Tilings with S-adic Rauzy fractals

Wolfgang Steiner
(joint work with Valérie Berthé and Jörg Thuswaldner)

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Codings of translations on the torus

Sturmian words are codings of rotations on $\mathbb{T}^1$

Rauzy'82:
Tribonacci word is the coding of a translation on $\mathbb{T}^2$

Conjecture by Arnoux and Rauzy:
each Arnoux-Rauzy word is the coding of a translation on $\mathbb{T}^2$

Cassaigne–Ferenczi–Zamboni'00:
there are Arnoux-Rauzy words that are not codings of translations

We show that almost all Arnoux-Rauzy words are codings of translations on $\mathbb{T}^2$

S-adic words

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of substitutions over the alphabet $A = \{1, 2, \ldots, d\}$. $\omega \in A^{\mathbb{N}}$ is a *limit word* of $(\sigma_n)_{n\in\mathbb{N}}$ if

$$\omega^{(0)} = \omega, \quad \omega^{(n)} = \sigma_n\big(\omega^{(n+1)}\big) \quad \text{for all } n \in \mathbb{N}.$$

for some words $\omega^{(n)}$; ω is an *S-adic word* with $S = \{\sigma_n : n \in \mathbb{N}\}$.

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Example

Arnoux-Rauzy words on $A = \{1, 2, 3\}$: $S = \{\alpha_1, \alpha_2, \alpha_3\}$,

$$\alpha_1: \begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 12 \\ 3 & \mapsto & 13 \end{array} \qquad \alpha_2: \begin{array}{ccc} 1 & \mapsto & 21 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 23 \end{array} \qquad \alpha_3: \begin{array}{ccc} 1 & \mapsto & 31 \\ 2 & \mapsto & 32 \\ 3 & \mapsto & 3 \end{array}$$

e.g.

$$\begin{aligned} \omega &= 1121131121112113112111211311211211311211121131121\cdots \\ &= \alpha_1(121312112131211213121213121121312 1\cdots) \\ &= \alpha_1\alpha_1(232123212322321232 1\cdots) \\ &= \alpha_1\alpha_1\alpha_2(3131323131\cdots) \\ &= \alpha_1\alpha_1\alpha_2\alpha_3(11211\cdots) \end{aligned}$$

Periodic case (fixed point of a substitution)

If $\omega^{(p)} = \omega$, then

$$\omega = \sigma_0 \sigma_1 \cdots \sigma_{p-1}(\omega)$$

has a periodic directive sequence $(\sigma_n)_{n \in \mathbb{N}}$.

Example

Tribonacci sequence

$$\omega = \alpha_1 \alpha_2 \alpha_3(\omega) = \tau(\omega)$$

$$\alpha_1\alpha_2\alpha_3 : \begin{array}{ccc} 1 & \mapsto & 1213121 \\ 2 & \mapsto & 121312 \\ 3 & \mapsto & 1213 \end{array} \qquad \tau : \begin{array}{ccc} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{array}$$

$$\alpha_1 \alpha_2 \alpha_3 = \tau^3$$

Broken line

The *abelianisation map* on A^* is

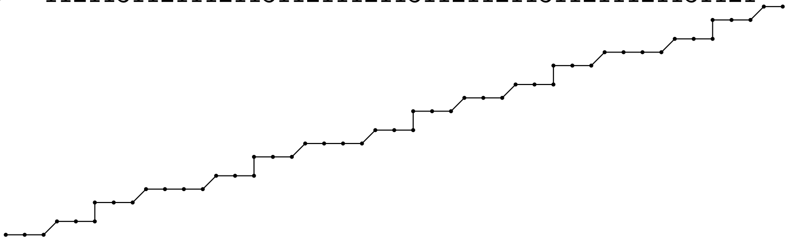
$$\boldsymbol{\ell} : \; A^* \to \mathbb{N}^d, \; w \mapsto {}^t(|w|_1, |w|_2, \ldots, |w|_d),$$

where $|w|_j$ denotes the number of occurrences of the letter j in w.
The *broken line* associated with $\omega = \omega_0\omega_1 \cdots \in A^{\mathbb{N}}$ has vertex set

$$\{\boldsymbol{\ell}(\omega_{[0,n)}) : n \in \mathbb{N}\}, \quad \text{where } \omega_{[0,n)} = \omega_0\omega_1 \cdots \omega_{n-1}.$$

