Invariant measures of rotational beta expansions

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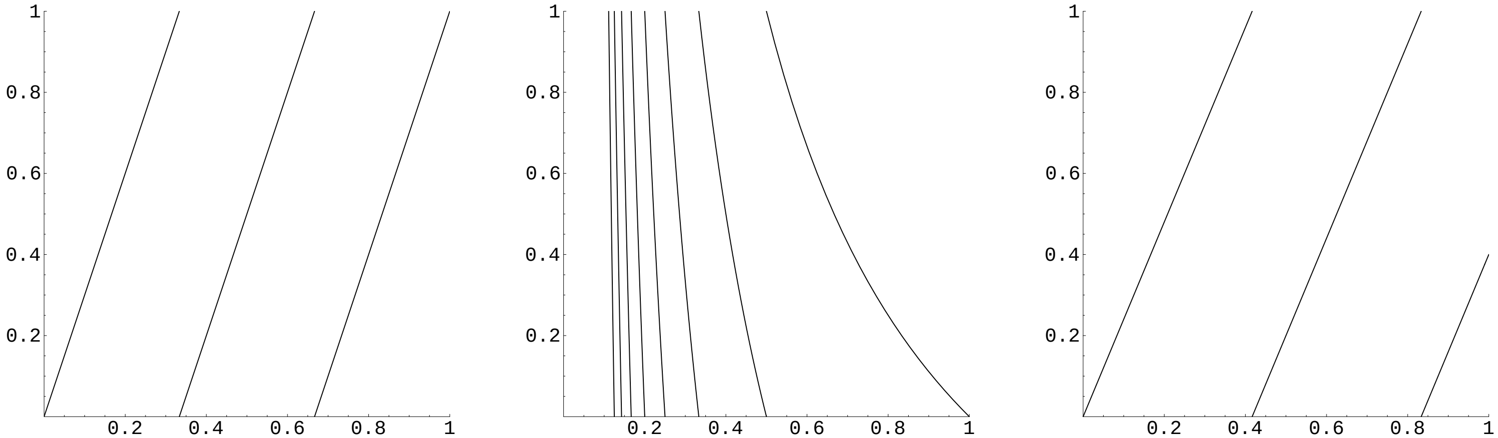
Let $y = f(x)$ be a positive real function. Consider a digital expansion of a real number x in a form:

$$x = \varepsilon_0 + f(\varepsilon_1 + f(\varepsilon_2 + f(\varepsilon_3 + \dots .$$

given by an algorithm with $\varepsilon_0 = \lfloor x \rfloor, r_0 = x - \lfloor x \rfloor$ and

$$\varepsilon_{n+1} = \lfloor f^{-1}(r_n) \rfloor, r_{n+1} = f^{-1}(r_n) - \lfloor f^{-1}(r_n) \rfloor .$$

This is **Rényi's f-expansion** ([16]). It is the usual b-adic expansion when $f(x) = x/b$ and the regular continued fraction when $f(x) = 1/x$. For $f(x) = x/\beta$ with a non-integer $\beta > 1$, it is called the **β-expansion**..



(a) 3-adic expansion (b) Continued fraction (c) Beta expansion

Figure 1: The graph of $f^{-1} \bmod 1$

Rényi studied ergodic properties of f-expansion for monotone function f under some axioms. **At that time, number theory and ergodic theory were very close.**

So we are given a piecewise smooth expanding function T on $[0,1]$. We wish to understood f-expansion in a framework of measure theoretical dynamical system $([0,1], \mathcal{B}, \mu, T)$, but we do not know the exact shape of μ. A measure μ on $[0,1]$ is called an **invariant measure**, if

$$\mu(T^{-1}(A)) = \mu(A)$$

for any A in the Borel σ-algebra of $[0,1]$. Take a random point on x and take a Dirac measure δ_x. Then

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i(x)} \quad n = 1, 2, 3, \ldots$$

has an accumulation point, which gives an invariant measure. Indeed, there are infinitely many choices of invariant measures. For e.g., choose a periodic orbit and distribute equal mass to the orbits.

