Sofic rotational beta expansions

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June 10, 2015

Recall

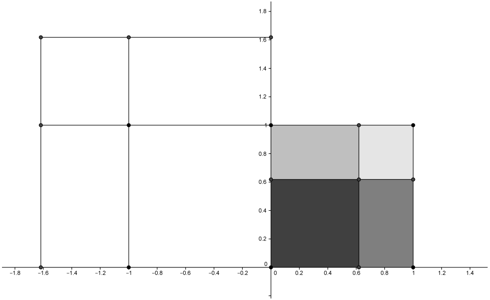
Let $1 < \beta \in \mathbb{R}$, $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $|\zeta| = 1$. Let $\eta_1, \eta_2, \xi \in \mathbb{C}$ such that $\eta_1/\eta_2 \notin \mathbb{R}$. Then $\mathcal{X} := \{\xi + x\eta_1 + y\eta_2 \mid x, y \in [0, 1)\}$ is a fundamental domain of the lattice $\mathcal{L} := \eta_1\mathbb{Z} + \eta_2\mathbb{Z}$ generated by η_1 and η_2 in $\mathbb{C}$.

Rotational beta transformation

A rotational beta transformation is a map $T : \mathcal{X} \to \mathcal{X}$ given by

$$T(z) = \beta\zeta z - d$$

where $d = d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta\zeta z \in \mathcal{X} + d$.



Rotational beta expansion

For $z \in \mathcal{X}$, we have

$$z = \sum_{i=1}^{\infty} \frac{d_i}{(\beta\zeta)^i},$$

where $d_i = d_i(z) = d(T^{i-1}(z))$. We say that the expansion of z wrt T is

$$d_T(z) := d_1 d_2 d_3 \ldots$$

Soficness

Denote by $\mathcal{A}$ the digit set $\{d(z)|z \in \mathcal{X}\}$ of T. We define $\mathcal{A}^*$ (resp. $\mathcal{A}^{\mathbb{Z}}$) as the set of all finite (resp. bi-infinite) words over $\mathcal{A}$. We say $w \in \mathcal{A}^*$ is admissible if w appears in the expansion $d_T(z)$ for some $z \in X$. Let

$$X_T := \{w \in \mathcal{A}^{\mathbb{Z}} | \text{ all subwords } w_i w_{i+1} ... w_j \text{ are admissible}\}.$$

The symbolic dynamical system associated to T is the topological dynamics $(\mathcal{X}_T, s)$ given by the shift operator $s((w_i)) = (w_{i+1})$. We say $(\mathcal{X}_T, s)$ (or simply, $(\mathcal{X}, T)$) is sofic if there is a finite directed graph G labeled by $\mathcal{A}$ such that for each $w \in \mathcal{X}_T$, there exists a bi-infinite path in G labeled w and vice versa.

Sofic (1-dim'l) beta expansions

Theorem

1. (Parry, 1960) The shift associated to a beta expansion is sofic if and only if the expansion of 1 is eventually periodic.
2. (Bertrand, 1977) If β is a Pisot number, then for every $x \in \mathbb{Q}(\beta) \cap \mathbb{R}^+$, the beta expansion of x is eventually periodic.

Theorem 1

Let $\partial(\mathcal{X})$ be the boundary of $\mathcal{X}$. The system $(\mathcal{X}, T)$ is sofic if and only if $\bigcup_{n=1}^{\infty} T^n(\partial(\mathcal{X}))$ is a finite union of segments.

Idea of the proof of Theorem 1

For $z \in \mathcal{X}$, we define the predecessor set as

$$P(z) := \bigcup_{n=1}^{\infty} \left\{ d(z')d(T(z')) \dots d(T^{n-1}(z')) \mid z' \in T^{-n}(z) \right\}.$$

The set $P(z)$ lists all trajectories going to z.

We define a relation on $\mathcal{X}$ by

$$z_1 \sim z_2 \iff P(z_1) = P(z_2).$$

Then (X, T) is sofic iff $\mathcal{X}/\sim$ contains finitely many equivalent classes.

For any $n \in \mathbb{N}$, the set $\mathcal{X} \setminus \bigcup_{i=1}^{n} T^i(\partial(\mathcal{X}))$ consists of finite number of open polygons. Hence, $\bigcup_{i=1}^{n} T^i(\partial(\mathcal{X}))$ induces a partition of $\mathcal{X}$.

If z_1 and z_2 are separated by a discontinuity segment of $\bigcup_{i=1}^{\infty} T^i(\partial(\mathcal{X}))$ (i.e., z_1 and z_2 belong to different partition cells), then $P(z_1) \neq P(z_2)$.

