Dynamical systems on $\mathbb{Z}_p$ generated by constant length substitutions

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$\mathbb{Z}_p$ is the inverse limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$, and is a discrete valuation ring with unique maximal ideal $p\mathbb{Z}_p$.

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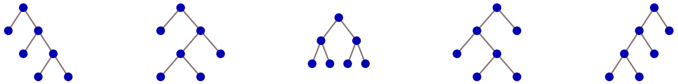

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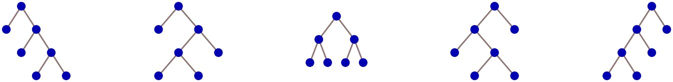

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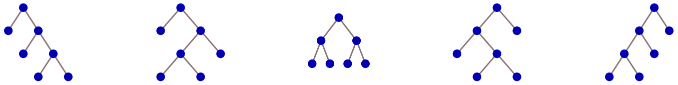

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Theorem (follows from Kummer 1852)

For all $n \geq 0$, $C(n)$ is odd if and only if $n+1$ is a power of 2.

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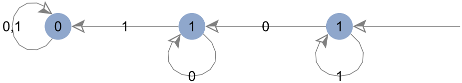

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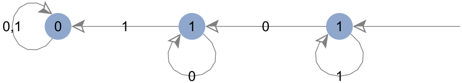

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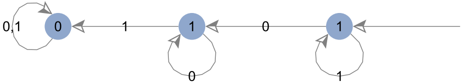

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Theorem (Cobham)

r-automatic sequences are precisely the letter-to-letter codings of fixed points of constant length-r substitutions.

Which combinatorial sequences?

Let $\mathcal{D}f$ denote the diagonal of a multivariate formal power series f:
The diagonal of a formal power series is

$$\mathcal{D}\left(\sum_{n_1,\ldots,n_k\geq 0} a_{n_1,\ldots,n_k} x_1^{n_1}\cdots x_k^{n_k}\right) := \sum_{n\geq 0} a_{n,\ldots,n} x^n.$$

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A sequence $(a_n)_{n\geq 0}$ of integers is algebraic if its generating function $\sum_{n\geq 0} a_n x^n$ is algebraic over $\mathbb{Q}(x)$. Equivalently: $P(x, \sum_{n\geq 0} a_n x^n) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y]$.

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Theorem (Furstenberg 1967, CKMFR 1980)

Let $(a_n)_{n\geq 0}$ be a sequence of elements in $\mathbb{F}_p$. Then $(a_n)_{n\geq 0}$ is automatic iff $(a_n)_{n\geq 0}$ is algebraic over $\mathbb{F}_p[x]$ iff $(a_n)_{n\geq 0}$ is the diagonal of a rational function $\frac{P(x,y)}{Q(x,y)} \in \mathbb{F}_p(x,y)$.

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Theorem (Denef–Lipshitz 1987)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be algebraic over $\mathbb{Z}_p[x]$. Then $f = \mathcal{D}(\frac{P(x,y)}{Q(x,y)})$ where $P(x,y), Q(x,y) \in \mathbb{Z}_p[x,y]$.

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If the sequence is algebraic over $\mathbb{Q}(x)$, then it projects mod p^α to (codings of) fixed points of length p substitutions for all α and all p.

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$\sum_{n\geq 1} C(n)x^n$ is the diagonal of