μ is **absolutely continuous** to the 1-dim Lebesgue measure m if $m(A) = 0$ implies $\mu(A) = 0$, which is denoted by $\mu \ll m$. If $\mu \ll m$ and $m \ll \mu$, then we say that μ and m are equivalent. An invariant Borel measure absolutely continuous to the Lebesgue measure is called physical measure. We call it an absolutely continuous invariant measure, ACIM. ACIM are the only invariant measures which have a certain statistical meaning.

Perron-Frobenius operator $P : L^1(m) \to L^1(m)$ is defined as

$$P(h) = \sum_{y \in T^{-1}(x)} \frac{h(y)}{|T'(y)|}.$$

If there is a fixed point $P(h) = h$ with $h \geq 0$, then hdm is the ACIM we wanted. The intuitive meaning of this fixed point equation is that the local mass is preserved under inverse image of T. Given a piecewise smooth function on $[0, 1]$ whose derivative is away from 1 in modulus, Lasota-Yorke [1] succeeded in proving the existence of ACIM and finiteness of the ergodic ACIM's.

Hereafter we restrict ourselves to **beta expansion**:

$$T(x) = \beta x - \lfloor \beta x \rfloor.$$

It is a generalization of binary and decimal expansion.

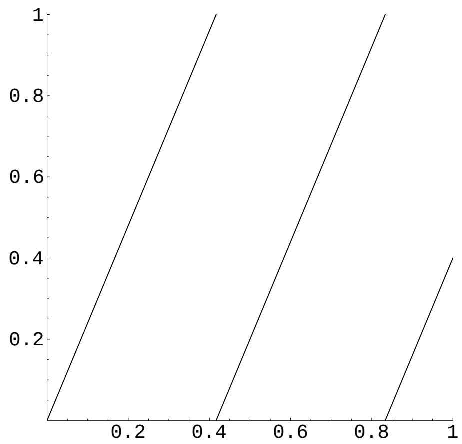


Figure 2: Beta expansion

Ergodic properties of beta expansion

Rényi [16] showed that this system is ergodic with respect to an ACIM. Parry [15] gave an explicit solution of the Perron Frobenius equation:

$$h(x) = \sum_{x<T^n(1)} \frac{1}{\beta^n}$$

giving an ACIM equivalent to the Lebesgue measure. Therefore the ACIM is unique and the system is ergodic w.r.t. this ACIM. Here $T^n(1)$ is the trajectory of the rightmost discontinuity.

This system has nice mixing properties. Moreover it is exact.

Smorodinski [18], Fischer [5], Ito-Takahashi [9].

The orbit $T^n(1)$ produced so called **expansion of one** which is a infinite sequence

$$d_\beta(1) = c_1 c_2 c_3 \dots$$

of letters in $\{0, 1, \dots, \lfloor \beta \rfloor\}$ satisfying:

$$1 = \frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \dots$$

where $T^n(1) = \sum_{i=1}^{\infty} c_{n+i-1} \beta^i$. Indeed, proving the fixed point equation, Parry used the property of $d_\beta(1)$.

Symbolic property of the beta expansion

To have a number theoretical result from beta expansion, we need a symbolic dynamical result related to the strings they produce.

If the orbit of discontinuity $(T^n(1))_{n=1,2,...}$ is finite, the system is sofic, i.e., the associated fundamental region satisfies GIFS. If β is a Pisot number, then the system is sofic, which is equivalent to say that $d_\beta(1)$ is eventually periodic. If $d_\beta(1)$ is purely periodic, then the associated symbolic system is SFT. A lot of open questions remain, see Blanchard [3].

Number theoretical property by its dynamics: return time, shrinking targets problems, orbits of 1 (J. Wu, B. Li, and many in Wuhan).

Under Pisot condition, a good natural extension characterizes periodic orbits: Ito-Rao [7], Berthé-Siegel [2].

Ito-Sadahiro [8] introduced the **negative beta expansion**

$$T : x \mapsto -\beta x - \lfloor -\beta x + \beta/(1+\beta) \rfloor$$

acting on $[-\beta/(1+\beta), 1/(1+\beta))$.