Example

$\omega = 112113112111211311211121131121121131121112113112\ldots$



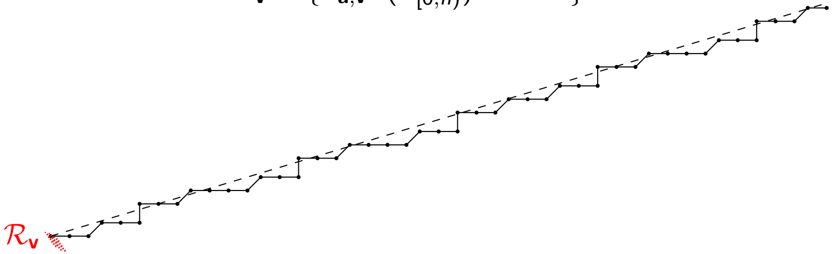
Rauzy fractal

If the letter frequencies $f_i = \lim_{n\to\infty} \frac{|\omega_{[0,n)}|_i}{n}$ exist, let

$$\mathbf{u} = {}^t(f_1, f_2, \ldots, f_d)$$

be the *frequency vector* of ω. Let $\pi_{\mathbf{u},\mathbf{v}}$ be the projection along $\mathbf{u}$ onto a hyperplane $\mathbf{v}^\perp$, $\mathbf{v} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$. The *Rauzy fractal* (in $\mathbf{v}^\perp$) is

$$\mathcal{R}_\mathbf{v} = \overline{\{\pi_{\mathbf{u},\mathbf{v}}\, \ell(\omega_{[0,n)}) : n \in \mathbb{N}\}}.$$

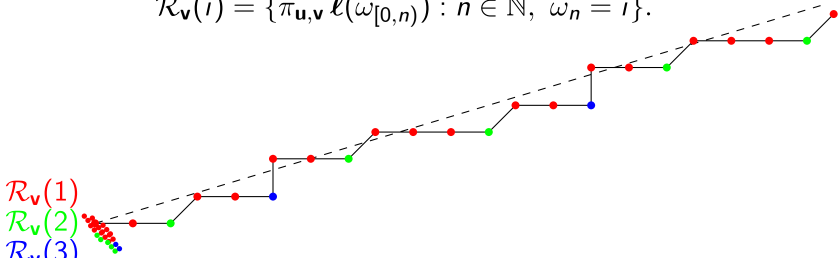


A particular role will be played by $\mathcal{R}_\mathbf{1}$, $\mathbf{1} = {}^t(1, \ldots, 1)$.

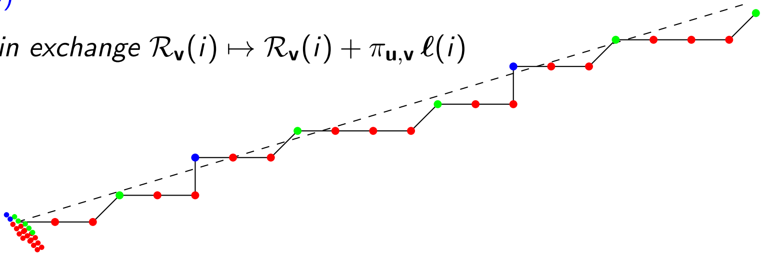
Subtiles and domain exchange

The Rauzy fractal has *subtiles*

$$\mathcal{R}_{\mathbf{v}}(i) = \overline{\{\pi_{\mathbf{u},\mathbf{v}}\,\boldsymbol{\ell}(\omega_{[0,n)}) : n \in \mathbb{N},\ \omega_n = i\}}.$$