$$\begin{aligned}\frac{y(2xy^2+2xy-1)}{xy^2+2xy+x-1} = \; & 0x^0y^0 + 1x^0y^1 + 0x^0y^2 + 0x^0y^3 + 0x^0y^4 + 0x^0y^5 + \cdots \\ & + 0x^1y^0 + 1x^1y^1 + 0x^1y^2 - 1x^1y^3 + 0x^1y^4 + 0x^1y^5 + \cdots \\ & + 0x^2y^0 + 1x^2y^1 + 2x^2y^2 + 0x^2y^3 - 2x^2y^4 - 1x^2y^5 + \cdots \\ & + 0x^3y^0 + 1x^3y^1 + 4x^3y^2 + 5x^3y^3 + 0x^3y^4 - 5x^3y^5 + \cdots \\ & + 0x^4y^0 + 1x^4y^1 + 6x^4y^2 + 14x^4y^3 + 14x^4y^4 + 0x^4y^5 + \cdots \\ & + 0x^5y^0 + 1x^5y^1 + 8x^5y^2 + 27x^5y^3 + 48x^5y^4 + 42x^5y^5 + \cdots \\ & + \cdots .\end{aligned}$$

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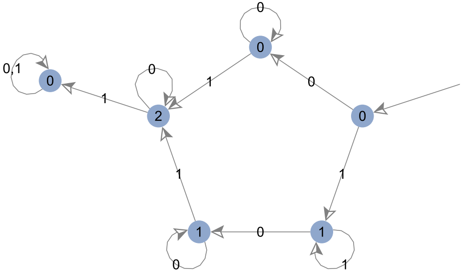

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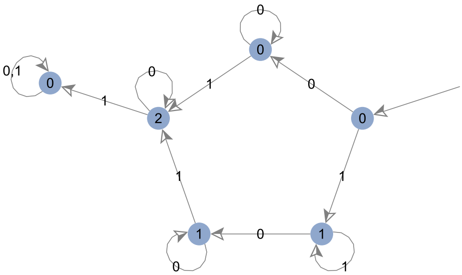

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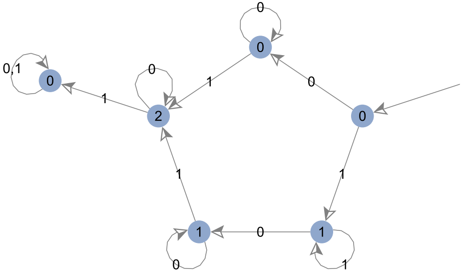

We look at combinatorial sequences of integers $(a_n)_{n\geq 0}$ as inverse limits of substitution sequences. What shifts do they generate in $\mathbb{Z}_p$?

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If a_α is p-automatic, let

$$\mathcal{M}_\alpha = (\mathcal{S}_\alpha, \Sigma_p, \delta_\alpha, s_0, \mathbb{Z}/(p^\alpha\mathbb{Z}), \tau_\alpha)$$

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Given a sequence u, let (X_u, σ) be the substitution shift generated by u.

Proposition (Rowland-Y, 2015)

Suppose that $a \in \mathbb{Z}_p^{\mathbb{N}}$ is such that for each $\alpha \geq 0$, a_α is p-automatic. There are projection maps $\pi^*_{\alpha,\alpha+1} : X_{u_{\alpha+1}} \to X_{u_\alpha}$, there is a shift (X_u, σ) and a letter-to-letter projection $\tau : X_u \to X_a$, there are length-p substitutions $(\theta_\alpha)_{\alpha \geq 0}$ and θ such that $\theta_\alpha(u_\alpha) = u_\alpha$ for each α, $\theta(u) = u$, and the following diagrams commute:

1.

$$\begin{array}{ccccccccccc}
\cdots & \overset{\pi^*_{\alpha-1,\alpha}}{\longleftarrow} & (X_{u_\alpha}, \sigma) & \overset{\pi^*_{\alpha,\alpha+1}}{\longleftarrow} & (X_{u_{\alpha+1}}, \sigma) & \overset{\pi^*_{\alpha+1,\alpha+2}}{\longleftarrow} & (X_{u_{\alpha+2}}, \sigma) & \cdots & (X_u, \sigma) \\
 & & \downarrow \tau_\alpha & & \downarrow \tau_{\alpha+1} & & \downarrow \tau_{\alpha+2} & & \downarrow \tau \\
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Theorem (Rowland-Y, 2015)

Suppose that for each $\alpha \geq 0$, the substitution θ_α is primitive. Then

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Part 2 follows from the fact that each (X_{u_α}, σ) is uniquely ergodic, and:

Theorem (Choksi 1958)

Let $\{(X_\alpha, \mathcal{B}_\alpha, \mu_\alpha), \pi_{\alpha,\alpha+1}, \alpha \geq 0\}$ be an inverse family of compact metric spaces. Then the inverse limit $(X, \mathcal{B}, \mu)$ exists. If $\mathcal{B}_\alpha$ is Borel for each α, then $\mathcal{B}$ is Borel.