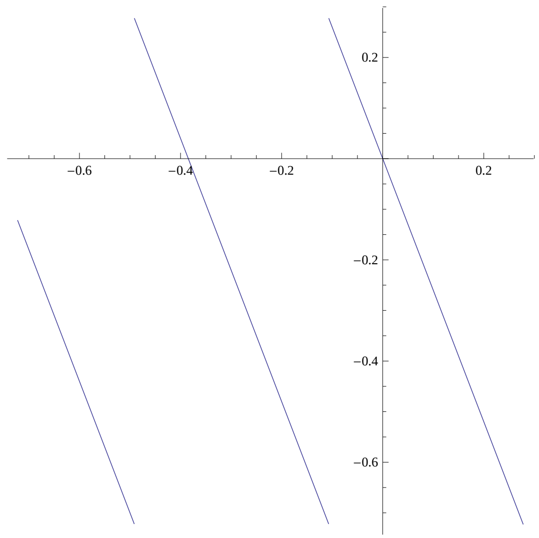


Figure 3: Negative Beta expansion for $\beta = 2.6$

The ACIM of T is unique. This follows from the result of Li-Yorke [13], which shows that the number of ergodic components does not exceeds the number of discontinuities. Its density is given by:

$$\sum_{x > T^n(-\beta/(1+\beta))} \frac{1}{(-\beta)^n}.$$

This expression is probably not intuitive. Liao-Steiner [14] proved that its ACIM is equivalent to the Lebesgue measure if and only if $\beta \geq (1+\sqrt{5})/2$. Symbolic dynamical study is parallel to the original beta expansion. Kalle [10] studied the isomorphisms between positive and negative beta expansions.

Rotational beta expansion

Let $1 < \beta \in \mathbb{R}, \zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| = 1$, $\xi, \eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1/\eta_2 \notin \mathbb{R}$. Then $\mathcal{X} = \{\xi + x\eta_1 + y\eta_2 \mid x \in [0,1), y \in [0,1)\}$ is a fundamental domain of the lattice $\mathcal{L} = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$ in $\mathbb{C}$ so

$$\mathbb{C} = \bigcup_{d \in \mathcal{L}} (\mathcal{X} + d)$$

is a disjoint partition of $\mathbb{C}$. Define a map $T : \mathcal{X} \to \mathcal{X}$ by $T(z) = \beta\zeta z - d$ where $d = d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta\zeta z \in \mathcal{X} + d$.

Given a point z in $\mathcal{X}$, we obtain an expansion

$$\begin{aligned} z &= \frac{d_1}{\beta\zeta} + \frac{T(z)}{\beta\zeta} \\ &= \frac{d_1}{\beta\zeta} + \frac{d_2}{(\beta\zeta)^2} + \frac{T^2(z)}{(\beta\zeta)^2} \\ &= \sum_{i=1}^{\infty} \frac{d_i}{(\beta\zeta)^i} \end{aligned}$$

with $d_i = d(T^{i-1}(z))$. In this case, we write $d_T(z) = d_1 d_2 \ldots$. We call T the **rotational beta transformation** and $d_T(z)$ the **expansion** of z with respect to T.

ACIM's are not unique !

Example 1. $\zeta = \sqrt{-1}, \beta = 1.039, \eta_1 = 2.92, \eta_2 = \exp(\pi\sqrt{-1}/3)$ and $\xi = 0$.