domain exchange $\mathcal{R}_{\mathbf{v}}(i) \mapsto \mathcal{R}_{\mathbf{v}}(i) + \pi_{\mathbf{u},\mathbf{v}}\,\boldsymbol{\ell}(i)$



$$\boldsymbol{\ell}(i) - \boldsymbol{\ell}(j) \in \mathbf{1}^{\perp} \cap \mathbb{Z}^d \quad \Rightarrow \quad \pi_{\mathbf{u},\mathbf{1}}\,\boldsymbol{\ell}(i) \equiv \pi_{\mathbf{u},\mathbf{1}}\,\boldsymbol{\ell}(j) \mod \mathbf{1}^{\perp} \cap \mathbb{Z}^d$$

Covering and tiling of $\mathbf{1}^\perp$

Lemma

Assume that $\mathcal{R}_\mathbf{1}$ is compact. Then

$$\left(\mathbf{1}^\perp \cap \mathbb{Z}^d\right) + \mathcal{R}_\mathbf{1} = \mathbf{1}^\perp$$

if and only if $f_1, f_2, \ldots, f_d$ are rationally independent.

$(0,-1,1)^t + \mathcal{R}_\mathbf{1}$

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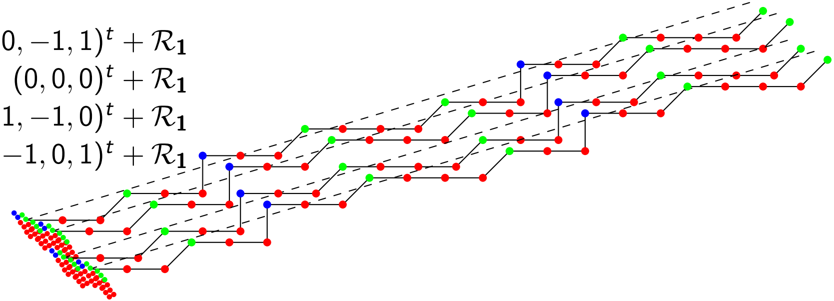

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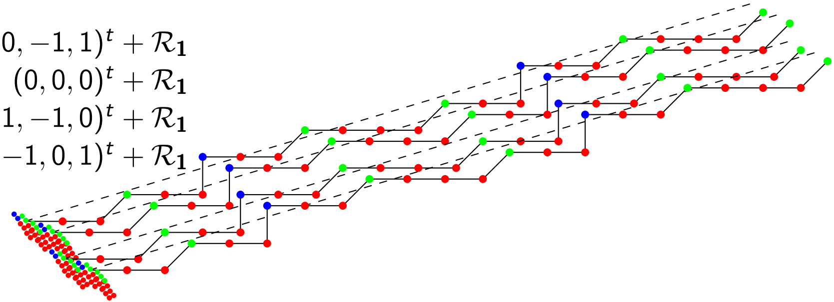
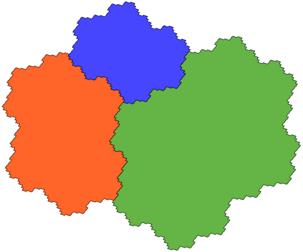


For which $\omega \in A^\mathbb{N}$, the collection

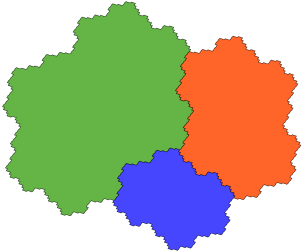
$$\mathcal{C}_\mathbf{1} = \left\{\mathbf{x} + \mathcal{R}_\mathbf{1}(i) : \mathbf{x} \in \mathbf{1}^\perp \cap \mathbb{Z}^d,\ i \in A\right\}$$

forms a *tiling* of $\mathbf{1}^\perp$?