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Open question

What is the point spectrum of (X_a, σ)? Dekking's theorem tells us that at the very least we have $(\mathbb{Z}_p, +1)$. Is there more?

Dynamical properties of (X_a, σ)

If a substitution θ on a finite alphabet defines the shift (X, σ), then the set $\mathcal{L} = \bigcap_{n \geq 0} \theta^n(X)$ contains only the periodic points of θ, of which there are finitely many. However, when we consider substitutions on an uncountable alphabet, $\mathcal{L}$ can possibly be larger.

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Example

Below we plot the binary expansion $C(2^n)$ for $0 \leq n \leq 20$, where 0 and 1 are represented by a white and black cell respectively:

suggesting that $C(2^n)_{n\geq 0}$ converges in $\mathbb{Z}_2$ to a point in $\pi\tau\mathcal{L}$, so

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$$\lim_n C(2^n) = \lim_n \pi\sigma^{2^n}\tau U = \lim_n \pi\tau\theta(\sigma^n(U)) \in \pi\tau\theta(X_U)$$

and taking subsequences $2^{2^{\cdots 2^n}}$ we see $\lim_n C(2^n) \in \pi\tau\mathcal{L}$.

Theorem (Michel,Miller,Rennie, 2014)

Let $C(n)_{n\geq 0}$ *be the sequence of Catalan numbers. Then for each* $k, r \in \mathbb{N}$, $\lim_{n\to\infty} C(kp^n + r)$ *exists in* $\mathbb{Z}_p$.

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A constant-recursive sequence $s(n)_{n\geq 0}$ is one that satisfies a recurrence

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Example

The Fibonacci sequence $F(n)_{n\geq 0} = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$ satisfies

$$F(n) = F(n-1) + F(n-2) = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}$$

where $\phi, \bar{\phi} = \frac{\sqrt{5}\pm 1}{2}$ are the roots of its characteristic polynomial $g(x) = x^2 - x - 1$.

Values of $F(3^n)$:

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These limits are $\pm\sqrt{\frac{2}{5}}$ in $\mathbb{Z}_3$. They are explained by interpolations.

If $s(n)_{n\geq 0}$ is a constant-recursive sequence with characteristic polynomial $g(x)$, then we can write

$$s(n) = \sum_{\beta} c_\beta(n)\beta^n$$

for some polynomials $c_\beta(x) \in K[x]$, where we sum over all roots $\beta_1, \ldots, \beta_\ell$ of $g(x)$, and $K = \mathbb{Z}_p(\beta_1, \ldots, \beta_\ell)$. Let O_K be the unit ball of K: there exists $f = f(p)$ such that all $p^f - 1$-st roots of unity lie in O_K. Let π be a uniformizer in O_K.
Given $x \not\equiv 0 \mod \pi$ in O_K, let $\omega(x)$ denote the $p^f - 1$-st root of unity such that $x \equiv \omega(x) \mod \pi$.