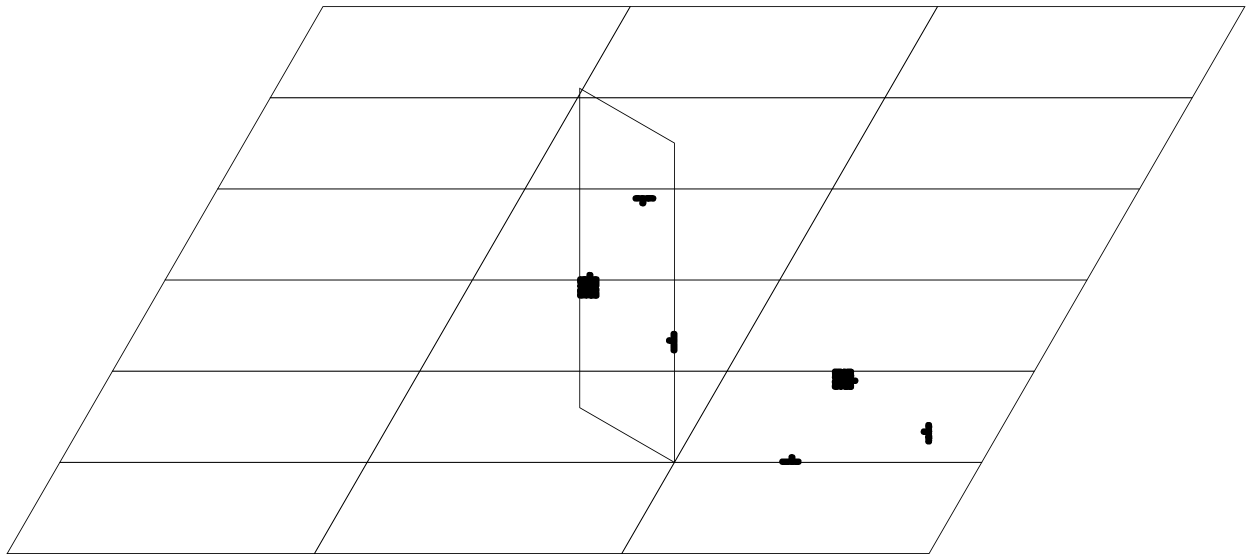
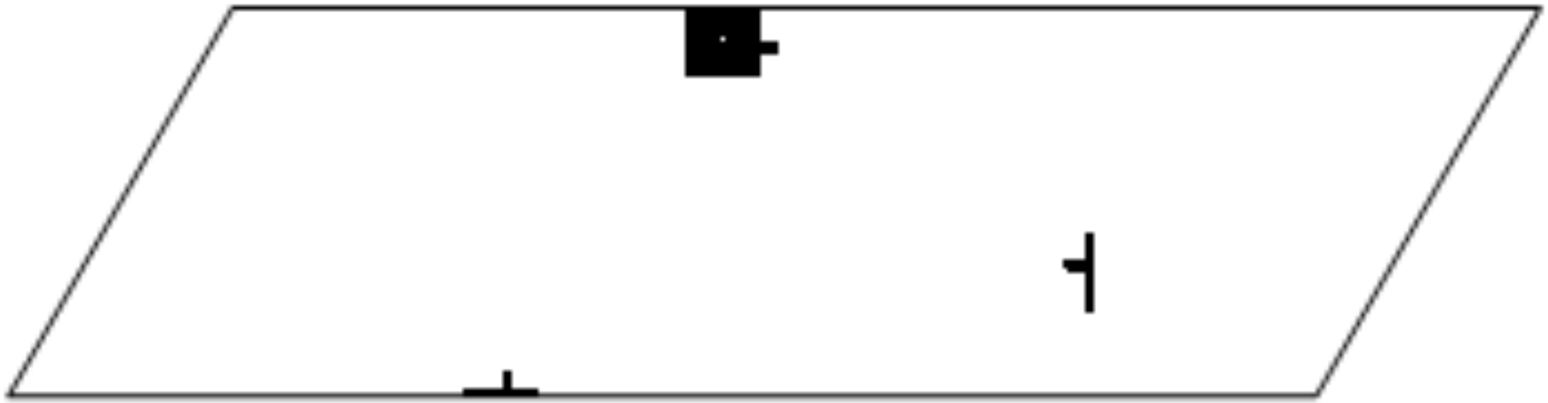
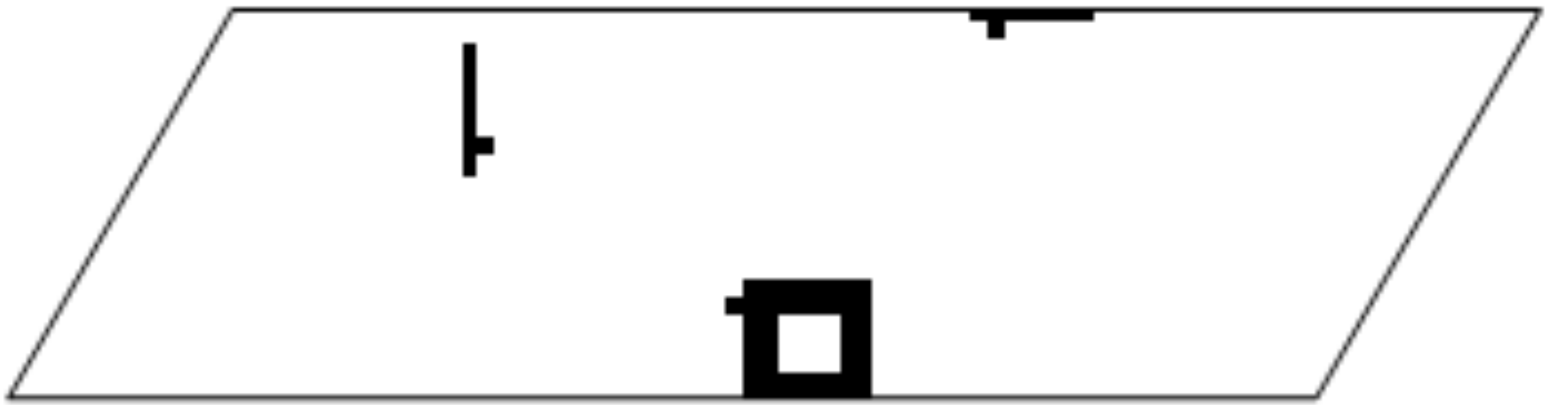