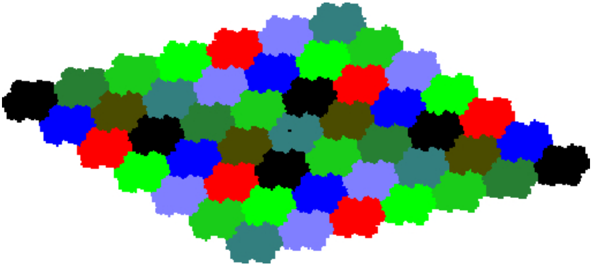
Domain exchange and tiling for the Tribonacci sequence



$\mathcal{R}_{\mathbf{v}}(1), \mathcal{R}_{\mathbf{v}}(2), \mathcal{R}_{\mathbf{v}}(3)$



$\mathcal{R}_{\mathbf{v}}(i) + \pi_{\mathbf{u},\mathbf{v}}\, \boldsymbol{\ell}(i)$



$\left\{ \mathbf{x} + \mathcal{R}_{\mathbf{1}} : \mathbf{x} \in \mathbf{1}^{\perp} \cap \mathbb{Z}^{d} \right\}$

Spectrum of the symbolic dynamical system

The *symbolic dynamical system generated by* $\omega \in A^{\mathbb{N}}$ is (X_ω, Σ), with Σ the shift on $A^{\mathbb{N}}$, i.e., $\Sigma((\omega_n)_{n\in\mathbb{N}}) = (\omega_{n+1})_{n\in\mathbb{N}}$, and

$$X_\omega = \overline{\{\Sigma^n(\omega) \mid n \in \mathbb{N}\}}$$

(closure w.r.t. product topology of discrete topology on A).
For most S-adic words ω, (X_ω, Σ) is uniquely ergodic.

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Define the representation map

$$\varphi: \; X_\omega \to \mathcal{R}_{\mathbf{v}}, \quad (\upsilon_n)_{n\in\mathbb{N}} \mapsto \bigcap_{k\in\mathbb{N}} \mathcal{R}_{\mathbf{v}}(\upsilon_{[0,k)}),$$

with $\mathcal{R}_{\mathbf{v}}(\upsilon_{[0,k)}) = \overline{\{\pi_{\mathbf{u},\mathbf{v}}\, \boldsymbol{\ell}(\omega_{[0,n)}) : n \in \mathbb{N},\ \omega_{[n,n+k)} = \upsilon_{[0,k)}\}}$,
if $\bigcap_{k\in\mathbb{N}} \mathcal{R}_{\mathbf{v}}(\upsilon_{[0,k)})$ is a single point for each $(\upsilon_n)_{n\in\mathbb{N}} \in X_\omega$.

Spectrum of the symbolic dynamical system

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Lemma

If $\mathcal{C}_{\mathbf{1}}$ forms a tiling of $\mathbf{1}^\perp$ and φ is well defined, then (X_ω, Σ) is measurably conjugate to $+\pi_{\mathbf{u},\mathbf{1}}\, \boldsymbol{\ell}(i)$ on $\mathbf{1}^\perp/(\mathbf{1}^\perp \cap \mathbb{Z}^d) \simeq \mathbb{T}^{d-1}$, hence (X_ω, Σ) has purely discrete spectrum. (cf. Pisot conjecture)

Balanced words

A pair of words $u, v \in A^*$ with $|u| = |v|$ is *C-balanced* if

$$-C \leq |u|_j - |v|_j \leq C \qquad \forall\, j \in A.$$

$\omega \in A^{\mathbb{N}}$ is *C-balanced* if each pair of factors u, v of ω with $|u| = |v|$ is C-balanced; ω is *balanced* if ω is C-balanced for some C.

Lemma

The Rauzy fractal $\mathcal{R}_{\mathbf{v}}$ is compact if and only if ω is balanced.

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Examples (Berthé–Cassaigne–St '13)

Let ω be an Arnoux-Rauzy word on $A = \{1, 2, 3\}$, $\sigma_n = \alpha_{t_n}$.