Suppose $\bigcup_{i=1}^{\infty} T^i(\partial(\mathcal{X}))$ is a finite collection of line segments. Let $P_1, \ldots, P_r$ be the polygons in the induced partition.

For $i = 1, ..., r$ for some $\mathcal{I} \subseteq \{1, \ldots, r\}$,

$$T(P_i) = \bigcup_{j \in \mathcal{I}} P_j.$$

For $d \in \mathcal{A}$, let $[d] := \{z \in \mathcal{X} | d_1(z) = d\}$. For some $\mathcal{I}^* \subseteq \mathcal{I}$,

$$T(P_i \cap [d]) = \bigcup_{j \in \mathcal{I}^*} P_j.$$

We construct the sofic graph G as follows. Set the vertex set as $V(G) = \{P_1, ..., P_r\}$. We draw an edge from P_i to P_j labeled $d \in \mathcal{A}$ if P_j is contained in $T(P_i \cap [d])$.

Main Results

Theorem 2

Let ζ be a q-th root of unity ($q > 2$) and β be a Pisot number. Let $\eta_1, \eta_2, \xi \in \mathbb{Q}(\zeta, \beta)$ such that $\eta_1/\eta_2 \notin \mathbb{R}$. If $\zeta + \zeta^{-1} \in \mathbb{Q}(\beta)$, then the system $(\mathcal{X}, T)$ is sofic.

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Corollary 3

If ζ is a $3rd$, $4th$ or $6th$ root of unity, then the system $(\mathcal{X}, T)$ is sofic for any Pisot number β.

Idea of the Proof of Theorem 2

Since $[\mathbb{Q}(\zeta,\beta) : \mathbb{Q}(\zeta+\zeta^{-1},\beta)] = 2$, there exist $a_{ij}, b_i \in \mathbb{Q}(\zeta+\zeta^{-1})$ such that

$$\zeta \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

and

$$(\beta\zeta - 1)\xi = b_1\eta_1 + b_2\eta_2.$$

We define an analog $U : [0,1)^2 \longrightarrow [0,1)^2$ of T by

$$U\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \beta(a_{11}x + a_{12}y) + b_1 - \lfloor \beta(a_{11}x + a_{12}y) + b_1 \rfloor \\ \beta(a_{21}x + a_{22}y) + b_2 - \lfloor \beta(a_{21}x + a_{22}y) + b_2 \rfloor \end{pmatrix}.$$

We keep track of the growth of $\bigcup_{i=1}^{K} U^i(\partial([0,1)^2))$ as K increases.

We identify a discontinuity segment with the line

$$f(X,Y) = (A,B)\begin{pmatrix} X \\ Y \end{pmatrix} + C,$$

$(0,0) \neq (A,B) \in \mathbb{R}^2$, containing it. Then we determine how the coefficients of $g \in U(f)$ evolve from (A,B,C).

If $g \in U(f)$, then

$$g(X, Y) = \frac{1}{\beta}(A, B) \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} X + c_1 \\ Y + c_2 \end{pmatrix} + C,$$

where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \Delta := \left\{ \begin{pmatrix} \lfloor \beta(a_{11}x + a_{12}y) + b_1 \rfloor - b_1 \\ \lfloor \beta(a_{21}x + a_{22}y) + b_2 \rfloor - b_2 \end{pmatrix} \middle| \, 0 \leq x, y < 1 \right\}.$$

Iterating U, we produce a sequence of coefficients

$$\left(A^{(n)}, B^{(n)}, C^{(n)}\right) \to \left(A^{(n+1)}, B^{(n+1)}, C^{(n+1)}\right),$$

where

$$\left(A^{(n+1)}, B^{(n+1)}\right) = \left(A^{(n)}, B^{(n)}\right) \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \qquad (1)$$

and

$$C^{(n+1)} = \beta C^{(n)} + \left(A^{(n)}, B^{(n)}\right) \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad (2)$$

with $\left(A^{(0)}, B^{(0)}, C^{(0)}\right) = (A, B, C)$.

Since ζ is a q-th root of unity, there are finitely many $(A^{(n)}, B^{(n)})$'s. We show that there are also finitely many $C^{(n)}$'s.

To this end, we look at $|\sigma_k\left(C^{(n)}\right)|$, where $\sigma_k : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_k)$ is the conjugate map that sends β to its conjugate β_k.

By the Pisot property of β, we can show that $|\sigma_k\left(C^{(n)}\right)|$ is bounded.

Theorem 4

Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/5)$. If $\beta > 2.90332$ such that $\sqrt{5} \notin \mathbb{Q}(\beta)$, then $(\mathcal{X}, T)$ is not a sofic system.