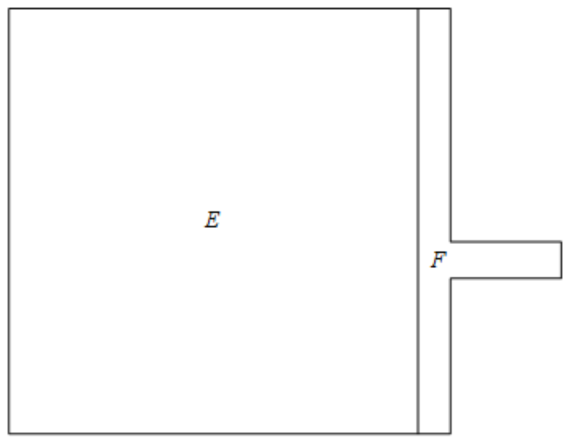
Figure 4: Non ergodic case



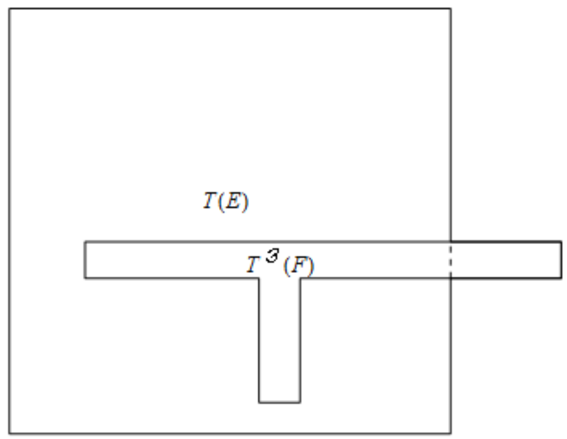
(a) First Component



(b) Second Component



(a) E and F



(b) Confirmation of the set equation

The same situation happens when β and η_1 satisfy

$$\frac{\sqrt{3}}{2}\beta + 1 + \frac{\sqrt{3}}{\beta} - \frac{\sqrt{3}}{2\beta^3} \leq \eta_1 \leq \frac{1}{2} + \frac{\sqrt{3}}{\beta} + \frac{\sqrt{3}}{2\beta^3}$$

while other parameters are fixed.