- If $\nexists\, n \in \mathbb{N}$ with $t_n = t_{n+1} = \cdots = t_{n+k}$ (i.e., weak partial quotients are bounded by h), then ω is $(2h+1)$-balanced.
- Let X be the set $\{1121, 1122, 12121, 12122\}$ together with the words obtained by permutations of letters.
  If $t_0 t_1 t_2 \cdots$ has no factor in X, then ω is 2-balanced.
- Cassaigne–Ferenczi–Zamboni '00 construct unbalanced Arnoux-Rauzy words on 3 letters.

Irrationality and contraction properties

Let $M_n = (|\sigma_n(j)|_i)_{i,j \in A}$ be the incidence matrix of σ_n,

$$\sigma_{[n,n+k)} = \sigma_n \sigma_{n+1} \cdots \sigma_{n+k-1}, \; M_{[n,n+k)} = M_n M_{n+1} \cdots M_{n+k-1}.$$

We call a sequence of substitutions $(\sigma_n)_{n \in \mathbb{N}}$

- *primitive* if $\forall n \in \mathbb{N} \; \exists k \in \mathbb{N} : M_{[n,n+k)} > 0$,
- *irreducible* if $\forall n \in \mathbb{N}$ the characteristic polynomial of $M_{[n,n+k)}$ is irreducible for all sufficiently large k.

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Lemma

Let $(\sigma_n)_{n\in\mathbb{N}}$ be an irreducible sequence of substitutions with balanced limit word ω. Then the coordinates of the frequency vector $\mathbf{u} = {}^t(f_1, f_2, \ldots, f_d)$ of ω are rationally independent.

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Lemma

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a primitive, irreducible, and recurrent sequence of substitutions with balanced limit word ω. Then

$$\lim_{n\to\infty} \pi_{\mathbf{u},\mathbf{v}}\, M_{[0,n)}\, \mathbf{x} = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

Main theorem

Let $(\sigma_n)_{n\in\mathbb{N}}$ *be a primitive and irreducible sequence of unimodular substitutions over the alphabet* A *with limit word* ω *such that*

$$\exists C \in \mathbb{N} : \forall k \in \mathbb{N}\ \exists n > 0 : \sigma_{[n,n+k)} = \sigma_{[0,k)},\ \omega^{(n+k)} \text{ is } C\text{-balanced.}$$

- (X_ω, Σ) *is minimal and uniquely ergodic.*
- $\mathcal{R}_{\mathbf{v}}$ *is compact and closure of its interior;* $\partial\mathcal{R}_{\mathbf{v}}$ *has measure* 0*.*
- $\mathcal{C}_{\mathbf{1}}$ *forms a multiple tiling of* $\mathbf{1}^\perp$*.*

Main theorem

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a primitive and irreducible sequence of unimodular substitutions over the alphabet A with limit word ω such that

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- $\mathcal{R}_{\mathbf{v}}$ *is compact and closure of its interior;* $\partial\mathcal{R}_{\mathbf{v}}$ *has measure* 0.
- $\mathcal{C}_{\mathbf{1}}$ *forms a multiple tiling of* $\mathbf{1}^\perp$.
- *If* $\exists k \in \mathbb{N} : \forall j_1, j_2 \in A\ \exists i \in A,\ p_1, p_2 \in A^*$*:*

  $$\ell(p_1) = \ell(p_2),\ \sigma_{[0,k)}(j_1) \in p_1 i A^*,\ \sigma_{[0,k)}(j_2) \in p_2 i A^*$$

  (strong coincidence condition), then the $\mathcal{R}_{\mathbf{v}}(i)$*,* $i \in A$*, are mutually disjoint in measure, and* (X_ω, Σ) *is measurably conjugate to an exchange of domains on* $\mathcal{R}_{\mathbf{v}}$.
- $\mathcal{C}_{\mathbf{1}}$ *forms a tiling of* $\mathbf{1}^\perp$ *iff the geometric coincidence condition holds. Then* (X_ω, Σ) *is measurably conjugate to a translation on* $\mathbb{T}^{d-1}$*; in particular, its spectrum is purely discrete.*