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Corollary (Rowland-Y, 2016)

Let p be a prime, and let $s(n)_{n\geq 0}$ be a constant-recursive sequence of p-adic integers with monic characteristic polynomial $g(x) \in \mathbb{Z}_p[x]$. Let $a, b \in \mathbb{Z}$ with $a \geq 1$. Then $\lim_{n\to\infty} s(ap^{fn} + b)$ exists in $\mathbb{Z}_p$ and is equal to

$$\lim_{n\to\infty} s(ap^{fn} + b) = \sum_{|\beta|_p=1} c_\beta(b)\omega(\beta)^a\beta^b .$$

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Example

If $p=3$, then $f=2$, and

$$F(3^{2n}) = \frac{\omega(\phi)-\omega(\overline{\phi})}{\sqrt{5}} = \sqrt{2/5},$$

and

$$F(3^{2n+1}) = \frac{\omega(\phi)^3-\omega(\overline{\phi})^3}{\sqrt{5}} = -\sqrt{2/5}.$$

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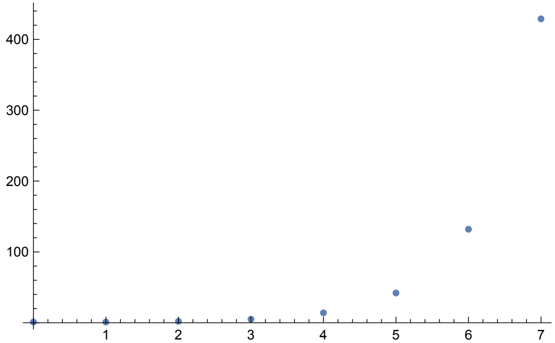
For $p=2$ one computes that

$$\lim_{n\to\infty} F(2^{2n}) = \sqrt{-\frac{3}{5}} \text{ and } \lim_{n\to\infty} F(p^{2n+1}) = -\sqrt{-\frac{3}{5}}.$$

Interpolation?



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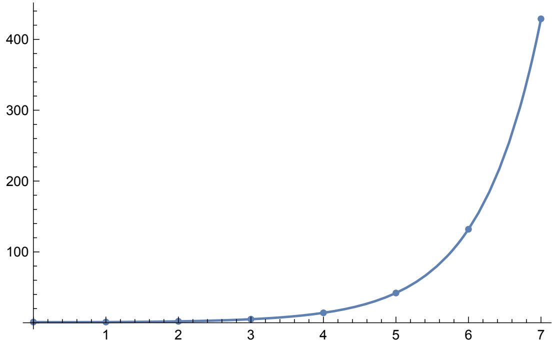

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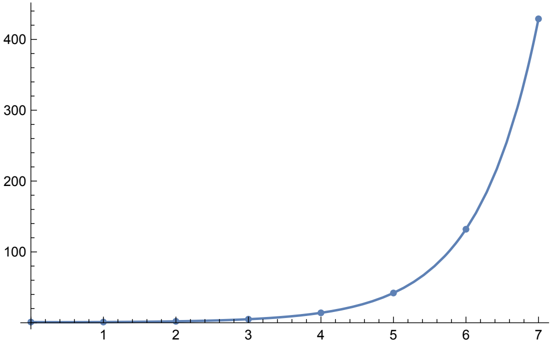

Can we interpolate a combinatorial sequence $s(n)$ to a continuous function $c(x)$ on $\mathbb{Z}_p$? Mostly No. Example:

$$\lim_{n\to\infty} C(2^n) \neq 1 = C(0), \text{ while } \lim_{n\to\infty} 2^n = 0.$$

But maybe we can interpolate subsequences.

Constant-recursive sequences

Theorem (Rowland–Y, 2016)

Let $s(n)_{n\geq 0}$ be a constant-recursive sequence with a monic characteristic polynomial. Then $s(n)_{n\geq 0}$ has an approximate twisted interpolation to $\mathbb{Z}_p$. That is, there exists q a power of p, a finite partition $\mathbb{N} = \bigcup_{j\in J} A_j$ with each A_j dense in $r + q\mathbb{Z}_p$ for some $0 \leq r \leq q-1$, finitely many continuous functions $s_j : \mathbb{Z}_p \to K$, and non-negative constants C, D with $D < 1$ such that

$$|s(n) - s_j(n)|_p \leq C \cdot D^n$$

for all $n \in A_j$ and $j \in J$.