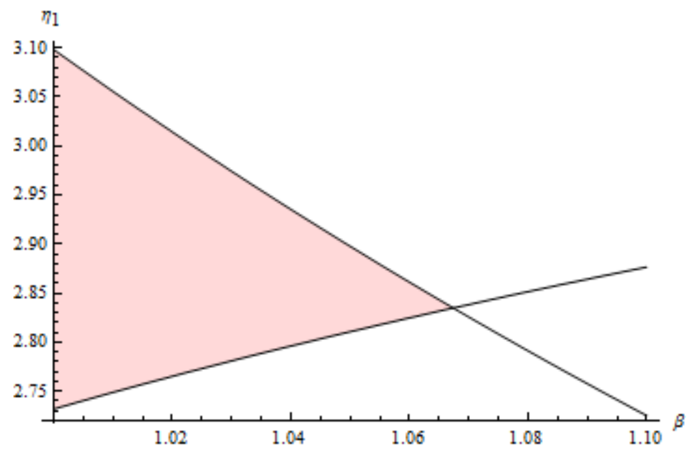


Figure 6: Non ergodic parameters

Denote by m the 2-dim Lebesgue measure. In this case, we have to study the Perron Frobenius operator equation:

$$P(h) = \sum_{y \in T^{-1}(x)} \frac{h(y)}{\mathrm{Jac}(T, y)}$$

acting on $L^1(\mathbb{R}^2, \mathbb{R})$. Then T is a very special case of piecewise expanding maps, studied by Keller, Gora-Boyarsky, Tsujii, Buzzi [11, 12, 6, 17, 19, 4, 20]. The main difficulty arises from the set of discontinuities. It becomes much more complicated than those in 1-dim.

We have to find a definition of total variation in higher dimension. The best one is found by Keller and used by Saussol [17]. Take a ball B and let

$$\operatorname{osc}(f, B) = \operatorname{esssup}_{x\in B} f(x) - \operatorname{essinf}_{x\in B} f(x),$$

the **oscillation** around B. Fix an $\varepsilon_0 > 0$ and put

$$\operatorname{Var}(f) = \sup_{0<\varepsilon\leq\varepsilon_0} \frac{1}{\varepsilon}\int \operatorname{osc}(f, B(x,\varepsilon))dx.$$

Then $\operatorname{Var}(f)$ is an analogy of the total variation and and the subspace $V = \{f \in L^1 \mid \operatorname{Var}(f) + \|f\| < \infty\}$ becomes relatively compact in $\mathcal{L}^1$.

Under some natural assumption on the piecewise expanding map, we can prove a Lasota-Yorke type inequality:

$$\mathrm{Var}(P^n(f)) < \eta \mathrm{Var}(f) + D\|f\|$$

with some $n \in \mathbb{N}$ and $0 < \eta < 1$. Iterating this inequality, from an infinite sequence

$$\frac{1}{m}\sum_{i=1}^{m} P^i(f), \quad m = 1, 2, \ldots$$

we can select a converging subsequence. This lead us to the unique limit, which satisfies $P(h) = h$.

We know that there exists an ACIM μ whose support contains a ball of positive density. This implies the number of ergodic components is finite and bounded by

$$\frac{1}{\pi}\left(\frac{D}{1-\eta}\right)^2.$$

However, the bound is not practically good since η is usually close to 1. We wish to find a more precise bound. Let

$\theta(\mathcal{X}) \in (0, \pi)$ be the angle between η_1 and η_2.

$$B_1 = \begin{cases} 2 & \text{if } \frac{1}{2} < \tan\left(\frac{\theta(\mathcal{X})}{2}\right) < 2 \\ 1 + \frac{2}{1+\sin\left(\frac{\theta(\mathcal{X})}{2}\right)} & \text{if } \sin(\theta(\mathcal{X})) < \sqrt{5} - 2 \\ \frac{3}{2} + \frac{1}{16}\cot^2\left(\frac{\theta(\mathcal{X})}{2}\right) + \tan^2\left(\frac{\theta(\mathcal{X})}{2}\right) & \text{otherwise} \end{cases}$$