(Multiple) tiling of $\mathbf{v}^\perp$ by (subtiles of) Rauzy fractals

The *discrete hyperplane* approximating $\mathbf{v}^\perp$ is

$$\Gamma(\mathbf{v}) = \{[\mathbf{x}, i] \in \mathbb{Z}^d \times A : 0 \le \langle \mathbf{v}, \mathbf{x} \rangle < \langle \mathbf{v}, \ell(i) \rangle\}.$$

Let

$$\mathcal{C}_\mathbf{v} = \{\pi_{\mathbf{u},\mathbf{v}}\, \mathbf{x} + \mathcal{R}_\mathbf{v}(i) : [\mathbf{x}, i] \in \Gamma(\mathbf{v})\}.$$

Since $\Gamma(\mathbf{1}) = \{[\mathbf{x}, i] \in \mathbb{Z}^d \times A : \mathbf{x} \in \mathbf{1}^\perp\}$ and $\pi_{\mathbf{u},\mathbf{v}}\, \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbf{v}^\perp$, this definition is consistent with that of $\mathcal{C}_\mathbf{1}$ above.

Theorem

Under the conditions of the main theorem, $\mathcal{C}_\mathbf{v}$ is a multiple tiling of $\mathbf{v}^\perp$ for all $\mathbf{v} \in \mathbb{R}^d_{\ge 0} \setminus \{\mathbf{0}\}$, and the covering degree of $\mathcal{C}_\mathbf{v}$ does not depend on $\mathbf{v}$. (In particular, we have aperiodic and periodic multiple tilings with same covering degree.)

Set equations

Let $\pi_n = \pi_{(M_{[0,n)})^{-1}\mathbf{u},\, \mathbf{v}^{(n)}}$ with $\mathbf{v}^{(n)} = {}^t(M_{[0,n)})\,\mathbf{v}$, and

$$\mathcal{R}_{\mathbf{v}}^{(n)}(i) = \overline{\left\{\pi_n\,\boldsymbol{\ell}\big(\omega_{[0,k)}^{(n)}\big) : k \in \mathbb{N},\, \omega_k^{(n)} = i\right\}}$$

be the subtile in $(\mathbf{v}^{(n)})^\perp = (M_{[0,n)})^{-1}\mathbf{v}^\perp$ corresponding to $\omega^{(n)}$. (Note that $(M_{[0,n)})^{-1}\mathbf{u}$ is the frequency vector of $\omega^{(n)}$.) Then

$$\begin{aligned}(M_{[0,n)})^{-1}\,\mathcal{R}_{\mathbf{v}}(i) &= \bigcup_{j\in A,\, p\in A^*:\sigma_{[0,n)}(j)\in piA^*} \big(\pi_n\,(M_{[0,n)})^{-1}\,\boldsymbol{\ell}(p) + \mathcal{R}_{\mathbf{v}}^{(n)}(j)\big) \\ &= \bigcup_{[\mathbf{y},j]\in E_1^*(\sigma_{[0,n)})(i)} \big(\pi_n\,\mathbf{y} + \mathcal{R}_{\mathbf{v}}^{(n)}(j)\big).\end{aligned}$$

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Let

$$\mathcal{C}_{\mathbf{v}}^{(n)} = \{\pi_n\,\mathbf{x} + \mathcal{R}_{\mathbf{v}}^{(n)}(i) : [\mathbf{x}, i] \in \Gamma(\mathbf{v}^{(n)})\}.$$

If $\sigma_{[0,n)}$ is unimodular, i.e., $|\det M_{[0,n)}| = 1$, then

$$\Gamma\big(\mathbf{v}^{(n)}\big) = E_1^*(\sigma_{[0,n)})\big(\Gamma(\mathbf{v})\big),$$

thus $(M_{[0,n)})^{-1}\,\mathcal{C}_{\mathbf{v}}$ is a collection of supertiles of $\mathcal{C}_{\mathbf{v}}^{(n)}$.