For instance, taking $\beta = 3, 4, 5$, we get a non-sofic system.

Idea of the Proof 4

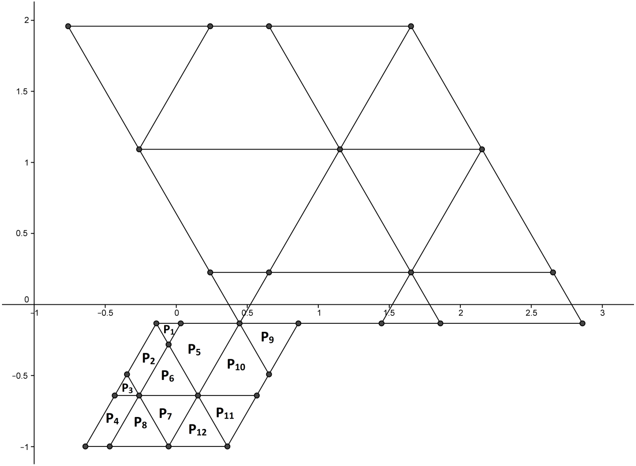
Let $\omega = \frac{1+\sqrt{5}}{2}$.

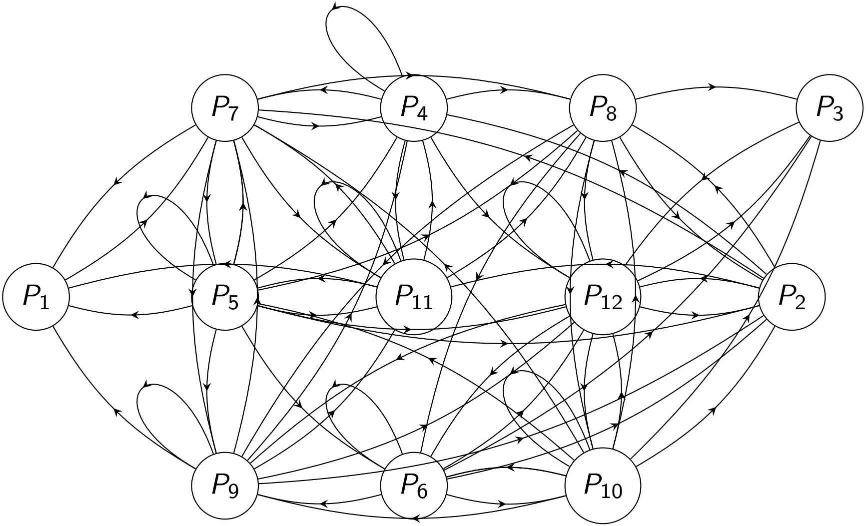
Since $\sqrt{5} \notin \mathbb{Q}(\beta)$, there exists a Galois map $\sigma \in Gal(\mathbb{Q}(\beta, \omega)/\mathbb{Q}(\beta))$ with $\sigma(\omega) = -1/\omega$.

We show that $\{|\sigma(C^{(n)})| | n \in \mathbb{N}\}$ diverges for some class of coefficients $C^{(n)}$.

Example: 3-fold

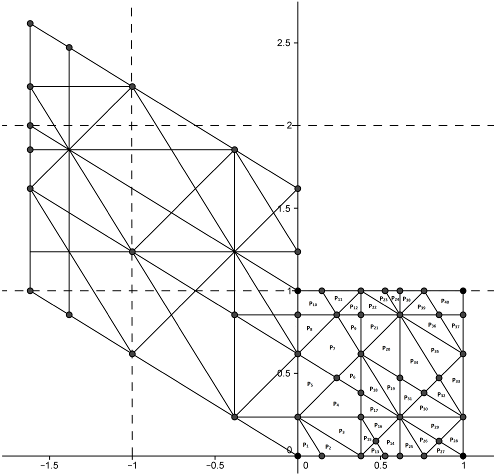
$$\beta = 1 + \sqrt{2},\ \eta_1 = 1,\ \eta_2 = \zeta^2 \text{ and } (\beta\zeta - 1)\xi = 3 - \beta$$




P_7
P_4
P_8
P_3
P_1
P_5
P_{11}
P_{12}
P_2
P_9
P_6
P_{10}

Example: 5-fold

$\xi = 0,\ \eta_1 = 1,\ \eta_2 = \zeta$ and $\beta = \frac{1+\sqrt{5}}{2}$.



Example: 7-fold

$\xi = 0$, $\eta_1 = 1$, $\eta_2 = \zeta$ and $\beta = 1 + 2\cos(2\pi/7)$, a cubic Pisot number whose minimum polynomial is $x^3 - 2x^2 - x + 1$