and

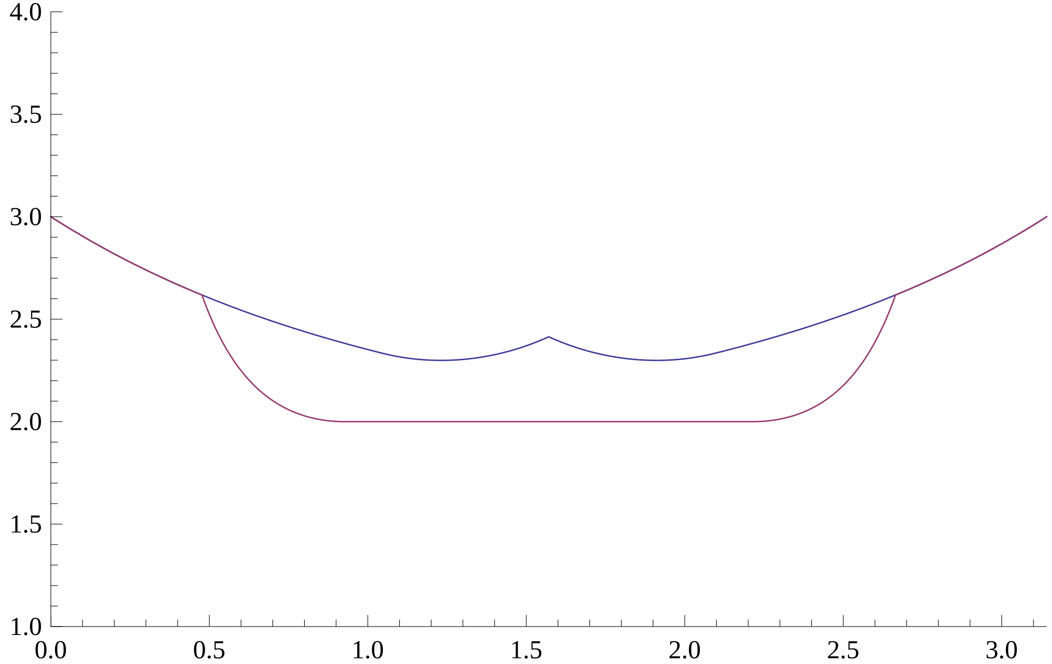
$$B_2 := \begin{cases} \frac{|\cos(\theta(\mathcal{X}))|+1}{2(|\cos(\theta(\mathcal{X}))|+\sin(\theta(\mathcal{X}))-1)} & \text{if } \frac{\pi}{3} < \theta(\mathcal{X}) < \frac{2\pi}{3} \\ 1 + \frac{2}{1+\sin\left(\frac{\theta(\mathcal{X})}{2}\right)} & \text{otherwise.} \end{cases}$$

Theorem 2. If $\beta > B_1$ then $(\mathcal{X}, T)$ has a unique absolutely continuous invariant probability measure μ. Moreover, if $\beta > B_2$ then μ is equivalent to the 2-dimensional Lebesgue measure restricted to $\mathcal{X}$.

This is an improvement of the result in the ArXiv:1502.01793, in particular if θ is small.

One can confirm the inequality $B_1 \leq B_2$ in Figure 7.

Figure 7: Comparison of B_1 and B_2



Idea of the proof.

Starting from a r-covering of $\mathcal{X}$, inductively we create a finer one by looking the inverse images of T. Then we can show that if β is large, then for any $\epsilon > 0$ and any point $z \in \mathcal{X}$ that $\bigcup_{n=1}^{m} T^{-n}(z)$ is an ϵ-covering. Assuming two ergodic ACIM, this fact means two ergodic components have non negligible communication, which gives a contradiction.

