of a nonsingular Sturmian word ω and $n \in \omega\big|_v$, the set $\omega\big|_v - n$ is an IP*-set.

Remark: In general the property of being an IP*-set is not translation invariant.

Proof is based on

Lemma

Let $\omega \in \{0,1\}^{\mathbb{N}}$ *be a nonsingular Sturmian word and* $p \in \beta\mathbb{N}$ *an idempotent ultrafilter. Then* $p^*(\omega) = \omega$.

Lemma

If $\omega, \omega' \in \Omega$ *are such that* $T^{n_0}(\omega) = T^{n_0}(\omega') = \tilde{\omega}$, *and let* $p \in \beta\mathbb{N}$ *be an idempotent ultrafilter. Then* $p^*(\omega) = p^*(\omega') \in \{\omega, \omega'\}$.

Other partitions

partitions defined by words generated by substitution rules

fixed points of generalized Thue-Morse substitution to an alphabet of size $r \geq 2$ give

Theorem (Bucci, P., Zamboni, 2011)

For each pair of positive integers r and N there exists a partition of

$$\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_r$$

such that

- $A_i - n$ *is an IP-set for each* $1 \leq i \leq r$ *and* $1 \leq n \leq N$.
- *For each* $n > N$, *exactly one of the sets* $\{A_1 - n, A_2 - n, \ldots, A_r - n\}$ *is an IP-set.*

we prove and use the fact that each fixed point of the generalized Thue-Morse substitution is distal (i.e., proximal only to itself)

Theorem (Bucci, P., Zamboni, 2011)

For each positive integer r there exists a partition of $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_r$ such that for each $1 \leq i \leq r$ and $n \geq 0$, the set $A_i - n$ is an IP-set.

words generating minimal topologically weak mixing subshifts
e.g., subshift generated by the substitution
$0 \mapsto 001$
$1 \mapsto 11001$

A minimal subshift (X, T) is *topologically weak mixing* if for every pair of factors $u, v \in \mathcal{F}_X$ the set

$$\{n \in \mathbb{N} \,|\, u\Sigma^n v \cap \mathcal{F}_X \neq \emptyset\}$$

is thick, i.e., for every positive integer N, the set contains N consecutive positive integers.

Infinite partition

Definition

We denote with $w^{(+)}$ the right palindromic closure of the word w, i.e., the shortest palindrome which has w as a prefix.

For example, $abaa^{(+)} = abaaba$.

Definition

The iterated palindromic operator ψ is defined inductively:

- $\psi(\varepsilon) = \varepsilon$,
- For any word w and any letter a, $\psi(wa) = (\psi(w)a)^{(+)}$.

For example, $\psi(aaba) = aabaaabaa$.

Infinite partition into IP-sets

Proposition [Bucci, P., Zamboni, 2011]

- Δ a right infinite word on an infinite alphabet Σ
- each letter $a \in \Sigma$ occurs in Δ an infinite number of times
- $\omega = \psi(\Delta)$

Then,

- for any $a \in \Sigma$, the set $a\omega\big|_a$ is an IP-set,
- $\{\omega\big|_a + 1\}_{a \in \Sigma}$ is an infinite partition of $\mathbb{N} - \{0\}$ into IP-sets

Central sets

$A \subset \mathbb{N}$

$\chi(A) \in \{0,1\}^{\mathbb{N}}$: $\chi_n = \begin{cases} 1, & \text{if } n \in A, \\ 0, & \text{otherwise.} \end{cases}$

Equivalently: $A = \chi(A)\big|_1$.

Definition

$A \subset \mathbb{N}$ is a central set if $\chi(A)$ is proximal to a uniformly recurrent word beginning with 1.