The Fibonacci sequence in $\mathbb{Z}_3$

Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ in $\mathbb{Q}_3(\sqrt{5})$.

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Corollary (Rowland–Y 2016)

For each $0 \leq i \leq 7$, define the function $F_i : \mathbb{Z}_3 \to \mathbb{Z}_3$ by

$$F_i(x) := \frac{\omega(\phi)^i \exp_3\left(x \log_3 \frac{\phi}{\omega(\phi)}\right) - \omega(\bar{\phi})^i \exp_3\left(x \log_3 \frac{\bar{\phi}}{\omega(\bar{\phi})}\right)}{\sqrt{5}}.$$

Then $F(n) = F_i(n)$ for all $n \equiv i \mod 8$.

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p-adic logarithm and exponential

The *p-adic logarithm*

$$\log_p x := \sum_{m \geq 1} (-1)^{m+1} \frac{(x-1)^m}{m}$$

converges for $x \in \mathbb{Z}_p$ such that $|x-1|_p < 1$.

The *p-adic exponential function*

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$$x = \exp_p \log_p x.$$

Proof outline for $\mathbb{Z}_3$

Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ in $\mathbb{Q}_3(\sqrt{5})$. Then

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What else lies in $\mathcal{L}$?

Catalan numbers modulo 2^α

Theorem

For all $n \geq 0$,

- $C(n) \not\equiv 9 \mod 16$,
- $C(n) \not\equiv 17, 21, 26 \mod 32$,
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Limiting density of attained residues; support of the σ-invariant measure.

Theorem (Burr, 1971)

For all $\alpha \geq 1$, $F(n)_{n\geq 0}$ *attains all residues modulo* 3^α *and* 5^α. *In other words for all* $\alpha \geq 1$,

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$$\lim_{\alpha\to\infty} \frac{|\{s(n) \bmod p^\alpha : n \geq 0\}|}{p^\alpha} = \mu\left(\mathbb{Z}_p \cap \bigcup_{i,r} s_{i,r}(r + q\mathbb{Z}_p)\right).$$

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The limiting density of residues attained by the Fibonacci sequence modulo 11^α *is*

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The case $p = 11$ *was easier to work out as* $\sqrt{5} \in \mathbb{Z}_{11}$*. Is a computation possible for all* p*?*

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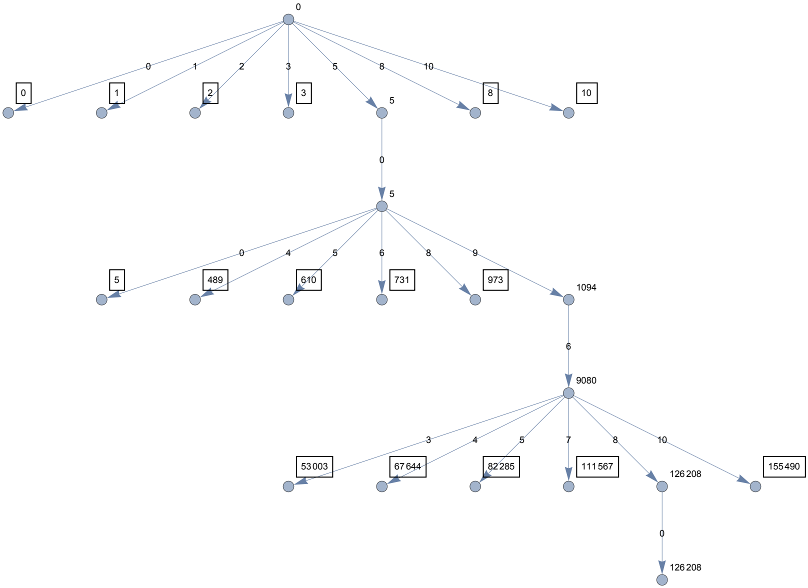

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