Convergence of $\mathcal{R}_{\mathbf{v}}^{(n_k)}$

Lemma

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a primitive and irreducible sequence of unimodular substitutions over the alphabet A with limit word ω such that

$$\exists C \in \mathbb{N} : \forall k \in \mathbb{N}\ \exists n > 0 : \sigma_{[n,n+k)} = \sigma_{[0,k)},\ \omega^{(n+k)} \text{ is } C\text{-balanced.}$$

Then $\exists\, \mathbf{v} \in \mathbb{R}^d_{\geq 0} \setminus \{\mathbf{0}\}$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{k\to\infty} \mathcal{R}_{\mathbf{v}}^{(n_k)}(i) = \mathcal{R}_{\mathbf{v}}(i) \qquad \forall i \in A$$

(w.r.t. Hausdorff metric).

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(w.r.t. Hausdorff metric). Using this equation, we show that

- *$\partial\mathcal{R}_{\mathbf{v}}(i)$ has zero measure,*
- *$\mathcal{C}_{\mathbf{v}}^{(n)}$ is a multiple tiling with covering degree independent of n,*
- *unions in the set equations are disjoint in measure,*
- *strong coincidence condition implies disjointness of the $\mathcal{R}_{\mathbf{v}}(i)$.*

Geometric coincidence condition

Theorem

Under the conditions of the main theorem, the following are equivalent:

- $\mathcal{C}_{\mathbf{v}}$ *is a tiling of* $\mathbf{v}^{\perp}$.
- $\exists n \in \mathbb{N},\ i \in A,\ [\mathbf{x}, h] \in \Gamma(\mathbf{1}^{(n)})$:

$$\left\{[\mathbf{y}, j] \in \Gamma(\mathbf{1}^{(n)}) : \|\pi_{(M_{[0,n)})^{-1}\mathbf{u},\mathbf{1}}(\mathbf{x}-\mathbf{y})\|_\infty \le 2C+1\right\} \subset E_1^*(\sigma_{[0,n)})[\mathbf{0}, i],$$

with $C \in \mathbb{N}$ *such that* $\omega^{(n)}$ *is* C*-balanced.*

Geometric coincidence condition

Theorem

Under the conditions of the main theorem, the following are equivalent:

- $\mathcal{C}_{\mathbf{v}}$ *is a tiling of* $\mathbf{v}^{\perp}$.
- $\exists n \in \mathbb{N},\ i \in A,\ [\mathbf{x}, h] \in \Gamma(\mathbf{1}^{(n)}):$
$$\big\{[\mathbf{y}, j] \in \Gamma(\mathbf{1}^{(n)}) : \|\pi_{(M_{[0,n)})^{-1}\mathbf{u},\mathbf{1}}(\mathbf{x}-\mathbf{y})\|_\infty \leq 2C+1\big\} \subset E_1^*(\sigma_{[0,n)})[\mathbf{0}, i],$$
with $C \in \mathbb{N}$ *such that* $\omega^{(n)}$ *is* C*-balanced.*
- *Strong coincidences and* $\exists n \in \mathbb{N},\ i \in A,\ [\mathbf{x}, h] \in \Gamma(\mathbf{1}^{(n)}):$
$$\big\{[\mathbf{y}, j] \in \Gamma(\mathbf{1}^{(n)}) : \|\pi_{(M_{[0,n)})^{-1}\mathbf{u},\mathbf{1}}(\mathbf{x}-\mathbf{y})\|_\infty \leq 2C+1\big\} \subset \bigcup_{i \in A} E_1^*(\sigma_{[0,n)})[\mathbf{0}, i],$$
with $C \in \mathbb{N}$ *such that* $\omega^{(n)}$ *is* C*-balanced.*

Arnoux-Rauzy words

Arnoux-Rauzy substitutions on $A = \{1, 2, \ldots, d\}$

$$\alpha_i : \; i \mapsto i, \; j \mapsto ij \text{ for } j \in A \setminus \{i\}.$$

Theorem (Avila–Delecroix, Delecroix–Hejda–St)

There is a constant $C(h)$ such that, for each directive sequence $(\alpha_{i_n})_{n\in\mathbb{N}}$ satisfying $\{i_n, \ldots, i_{n+h}\} = A$ for all $n \in \mathbb{N}$ (i.e., strong partial quotients bounded by h), the limit word ω is $C(h)$-balanced.