- Every central set is an IP-set.
- Some IP-sets are not central.
  - Example: the set $\mathbf{s}\big|_0$, where $\mathbf{s}$ is a Sierpinski word $\mathbf{s} = 010111010111111111010111010\cdots$ (a fixed point of $0 \to 010$, $1 \to 111$). $\mathbf{s}\big|_1$ is central
- Originally defined by Furstenberg in terms of topological dynamics.
- most of our results on IP-sets apply also to central sets

Central sets: equivalent definition

$(\mathcal{S}, +)$ a semigroup

- $\mathcal{I} \subseteq \mathcal{S}$ is a *right (resp. left) ideal* if $\mathcal{I} + \mathcal{S} \subseteq \mathcal{I}$ (resp. $\mathcal{S} + \mathcal{I} \subseteq \mathcal{I}$).
- It is a *two sided ideal* if it is both a left and right ideal.
- A right (resp. left) ideal $\mathcal{I}$ is *minimal* if every right (resp. left) ideal $\mathcal{J}$ included in $\mathcal{I}$ coincides with $\mathcal{I}$.
- every compact Hausdorff left-topological semigroup $\mathcal{S}$ (e.g., $\beta\mathbb{N}$) admits the smallest two sided ideal $K(\mathcal{S})$
- idempotents in $K(\mathcal{S})$ are called minimal

Definition

A subset $A \subset \mathbb{N}$ is called central if it is a member of some minimal idempotent in $\beta\mathbb{N}$.

Equivalence: Bergelson and Hindman, 1990

Pisot conjecture and central sets

Theorem (M. Barge, L. Zamboni, 2013)

Let τ be an irreducible primitive substitution of Pisot type. Then for any pair of fixed points x and y of τ the following are equivalent:

1. *x and y are strongly coincident.*
2. *x and y are proximal.*
3. *There exists a minimal idempotent $p \in \beta\mathbb{N}$ such that $y = p^*(x)$.*
4. *For any prefix u of y, the set $x\big|_u$ is a central set.*

Pisot substitutions and strong coincidence condition

- $\Sigma = \{1, 2, \ldots, n\}$, $n \geq 2$; $\tau : \Sigma \to \Sigma^+$ a substitution
- the *Abelianization* of τ is the square matrix M_τ: $m_{ij} = |\tau(j)|_i$
- τ is *primitive* if all the entries of M_τ^n are strictly positive.
- In this case M_τ has a simple positive Perron-Frobenius eigenvalue called the *dilation* of τ.
- τ is *irreducible* if the minimal polynomial of its dilation is equal to the characteristic polynomial of M_τ.
- τ is of *Pisot type* if its dilation is a Pisot number.
- a Pisot number is an algebraic integer greater than 1 all of whose algebraic conjugates lie strictly inside the unit circle
- A primitive substitution τ satisfies the *strong coincidence condition* if and only if any pair of fixed points x and y are *strongly coincident*, i.e., we can write $x = scx'$, and $y = tcy'$ for some $s, t \in \Sigma^+$, $c \in \Sigma$, and $x', y' \in \Sigma^\infty$ with $s \sim_{\text{ab}} t$.
- Conjecture: if τ is an *irreducible* primitive substitution of Pisot type, then τ satisfies the strong coincidence condition.

References

- M. Bucci, N. Hindman, S. P., L. Q. Zamboni *Additive properties of sets defined by the Thue-Morse word*. Journal of Combinatorial Theory, Series A, 120 (2013), 1235–1245.
- M. Bucci, S. P. and L.Q. Zamboni, *Central sets generated by uniformly recurrent words.* to appear in Ergodic Theory and Dynamical Systems, doi:10.1017/etds.2013.69.
- S. P. and L.Q. Zamboni, *Additive properties of sets and substitutive dynamics.* To appear in a forthcoming book "Recent Mathematical Developments in Aperiodic Order" edited by J. Kellendonk, D. Lenz and J. Savinien.
- M. Barge and L.Q. Zamboni, *Central sets and substitutive dynamical systems.* Adv. Math. 248 (2013), 308-323.
- N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: Theory and applications,* 2nd edition, Walter de Gruyter & Co., Berlin, 2012.