Summary of our results

	unique	Lebesgue	density	sofic
Beta	Yes	Yes	Yes	β: Pisot
Negative	Yes	$\beta \geq \frac{1+\sqrt{5}}{2}$	Yes	β: Pisot
Rotation	$\beta > B_1$	$\beta > B_2$	?	Pisot & $\cos(2\pi/q) \in \mathbb{Q}(\beta)$

Open questions

- Improve the constants B_1 and B_2. They seem not optimal.
- Make explicit the density of ACIM. Possible in sofic cases.

d-dimensional beta expansion

Our discussion readily generalizes to d-dim beta expansions with $d \geq 3$. Let $\beta > 1$ and $M \in O(d)$. Take a fundamental domain $\mathcal{X}$ of a lattice $\mathcal{L}$ in $\mathbb{R}^d$. Then the d-dimensional rotational beta expansion is defined by

$$T(z) = \beta M x - d(x)$$

where $d(x)$ is a unique element of $\mathcal{L}$ that the right side is in $\mathcal{X}$. We can show that

Theorem 3. If $\beta > \sqrt{\frac{2d^3}{\pi}}$ then there is an ACIM of T equivalent to the d-dim Lebesgue measure.

A stupid question.

Let B a d-dimensional unit ball, $w_1, w_2, \ldots, w_\ell > 0$ and $w_1 + \cdots + w_\ell < 2$. Can we cover B by stripes of width w_i for $i = 1, \ldots, \ell$?

Negative answer to this Q would improve the last bound to $d + 1$. It is valid for $d \leq 3$.

Note:

On the last day of the meeting we got to know from Wöden Kusner at Graz that this problem is due to Tarski and it is negatively solved by Bang in the 50th by an ingenious proof.

References

[1] A.Lasota and J.A.Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. of A.M.S. **186** (1973), 481–488.

[2] V. Berthé and A. Siegel, *Purely periodic β-expansions in the Pisot non-unit case*, J. Number Theory **127** (2007), no. 2, 153–172.

[3] F. Blanchard, *β-expansions and symbolic dynamics*, Theoret. Comput. Sci. **65** (1989), no. 2, 131–141.

[4] J. Buzzi and G. Keller, *Zeta functions and transfer operators for multidimensional piecewise affine and expanding maps*, Ergodic Theory Dynam. Systems **21** (2001), no. 3, 689–716.

[5] R. Fischer, *Ergodische Theorie von Ziffernentwicklungen in Wahrscheinlichkeitsräumen*, Math. Z. **128** (1972), 217–230.

[6] P. Góra and A. Boyarsky, *Absolutely continuous invariant measures for piecewise expanding C^2 transformation in* $\mathbf{R}^N$, Israel J. Math. **67** (1989), no. 3, 272–286.

[7] Sh. Ito and H. Rao, *Purely periodic β-expansions with Pisot unit base*, Proc. Amer. Math. Soc. **133** (2005), no. 4, 953–964.

[8] Sh. Ito and T. Sadahiro, *Beta-expansions with negative bases*, Integers **9** (2009), A22, 239–259.

[9] Sh. Ito and Y. Takahashi, *Markov subshifts and realization of β-expansions*, J. Math. Soc. Japan **26** (1974), no. 1, 33–55.

[10] C. Kalle, *Isomorphisms between positive and negative*

β-transformations, Ergodic Theory Dynam. Systems **34** (2014), no. 1, 153–170.

[11] G. Keller, *Ergodicité et mesures invariantes pour les transformations dilatantes par morceaux d'une région bornée du plan*, C. R. Acad. Sci. Paris Sér. A-B **289** (1979), no. 12, A625–A627.

[12] ________, *Generalized bounded variation and applications to piecewise monotonic transformations*, Z. Wahrsch. Verw. Gebiete **69** (1985), no. 3, 461–478.

[13] T.Y. Li and J. A. Yorke, *Ergodic transformations from an*

interval into itself, Trans. Amer. Math. Soc. **235** (1978), 183–192.

[14] L. Liao and W. Steiner, *Dynamical properties of the negative beta-transformation*, Ergodic Theory Dynam. Systems **32** (2012), no. 5, 1673–1690.

[15] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[16] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

[17] B. Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps*, Israel J. Math. **116** (2000), 223–248.

[18] M. Smorodinsky, *β-automorphisms are Bernoulli shifts*, Acta Math. Acad. Sci. Hungar. **24** (1973), 273–278.

[19] M. Tsujii, *Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane*, Comm. Math. Phys. **208** (2000), no. 3, 605–622.

[20] ———, *Absolutely continuous invariant measures for*

expanding piecewise linear maps, Invent. Math. **143** (2001), no. 2, 349–373.