Let μ be an ergodic invariant probability measure for the Arnoux-Rauzy algorithm such that $\mu([w]) > 0$ for the cylinder corresponding to a word $w = w_{[0,n)} \in A^*$ with $\{w_0, \ldots, w_{n-1}\} = A$. Then, for μ-almost every $\mathbf{u}$ in the Rauzy gasket, the Arnoux-Rauzy word with frequency vector $\mathbf{u}$ is balanced.

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Theorem

(X_ω, Σ) is conjugate to an exchange of domains on $\mathcal{R}_\mathbf{v}$

- *for μ-almost every sequence $(\alpha_{i_n})_{n\in\mathbb{N}}$,*
- *for each recurrent sequence $(\alpha_{i_n})_{n\in\mathbb{N}}$ with bounded strong partial quotients ($\exists h \in \mathbb{N} : \{i_n, \ldots, i_{n+h}\} = A \; \forall n \in \mathbb{N}$),*
- *for each recurrent sequence $(\alpha_{i_n})_{n\in\mathbb{N}}$ on $A = \{1, 2, 3\}$ with bounded weak p.q. ($\exists h \in \mathbb{N} : \#\{i_n, \ldots, i_{n+h}\} > 1 \; \forall n \in \mathbb{N}$).*

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Theorem (Berthé–Jolivet–Siegel'12)

Each directive sequence $(\alpha_{i_n})_{n\in\mathbb{N}}$ of an Arnoux-Rauzy word on 3 letters satisfies the geometric coincidence condition.

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(X_ω, Σ) is conjugate to a rotation on $\mathbb{T}^2$

- *for μ-almost every sequence $(\alpha_{i_n})_{n\in\mathbb{N}}$ on $A = \{1, 2, 3\}$,*
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Brun words

Brun substitutions on $A = \{1, 2, 3\}$

$$\beta_{ij} : \; j \mapsto ij, \; k \mapsto k \text{ for } k \in A \setminus \{j\},$$

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For Lebesgue almost every frequency vector $\mathbf{u} \in \mathbb{R}^3_+$, the Brun word with frequency vector $\mathbf{u}$ is balanced.

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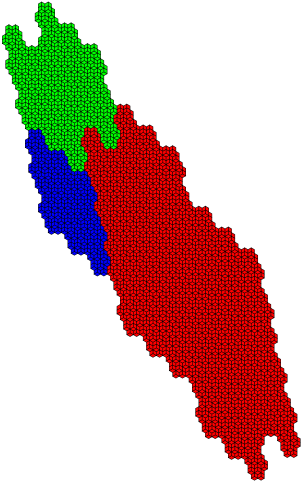
Theorem

(X_ω, Σ) is conjugate to a rotation on $\mathbb{T}^2$

- *for the Brun word corresponding to Lebesgue a.e. direction,*
- *for each recurrent sequence $(\beta_{i_n,j_n})_{n\in\mathbb{N}}$ with bounded strong partial quotients ($\exists h \in \mathbb{N} : \{i_n, \ldots, i_{n+h}\} = A \; \forall n \in \mathbb{N}$).*

An example



Arnoux-Rauzy on $A = \{1, 2, 3\}$ with
$\sigma_n = \alpha_{t_n}$, $t_0 t_1 \cdots = \lim_{k\to\infty} \varphi^k(1)$
$\varphi : 1 \mapsto 1123, 2 \mapsto 23, 3 \mapsto 123$ (Chacon)

Berthé–Cassaigne–St '13:
$\omega^{(n)}$ is 2-balanced $\forall\, n \in \mathbb{N}$

$E_1^*(\sigma_{[0,13)})[\mathbf{0}, i]$ for $i = 1$ (red), $i = 2$ (green), $i = 3$ (blue)